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Left (Right) Regular and Transposition Regular Semigroups and Their Structures

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Abstract: Regular semigroups and their structures are the most wonderful part of semigroup theory, and the contents are very rich. In order to explore more regular semigroups, this paper extends the relevant classical conclusions from a new perspective: by transforming the positions of the elements in the regularity conditions, some new regularity conditions (collectively referred to as transposition regularity) are obtained, and the concepts of various transposition regular semigroups are introduced (L1/L2/L3, R1/R2/R3-transposition regular semigroups, etc.). Their relations with completely regular semigroups and left (right) regular semigroups, proposed by Clifford and Preston, are analyzed. Their properties and structures are studied from the aspects of idempotents, local identity elements, local inverse elements, subsemigroups and so on. Their decomposition theorems are proved respectively, and some new necessary and sufficient conditions for semigroups to become completely regular semigroups are obtained.

Keywords: regular semigroups; transposition regular semigroups; decomposition theorems; completely regular semigroups

MSC: 20M10 and 20M17

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1. Introduction

Regularity was first proposed by J. V. Neumann when studying ring theory (see [1]); W. D. Munn and R. Penrose formally proposed the concepts of regular semigroups (see [2]), and then regular semigroups became an important research direction (see [3–12]). Many important subclasses of regular semigroups (for example, completely regular semigroups, inverse semigroups, orthodox semigroups, locally inverse semigroups, etc.) (see [13–22]) have been proposed one after another, and their structures have been deeply revealed. In particular, as a group union semigroup, completely regular semigroups have been deeply studied and widely used, and have become the most wonderful content of regular semigroups.

In 1961, Clifford and Preston proposed the concept of left (right) regular semigroups, which are the generalizations of regular semigroups in their monograph (see [4]). In this paper, it is proved that the necessary and sufficient condition for a semigroup to be a completely regular semigroup is that it is both a left regular semigroup and a right regular semigroup. Kiss generalized left (right) regular element of semigroups in 1972 (see [23]). Anjaneyulu proved that in a duo semigroup S, the set of all left regular elements and the set of all right regular elements coincide (see [24]). In [25], the ideals and principal ideals of left (right) regular semigroups were studied. In addition, regular semigroups have many forms of generalization, which are collectively referred to as generalized regular semigroups, such as eventually regular semigroups (or π-regular semigroups; see [26]), abundant semigroups (see [27,28]), superabundant semigroups (see [29]), and so on. The following figure shows that the relationships among existing associative structures to have a clearer understanding of the existing algebraic structure.

In Figure 1, the yellow triangle represents a regular semigroup, the red triangle represents a left regular semigroup and the black triangle represents a right regular semigroup. Their common part is the completely regular semigroup.

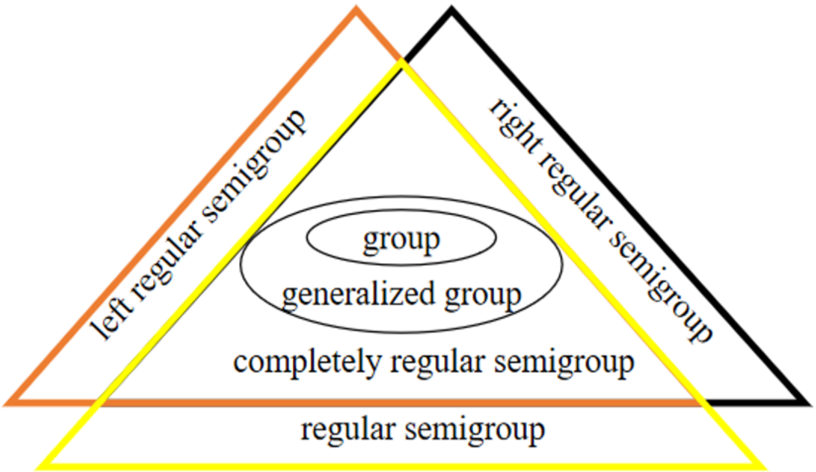

Figure 1. The relationship among various associative algebras.

Clifford and Preston proposed $xaa = a$, which is the concept of the left regular element, and $aax = a$, which is the concept of the right regular element; the structures of the regular semigroup, left (right) regular semigroup and completely regular semigroup are very clear. Firstly, we combine the left regular element and regular element, and an equation is obtained: $xaa = a = axa$. In terms of the condition, this equation is stronger than the completely regular element. However, from the definition point of view, it is uncertain whether there are completely regular elements that do not satisfy this equation. In this paper, we define this equation as a L1-transposition regular element, and L1-transposition regular semigroups are obtained: the composition theorem of the L1-transposition regular semigroup is proved, and it is equivalent to the completely regular semigroup. That is to say, there is no case that the completely regular element does not satisfy that equation. So far, a new equation expressing the completely regular elements appears. In a similar way, we obtain the R1-transposition regular semigroup and LR-transposition regular semigroup.

By exchanging the regularity equation and combining the regularity condition and uniqueness, we obtain a new equation: there exists unique x such that $axa = a$ and $xxaa = xa$. According to the condition, this equation is stronger than the completely regular element. However, it is uncertain whether there are completely regular elements that do not satisfy this equation from the definition. In this paper, we define this equation as the L2-transposition regular element, and the structure of the L2-transposition regular semigroup is characterized: It is equivalent to a group. That is, there are completely regular elements that do not satisfy this equation. We obtain a new equation representing the group. Similarly, we obtain the R2-transposition regular semigroup.

Finally, an equation is acquired by adding uniqueness on the left regular condition: there exists unique x such that $xaa = a$. Clearly, it is stronger than the left regular condition. The completely regular condition is stronger than left regular condition. However, it is not clear whether this equation is stronger than the completely regular condition. So we define this equation as an L3-transposition regular element. According to the structure theorem of the L3-transposition regular semigroup, the L3-transposition regular semigroup is stronger than the completely semigroup and generalized group. So far, a subclass of completely regular semigroups is obtained. In the same measure, the R3-transposition regular semigroup is acquired. The conditions satisfied by each transposition regular semigroup are shown in Table 1.

Table 1. These are various transposition regularities.

Transposition Regularity	$axa = a$	$xaa = a$	$aax = a$	$(x(xa))a = xa$	$a((ax)x) = ax$	Uniqueness
L1	√	√				
R1	√		√			
LR		√	√			
L2	√			√		√
R2	√				√	√
L3		√				√
R3			√			√

2. Preliminaries

Firstly, we introduce Green's equivalences of a semigroup.

If a is an element of semigroup S, the smallest left ideal of S containing a is $Sa \cup \{a\}$, denoted by S^1a. We shall call it the principal left ideal generated by a. An equivalence $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$ if, and only if, a and b generate the same principal left ideal, that is, if and only if $S^1a = S^1b$.

Similarly, we define the equivalence $\mathcal{R}$ by the rule that $a\mathcal{R}b$ if, and only if, $aS^1 = bS^1$.

An alternative characterization, making the "mutual divisibility" aspect of these equivalences more apparent, is given in the following proposition:

Proposition 1 ([3]). *Let a and b be elements of a semigroup S. Then, $a\mathcal{L}b$ if, and only if, there exists x and y in S^1 such that $xa = b$ and $yb = a$. Additionally, $a\mathcal{R}b$ if, and only if, there exists u and v in S^1 such that $au = b$ and $bv = a$.*

Since the intersection of $\mathcal{L}$ and $\mathcal{R}$ is of great importance in the development of the theory, we reserve for it the letter $\mathcal{H}$.

We now define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

We refer to S^1 as the monoid obtained from S by adjointing an identity if necessary.

Definition 1 ([1]). *Element a of a semigroup S is said to be regular if there exists x in S such that $axa = a$. The semigroup S is said to be regular if all its elements are regular.*

Definition 2 ([3]). *A semigroup S is said to be completely regular if there exists a unary operation $a \mapsto a^{-1}$ on S with the properties*

$$(a^{-1})^{-1} = a, aa^{-1}a = a, aa^{-1} = a^{-1}a.$$

Theorem 1 ([3]). *Let S be a semigroup. Then, the following statements are equivalent:*

(1) S is completely regular;
(2) Every element of S lies in a subgroup of S;
(3) Every $\mathcal{H}$-class in S is a group.

Definition 3 ([3]). *A Clifford semigroup is defined as a completely regular semigroup $(S, \mu, ^{-1})$ in which, for all x,y in S,*

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

In an arbitrary semigroup S, let us say that an element c is central if $sc = cs$ for every s in S. The set of central elements forms a subsemigroup of S, which is said to be the center of S.

Definition 4 ([30]). *A generalized group $(G, *)$ is a non-empty set admitting a binary operation $*$ said to be multiplication subject to the set of rules given below:*

(1) $(x * y) * z = x * (y * z)$ *for all* $x,y,z \in G$*;*
(2) *For each* $x \in G$*, there exists a unique* $e(x) \in G$ *such that* $x * e(x) = e(x) * x = x$*;*
(3) *For each* $x \in G$*, there exists* $x^{-1} \in G$ *such that* $x * x^{-1} = x^{-1} * x = e(x)$*.*

Theorem 2 ([30]). *For each element x in a generalized group $(G, *)$, there exists a unique $x^{-1} \in G$.*

Theorem 3 ([30]). *Let $(G, *)$ be a generalized group. If $x * y$=$y * x$ for all $x, y \in G$, then G is a group.*

Definition 5 ([1]). *An element a of a semigroup S is said to be left regular if there exists x in S such that $xaa = a$. The semigroup S is said to be left regular if all its elements are left regular.*

Definition 6 ([1]). *An element a of a semigroup S is said to be right regular if there exists x in S such that $aax = a$. The semigroup S is said to be right regular if all its elements are right regular.*

Theorem 4 ([25]). *Let S be left(right) regular semigroup. Then the following conditions are equivalent:*

(1) *S is completely regular semigroup;*
(2) *S is regular semigroup;*
(3) *S is π-regular semigroup;*
(4) *S is completely π-regular semigroup;*
(5) *S is right(left) regular semigroup.*

3. L1-Transposition Regular Semigroup and R1-Transposition Regular Semigroup

Definition 7. *Let G be a groupoid, $a \in G$.*

(1) *If there exists $e \in G$ such that $ea = a(ae = a)$, e is said to be a local left (right) identity element of a. e is said to be a local identity element if e is both a local left identity element and local right identity element.*
(2) *Let e be a local left identity element/right identity element/identity element of a. If there exists b such that $ba = e(ab = e)$, b is said to be a local left(right) inverse element of a relative to e. b is a local inverse element of a relative to e if b is both a local left inverse element of a relative to e and local right inverse element of a relative to e.*

Definition 8. *Let G be a semigroup, $a \in G$. a is a L1-transposition regular element of G if $\exists p \in G$ s.t. $paa = a = apa$. The semigroup G is said to be L1-transposition regular if all its elements are L1-transposition regular.*

Remark 1. *The L1-transposition regular semigroup is both a left regular semigroup and right regular semigroup. According to Theorem 4, the L1-transposition regular semigroup is a completely regular semigroup.*

Definition 9. *Let G be a semigroup, $a \in G$. a is a strong L1-transposition regular element of G if a is a L1-transposition regular element and $\exists x \in G$, s.t. $ax = pa$. a is a stubborn L1-transposition regular element of G if a is a L1-transposition regular element and $\forall x \in G$, $ax \neq pa$.*

Example 1 shows that not every L1-transposition regular element is a strong L1-transposition regular element.

Example 1. *Let $X = N^*$ (N^* represents positive integer set) and G be the set of all mappings of X; the operation on G is the composition operation of mappings. Clearly, the identity mapping is the identity of G. Let*

$$f : X \to X$$

$$x \mapsto x^2, \forall x \in X.$$

Then $f \in G$. Let

$$g : X \to X$$
$$x \mapsto [\sqrt{x}], \forall x \in X.$$

where [] is the rounding function, then g is the surjection, and $g \in G$. $\forall x \in X$, there is

$$(gf)(x) = g(f(x)) = [\sqrt{x^2}] = x$$

which is the identity mapping, and $gff = f = fgf$; then, f is a L1-transposition regular element. However, $\forall x \in X$, there does not exist a mapping $h(x) \in G$ such that $(fh)(x) = (gf)(x)$, that is, f is not a strong L1-transposition regular element.

Proposition 2. *Let G be a L1-transposition regular semigroup. $\forall a \in G, \exists p \in G$ s.t. $paa = a = apa$. Let $e = pa$. Then $\forall n \in N^*$,*

(1) $ea^n = a^n e = a^n$, $p^n a^n = e$;
(2) e is idempotent.

Proof.

(1) According to $paa = a = apa$,$e = pa$, we know $ea = a = ae$. So

$$ea^n = (ea)a^{n-1} = aa^{n-1} = a^n, a^n e = a^{n-1}(ae) = a^{n-1}a = a^n.$$

$$p^n a^n = p^{n-1}(paa)a^{n-2} = p^{n-1}aa^{n-2} = p^{n-1}a^{n-1} = ... = p^2a^2 = ppaa = pa = e.$$

(2) According to associative law,

$$ee = (pa)e = p(ae) = pa = e.$$

So e is idempotent. □

Proposition 3. *Let G be a L1-transposition regular semigroup, then e is an idempotent of G. Let*

$$Ge=\{a \in G | ea = a = ae, \text{ and } \exists p \in G \text{ s.t. } e = pa\}$$

Then

(1) Ge is a submonoid of G;

(2) If Ge has a finite number of elements. Then Ge is a subgroup of G.

Proof.

(1) Clearly, $e \in Ge$, that is, Ge is non-empty.
Let $a, b \in Ge$. Then $ea = a = ae$, $eb = b = be$. And $\exists p, q \in G$, s.t. $pa = e$, $qb = e$. Then

$$(ab)e = a(be) = ab, e(ab) = (ea)b = ab;$$

$$(qp)(ab) = q(pa)b = qeb = q(eb) = qb = e.$$

That means $ab \in Ge$, then Ge is a submonoid of G.

(2) Let G have a finite number of elements. If G has an element, then G = {e} is a singleton group.

If $|G| > 1$, for any $a \in Ge$, and $a \neq e$, then according to (1), $a, aa, aaa, ...a^n \in Ge$ ($\forall n \in N^*$). Because Ge has a finite number of elements, then there must exist $n, k \in N^*$ such that $a^n = a^{n+k}$. According to the definition of Ge, $\exists x \in G$ s.t. $xa^n = e$. So

$$e = xa^n = xa^{n+k} = (xa^n)a^k = ea^k = a^k.$$

Because $a \neq e$, then $k \neq 1$. So, $e = a^k = aa^{k-1}$. This means that for any $a \in Ge$, $a \neq e$, $a = ae$ and there is a right inverse element of a. So Ge is a group. $\square$

Proposition 4. *S is a semigroup, and a is a L1-transposition regular element in S. Let*

$$P(a) = \{p | paa = a = apa, p \in S\}$$

Then the following conditions are equivalent:

(1) *There is an idempotent element in $P(a)$;*
(2) *$\exists p \in P(a)$ such that $pa = a$;*
(3) *a is idempotent;*
(4) *$P(a)$ is a subsemigroup of S.*

Proof. (1)⇒(2) Let $p \in P(a)$ and $p^2 = p$. Then $paa = a$. Multiply both ends left by p, and there is $ppaa = pa$. Because p is idempotent, $ppaa = paa = pa$, that is, $pa = a$.

(2)⇒(3) Let $p \in P(a)$ and $pa = a$, then $paa = a$, that is, $(pa)a = aa = a$. So $a^2 = a$.

(3)⇒(4) $\forall p, q \in P(a)$, then

$$(pq)aa = p(qaa) = pa = paa = a,$$

$$a(pq)a = apqaa = ap(qaa) = apa = a.$$

Then $pq \in P(a)$. That is, $P(a)$ is a subsemigroup of S.

(4)⇒(1) $\forall p, q \in P(a)$, then $pq \in P(a)$, so $a = pqaa = p(qaa) = pa.aa = (pa)a = a$. Then a is idempotent, that is, there is an idempotent element in $P(a)$. $\square$

Theorem 5. *Let G be a L1-transposition regular semigroup. Define the binary operation $\approx$ on G as follows:*

$$a \approx b \Leftrightarrow e_a = e_b, \forall a, b \in G,$$

where e_a is a local identity element of a. Then we have the following:

(1) *The binary operation $\approx$ on G is the equivalence relation, and we denote the equivalence class contained x by $[x]_\approx$;*
(2) *$\forall x \in G$, $[x]_\approx$ is a subgroup;*
(3) *$G = \bigcup_{x \in G} [x]_\approx$, that is, every L1-transposition regular semigroup is the disjoint union of subgroups;*
(4) *G is a completely regular semigroup.*

Proof.

(1) Clearly, $\forall x \in G$, $e_x = e_x$. That is, $x \approx x$.

Assume that $x \approx y$, then $e_x = e_y$, and $e_y = e_x$. So $y \approx x$.

If $x \approx y$ and $y \approx z$, then $e_x = e_y$, and $e_x = e_y$. Clearly, $e_x = e_z$. That is, $x \approx z$. So $\approx$ is an equivalence relation on G.

(2) $\forall a, b \in [x]_\approx$, assume that $e_a = e_b = e$. Assume that there exist $p, q \in G$ such that $paa = a = apa$, $qbb = b = bqb$. Then, $ea = a = ae$, $pa = e$, $eb = b = be$, and $qb = e$. So

$$e(ab) = (ea)b = ab, (ab)e = a(be) = ab.$$

That is, $ab \in [x]_\approx$.

Because $a \in [x]_\approx$, and $ee = e$. Then $e \in [x]_\approx$.

Because

$$(epe)e = ep(ee) = epe, e(epe) = (ee)pe = epe.$$

Then $epe \in [x]_\approx$.

$(epe)a = ep(ea) = epa = e(pa) = ee = e$, that is, epe is a local left inverse element of a relative to e. According to the definition of group, $[x]_{\approx}$ is a subgroup of G.

(3) According to (2), epe is a local right inverse element of a relative to e, that is, $a(epe) = e$. Let $q = epe$, then $ea = a = ae, aq = e = qa$. Assume that the local identity element of a is not unique, and there exist $f, m \in G$ such that $fa = a = af, ma = f = am$. Then

$$ef = (qa)f = q(af) = qa = e, ef = e(am) = (ea)m = am = f.$$

That is, $e = f$. That is to say, the local identity element of a is unique. So $\bigcap_{x \in G} [x]_{\approx} = \varnothing$, and $G = \bigcup_{x \in G} [x]_{\approx}$. That is, every L1-transposition regular semigroup is the disjoint union of subgroups.

(4) According to (3), G is the disjoint union of groups. According to Theorem 1, G is the completely regular semigroup.

□

Definition 10. *Let G be a semigroup, $\exists a \in G$. a is a R1-transposition regular element of G if $\exists p \in G$ s.t. $apa = a = aap$. The semigroup G is said to be R1-transposition regular if all its elements are R1-transposition regular.*

Remark 2. *The R1-transposition regular semigroup is both a right regular semigroup and regular semigroup. According to Theorem 4, the R1-transposition regular semigroup is a completely regular semigroup.*

Definition 11. *Let G be a semigroup, $a \in G$. a is a strong R1-transposition regular element of G if a is a R1-transposition regular element and $\exists x \in G$, s.t. $xa = ap$. a is a stubborn R1-transposition regular element of G if a is a R1-transposition regular element and $\forall x \in G$, $xa \neq ap$.*

Example 2 shows not every R1-transposition regular element is a strong R1-transposition regular element.

Example 2. *In Example 1,*

$$(gf)(x) = g(f(x)) = [\sqrt{x^2}] = x$$

is the identity mapping, and $ggf = g = gfg$, then g is a R1-transposition regular element. However, $\forall x \in X$, there does not exist a mapping $l(x) \in G$ such that $(lg)(x) = (gf)(x)$, that is, g is not a strong R1-transposition regular element.

Proposition 5. *Let G be a R1-transposition regular semigroup. $\forall a \in G, \exists p \in G$ s.t. $apa = a = aap$. Let $e = pa$. Then $\forall n \in N^*$,*

(1) $ea^n = a^n e = a^n$, $a^n p^n = e$;

(2) e is idempotent.

Proof.

(1) According to $apa = a = aap$, $e = pa$, we know $ea = a = ae$. So

$$ea^n = (ea)a^{n-1} = aa^{n-1} = a^n, a^n e = a^{n-1}(ae) = a^{n-1}a = a^n.$$

$$a^n p^n = a^{n-2}(aap)p^{n-1} = a^{n-2}ap^{n-1} = a^{n-1}p^{n-1} = ... = a^2p^2 = aapp = ap = e.$$

(2) According to the associative law,

$$ee = e(ap) = (ea)p = ap = e$$

So e is idempotent. □

Theorem 6. *Let G be a R1-transposition regular semigroup. Define the binary operation $\approx$ on G as follows:*

$$a \approx b \Leftrightarrow e_a = e_b, \forall a, b \in G,$$

where e_a is the local identity element of a. Then we have the following:

(1) The binary operation $\approx$ on G is equivalence relation, and we denote the equivalence class contained x by $[x]_{\approx}$;
(2) $\forall x \in G$, $[x]_{\approx}$ is a subgroup;
(3) $G = \bigcup\limits_{x \in G} [x]_{\approx}$, that is, every R1-transposition regular semigroup is the disjoint union of subgroups;
(4) G is a completely regular semigroup.

Proof.

(1) Clearly, $\forall x \in G$, $e_x = e_x$. That is, $x \approx x$.

Assume that $x \approx y$, then $e_x = e_y$, and $e_y = e_x$. So $y \approx x$.

If $x \approx y$ and $y \approx z$, then $e_x = e_y$, and $e_x = e_y$. Clearly, $e_x = e_z$. That is, $x \approx z$. So $\approx$ is a equivalence relation on G.

(2) $\forall a, b \in [x]_{\approx}$, assume that $e_a = e_b = e$. Assume that there exist $p, q \in G$ such that $apa = a = aap, bqb = b = bbq$. Then $ea = a = ae$, $ap = e$, $eb = b = be$, and $bq = e$. So

$$e(ab) = (ea)b = ab, (ab)e = a(be) = ab.$$

That is, $ab \in [x]_{\approx}$.

Because $\forall a \in [x]_{\approx}$, and $ee = e(ap) = (ea)p = ap = e$. Then $e \in [x]_{\approx}$.

Because

$$(epe)e = ep(ee) = epe, e(epe) = (ee)pe = epe.$$

Then $epe \in [x]_{\approx}$.

$a(epe) = (ae)pe = ape = (ap)e = ee = e$, that is, epe is the local right inverse element of a relative to e. According to the definition of the group, $[x]_{\approx}$ is the subgroup of G.

(3) According to (2), epe is the local left inverse element of a relative to e, that is, $(epe)a = e$. Let $q = epe$, then $ea = a = ae, aq = e = qa$. Assume that the local identity element of a is not unique, and there exist $f, m \in G$ such that $fa = a = af, ma = f = am$. Then

$$ef = (qa)f = q(af) = qa = e, ef = e(am) = (ea)m = am = f.$$

That is, $e = f$. That is to say, the local identity element of a is unique. So $\bigcap\limits_{x \in G} [x]_{\approx} = \varnothing$, and $G = \bigcup\limits_{x \in G} [x]_{\approx}$. That is, every R1-transposition regular semigroup is the disjoint union of subgroups.

(4) According to (3), G is the disjoint union of groups. According to Theorem 1, G is a completely regular semigroup.

□

4. LR-Transposition Regular Semigroup and Completely Regular Semigroup

Definition 12. *Let G be a semigroup, $\exists a \in G$. a is a LR-transposition regular element of G if $\exists p, q \in G$ s.t. $paa = a = aaq$. The semigroup G is said to be LR-transposition regular if all its elements are LR-transposition regular.*

Remark 3. *The LR-transposition regular semigroup is both a left regular semigroup and right regular semigroup. According to Theorem 4, the LR-transposition regular semigroup is a completely regular semigroup.*

Proposition 6. *Let G be a LR-transposition regular semigroup. For any element a in G, $\exists p, q \in G$ such that $paa = a = aaq$. Then $pa = aq$.*

Proof. Let G be a LR-transposition regular semigroup. For any element a in G, $pa = p(aaq) = (paa)q = aq$. □

Theorem 7. *Let G be a LR-transposition regular semigroup. For any a in G, we have the following:*

(1) The local identity element of a is idempotent;
(2) The local identity element of a is unique.

Proof.

(1) Let G be a LR-transposition regular semigroup. For any a in G, $\exists p, q \in G$, s.t. $paa = a = aaq$. According to Proposition 6, $pa = aq$. Let $pa = e = aq$, there is

$$ee = (pa)e = p(ae) = pa = e.$$

That is to say, the local identity element of a is idempotent.

(2) Assume that local identity element of a is not unique, and there exist $e, p, q, f, m, n \in G$ such that $ea = a = ae, pa = e = aq.fa = a = af, ma = f = an$. Additionally,

$$fe = (ma)e = m(ae) = ma = f, fe = f(aq) = (fa)q = aq = e.$$

That is, $e = f$. That is to say, the local identity element of a is unique. □

Theorem 8. *Let S be a semigroup. Then the following conditions are equivalent:*

(1) a is a strong L1-transposition regular element, $a \in S$;
(2) a is a strong R1-transposition regular element, $a \in S$;
(3) a is a LR-transposition regular element, $a \in S$.

Proof. (1)⇒(2) Let S be semigroup, $a \in S$. Assume that a is a strong L1-transposition regular element, then $\exists p \in S$ s.t. $paa = a = apa$, and $\exists x \in S$ s.t. $ax = pa$. That is, $axa = a = aax$. According to Definitions 10 and 11, a is a strong R1-transposition regular element.

(2)⇒(3) Let S be semigroup, $a \in S$. Assume that a is a strong R1-transposition regular element, then $\exists p \in S$ s.t. $apa = a = aap$, and $\exists x \in S$ s.t. $xa = ap$. That is, $xaa = (ap)a = a$. That is, $xaa = a = aap$. According to Definition 12, a is a LR-transposition regular element.

(3)⇒(1) Let S be semigroup, $a \in S$. Assume that a is a LR-transposition regular element, then $\exists p, q \in S$ s.t. $paa = a = aaq$, and $pa = p(aaq) = (paa)q = aq$. Then $(pa)a = a = a(pa)$, and $pa = aq$. According to Definition 8 and Definition 9, a is a strong L1-transposition regular element. □

According to Theorem 4, the L1-transposition regular semigroup, R1-transposition regular semigroup, LR-transposition regular semigroup and completely regular semigroup are equivalent to one another. The following theorem starts with the elements and proves their equivalence.

Theorem 9. *Let S be a semigroup. Then the following conditions are equivalent:*

(1) S is a L1-transposition regular semigroup;
(2) S is a R1-transposition regular semigroup;
(3) S is a LR-transposition regular semigroup;
(4) S is a completely regular semigroup.

Proof. (1)⇒(2) Let S be a L1-transposition regular semigroup. According to Theorem 5, for any a in S, $\exists p, e \in S$ s.t. $ea = a = ae, (epe)a = e = a(epe)$. Let $q = epe$, then

$qa = e = aq$, that is to say, $aqa = a = aaq$. According to Definition 10, S is a R1-transposition regular semigroup.

(2)⇒(3) Let S be a R1-transposition regular semigroup. According to Theorem 6, for any a in S, $\exists p, e \in S$ s.t. $ea = a = ae$, $(epe)a = e = a(epe)$. Let $q = epe$, then $qa = e = aq$ and $qaa = a = aaq$. According to Definition 12, S is a LR-transposition regular semigroup.

(3)⇒(4) Let S be a LR-transposition regular semigroup. For any a in S, $\exists p, q \in S$ such that $paa = a = aaq$, then $pa = p(aaq) = (paa)q = aq$.

Let $a^{-1} = paq$,

$$aa^{-1}a = a(paq)a = a(aqp)a = apa = aaq = a,$$

$$a^{-1}aa^{-1} = (paq)a(paq) = p(aq)apaq = ppaapaq = papaq = paaqq = paq = a^{-1},$$
$$aa^{-1} = a(paq) = apaq = paaq = aq, a^{-1}a = (paq)a = ppaa = pa.$$

Because $pa = aq$, then $aa^{-1} = a^{-1}a$. According to Definition 2, S is a completely regular semigroup.

(4)⇒(1) Let S be a completely regular semigroup. For any a in S, $\exists a^{-1} \in S$ s.t. $aa^{-1}a = a$, $(a^{-1})^{-1} = a$ and $aa^{-1} = a^{-1}a$. Let $p = a^{-1}$. Then

$$paa = a^{-1}aa = aa^{-1}a = a,$$

$$apa = aa^{-1}a = a.$$

That is, $paa = a = apa$. According to Definition 8, S is a L1-transposition regular semigroup. □

According to Theorems 8 and 9, L1, strong L1, R1, strong R1, and LR-transposition regular semigroups are equivalent to completely regular semigroups. However, not every L1(R1)-transposition regular element is a strong L1(R1)-transposition regular element; see Examples 1 and 2.

According to Definition 4, the generalized group is the L1/R1/LR-transposition regular semigroup. However, not every L1/R1/LR-transposition regular semigroup is the generalized group; see Example 3.

Example 3. *Let $G = \{a, b, c, d\}$. The operation on G is shown in Table 2. Clearly, G is the L1/R1/LR-transposition regular semigroup since $a * a = a$, $b * b = b$, $c * c = c$, and $d * d = d$.*

Table 2. This is an L1/R1/LR-transposition regular semigroup.

$*$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

However, G is not the generalized group since $c * d = d = d * c$, $d * d = d$ and $c \neq d$.

Proposition 7. *Let a be a L1(R1/LR)-transposition regular element of semigroup S. Then H_a is a subgroup of S.*

Proof. Let S be a semigroup and a be a L1-transposition regular element of S. Then there exist $p \in S$ s.t. $paa = a = apa$. $\forall b \in H_a$, there exists $x, y, u, v \in S^1$ s.t. $ua = b, vb = a, ax = b$, $by = a$. Thus,

$$b = ua = uapa = bpa,$$

$$b = ax = paax = pab.$$

$\forall b \in H_a$, pa is an identity element of b. Then

$$b = ua = uapa = bpa = bp(vb) = b(pv)b,$$

$$b = ax = paax = pab = p(vb)b = (pv)bb.$$

So b is a L1-transposition regular semigroup. Because b is arbitrary, every element of H_a is a L1-transposition regular element. That is to say, H_a is a L1-transposition regular semigroup. According to the above, there exists identity element pa of H_a, and $pa = p(vb) = (pv)b$. That is, $\forall b \in H_a$, there exists left inverse element pv s.t. $(pv)b = pa$. According to the definition of group, H_a is a subgroup of S. In a similar way, if a is a R1/LR-transposition regular element, the same conclusions are obtained. $\square$

5. L2-Transposition Regular Semigroup and R2-Transposition Regular Semigroup

Definition 13. *Let G be a semigroup, $a \in G$. a is a L2-transposition regular element of G if there exists a unique $x \in G$ s.t.*

$$axa = a \text{ and } xxaa = xa.$$

The semigroup G is said to be L2-transposition regular if all its elements are L2-transposition regular. a is a R2-transposition regular element of G if there exists a unique $x \in G$ s.t.

$$axa = a \text{ and } aaxx = ax.$$

The semigroup G is said to be a R2-transposition regular if all its elements are R2-transposition regular.

Proposition 8. *Let G be a L2-transposition regular semigroup. $\forall a \in G$, there exists a unique $x \in G$ s.t. $axa = a$ and $xxaa = xa$. Let $xa = e$. Then $xe = x = ex$.*

Proof. According to $xa = e$, $ae = e$, $xxaa = x(xa)a = xea = xa = e$. So

$$ee = (xa)e = x(ae) = xa = e.$$

That is, the local right identity element e of a is idempotent.

Additionally,

$$a(xe)a = a(xea) = axa = a, \text{ and } (xe)(xe)aa = (xe)(xea)a = (xe)ea = x(ee)a = (xe)a.$$

Then $a(xe)a = a$, and $(xe)(xe)aa = (xe)a$. Then $xe = x$ since x is unique.

$$a(ex)a = (ae)xa = axa = a, \text{ and } (ex)(ex)aa = e(xe)xaa = e(xxaa) = e(xa) = ee = e.$$

Then $a(ex)a = a$, $(ex)(ex)aa = (ex)a$. Then $ex = x$ since x is unique. That is, $ex = x = xe$. $\square$

Proposition 9. *Let G be a R2-transposition regular semigroup. $\forall a \in G$, there exists a unique $x \in G$ s.t. $axa = a$ and $aaxx = ax$. Let $ax = e$. Then $xe = x = ex$.*

The proof is similar to Proposition 8.

Theorem 10. *Let G be a semigroup. It is a L2-transposition regular semigroup or R2-transposition regular semigroup if, and only if, it is a group.*

Proof. ($\Rightarrow$) Let G be a L2-transposition regular semigroup. $\forall a \in G$, there exists a unique $x \in G$ s.t. $axa = a$ and $xxaa = xa$. Let $xa = e$. Then $ae = a, xea = xa = e$.

Let G_e be a subset of all elements of G whose local right identity element is e.

Let $a, b \in G_e$, then $ae = a, be = b$. Then

$$(ab)e = a(be) = ab.$$

That is, $ab \in G_e$.

Clearly, G_e satisfies the associative law, that is, G_e is a subgroup of G.

Because

$$ee = (xa)e = x(ae) = xa = e,$$

$e \in G_e$.

Let $a \in G_e$, there exists a unique $x \in G$ such that $axa = a$ and $xxaa = xa$. According to the proposition, $xe = x$. So $x \in G_e$.

Because x is unique, $xa = e$ is unique. $\forall a \in G_e$, $\exists a'$, $a'' \in G_e$, s.t. $ae = a$, $a'a = e$. $a'e = a'$, $a''a' = e$. $a''e = a''$. Then

$$ea = a''a'a = a''(a'a) = a''e = a''.$$

Then

$$a'(aa') = (a'a)a' = ea' = a''a'a' = eaa'a',$$

So $ea' = eaa'a'$.

Multiply both ends right by a,

$$\text{left}{=}ea'a = e(a'a) = ee = e.$$

$$\text{right}{=}eaa'a'a = eaa'e = ea(a'e) = eaa'.$$

Then, because e is unique, $aa' = e$.

At the same time,

$$ea = aa'a = a(a'a) = ae = a.$$

So in G_e, $ea = a = ae$, $a'a = e = aa'$. Then G_e is a subgroup of G. Because the identity element is unique, then $\forall i, j \in I$, $G_i \bigcap\limits_{i \neq j} G_j = \varnothing$, where I is the index set. Then G is the disjoint union of groups, according to Theorem 1, and G is a completely regular semigroup.

Because G is a L2-transposition regular semigroup, $G = \bigcup\limits_{n \in I} G_n$, and $\forall i, j \in I$, $G_i \bigcap\limits_{i \neq j} G_j = \varnothing$, where G_n is a subgroup of G, I is the index set. $\forall m, n \in I$ and $m \neq n$, G_m, G_n is a subgroup of G, respectively, e_m, e_n is the identity element of G_m, G_n respectively, then $e_m e_m = e_m, e_n e_n = e_n$. Assume that $e_m e_n = p$, where $p \in G_e$, e is an identity element of G_e, and p^{-1} is an inverse element of p relative to identity element e. Then,

$$ee = e, pe = p = ep, pp^{-1} = e = p^{-1}p, p^{-1}e = p^{-1} = ep^{-1}.$$

So $p = e_m e_n = e_m e_m e_n = e_m p, e_m e = e_m(pp^{-1}) = (e_m p)p^{-1} = pp^{-1} = e$.

Additionally, $p = e_m e_n = e_m e_n e_n = pe_n, ee_n = (p^{-1}p)e_n = p^{-1}(pe_n) = p^{-1}p = e$.

Then $ee_m e = e(e_m e) = ee = e, e_m e_m ee = e_m e$. and $ee_n e = (ee_n)e = ee = e, e_n e_n ee = e_n e$ because $eee = ee = e$, and $eeee = ee$. According to the definition of the L2-transposition regular semigroup, $e = e_m = e_n$. Then identity elements of all groups in G are equal, that is to say, there is only an identity element in G. For any elements in G, there exists a unique inverse element. So G is a group.

In a similar way, if G is a R2-transposition regular semigroup, then G is a group.

($\Leftarrow$) Let G be a group, and e be an identity of G. $\forall a \in G$, there exists a unique $a^{-1} \in G$ s.t. $aa^{-1}a = ea = a, a^{-1}a^{-1}aa = a^{-1}(a^{-1}a)a = (a^{-1}e)a = a^{-1}a$. Then G is a L2-transposition regular semigroup. In a similar way, G is a R2-transposition regular semigroup. □

6. L3-Transposition Regular Semigroup and R3-Transposition Regular Semigroup

Definition 14. *Let G be a semigroup, $a \in G$. a is said to be a L3-transposition regular element of G if there exists a unique $x \in G$ s.t.*

$$xaa = a.$$

The semigroup G is said to be L3-transposition regular if all its elements are L3-transposition regular.

Definition 15. *a is said to be a R3-transposition regular element of G if there exists a unique $x \in G$ s.t.*

$$aax = a.$$

The semigroup G is said to be R3-transposition regular if all its elements are R3-transposition regular.

Proposition 10. *Let G be a semigroup, $a \in G$. Additionally,*

$$xaa = a, x \in G.$$

Then for any positive integer m, there is $x^m a^m = xa$.

Proof. Because $xaa = a$, $x^2a^2 = (xx)(aa) = x(xaa) = xa$. Assume that $x^m a^m = xa$ and $m > 2$, then

$$x^{m+1}a^{m+1} = (x^m x)(aa^m) = (x^m x)(aaa^{m-1}) = x^m(xaa)a^{m-1} = x^m aa^{m-1} = x^m a^m = xa.$$

According to the mathematical induction, for any positive integer, $x^m a^m = xa$ hold. □

Proposition 11. *Let G be a L3-transposition regular semigroup. $\forall a \in G$, there exists a unique $x \in G$ s.t. $axa = a$ and $xaa = a$. Let $xa = e$. Then $xe = x = ex$.*

Proof. According to $xa = e$, $ea = a$, and $(xe)aa = x(ea)a = xaa = a$. Then, $xe = x$ since x is unique.

Then $(ex)aa = e(xaa) = ea = a$. Then $ex = x$ since x is unique. That is, $ex = x = xe$. □

Proposition 12. *Let G be a R3-transposition regular semigroup. $\forall a \in G$, there exists a unique $x \in G$ s.t. $aax = a$. Let $ax = e$. Then $xe = x = ex$.*

The proof is similar to Proposition 11.

Theorem 11. *Let G be a semigroup. If G is a L3-transposition regular semigroup, then it is a generalized group.*

Proof. Let G be a L3-transposition regular semigroup. Then $\forall a \in G$, there exists $x \in G$ s.t. $xaa = a$.

Let G_e be a subset of all elements of G whose local left identity element is e.

(1) Let $a, b \in G_e$. Then $ea = a$, $eb = b$, and $e(ab) = (ea)b = ab$. So $ab \in G_e$.
(2) Because G_e satisfies the associative law, G_e is a subsemigroup of G.
(3) $ee = e(xa) = (ex)a = xa = e$. Then $e \in G_e$.
(4) According to Proposition 11, $ex = x$, that is, $x \in G_e$, and $xa = e$.

Then G_e is a subgroup of G.

Because x is unique, then $xa = e$ is unique. So G is a union of a disjoint group, according to Theorem 1, G is a completely regular semigroup. Because the local left inverse element of a is unique, the local inverse element of a is unique, then G is a generalized group. □

Example 4 shows that not every generalized group is a L3-transposition regular semigroup.

Example 4. *Let $G = \{a, b, c, d, e\}$. The operation on G is shown in Table 3. Clearly, $(G, *)$ is a generalized group.*

Table 3. This is a generalized group.

$*$	a	b	c	d	e
a	a	b	c	d	e
b	a	b	c	d	e
c	a	b	c	d	e
d	a	b	c	d	e
e	a	b	c	d	e

However, it is not a L3-transposition regular semigroup since $a * a * a = a * a = a$, $b * a * a = a * a = a$, *and* $a \neq b$.

Theorem 12. *Let G be a semigroup. If G is a R3-transposition regular semigroup, then it is a generalized group.*

Proof. Let G be a R3-transposition regular semigroup. Then $\forall a \in G$, there exists $x \in G$ s.t. $aax = a$.

Let G_e be a subset of all elements of G whose local right identity element is e.

(1) Let $a, b \in G_e$. Then $ae = a$, $be = b$, and $(ab)e = a(be) = ab$. So $ab \in G_e$.
(2) Because G_e satisfies the associative law, G_e is a subsemigroup of G.
(3) $ee = (ax)e = a(xe) = ax = e$. Then $e \in G_e$.
(4) According to Proposition 12, $xe = x$, that is, $x \in G_e$, and $ax = e$.

Then G_e is a subgroup of G.

Because x is unique, then $ax = e$ is unique. So G is a union of the disjoint group, according to Theorem 1, G is a completely regular semigroup. Because the local left inverse element of a is unique, the local inverse element of a is unique, then G is a generalized group. □

Example 5 shows that not every generalized group is a R3-transposition regular semigroup.

Example 5. *Let $G = \{a, b, c, d, e\}$. The operation on G is shown in Table 4. Clearly, $(G, *)$ is a generalized group.*

Table 4. This is a generalized group.

$*$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

However, it is not a R3-transposition regular semigroup since $a * a * a = a * a = a$, $a * a * b = a * b = a$, *and* $a \neq b$.

Theorem 13. *Let G be a semigroup. G is both a L3-transposition regular semigroup and R3-transposition regular semigroup if, and only if, it is a group.*

Proof. Let G be a L3-transposition regular semigroup and R3-transposition regular semigroup, $a \in G$. There exist unique $x, y \in G$ s.t. $xaa = a = aay$. According to Theorems 11 and 12, $xa = ax$ and $ay = ya$. There is

$$xayaa = xa(ya)a = xa(ay)a = x(aay)a = xaa = a,$$

$$aaxay = a(ax)ay = a(xa)ay = a(xaa)y = aay = a,$$

So $xay = x = y$. That is, there exists unique $x \in G$ s.t. $xaa = a = aax$. Because identity element is unique, $\forall i, j \in I$, $G_i \bigcap\limits_{i \neq j} G_j = \varnothing$, where I is the index set. Then G is the union of the disjoint group, according to Theorem 1, and G is a completely regular semigroup.

Because G is a L3-transposition regular semigroup and R3-transposition regular semigroup, let $G = \bigcup\limits_{n \in I} G_n$, $\forall i, j \in I$, $G_i \bigcap\limits_{i \neq j} G_j = \varnothing$, where G_n is a subgroup of G, and I is the index set. $\forall m, n \in I$ and $m \neq n$, G_m, G_n is subgroup of G, respectively, and e_m, e_n is the identity element of G_m, G_n respectively, that is, $e_m e_m = e_m, e_n e_n = e_n$. Assume that $e_m e_n = p$, where $p \in G_e$, e is the identity element of G_e, and p^{-1} is the inverse element of p relative to e. That is,

$$ee = e, pe = p = ep, pp^{-1} = e = p^{-1}p, p^{-1}e = p^{-1} = ep^{-1}.$$

Then $p = e_m e_n = e_m e_m e_n = e_m p, e_m e = e_m(pp^{-1}) = (e_m p)p^{-1} = pp^{-1} = e$.

Additionally, $p = e_m e_n = e_m e_n e_n = p e_n, e e_n = (p^{-1}p)e_n = p^{-1}(p e_n) = p^{-1}p = e$.

Then $e_m ee = e_m e = e, eee_n = ee_n = e$. and $ee_n e = (ee_n)e = ee = e, e_n e_n ee = e_n e$ because $eee = ee = e$. Then $e = e_m = e_n$. The identity elements of all groups in G are equal, that is to say, there is only an identity element in G. For any elements in G, there exists a unique inverse element. So G is a group.

($\Leftarrow$) Let G be a group and e be an identity of G. $\forall a \in G$, there exists a unique $a^{-1} \in G$ s.t. $a^{-1}a = e = aa^{-1}$. Then $a^{-1}aa = ea = a, aaa^{-1} = a(aa^{-1}) = ae = a$. Then G is a L3-transposition regular semigroup and R3-transposition regular semigroup. □

Above all, the L1/R1/LR-transposition regular semigroup is a completely regular semigroup, the L2/R2-transposition regular semigroup is a group and semigroup which are both L3-transposition regular semigroups, and the R3-transposition regular semigroup is a group. Figure 2 shows the relationships among various transposition regular semigroups.

Example 4 shows that not every generalized group is a L3-transposition semigroup.

Example 5 shows that not every generalized group is a R3-transposition semigroup.

Example 3 shows that not every L1/R1/LR-transposition semigroup is a generalized group.

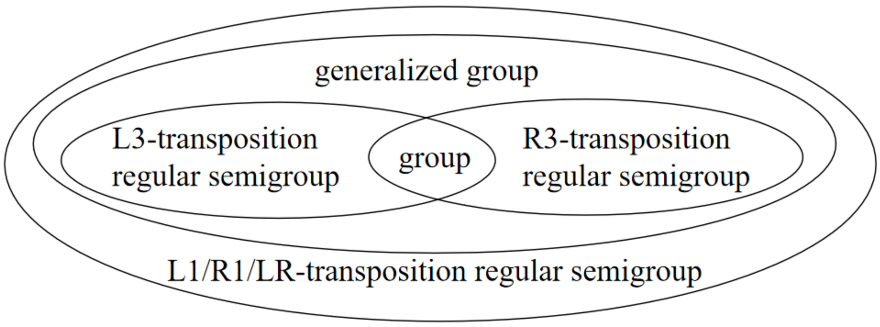

Figure 2. The relationship among various transposition regular semigroups.

7. Discussion

In this paper, some concepts of transposition regular elements and transposition regular semigroups are introduced, some necessary and sufficient conditions of completely regular semigroups are obtained, related decomposition theorems of transposition regular semigroups are given, and some important results are proved: (1) the necessary and sufficient condition for a semigroup to be a completely regular semigroup is that it is a L1/R1/LR-transposition regular semigroup; (2) the L2/R2-transposition regular semigroups are equivalent to groups; (3) the decomposition theorem of the L3/R3-transposition regular semigroup is proved—every L3/R3-transposition regular semigroup is a union of subgroups, and they are generalized groups; and (4) a semigroup which is both a L3-transposition regular semigroup and R3-transposition regular semigroup is a group.

In Ref. [25], they proved that a semigroup which is left regular semigroup and regular semigroup is a completely regular semigroup through elements. However, in this paper, we firstly prove the composition theorem of L1-transposition regular to prove that it is equivalent to a completely regular semigroup. This method helps us to understand their structures clearly. We give some new equation description of a completely regular semigroup. This helps us to prove that an algebraic structure is a completely regular semigroup, which requires fewer steps and is more convenient. As the next research topic, we can explore the relationships among transposition regular semigroups and hypersemigroups and non-classical logical algebras (see [31–33]).

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