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Estimation of the Generalized Logarithmic Transformation Exponential Distribution under Progressively Type-II Censored Data with Application to the COVID-19 Mortality Rates

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Abstract: In this paper, classical and Bayesian estimation for the parameters and the reliability function for the generalized logarithmic transformation exponential (GLTE) distribution has been proposed when the life-times are progressively censored. The maximum likelihood estimator of unknown parameters and their corresponding reliability function are obtained under the classical setup. The Bayes estimators are obtained for symmetric (squared error) and asymmetric (LINEX and general entropy) loss functions. This was achieved by considering discrete prior for the scale parameter and conditional gamma prior for the shape parameter. Interval estimation of the unknown parameters and reliability function for classical and Bayesian schemes is also considered. The performances of various derived estimators are recorded using simulation study for different sample sizes and progressive censoring schemes. Finally, the COVID-19 mortality data sets are provided to illustrate the computation of various estimators.

Keywords: GLTE distribution; progressive type-II censoring; Bayesian estimation; maximum likelihood estimation; highest posterior density intervals; COVID-19

MSC: 62F15; 62F10; 65C05

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1. Introduction

In statistical theory, there are several lifetime probability models which are used to analyse the uncertainty of the random phenomena in various fields, for example, engineering, medical sciences, financial affairs, etc. The most commonly used lifetime model is exponential distribution. This distribution has a simple mathematical form and constant hazard rate which makes this distribution quite useful in real life testing and reliability problems. However, having constant hazard rate sometimes become a hurdle for analysing some situations. To tackle this issue, different probability models were discussed such as Weibull, gamma, Burr, Lindley distribution, etc. Nowadays, we are seeing problems which are complex in nature and the statistical behaviour of such problem change rapidly. So, in such situations, we must have a general form of distributions which can also transform according to our need. In this direction, various generalized family of distributions are proposed by academicians (see [1–5]). Ref. [6] proposed a logarithm transformation (LT) method for obtaining the new family of distributions. The family of distributions proposed by [6] has the following cumulative distribution function (cdf):

$$F(x) = 1 - \frac{\ln(2 - G(x))}{\ln 2}, \quad x > 0. \tag{1}$$

Proceeding in the same manner, Ref. [7] generalized the results of [6] by introducing a shape parameter in the baseline cdf of Equation (1). The family of distributions proposed by [7] has the following cdf:

$$F(x) = 1 - \frac{\ln(2 - G^{\alpha}(x))}{\ln 2}, \quad x > 0, \ \alpha > 0. \tag{2}$$

To study the properties of (2), the authors considered $G(x) = 1 - e^{-\theta x}$, $x > 0, \theta > 0$ (exponential baseline distribution) and called it generalized logarithm transformation exponential (GLTE) distribution. The pdf and cdf of GLTE distribution are given as

$$f(x) = \begin{cases} \dfrac{\alpha\theta e^{-\theta x}\left(1 - e^{-\theta x}\right)^{\alpha-1}}{(2 - (1 - e^{-\theta x})^{\alpha})\ln 2} & , \quad x > 0, \quad \theta, \alpha > 0 \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{3}$$

$$F(x) = 1 - \frac{\ln(2 - (1 - e^{-\theta x})^{\alpha})}{\ln 2}, \quad x > 0, \ \theta, \alpha > 0. \tag{4}$$

respectively. The reliability function and hazard rate function of GLTE distribution are given as

$$R(t) = 1 - F(t) = \frac{\ln(2 - (1 - e^{-\theta t})^{\alpha})}{\ln 2}, \quad t > 0, \ \theta, \alpha > 0. \tag{5}$$

and

$$H(t) = \frac{f(t)}{R(t)} = \frac{\alpha\theta e^{-\theta x}\left(1 - e^{-\theta x}\right)^{\alpha-1}}{(2 - (1 - e^{-\theta x})^{\alpha})\ln(2 - (1 - e^{-\theta t})^{\alpha})}, \quad t > 0, \ \theta, \alpha > 0. \tag{6}$$

The authors of [7] have extensively discussed the behaviour of hazard rate function for different values of the parameters. The authors observed that for $\alpha \geq 1$ GLTE distribution is depicting increasing hazard rate, for $\alpha \geq 0.5$, GLTE distribution is depicting decreasing hazard rate. The authors also emphasized that the distribution has a bathtub hazard rate. These results are depicted in Figure 2 of [7].

Ref. [7] considered classical and Bayesian estimation of GLTE distributions of unknown parameters under type-II censoring scheme. To the best of our knowledge, until now no attempt has been made to estimate the parameters and reliability characteristic for GLTE distribution other than the type-II censoring scheme. This is why we have considered progressive type-II censoring scheme for our study which is a generalize form of type-II censoring scheme. Hence, the results of this paper are of more general in nature. In addition to that the GLTE distribution provides a better fit to the considered demographic data (mortality rates) over other lifetime distributions e.g., Weibull, Chen, etc. (see Section 5). Keeping these points in mind, we have discussed the classical as well as the Bayesian estimation and reliability characteristics for the GLTE distribution under the progressive type-II censored data.

In any life-testing experiment, it is very cumbersome to complete the experiment for a long time period due to time and cost constraints. There are various types of censoring schemes which have been introduced in the literature to reduce the time and cost involved into the experiments. Type-I and Type-II are the most common censoring schemes among the various censoring schemes. The number of observed failures in the type-I censoring scheme is random in nature, whereas the termination time of the experiments is random in the type-II censoring scheme. Surviving units cannot be withdrawn during the experimentation of these two schemes. The main advantage of progressive censoring is that it saves a lot of energy and money due to its ability to drop live units from the experiment in practical failure time experiment. Initially, the progressive censoring scheme was proposed by [8].

Let n units are put on test at the same time. If first failure occurs at the time X_1, r_1 surviving units are randomly removed from the experiment. At the second failure time X_2, r_2 surviving units are randomly removed from the remaining surviving units. The experiment continues until the m^{th} failure time X_m observed. At this time, the test can be terminated with removing all the surviving units from the experiment. Here, $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ is a set of observed life-time, referred to as progressively type-II censored sample (see, Figure 1). In the case of $r_1 = r_2 = \ldots = r_{m-1} = 0$, this scheme becomes type-II censoring scheme. It is also observed that for $r_1 = r_2 = \ldots = r_m = 0$, this scheme transformed into the case of complete sample.

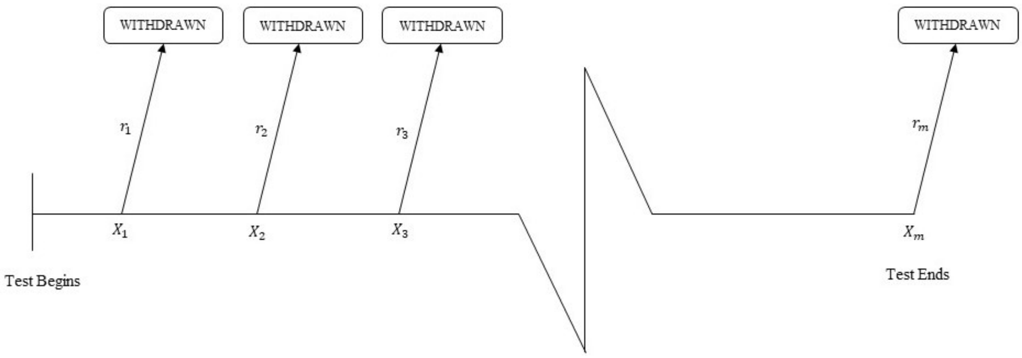

Figure 1. Progressive Type-II Censoring Scheme.

In the last few years, progressive censoring scheme has received considerable attention (see [9–11]). Ref. [12] has made a remarkable contribution to developments, implementations and other dimensions of progressive censoring. The progressive censored data for the Burr model was discussed by [13]. The Bayesian inference for the Weibull distribution under progressive censoring scheme was discussed by [14]. The progressive type-II censoring scheme was discussed by [15] for exponential Weibull model and for the inverted exponential model was discussed by [16]. The scheme of progressive censoring for the Burr type-XII and Inverse Weibull were proposed by [17,18], respectively. Ref. [19] have developed a method of estimation for xgamma distribution under progressive type-II censoring approach. Ref. [20] have discussed progressive censoring scheme for lognormal distribution and [21] have discussed Bayesian inference of Hjorth distribution under the progressive type-II censoring scheme. Recently, Ref. [22] addressed the problem of Bayesian reliability estimation for the Topp–Leone distribution based on progressive type-II censoring scheme. The authors obtained Bayes estimators using approximation techniques such as Lindley's approximation, MCMC and Tierney-Kadane method ([23]).

In this paper, we address the problem of classical and Bayesian estimation of GLTE distribution under the progressive type-II censored data. For the scale parameter, a discrete prior is considered, whereas a conditional gamma prior for the shape parameter is considered. The rest of the paper, we organized as follows: In Section 2, the Maximum likelihood estimators and asymptotic confidence intervals are obtained for the parameters and the reliability function of GLTE distribution. The Bayes estimators under symmetric, asymmetric loss functions and highest posterior density intervals are obtained in Section 3. In Section 4, a simulation study is presented to report the performances of the point and interval estimators. In Section 5, the mortality data sets of Italy and The Netherlands due to COVID-19 are provided to illustrate the computation of various derived results. Finally, the conclusions appear in Section 6.

2. Maximum Likelihood Estimation and Asymptotic Confidence Interval

Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ is a progressive type-II censored sample from a life test on n units having the GLTE (θ, α) distribution with density given in (1) and $r_1, r_2, \ldots, r_m$

denote the corresponding number of units removed from the test. The likelihood function based on the progressively type-II censored sample is given by

$$l(\theta,\alpha|\mathbf{x}) = A\prod_{i=1}^{m} f(x_i;\theta,\alpha)[1-F(x_i;\theta,\alpha)]^{r_i}, \quad (7)$$

where, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $A = n(n-1-r_1)(n-2-r_1-r_2)\ldots(n-\sum_{i=1}^{m-1}(r_i+1))$; and $f(x)$ and $F(x)$ are given respectively by (1) and (2).

Substituting (1) and (2), into (7), the likelihood function is

$$l(\theta,\alpha|\mathbf{x}) = A\prod_{i=1}^{m}\left\{\frac{\alpha\theta e^{-\theta x_i}(1-e^{-\theta x_i})^{\alpha-1}}{(2-(1-e^{-\theta x_i})^{\alpha})\ln 2}\right\}\left\{\frac{\ln(2-(1-e^{-\theta x_i})^{\alpha})}{\ln 2}\right\}^{r_i}. \quad (8)$$

The log-likelihood function can be written as

$$L = \ln l(\theta,\alpha|\mathbf{x}) = \ln A + m\ln\frac{\alpha\theta}{\ln 2} + (\alpha-1)\sum_{i=1}^{m}\ln(1-e^{-\theta x_i}) - \theta\sum_{i=1}^{m} x_i \quad (9)$$
$$-\sum_{i=1}^{m}\ln(2-(1-e^{-\theta x_i})^{\alpha}) + \sum_{i=1}^{m} r_i\left\{\frac{\ln(2-(1-e^{-\theta x_i})^{\alpha})}{\ln 2}\right\},$$

$$\frac{\partial L}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\ln(1-e^{-\theta x_i}) + \sum_{i=1}^{m}\frac{(1-e^{-\theta x_i})^{\alpha}\ln(1-e^{-\theta x_i})}{(2-(1-e^{-\theta x_i})^{\alpha})} + \sum_{i=1}^{m} r_i\frac{(1-e^{-\theta x_i})^{\alpha}\ln(1-e^{-\theta x_i})}{(2-(1-e^{-\theta x_i})^{\alpha})\ln 2} = 0, \quad (10)$$

and

$$\frac{\partial L}{\partial\theta} = \frac{m}{\theta} - \sum_{i=1}^{m} x_i + (\alpha-1)\sum_{i=1}^{m}\frac{x_i e^{-\theta x_i}}{1-e^{-\theta x_i}} + \frac{\alpha x_i(1-e^{-\theta x_i})^{\alpha-1}(1-e^{-\theta x_i})}{(2-(1-e^{-\theta x_i})^{\alpha})} + \quad (11)$$
$$\sum_{i=1}^{m} r_i\frac{\alpha x_i(1-e^{-\theta x_i})^{\alpha-1}(1-e^{-\theta x_i})}{(2-(1-e^{-\theta x_i})^{\alpha})\ln 2} = 0.$$

It is clear that the Equations (10) and (11) are of implicit forms and are non-linear in nature, so, MLEs of unknown parameters are tedious to obtain, analytically. Thus, one may use any numerical approximation techniques, such as Newton–Raphson (N-R) method, fixed-point iterations, etc. Here, we used the N-R method to evaluate the MLE of the parameters with the help of R Software. Once the MLEs of the parameters are obtained from Equations (10) and (11), the MLE of reliability function is evaluated using the invariance property and is given in (12)

$$\hat{R(t)} = \frac{\ln(2-(1-e^{-\hat{\theta}t})^{\hat{\alpha}})}{\ln 2}. \quad (12)$$

Further, using the property MLE, we obtained the asymptotic confidence intervals (ACIs) for α, θ and $R(t)$. The ACIs are obtained from the diagonal elements of the inverse Fisher information matrix that gives the asymptotic variance for the parameters α and θ respectively. Thus, the $100(1-\eta)\%$ confidence interval for α , θ and $R(t)$ can be defined respectively, as $\left(\hat{\alpha} - Z_{\eta/2}\sqrt{var(\hat{\alpha})}, \hat{\alpha} + Z_{\eta/2}\sqrt{var(\hat{\alpha})}\right)$, $\left(\hat{\theta} - Z_{\eta/2}\sqrt{var(\hat{\theta})}, \hat{\theta} + Z_{\eta/2}\sqrt{var(\hat{\theta})}\right)$, $\left(\widehat{R(t)} - Z_{\eta/2}\sqrt{var\left(\widehat{R(t)}\right)}, \widehat{R(t)} + Z_{\eta/2}\sqrt{var\left(\widehat{R(t)}\right)}\right)$, where $Z_{\eta/2}$ is a standard normal variate.

3. Bayesian Estimation

In this section, we consider both parameters shape as well as scale as unknown for GLTE distribution. To estimate these parameters, we adopt the method proposed by [24]. Several researchers also used this method (see, [25,26]). Here, the scale parameter

θ is restricted to a finite number of values $\theta_1, \theta_2, \ldots, \theta_N$ with probabilities $\eta_1, \eta_2, \ldots, \eta_N$ respectively, the prior distribution for θ is given by

$$\sum_{j=1}^{N} \eta_j = 1\,, \qquad 0 \le \eta_j \le 1,$$
$$Pr(\theta = \theta_j) = \eta_j\,, \quad j = 1, 2, \ldots, N.$$

Further, we consider conditional gamma prior for parameter α over $\theta = \theta_j$ with hyper parameters a_j and b_j,

$$\pi(\alpha|\theta_j, \mathbf{x}) = \frac{b_j^{a_j} \alpha^{a_j - 1} e^{-b_j \alpha}}{\Gamma(a_j)}\,, \qquad \alpha > 0,\ a_j, b_j > 0. \tag{13}$$

Here, we consider symmetric (squared error) and asymmetric (LINEX and general entropy) loss functions to obtain the Bayes estimators of unknown parameters and reliability function of the distribution. The likelihood function (8) in terms of continuous parameter α and discrete parameter θ_j is

$$l(\theta_j, \alpha|\mathbf{x}) = A \prod_{i=1}^{m} \left\{ \frac{\alpha \theta_j e^{-\theta_j x_i} (1 - e^{-\theta_j x_i})^{\alpha - 1}}{(2 - (1 - e^{-\theta_j x_i})^{\alpha}) \ln 2} \right\} \left\{ \frac{\ln(2 - (1 - e^{-\theta_j x_i})^{\alpha})}{\ln 2} \right\}^{r_i}. \tag{14}$$

The joint posterior of α and $\theta = \theta_j$ is

$$\pi^*(\alpha, \theta_j|\mathbf{x}) = \frac{\pi(\alpha|\theta_j)\ Pr(\theta = \theta_j)\ l(\theta_j, \alpha|\mathbf{x})}{\int_0^\infty \sum_{j=1}^{N} \pi(\alpha|\theta_j)\ Pr(\theta = \theta_j)\ l(\theta_j, \alpha|\mathbf{x}) d\alpha} = \frac{T_1}{T_0} \tag{15}$$

where,

$$T_1 = \prod_{i=1}^{m} \left\{ \frac{\alpha \theta_j e^{-\theta_j x_i} (1 - e^{-\theta_j x_i})^{\alpha - 1}}{(2 - (1 - e^{-\theta_j x_i})^{\alpha}) \ln 2} \right\} \left\{ \frac{\ln(2 - (1 - e^{-\theta_j x_i})^{\alpha})}{\ln 2} \right\}^{r_i} \frac{b_j^{a_j} \alpha^{a_j - 1} e^{-b_j \alpha}}{\Gamma(a_j)} \eta_j,$$

and

$$T_0 = \int_0^\infty \sum_{j=1}^{N} T_1 d\alpha.$$

The marginal posterior density of θ_j is obtained by integrating (15) with respect to α, we get

$$P_j = \int_0^\infty \pi^*(\alpha, \theta_j|\mathbf{x}) d\alpha. \tag{16}$$

Combining the likelihood function (14) and prior density (13), we obtain the posterior density of α

$$\pi^*(\alpha|\theta_j, \mathbf{x}) = \frac{\pi(\alpha|\theta_j)\ l(\theta_j, \alpha|\mathbf{x})}{\int_0^\infty \pi(\alpha|\theta_j)\ l(\theta_j, \alpha|\mathbf{x}) d\alpha} = \frac{K_1}{K_0} \tag{17}$$

where,

$$K_1 = \prod_{i=1}^{m} \left\{ \frac{\alpha \theta_j e^{-\theta_j x_i (1 - e^{-\theta_j x_i})^{\alpha - 1}}}{(2 - (1 - e^{-\theta_j x_i})^{\alpha}) \ln 2} \right\} \left\{ \frac{\ln(2 - (1 - e^{-\theta_j x_i})^{\alpha})}{\ln 2} \right\}^{r_i} \frac{b_j^{a_j} \alpha^{a_j - 1} e^{-b_j \alpha}}{\Gamma(a_j)},$$

and

$$K_0 = \int_0^\infty K_1 d\alpha.$$

3.1. Bayes Estimation under Symmetric Loss Function

Squared Error Loss Function (SELF): In the SELF, the magnitude of underestimation and overestimation are equal. It is also known as Quadratic loss function. In the SELF, Bayes estimator is represented by the posterior mean. The squared error loss function is defined as

$$L(\hat{\delta}, \delta) = (\hat{\delta} - \delta)^2, \quad \hat{\delta} \in \mathbb{D}, \ \delta \in \Theta,$$

where, $\hat{\delta}$ is the Bayes estimator of δ. $\mathbb{D}$ is a decision space and Θ is parameter space.

The Bayes estimators $\tilde{\alpha}_{BS}$ and $\tilde{\theta}_{BS}$ of parameters α and θ, respectively, are

$$\begin{aligned}
\tilde{\alpha}_{BS} &= \int_0^\infty \sum_{j=1}^{N} P_j \alpha \pi^*(\alpha|\theta_j, \mathbf{x}) d\alpha \\
&= \int_0^\infty \sum_{j=1}^{N} P_j \alpha \frac{K_1}{K_0} d\alpha \\
\tilde{\alpha}_{BS} &= \frac{1}{K_0} \int_0^\infty \alpha \left(\sum_{j=1}^{N} P_j K_1 \right) d\alpha. \\
\tilde{\theta}_{BS} &= \sum_{j=1}^{N} \theta_j P_j.
\end{aligned}$$

The Bayes estimator $\tilde{R}_{BS}(t)$ of the reliability function $R(t)$ is

$$\begin{aligned}
\tilde{R}_{BS}(t) &= \int_0^\infty \sum_{j=1}^{N} P_j \frac{\ln(2 - (1 - e^{-\theta_j t})^\alpha)}{\ln 2} \pi^*(\alpha|\theta_j, \mathbf{x}) d\alpha \\
&= \int_0^\infty \sum_{j=1}^{N} P_j \frac{\ln(2 - (1 - e^{-\theta_j t})^\alpha)}{\ln 2} \frac{K_1}{K_0} d\alpha \\
&= \frac{1}{K_0} \int_0^\infty \left(\sum_{j=1}^{N} P_j K_1 \right) \frac{\ln(2 - (1 - e^{-\theta_j t})^\alpha)}{\ln 2} \, d\alpha.
\end{aligned}$$

3.2. Bayes Estimation under Asymmetric Loss Function

When the magnitude of overestimation and underestimation are not equal, then we used the asymmetric loss function. In the asymmetric loss function, we consider LINEX loss function and General Entropy loss function.

LINEX Loss function: The LINEX loss function is defined as

$$L(\hat{\delta} - \delta) \propto e^{c(\hat{\delta} - \delta)} - c(\hat{\delta} - \delta) - 1, \qquad c \neq 0, \ \hat{\delta} \in \mathbb{D}, \ \delta \in \Theta.$$

The Bayes estimator $\hat{\delta}_{BL}$ of δ under the LINEX loss function is

$$\hat{\delta}_{BL} = -\frac{1}{c} \ln[E_\delta(exp(-c\delta))], \tag{18}$$

provided, $E_\delta(exp(-c\delta))$ exists and finite. The Bayes estimators $\tilde{\alpha}_{BL}$ and $\tilde{\theta}_{BL}$ of parameters α and θ under LINEX loss function (18), respectively, are

$$
\begin{aligned}
\tilde{\alpha}_{BL} &= -\frac{1}{c}\ln\left[\int_0^\infty \sum_{j=1}^{N} P_j exp(-c\alpha)\frac{K_1}{K_0}d\alpha\right] \\
\tilde{\alpha}_{BL} &= -\frac{1}{aK_0}\ln\left[\int_0^\infty \left(\sum_{j=1}^{N} P_j K_1\right) exp(-c\alpha)d\alpha\right]. \\
\tilde{\theta}_{BL} &= -\frac{1}{c}\ln\left[\sum_{j=1}^{N} P_j exp(-c\theta_j)\right].
\end{aligned}
$$

The Bayes estimators $\tilde{R}_{BL}(t)$ of the reliability function $R(t)$ is

$$
\begin{aligned}
\tilde{R}_{BL}(t) &= -\frac{1}{c}\ln\left[\int_0^\infty \sum_{j=1}^{N} P_j exp\left(-c\left(\frac{\ln(2-(1-e^{-\theta_j t})^\alpha)}{\ln 2}\right)\right)\frac{K_1}{K_0}d\alpha\right] \\
&= -\frac{1}{cK_0}\ln\left[\int_0^\infty \left(\sum_{j=1}^{N} P_j K_1\right) exp\left(-c\left(\frac{\ln(2-(1-e^{-\theta_j t})^\alpha)}{\ln 2}\right)\right)d\alpha\right].
\end{aligned}
$$

General Entropy Loss Function: The General Entropy loss function is defined as

$$
L(\hat{\delta},\delta) \propto \left(\frac{\hat{\delta}}{\delta}\right)^q - q\ln\left(\frac{\hat{\delta}}{\delta}\right) - 1, \qquad q \neq 0,\ \hat{\delta} \in \mathbb{D},\ \delta \in \Theta.
$$

The Bayes estimator $\hat{\delta}_{BG}$ of δ under GE loss function is

$$
\hat{\delta}_{BG} = \left[E_\delta(\delta^{-q})\right]^{-\frac{1}{q}}, \tag{19}
$$

provided, $E_\delta(\delta^{-q})$ exists and is finite.

The Bayes estimators $\tilde{\alpha}_{BG}$ and $\tilde{\theta}_{BG}$ of parameters α and θ under General Entropy loss function (19), respectively, are

$$
\begin{aligned}
\tilde{\alpha}_{BG} &= \left[\int_0^\infty \sum_{j=1}^{N} P_j \alpha^{-q}\frac{K_1}{K_0}d\alpha\right]^{-\frac{1}{q}} \\
\tilde{\alpha}_{BG} &= \left[\frac{1}{K_0}\int_0^\infty \alpha^{-q}\left(\sum_{j=1}^{N} P_j K_1\right)d\alpha\right]^{-\frac{1}{q}}. \\
\tilde{\theta}_{BG} &= \left[\sum_{j=1}^{N} P_j \theta_j^{-q}\right]^{-\frac{1}{q}}.
\end{aligned}
$$

The Bayes estimator $\tilde{R}_{BG}(t)$ of the reliability function $R(t)$ is

$$
\begin{aligned}
\tilde{R}_{BG}(t) &= \left[\int_0^\infty \sum_{j=1}^{N} P_j \left\{\frac{\ln(2-(1-e^{-\theta_j t})^\alpha)}{\ln 2}\right\}^{-q}\frac{K_1}{K_0}d\alpha\right]^{-\frac{1}{q}} \\
&= \left[\frac{1}{K_0}\int_0^\infty \left(\sum_{j=1}^{N} P_j K_1\right)\left\{\frac{\ln(2-(1-e^{-\theta_j t})^\alpha)}{\ln 2}\right\}^{-q}d\alpha\right]^{-\frac{1}{q}}.
\end{aligned}
$$

3.3. Bayesian Credible Interval

In the Bayesian paradigm, parameter τ is a random variable and the probability for this parameter τ lies within the specified intervals. The highest posterior density (HPD) interval was discussed by [27]. The HPD interval is the shortest interval among all Bayesian intervals. The HPD interval for parameter τ based on the samples from simulation method ie., $\tau_{(1)}, \tau_{(2)}, \ldots, \tau_{(M)}$ was discussed by [28] (see [29,30]). The algorithm to obtain the HPD interval is as follows

(i) In the first step, we generate a random censored data from GLTE distribution using Equation (4) for some fixed values of the parameters using different censoring schemes.

(ii) After this, we use the results of MLEs and Bayes estimators derived in Sections 2 and 3, respectively, to calculate the required estimators (say τ) with the help of the generated data in step **(i)**.

(iii) We repeat the steps **(i–ii)** N times and obtain the N values of the estimators i.e., $\tau_1, \tau_2, \ldots, \tau_N$.

(iv) Now we apply the method of [28] and obtain the Bayes credible interval.

4. Simulation Study

In this section, we perform a simulation study for the GLTE distribution under the progressive type-II censored sample. This simulation study is conducted to measure the performances of various estimators obtained in this article. The algorithm for generation of progressive type-II censored sample was given by [31]. The algorithm is modified according to our problem and is given as:

- Generate m *iid* random numbers $W_1, W_2, \ldots, W_m$ from $U(0,1)$.
- Determine the values of the censored scheme r_i, for $i = 1, 2, \ldots, m$.
- Set $V_i = W_i^{1/(i+\sum_{j=m-i+1}^{m} r_j)}$ for $i = 1, 2, \ldots, m$.
- Set $U_i = 1 - V_m V_{m-1} \ldots V_{m-i+1}$, $i = 1, 2, \ldots m$. Then $\{U_i,\ i = 1, 2, \ldots, m\}$ is the progressive type-II censored sample from $U(0,1)$.
- Set
$$X_i = F^{-1}(U_i) = -\frac{1}{\theta} \ln[1 - (2 - 2^{(1-U_i)})^{1/\alpha}], \qquad i = 1, 2, \ldots, m.$$
Now, $X_1, X_2, \ldots, X_m$ is the progressive type-II censored sample from GLTE distribution.

The performances of estimators are measured by using the criteria of mean square error (MSE) and expected risk (ER). The values of MSEs and ERs are calculated for various configuration of parameters (α, θ) i.e., $(1, 1.5), (1.2, 1.5)$. We have also considered few censoring schemes (C.S.) which are defined

* $R_1 = r_1, r_2, \ldots, r_{m-1} = 1$, $r_m = n - 2m + 1$.
* $R_2 = r_1 = n - 2m + 1$, $r_2 = r_3 = \ldots = r_m = 1$.
* $R_3 = r_1 = n - m$, $r_2 = r_3 = \ldots r_m = 0$.

In order to calculate MSEs and ERs, we replicated our results 1000 times. All the calculated results are reported in Tables 1–10. The following observations are made from the calculated tables.

(1) Tables 1 and 3 show MSEs of the α, θ and $R(t)$ for the configuration of $(\alpha, \theta) = (1, 1.5)$. It can be observed that Bayes estimates of α, θ and $R(t)$ show less MSEs in comparison to MLEs. Furthermore, we observe that Bayes estimator under LINEX loss function for $c = 0.5$ performs better among other Bayes estimators. In terms of $R(t)$, all Bayes estimators perform approximately the same with small MSEs.

(2) Tables 2 and 4 show MSEs of the α, θ and $R(t)$ for the configuration of $(\alpha, \theta) = (1.2, 1.5)$. It is observed that Bayes estimates of α, θ and $R(t)$ show less MSEs in comparison to MLEs. Moreover, we observe that Bayes estimator under LINEX loss function performs better among other Bayes estimators. In terms of $R(t)$, all estimators perform approximately the same with small MSEs.

(3) Tables 5 and 7 report ERs of various Bayes estimators for parameters (α, θ) = (1,1.5), respectively. From Tables, we observe that Bayes estimator for LINEX loss function at $c = -0.5$ exhibits lowest MSEs among other Bayes estimators. In terms of $R(t)$, all estimators perform approximately the same with small MSEs.

(4) Tables 6 and 8 report ERs of various Bayes estimators for parameters (α, θ) = (1.2,1.5), respectively. We can observe that Bayes estimator for LINEX loss function at $c = -0.5$ and for GE loss function at $q = -0.5$ exhibits lower ERs among other Bayes estimates for the parameters α and θ, respectively. In terms of $R(t)$, all estimators perform approximately the same with small MSEs, but Bayes estimators seem to perform slightly better.

(5) Tables 9 and 10 show the average length of ACIs and HPD intervals of parameters α, θ and $R(t)$, respectively. It is observed that the average lengths of the intervals decreases when the different choices of (n, r) increases for both classical and Bayesian estimation. We also observe that HPD intervals of the estimators under LINEX loss function are mostly better among other estimators.

From Tables 1–10, we conclude that, for different choices of the parameters (α, θ) = $\{(1, 1.5), (1.2, 1.5)\}$, the Bayes estimator under asymmetric loss function i.e., LINEX loss function shows less MSEs and ERs. Hence, it performs better than other estimators. It is worth mentioning here that for very few cases, the Bayes estimator under GE loss function performs better in terms of ERs. In terms of $R(t)$, we observe that the MLEs and Bayes estimators perform approximately the same. On the other hand, when we consider the performances of reliability estimates in term of ERs, we see that Bayes estimators under asymmetric loss functions exhibit lower values in comparison to symmetric loss function and MLE. The censoring scheme R_3 performs better among all the censoring schemes (R_1 and R_2). All the above observations are made for increasing values of n and m.

Table 1. MSE of the different estimates of the parameters $(\alpha, \theta) = (1, 1.5)$.

						BL		BG	
n	m	$C.S.$	Parameter	ML	BS	$c = -0.5$	$c = 0.5$	$q = -0.5$	$q = 0.5$
50	20	R_1	α	0.1285	0.0329	0.0333	0.0329	0.0333	0.0341
			θ	0.4923	0.3996	0.4057	0.3943	0.4001	0.4024
		R_2	α	0.1078	0.0246	0.0252	0.0244	0.0246	0.0251
			θ	0.3624	0.2931	0.2989	0.2881	0.2927	0.2931
		R_3	α	0.0947	0.0181	0.0187	0.0176	0.0181	0.0177
			θ	1.6345	0.1944	0.1992	0.1902	0.1936	0.1929
50	25	R_1	α	0.0956	0.0233	0.0241	0.0228	0.0233	0.0227
			θ	0.2823	0.2193	0.2210	0.2152	0.2188	0.2189
		R_2	α	0.0956	0.0233	0.0241	0.0228	0.0233	0.0227
			θ	0.2823	0.2193	0.2210	0.2152	0.2188	0.2189
		R_3	α	0.0873	0.0176	0.0184	0.0171	0.0174	0.0172
			θ	0.1881	0.1472	0.1512	0.1437	0.1463	0.1454
70	25	R_1	α	0.0931	0.0305	0.0309	0.0312	0.0309	0.0322
			θ	0.4090	0.3262	0.3311	0.3222	0.3270	0.3298
		R_2	α	0.0750	0.0214	0.0221	0.0216	0.0215	0.0218
			θ	0.2647	0.2102	0.2149	0.2062	0.2097	0.2098
		R_3	α	0.0701	0.0172	0.0179	0.0166	0.0171	0.0165
			θ	0.1799	0.1437	0.1476	0.1403	0.1429	0.1421
100	30	R_1	α	0.0802	0.0312	0.0317	0.0316	0.0317	0.0320
			θ	0.3832	0.2999	0.3042	0.2965	0.3009	0.3041
		R_2	α	0.0608	0.0205	0.0214	0.0198	0.0205	0.0201
			θ	0.2024	0.1574	0.1614	0.1541	0.1569	0.1566
		R_3	α	0.0587	0.0170	0.0177	0.0166	0.0169	0.0165
			θ	0.1399	0.1105	0.1138	0.1076	0.1097	0.1087
100	50	R_1	α	0.0354	0.0147	0.0151	0.0151	0.0145	0.0151
			θ	0.1025	0.0774	0.0795	0.0757	0.0771	0.0767
		R_2	α	0.0354	0.0147	0.0151	0.0151	0.0145	0.0151
			θ	0.1025	0.0774	0.0795	0.0757	0.0771	0.0767
		R_3	α	0.0380	0.0137	0.0142	0.0133	0.0138	0.0133
			θ	0.0732	0.0574	0.0593	0.0558	0.0568	0.0561

Table 2. MSE of the different estimates of the parameters $(\alpha, \theta) = (1.2, 1.5)$.

n	m	$C.S.$	Parameter	ML	BS	BL $c = -0.5$	BL $c = 0.5$	BG $q = -0.5$	BG $q = 0.5$
50	20	R_1	α	0.2064	0.0520	0.0511	0.0533	0.0539	0.0575
			θ	0.4037	0.3022	0.3047	0.3004	0.3040	0.3087
		R_2	α	0.1724	0.0367	0.0358	0.0379	0.0381	0.0413
			θ	0.3062	0.2264	0.2292	0.2243	0.2272	0.2299
		R_3	α	0.1484	0.0288	0.0276	0.0301	0.0302	0.0335
			θ	0.2100	0.1584	0.1611	0.1564	0.1587	0.1601
50	25	R_1	α	0.1536	0.0333	0.0325	0.0345	0.0346	0.0371
			θ	0.2419	0.1696	0.1715	0.1683	0.1704	0.1729
		R_2	α	0.1536	0.0333	0.0325	0.0345	0.0346	0.0371
			θ	0.2419	0.1696	0.1715	0.1683	0.1704	0.1729
		R_3	α	0.1378	0.0265	0.0256	0.0276	0.0279	0.0304
			θ	0.1677	0.1196	0.1215	0.1181	0.1198	0.1209
70	25	R_1	α	0.1493	0.0484	0.0477	0.0498	0.0498	0.0527
			θ	0.3347	0.2463	0.2477	0.2455	0.2483	0.2534
		R_2	α	0.1196	0.0314	0.0307	0.0326	0.0327	0.0352
			θ	0.2258	0.1625	0.1644	0.1611	0.1632	0.1656
		R_3	α	0.1100	0.0258	0.0251	0.0268	0.0269	0.0293
			θ	0.1598	0.1169	0.1189	0.1154	0.1171	0.1182
100	30	R_1	α	0.1276	0.0471	0.0468	0.0491	0.0487	0.0511
			θ	0.3108	0.2242	0.2250	0.2241	0.2265	0.2320
		R_2	α	0.0959	0.0278	0.0246	0.0285	0.0289	0.0306
			θ	0.1739	0.1213	0.1229	0.1203	0.1219	0.1238
		R_3	α	0.0910	0.0231	0.0227	0.0241	0.0243	0.0260
			θ	0.1247	0.0894	0.091	0.0883	0.0896	0.0904
100	50	R_1	α	0.0555	0.0208	0.0208	0.0217	0.0221	0.0228
			θ	0.0893	0.0605	0.0608	0.0604	0.0611	0.0625
		R_2	α	0.0555	0.0208	0.0208	0.0217	0.0221	0.0228
			θ	0.0893	0.0605	0.0608	0.0604	0.0611	0.0625
		R_3	α	0.0585	0.0189	0.0191	0.0197	0.0201	0.0213
			θ	0.0732	0.0466	0.0471	0.0463	0.0468	0.0474

Table 3. MSE of the different estimates of $R(t)$ with parameters $(\alpha, \theta) = (1, 1.5)$ and $R(t = 0.1) = 0.5581$.

n	m	$C.S.$	ML	BS	BL $c = -0.5$	BL $c = 0.5$	BG $q = -0.5$	BG $q = 0.5$
50	20	R_1	0.0013	0.0014	0.0015	0.0015	0.0015	0.0016
		R_2	0.0014	0.0010	0.0011	0.0011	0.0011	0.0011
		R_3	0.0019	0.0009	0.0009	0.0009	0.0009	0.0010
50	25	R_1	0.0012	0.0008	0.0010	0.0009	0.0009	0.0009
		R_2	0.0012	0.0008	0.0010	0.0009	0.0009	0.0009
		R_3	0.0016	0.0007	0.0008	0.0008	0.0008	0.0008
70	25	R_1	0.0009	0.0011	0.0013	0.0013	0.0012	0.0013
		R_2	0.0011	0.0007	0.0009	0.0010	0.0009	0.0009
		R_3	0.0015	0.0007	0.0008	0.0008	0.0008	0.0008
100	30	R_1	0.0006	0.0009	0.0012	0.0013	0.0011	0.0012
		R_2	0.0009	0.0006	0.0009	0.0009	0.0009	0.0008
		R_3	0.0012	0.0006	0.0008	0.0008	0.0008	0.0008
100	50	R_1	0.0006	0.0005	0.0007	0.0009	0.0006	0.0008
		R_2	0.0006	0.0005	0.0007	0.0009	0.0006	0.0008
		R_3	0.0009	0.0005	0.0007	0.0007	0.0006	0.0007

Table 4. MSE of the different estimates of $R(t)$ with parameters $(\alpha, \theta) = (1.2, 1.5)$ and $R(t = 0.1) = 0.6188$.

n	m	$C.S.$	ML	BS	BL $c = -0.5$	BL $c = 0.5$	BG $q = -0.5$	BG $q = 0.5$
50	20	R_1	0.0008	0.0014	0.0015	0.0015	0.0015	0.0015
		R_2	0.0009	0.0011	0.0012	0.0012	0.0012	0.0012
		R_3	0.0012	0.0011	0.0011	0.0011	0.0011	0.0012
50	25	R_1	0.0008	0.0008	0.0010	0.0009	0.0009	0.0009
		R_2	0.0008	0.0008	0.0010	0.0009	0.0009	0.0009
		R_3	0.0010	0.0009	0.0010	0.0010	0.0010	0.0010
70	25	R_1	0.0006	0.0011	0.0013	0.0013	0.0012	0.0013
		R_2	0.0007	0.0008	0.0010	0.0011	0.0010	0.0010
		R_3	0.0009	0.0009	0.0010	0.0010	0.0010	0.0010
100	30	R_1	0.0006	0.0008	0.0011	0.0012	0.0010	0.0011
		R_2	0.0007	0.0006	0.0009	0.0009	0.0009	0.0008
		R_3	0.0008	0.0007	0.0009	0.0009	0.0009	0.0009
100	50	R_1	0.0004	0.0006	0.0008	0.0010	0.0007	0.0009
		R_2	0.0004	0.0006	0.0008	0.0010	0.0007	0.0009
		R_3	0.0005	0.0006	0.0008	0.0008	0.0007	0.0008

Table 5. ER of the different estimates of the parameters $(\alpha, \theta) = (1, 1.5)$.

n	m	$C.S.$	Parameter	BS	BL $c = -0.5$	BL $c = 0.5$	BG $q = -0.5$	BG $q = 0.5$
50	20	R_1	α	0.0329	0.0021	0.0022	0.0047	0.0047
			θ	0.3996	0.0419	0.0641	0.0147	0.0163
		R_2	α	0.0246	0.0015	0.0015	0.0032	0.0032
			θ	0.2931	0.0316	0.0447	0.0109	0.0121
		R_3	α	0.0181	0.0015	0.0014	0.0022	0.0023
			θ	0.1944	0.0218	0.0278	0.0077	0.0084
70	25	R_1	α	0.0305	0.0019	0.0019	0.0042	0.0043
			θ	0.3262	0.0354	0.0492	0.0131	0.0143
		R_2	α	0.0214	0.0013	0.0014	0.0027	0.0028
			θ	0.2102	0.0354	0.0299	0.0085	0.0093
		R_3	α	0.0172	0.0014	0.0013	0.0021	0.0021
			θ	0.1437	0.0167	0.0195	0.0061	0.0066
100	30	R_1	α	0.0312	0.0017	0.0017	0.0041	0.0034
			θ	0.2999	0.0333	0.0436	0.0131	0.0141
		R_2	α	0.0205	0.0013	0.0013	0.0024	0.0026
			θ	0.1574	0.0183	0.0214	0.0068	0.0073
		R_3	α	0.0170	0.0015	0.0014	0.0019	0.0020
			θ	0.1105	0.0131	0.0146	0.0049	0.0052
100	50	R_1	α	0.0147	0.0013	0.0012	0.0017	0.0018
			θ	0.0774	0.0093	0.0101	0.0037	0.0039
		R_2	α	0.0147	0.0013	0.0012	0.0017	0.0018
			θ	0.0774	0.0093	0.0101	0.0037	0.0039
		R_3	α	0.0137	0.0014	0.0013	0.0016	0.0016
			θ	0.0574	0.0071	0.0073	0.0027	0.0028

Table 6. ER of the different estimates of the parameters $(\alpha, \theta) = (1.2, 1.5)$.

n	m	C.S.	Parameter	BS	BL $c=-0.5$	BL $c=0.5$	BG $q=-0.5$	BG $q=0.5$
50	25	R_1	α	0.0333	0.0031	0.0032	0.0037	0.0038
			θ	0.1696	0.0194	0.0235	0.0078	0.0084
		R_2	α	0.0333	0.0031	0.0032	0.0037	0.0038
			θ	0.1696	0.0194	0.0235	0.0078	0.0084
		R_3	α	0.0265	0.0028	0.0029	0.0031	0.0031
			θ	0.1196	0.0141	0.0159	0.0057	0.0061
70	30	R_1	α	0.0383	0.0034	0.0036	0.0042	0.0043
			θ	0.1541	0.0181	0.0208	0.0078	0.0083
		R_2	α	0.0295	0.0026	0.0027	0.0031	0.0032
			θ	0.1259	0.0148	0.0168	0.0062	0.0066
		R_3	α	0.0241	0.0025	0.0026	0.0026	0.0027
			θ	0.0911	0.0109	0.0118	0.0046	0.0048
100	35	R_1	α	0.0392	0.0033	0.0034	0.0042	0.0043
			θ	0.1529	0.0179	0.0206	0.0082	0.0086
		R_2	α	0.0274	0.0024	0.0025	0.0028	0.0029
			θ	0.1021	0.0121	0.0135	0.0052	0.0055
		R_3	α	0.0218	0.0023	0.0024	0.0023	0.0024
			θ	0.0725	0.0088	0.0093	0.0038	0.0039
100	50	R_1	α	0.0208	0.0021	0.0022	0.0022	0.0022
			θ	0.0605	0.0074	0.0077	0.0034	0.0035
		R_2	α	0.0208	0.0021	0.0022	0.0022	0.0022
			θ	0.0605	0.0074	0.0077	0.0034	0.0035
		R_3	α	0.0189	0.0021	0.0022	0.0021	0.0021
			θ	0.0466	0.0057	0.0059	0.0025	0.0026

Table 7. ER of the different estimates of $R(t)$ with parameters $(\alpha, \theta) = (1, 1.5)$.

n	m	C.S.	BS	BL $c=-0.5$	BL $c=0.5$	BG $q=-0.5$	BG $q=0.5$
50	20	R_1	0.0014	0.0001	0.0001	0.0002	0.0002
		R_2	0.0010	0.0002	0.0001	0.0002	0.0002
		R_3	0.0009	0.0001	0.0001	0.0001	0.0001
50	25	R_1	0.0008	0.0001	0.0001	0.0001	0.0002
		R_2	0.0008	0.0001	0.0001	0.0001	0.0002
		R_3	0.0007	0.0001	0.0001	0.0001	0.0001
70	25	R_1	0.0011	0.0001	0.0001	0.0001	0.0001
		R_2	0.0007	0.0002	0.0001	0.0002	0.0002
		R_3	0.0007	0.0001	0.0001	0.0001	0.0001
70	30	R_1	0.0008	0.0001	0.0001	0.0002	0.0002
		R_2	0.0007	0.0001	0.0001	0.0002	0.0001
		R_3	0.0006	0.0001	0.0001	0.0001	0.0001
100	30	R_1	0.0009	0.0002	0.0001	0.0002	0.0002
		R_2	0.0006	0.0001	0.0001	0.0001	0.0001
		R_3	0.0006	0.0001	0.0001	0.0001	0.0001
100	35	R_1	0.0007	0.0001	0.0001	0.0002	0.0002
		R_2	0.0006	0.0001	0.0001	0.0001	0.0001
		R_3	0.0006	0.0001	0.0001	0.0001	0.0001
100	50	R_1	0.0005	0.0001	0.0001	0.0001	0.0001
		R_2	0.0005	0.0001	0.0001	0.0001	0.0001
		R_3	0.0005	0.0001	0.0001	0.0001	0.0001

Table 8. ER of the different estimates of $R(t)$ with parameters $(\alpha, \theta) = (1.2, 1.5)$.

n	m	$C.S.$	BS	BL $c = -0.5$	BL $c = 0.5$	BG $q = -0.5$	BG $q = 0.5$
50	20	R_1	0.0014	0.0001	0.0001	0.0002	0.0002
		R_2	0.0011	0.0002	0.0001	0.0002	0.0001
		R_3	0.0011	0.0001	0.0001	0.0001	0.0001
50	25	R_1	0.0008	0.0001	0.0001	0.0001	0.0001
		R_2	0.0008	0.0001	0.0002	0.0001	0.0002
		R_3	0.0009	0.0001	0.0001	0.0001	0.0001
70	25	R_1	0.0011	0.0001	0.0001	0.0002	0.0002
		R_2	0.0008	0.0001	0.0001	0.0002	0.0002
		R_3	0.0009	0.0001	0.0001	0.0001	0.0001
70	30	R_1	0.0008	0.0001	0.0001	0.0002	0.0002
		R_2	0.0007	0.0001	0.0001	0.0002	0.0001
		R_3	0.0007	0.0001	0.0001	0.0001	0.0001
100	30	R_1	0.0008	0.0002	0.0001	0.0002	0.0002
		R_2	0.0006	0.0001	0.0001	0.0001	0.0001
		R_3	0.0007	0.0001	0.0001	0.0001	0.0001
100	35	R_1	0.0007	0.0002	0.0001	0.0002	0.0002
		R_2	0.0006	0.0001	0.0001	0.0001	0.0001
		R_3	0.0006	0.0001	0.0001	0.0001	0.0001
100	50	R_1	0.0006	0.0001	0.0001	0.0001	0.0001
		R_2	0.0006	0.0001	0.0001	0.0001	0.0001
		R_3	0.0006	0.0001	0.0001	0.0001	0.0001

Table 9. Classical and Bayesian Interval estimation for α and θ under progressive type-II censoring scheme.

n	m	$C.S.$	Parameters	ACI	HPD BS	HPD BL $c = -0.5$	HPD BL $c = 0.5$	HPD BG $q = -0.5$	HPD BG $q = 0.5$
50	25	R_1	α	0.5269	0.2944	0.2976	0.2935	0.2952	0.2954
			θ	0.8533	0.7760	0.7796	0.7726	0.2409	0.2402
		R_2	α	0.5269	0.2944	0.2976	0.2935	0.2952	0.2954
			θ	0.8533	0.7760	0.7796	0.7726	0.2909	0.2402
		R_3	α	0.5346	0.2611	0.2625	0.2594	0.2687	0.2536
			θ	0.7222	0.6438	0.6401	0.6397	0.2634	0.2646
70	25	R_1	α	0.5197	0.2861	0.2933	0.2894	0.2896	0.2872
			θ	1.0161	0.7429	0.7462	0.7408	0.2422	0.2397
		R_2	α	0.4844	0.2867	0.2936	0.2881	0.2874	0.2892
			θ	0.8357	0.7579	0.7607	0.7561	0.2439	0.2290
		R_3	α	0.4934	0.2583	0.2626	0.2567	0.2603	0.2597
			θ	0.7117	0.6362	0.6401	0.6326	0.2650	0.2662
100	30	R_1	α	0.4558	0.3067	0.3174	0.3052	0.3066	0.3068
			θ	0.9916	0.8890	0.8915	0.8887	0.2250	0.2216
		R_2	α	0.4197	0.2609	0.2667	0.2420	0.2620	0.2573
			θ	0.7483	0.6562	0.6599	0.6526	0.2494	0.2542
		R_3	α	0.4351	0.2419	0.2450	0.2429	0.2421	0.2403
			θ	0.6423	0.5540	0.5580	0.5500	0.2662	0.2669
100	50	R_1	α	0.3478	0.2293	0.2334	0.2211	0.2307	0.2223
			θ	0.5841	0.5051	0.5084	0.5020	0.2631	0.2647
		R_2	α	0.3478	0.2293	0.2334	0.2211	0.2307	0.2223
			θ	0.5841	0.5051	0.5084	0.5020	0.2631	0.2647
		R_3	α	0.3722	0.2214	0.2250	0.2230	0.2198	0.2243
			θ	0.5008	0.4345	0.4380	0.4311	0.2680	0.2702

Table 10. Classical and Bayesian Interval estimation for $R(t)$ under progressive type-II censoring scheme.

n	m	C.S.	ACI	HPD				
					BL		BG	
				BS	$c=-0.5$	$c=0.5$	$q=-0.5$	$q=0.5$
50	25	R_1	0.8975	0.0512	0.0569	0.0541	0.0563	0.0545
		R_2	0.8975	0.0512	0.0569	0.0541	0.0563	0.0545
		R_3	0.8948	0.0526	0.0557	0.0539	0.0557	0.0531
70	25	R_1	0.8973	0.0509	0.0586	0.0605	0.0561	0.0567
		R_2	0.8961	0.0498	0.0527	0.0575	0.0528	0.0565
		R_3	0.8936	0.0512	0.0543	0.0555	0.0542	0.0558
100	30	R_1	0.8962	0.0494	0.0579	0.0606	0.0546	0.0604
		R_2	0.8949	0.0498	0.0553	0.0555	0.0550	0.0552
		R_3	0.8928	0.0503	0.0539	0.0527	0.0526	0.0558
100	50	R_1	0.8959	0.0423	0.0472	0.0462	0.0466	0.0447
		R_2	0.8959	0.0423	0.0472	0.0462	0.0466	0.0447
		R_3	0.8947	0.0455	0.0482	0.0483	0.0478	0.0482

5. Real Data

In this section, we consider the mortality data sets of Italy and The Netherlands due to COVID-19. The COVID-19 (coronavirus disease) was declared as a pandemic by World Health Organization (WHO) in 2020. It is the third-highest cause of deaths in 2020, as revealed by the US Centers for Disease Control and Prevention (CDC). The mortality rate was calculated through the ratio of number of deaths and total number of cases (reported cases per 100,000). The mortality rate due to COVID-19 increased by 15.9% from 2019 (see, https://www.pharmaceutical-technology.com/comment/covid-19-cause-death-2020/ (accessed on 26 January 2022)). Here, these two COVID-19 data sets represent the mortality rates for 59 days and 30 days of Italy and The Netherlands, respectively (see, https://covid19.who.int/ (accessed on 26 January 2022)). The mortality rates of Italy recorded from 27 February to 27 April 2020 is given in Table 11. Table 12 represents the mortality rates of The Netherlands recorded from 31 March to 30 April 2020 (also see [32]).

Table 11. Data set of 59 days of COVID-19 mortality rates of Italy.

4.571	7.201	3.606	8.479	11.410	8.961	10.919	10.908	6.503	18.474
11.010	17.337	16.561	13.226	15.137	8.697	15.787	13.333	11.822	14.242
11.273	14.330	16.046	11.950	10.282	11.775	10.138	9.037	12.396	10.644
8.64	8.905	8.906	7.407	7.445	7.214	6.194	4.640	5.452	5.073
4.416	4.859	4.408	4.639	3.148	4.040	4.253	4.011	3.564	3.827
3.134	2.780	2.881	3.341	2.686	2.814	2.508	2.450	1.518	

Table 12. Data set of 30 days of COVID-19 mortality rates of The Netherlands.

14.918	10.656	12.274	10.289	10.832	7.099	5.928	13.211	7.968	7.584
5.555	6.027	4.097	3.611	4.960	7.498	6.940	5.307	5.048	2.857
2.254	5.431	4.462	3.883	3.461	3.647	1.974	1.273	1.416	4.235

In the terms of suitable fitting of the distribution, the GLTE distribution is compared with other related distributions such as exponential distribution, Weibull distribution, and Chen distribution (see Tables 13 and 14). The considered real data set was measured on the basis of -Log L which is the negative of the logarithmic value of likelihood, and Kolmogorov–Smirnov (K-S) test statistic for the distribution selection criterion. We also used Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The AIC and BIC are defined as

$$AIC = 2 \times \ k - 2 \times \ log\hat{L}, \qquad BIC = k \times \ log(n) \ - 2 \times \ log\hat{L}$$

where k = Number of parameters, n = Sample size and $\hat{L}$ = value of the maximum likelihood for the considered distribution. The K-S test statistic D is defined as

$$D = Sup|F_n(x) - F(x)| \ ,$$

where, $F_n(x)$ = Empirical distribution function. From Tables 13 and 14, we observe that the value of AIC, BIC and -Log L of the GLTE distribution is minimum among other distributions. The minimum value of AIC, BIC and -Log L shows that the GLTE distribution better fits for the considered real data sets rather than the other distributions. We observe that Italy's mortality rates for 59 days supports GLTE distribution with the K-S distance as 0.1137 and p-value as 0.3998 for $\alpha = 3.3501$ and $\theta = 0.2626$ (see Figure 2). The Netherlands's mortality rates for 30 days also supports GLTE distribution with the K-S distance as 0.0790 and p-value as 0.9845 for $\alpha = 3.4564$ and $\theta = 0.3542$ (see Figure 3).

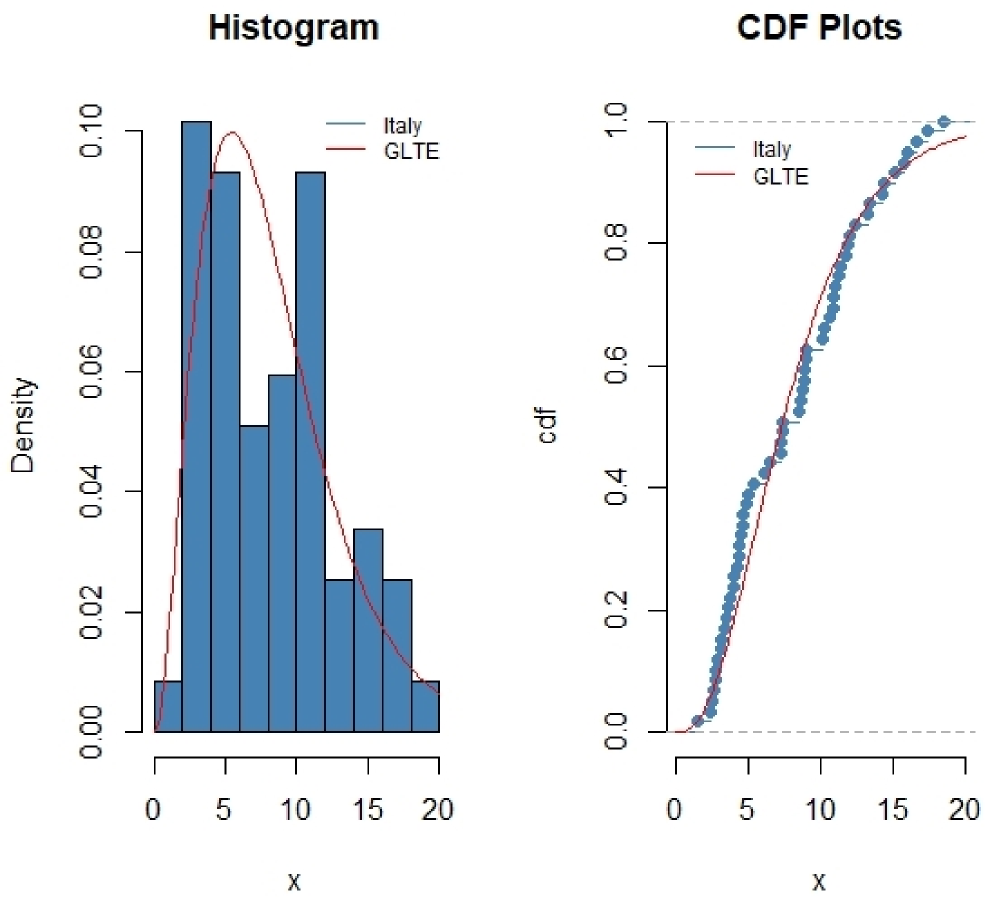

Figure 2. Histogram and ecdf plot for GLTE distribution and the mortality rates of Italy.

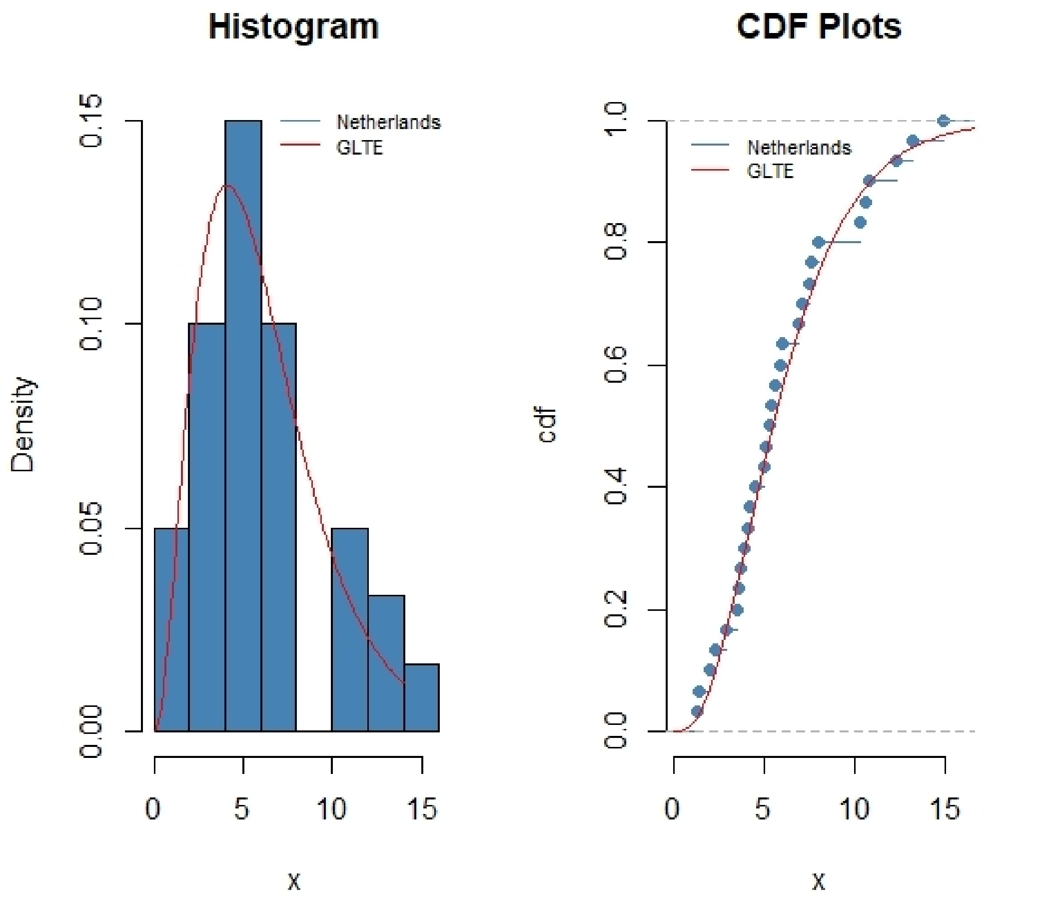

Figure 3. Histogram and ecdf plot for GLTE distribution and the mortality rates of The Netherlands.

Table 13. ML estimates of the parameters, -Log L, K-S distance, AIC and BIC for the fitted models for Italy's mortality rates for 59 days.

Models	Estimates		-Log L	K-S	AIC	BIC
	α	θ				
Exponential	-	1.6880	182.8277	0.2425	367.6554	369.7329
Weibull	0.8522	8.1531	182.0319	0.2089	368.0639	372.2190
GLTE	3.3501	0.2626	167.8433	0.1137	339.6866	343.8417
Chen	0.5528	0.0302	170.3529	0.1148	344.7058	348.8608

Table 14. ML estimates of the parameters, -Log L, K-S distance, AIC and BIC for the fitted models for The Netherlands's mortality rates for 30 days.

Models	Estimates		-Log L	K-S	AIC	BIC
	α	θ				
Exponential	-	1.6732	84.5252	0.2633	171.0505	172.4517
Weibull	0.8543	6.1556	84.1336	0.2147	172.2672	175.0696
GLTE	3.4563	0.3542	76.7005	0.0790	157.4011	160.2035
Chen	0.5674	0.0467	79.3747	0.1298	162.7495	165.5519

Now, we consider censoring schemes (R_1, R_2, R_3) which are used in the simulation study to generate progressive type-II censored samples from the data set of 59 days of mortality rates of Italy of size $m = 15$ and $m = 25$. For the data set of 30 days of mortality rates of The Netherlands of size $m = 5$ and $m = 15$ for the same censoring schemes.

For both the data sets of COVID-19, we calculate the Bayes estimators under symmetric and asymmetric loss functions. For asymmetric loss function, we consider $c = \{-0.5, 0.5\}$ in LINEX loss function and $q = \{-0.5, 0.5\}$ in GE loss functions. The calculated Bayes estimates of α, θ and $R(t)$ as $t = 0.1$ are given in Tables 15 and 16.

Table 15. Bayes estimates under the progressive type-II censored data for the COVID-19 data set of the mortality rates of Italy.

n	m	$C.S.$	Parameter	BS	BL $c = -0.5$	$c = 0.5$	BG $q = -0.5$	$q = 0.5$
59	15	R_1	α	1.4681	1.4786	1.4578	1.4611	1.4472
			θ	0.1006	0.1006	0.1006	0.1006	0.1004
			$R(t)$	0.9987	0.9987	0.9987	0.9987	0.9987
		R_2	α	1.6403	1.6518	1.6289	1.6333	1.6192
			θ	0.1000	0.1000	0.1000	0.1000	0.1000
			$R(t)$	0.9994	0.9994	0.9994	0.9994	0.9994
		R_3	α	1.6385	1.6500	1.6272	1.6316	1.6176
			θ	0.1000	0.1000	0.1000	0.1000	0.1000
			$R(t)$	0.9994	0.9994	0.9994	0.9994	0.9994
59	25	R_1	α	1.3916	1.4003	1.3832	1.3856	1.3735
			θ	0.1007	0.1007	0.1006	0.1006	0.1005
			$R(t)$	0.9983	0.9983	0.9983	0.9983	0.9983
		R_2	α	1.3821	1.3913	1.3732	1.3757	1.3630
			θ	0.1015	0.1015	0.1014	0.1013	0.1011
			$R(t)$	0.9982	0.9982	0.9982	0.9982	0.9982
		R_3	α	1.5243	1.5336	1.5154	1.5184	1.5068
			θ	0.1004	0.1004	0.1004	0.1004	0.1003
			$R(t)$	0.9990	0.9991	0.9991	0.9991	0.9991

Table 16. Bayes estimates under the progressive type-II censored data for the COVID-19 data set of the mortality rates of The Netherlands.

n	m	$C.S.$	Parameter	BS	BL $c = -0.5$	$c = 0.5$	BG $q = -0.5$	$q = 0.5$
30	5	R_1	α	1.3204	1.3331	1.3081	1.3110	1.2923
			θ	0.1012	0.1012	0.1011	0.1010	0.1008
			$R(t)$	0.9973	0.9973	0.9973	0.9973	0.9973
		R_2	α	1.4758	1.4903	1.4617	1.4662	1.4469
			θ	0.1000	0.1000	0.1000	0.1000	0.1000
			$R(t)$	0.9986	0.9986	0.9986	0.9986	0.9986
		R_3	α	1.4918	1.5065	1.4775	1.4821	1.4627
			θ	0.1000	0.1000	0.1000	0.1000	0.1000
			$R(t)$	0.9987	0.9987	0.9987	0.9987	0.9987
30	15	R_1	α	0.9688	0.9797	0.9590	0.9587	0.9388
			θ	0.1268	0.1271	0.1266	0.1249	0.1214
			$R(t)$	0.9863	0.9863	0.9863	0.9863	0.9862
		R_2	α	0.9688	0.9791	0.9590	0.9587	0.9388
			θ	0.1268	0.1271	0.1266	0.1249	0.1214
			$R(t)$	0.9863	0.9863	0.9863	0.9863	0.9862
		R_3	α	0.9631	0.9701	0.9563	0.9560	0.9419
			θ	0.1017	0.1017	0.1017	0.1015	0.1012
			$R(t)$	0.9887	0.9887	0.9887	0.9887	0.9887

6. Conclusions

In this paper, we consider the classical and Bayesian estimation of the unknown parameters and reliability function of the GLTE distribution when the data are progressively type-II censored. The MLEs and Bayes estimators of the parameters and reliability function are obtained. The Bayes estimators were computed using discrete prior for scale parameter and a conditional gamma prior for the shape parameter. We used symmetric (squared error) and asymmetric (LINEX, General Entropy) loss functions to compute the MSEs and ERs. We also address the problem of interval estimation under classical and Bayesian scheme and we derived asymptotic confidence and highest posterior density intervals. The simulation study was done for the different choices of parameter combinations to report the performances of the various estimators, along with the different sample sizes and censoring schemes.

The significance of GLTE distribution over other lifetime distributions such as Weibull, exponential and Chen are discussed in this paper. For this purpose, we considered two real data sets reporting mortality rates of two countries i.e., The Netherlands and Italy. We used the Akaike information and Bayesian information criteria to show the comparison among the probability distributions. The dominance of GLTE distribution over other probability distributions is discussed for the considered data sets of this study. From the simulation study, it is observed that Bayes estimator under LINEX loss function is performing better in most of the cases. It is also worth mentioning that the censoring scheme R_3 works quite well among other schemes considered in this study. So, in real-life situation we should incline our study towards the asymmetric loss function i.e., LINEX and censoring scheme R_3. All the results and conclusions are based on the problem which is considered in this study.

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