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The Basic Locally Primitive Graphs of Order Twice a Prime Square

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Abstract: A graph Γ is called G-basic if G is quasiprimitive or bi-quasiprimitive on the vertex set of Γ, where $G \leqslant \mathrm{Aut}(\Gamma)$. It is known that locally primitive vertex-transitive graphs are normal covers of basic ones. In this paper, a complete classification of the basic locally primitive vertex-transitive graph of order $2p^2$ is given, where p is an odd prime.

Keywords: locally primitive graphs; vertex-transitive graphs; basic graphs

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1. Introduction

Throughout this paper, graphs are assumed to be connected, undirected and simple unless otherwise stated, and groups are assumed to be finite. For a graph Γ, the notation $V\Gamma$, $E\Gamma$ and $\mathrm{Aut}(\Gamma)$ are denoted by its vertex set, edge set, and full group of automorphism respectively. Let $G \leq \mathrm{Aut}(\Gamma)$ be a group of automorphism of Γ. Then, Γ is called *G-vertex-transitive* or *G-edge-transitive* if G is transitive on $V\Gamma$ or $E\Gamma$ respectively.

An arc in Γ is an ordered pair of edges. The graph Γ is called G-arc transitive if G acts transitively on the set of all arcs in Γ. For each $\alpha \in V\Gamma$, let $\Gamma(\alpha) = \{\beta \in V\Gamma | \{\alpha, \beta\} \in E\Gamma\}$ be the set of vertices to which α is adjacent. Then, Γ is called *G-locally primitive* if the stabilizer $G_\alpha = \{x \in G | \alpha^x = \alpha\}$ acts primitively on $\Gamma(\alpha)$.

A graph Γ is called *(G, s)-arc-transitive* for a positive integer s if G acts transitively on the set of s-arcs of Γ. Then, the Γ is called *G-arc-transitive* (namely *symmetric graph*) when $s = 1$. Γ is called *(G, s)-transitive* if it is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive. We know that the (G, s)-arc-transitive graphs with $s \geq 2$ and the arc-transitive graphs with prime valency are both locally primitive. If Γ is G-locally primitive, then it is G-edge transitive.

Moreover, if Γ is both G-vertex transitive and G-locally primitive, then Γ is also G-arc transitive; in this case, Γ is called G-locally primitive arc-transitive. A permutation group G on a set Ω is called quasiprimitive if each nontrivial normal subgroup of G is transitive on Ω. The group G is called bi-quasiprimitive if each nontrivial normal subgroup of G has at most two orbits and there exists at least one normal subgroup of G that has exactly two orbits. A graph Γ is called G-basic if G is quasiprimitive or bi-quasiprimitive on $V\Gamma$ for some $G \leq \mathrm{Aut}\Gamma$.

The study of locally primitive graphs has a long and rich history and has been one of the central topics in algebraic graph theory for decades, see for example [1,2]. The main approach to study locally primitive graphs is *global-action analysis*, which was first systematically investigated by Cheryl Praeger in 1992 [2]. It proved that if a graph Γ is G-locally primitive arc-transitive then either Γ is a G-basic graph or a normal cover of the basic graphs. In this paper, we mainly study the basic locally primitive arc-transitive graphs of order $2p^2$. The classification of some special symmetric graphs of order $2p^2$ has received much attention in the literature.

For instance, references [3–6] gave a classification of arc-transitive graphs of order $2p^2$ with valency 3, 5 and 7. Reference [7] showed that if a graph of order $2p^2$ is both vertex transitive and edge transitive, then it must be arc transitive. Recently, reference [8] gave a classification of tetravalent non-normal Cayley graphs of order $2p^2$. Here, we characterised the locally primitive arc-transitive graphs of order $2p^2$. There are many typical examples, including:

(i) the complete graph K_{2p^2};
(ii) the complete bipartite graph K_{p^2,p^2};
(iii) the graph $\mathsf{K}_{p^2,p^2} - p^2\mathsf{K}_2$ obtained by deleting a 1-factor from K_{p^2,p^2};
(iv) the incidence graph $\mathrm{PH}(d,q)$ and the nonincidence graph $\overline{\mathrm{PH}}(n,q)$ of the projective geometry $\mathrm{PG}(d-1,q)$, where $n \geq 3$ and $\frac{q^n-1}{q-1} = p^2$;
(v) the bidirect square of the incidence graph $\mathrm{D}_2^1(11,5)$ and the nonincidence graph $\overline{\mathrm{D}}_2^1(11,5)$ of the $2-(11,5,1)$-design; and
(vi) the bidirect square of $\mathrm{PH}(d,q)$ and $\overline{\mathrm{PH}}(n,q)$, where $\frac{q^n-1}{q-1} = p$.

This paper gives a classification of vertex quasiprimitive or bi-quasiprimitive locally primitive graph of order $2p^2$. The case when $p = 2$ is characterised in [9]. The main result of the paper is stated as follows.

Theorem 1. *Let Γ be a G-locally primitive graph of order $2p^2$ with valency at least three, where $G \leq \mathsf{Aut}(\Gamma)$ and p is an odd prime. Assume that G is quasiprimitive or biquasiprimitive on the vertices of Γ. Then, Γ is either the bi-normal Cayley graph of the generalized dihedral group $\mathrm{Dih}(\mathbb{Z}_p^2)$, or one of the following graphs:*

(1) $\Gamma \cong \mathsf{K}_{2p^2}, \mathsf{K}_{p^2,p^2}, \mathsf{K}_{p^2,p^2} - p^2\mathsf{K}_2$;
(2) $\Gamma \cong \mathrm{HS}(50)$ *is Hoffman–Singleton graph, and* $G = \mathrm{PSU}(3,5).\mathbb{Z}_2$;
(3) $\mathrm{PH}(n,q)$ *or* $\overline{\mathrm{PH}}(n,q)$, *where* $n \geq 3$ *and* $\frac{q^n-1}{q-1} = p^2$;
(4) *the standard double cover of* $\Sigma^{\times 2}$, *where* $\Sigma = \mathsf{K}_p$; *or*
(5) $\Gamma \cong \Sigma^{\times_{bi} 2}$, *a bidirect square of* Σ, *where* $\Sigma = \mathrm{D}_2^1(11,5)$ *or* $\overline{\mathrm{D}}_2^1(11,5)$, $\mathrm{PH}(n,q)$ *or* $\overline{\mathrm{PH}}(n,q)$ *with* $\frac{q^n-1}{q-1} = p$.

After the introduction, we give some preliminary results in Section 2. In Section 3, we study the basic graphs and complete the proof of Theorem 1.

2. Preliminary Results

First, we collect the description of the eight types of quasiprimitive permutation groups. Let G be a quasiprimitive permutation group on Ω and let $N = \mathsf{soc}(G)$, the *socle* of G. Then, either N is the unique minimal normal subgroup of G or N is the product of two isomorphic and nonabelian minimal normal subgroups of G. Thus, $N = T_1 \times \cdots \times T_k$, where $k \geq 1$ and T is simple. Quasiprimitive permutation group G is divided into eight different types according to the structure and the action of N by O'Nan–Scott's theorem. This was obtained by Praeger in 1992; see [2].

Theorem 2. *Let G be a quasiprimitive permutation group on Ω and $N = \mathsf{soc}(G)$. Then, G is one of the eight types as follows:*

(1) HA : *N is abelian, and thus $N = \mathbb{Z}_p^k$ is regular on Ω and $G \leq Hol(N) = AGL(d,p)$, where p is a prime and $k \geq 1$;*
(2) HS : *$N = M \times L$ such that $M \cong L \cong T$ are nonabelian simple and regular on Ω, and $G \leq Hol(T) = T{:}\mathsf{Aut}(T)$;*
(3) HC : *$N = M \times L$ such that $M \cong L \cong T^l$ with $l \geq 2$ and T nonabelian simple, and $G \leq Hol(M) = M{:}\mathsf{Aut}(M)$;*
(4) AS : *$N = T$ is a nonabelian simple group, and $T \leq G \leq \mathsf{Aut}(T)$;*

(5) SD : $N = T^k$ *with* $k \geq 2$ *and* T *nonabelian simple, and* $N_\omega = \{(t, t, \cdots, t) | t \in T\} \cong T$ *for each* $\omega \in \Omega$*;*
(6) CD : $N = T^k$ *with* $k \geq 2$ *and* T *nonabelian simple, and* $N_\omega \cong T^l$ *with* $l \geq 2$ *and* $l \mid k$*, where* $\omega \in \Omega$*;*
(7) TW : N *is nonabelia, non-simple and minimal normal in* G *acting regularly on* Ω*; and*
(8) PA : N *is a nonabelian minimal normal subgroup that has no regular normal subgroup.*

Let a and d be positive integers. A prime r is called a primitive prime divisor of $a^d - 1$ if r divides $a^d - 1$ but not $a^i - 1$ for $1 \leq i < d$. The following lemma is a well-known result called the Zsigmondy theorem.

Lemma 1 ([10], p. 508)**.** *For any positive integers* a *and* d*, either* $a^d - 1$ *has a primitive prime divisor, or* $(d, a) = (6, 2)$ *or* $(2, 2^m - 1)$*, where* $m \geq 2$*.*

The next lemma can be easily obtained by Lemma 1.

Lemma 2. *Let* $q = r^f$ *with* r *a prime and* f *a positive integer. Assume that* p *is an odd prime and* n*,* m*,* s *are positive integers. Then, the following statements hold.*

(1) *If* $\frac{q^n-1}{q-1} = p^m$*, then* n *is a prime.*
(2) *If* $\frac{q^n-1}{q-1} = 2p^s$*, then* $n = 2$*.*

The following lemma may be deduced from the classification of permutation groups of the degree of a product of two prime powers (refer to [11]).

Lemma 3. *Let* T *be a nonabelian simple group that has a subgroup* H *of index* $2p^2$ *with* p *a prime. Then,* T*,* H *and* $|T{:}H|$ *are as in Table 1.*

Table 1. Non-abelian simple group containing subgroups with index $2p^2$.

Row	T	H	$\lvert T{:}H \rvert$	Remark
1	A_{2p^2}	A_{2p^2-1}	$2p^2$	
2	$\mathrm{PSU}(3,5)$	A_7	$2 \cdot 5^2$	
3	$\mathrm{PSL}(n,q)$	P_1	$2p^2 = \frac{q^n-1}{q-1}$	$n = 2$, and $q = 2^e$
4	$\mathrm{PSL}(n,q)$	H_1	$2p^2 = 2\frac{q^n-1}{q-1}$	$q = 2^e$

Remark 1. $P_1 = [q^{n-1}] : (\mathbb{Z}_{\frac{q-1}{(n,q-1)}}.\mathrm{PSL}(n-1,q)) \cong H_1.\mathbb{Z}_4$*, which is the parabolic subgroup of* $\mathrm{PSL}(n,q)$*.*

The following lemma presents the non-abelian simple groups with a subgroup of prime-power index.

Lemma 4 (Guralnick [12])**.** *Let* T *be a non-abelian simple group with a subgroup* H *of index* p^e*. Then,* T *and* H *are listed in Table 2. Further, either* T *is* 2*-transitive on* $[T:H]$ *or* $T = \mathrm{PSU}(4,2)$*.*

Table 2. Non-abelian simple group containing subgroups with index p^e.

Row	***T***	***H***	$\|T : H\|$
1	A_{p^e}	A_{p^e-1}	p^e
2	$\mathrm{PSL}(n,q)$	P_1	$p^e = \frac{q^n-1}{q-1}$
3	$\mathrm{PSL}(2,11)$	A_5	11
4	M_{11}	M_{10}	11
5	M_{23}	M_{22}	23
6	$\mathrm{PSU}(4,2)$	$\mathbb{Z}_2^4 : \mathsf{A}_5$	27

A group X is called a generalized dihedral group, if there exists an abelian subgroup H and an involution τ such that $X = H{:}\langle\tau\rangle$ and $h^\tau = h^{-1}$ for each $h \in H$. This group is denoted by $\mathrm{Dih}(H)$. Locally primitive graphs must be edge-transitive. The following result can be easily obtained from ([13] Lemma 2.4).

Lemma 5. *Let Γ be a G-locally primitive graph of valency k, where $G \leq \mathsf{Aut}(\Gamma)$. Assume that G contains an abelian normal subgroup N that has exactly two orbits on $V\Gamma$. Then, Γ is a bi-normal Cayley graph of the generalized dihegral group* $\mathrm{Dih}(\mathrm{N})$.

Let Σ_i be a connected graph with vertex set V_i, where $i = 1$ or 2. Recall that the direct product $\Sigma_1 \times \Sigma_2$ is the graph with vertex set $V_1 \times V_2$ such that two vertices (v_1, v_2) and (v_1', v_2') are adjacent if and only if v_i and v_i' are adjacent in Σ_i for $i = 1$ and 2. For convenience, we denote $\Sigma^{\times 2} = \Sigma \times \Sigma$.

For a graph Σ with vertex set V, the standard double cover is defined to be the graph $\bar{\Sigma}$ with the vertex set $V \times \{1,2\}$ and two vertices (α, i) and (β, j) are adjacent if and only if $i \neq j$ and α and β are adjacent in Σ. It is easily shown that $\bar{\Sigma} = \Sigma \times \mathsf{K}_2$, a bipartite graph with biparts $V \times \{1\}$ and $V \times \{2\}$. Clearly, the standard double cover of K_n is $\mathsf{K}_{n,n} - n\mathsf{K}_2$.

Lemma 6 ([14] Lemma 3.3)**.** *Let $\Gamma = (V, E)$ be a connected bipartite graph with biparts B_1 and B_2. Assume that $G \leq \mathsf{Aut}(\Gamma)$ is transitive on E and intransitive on V such that G_α and G_β are conjugate in G, where $\alpha \in B_1$ and $\beta \in B_2$. Then, Γ is the standard double cover of the orbital graph Σ of G acting on B_2. Furthermore, Γ is G-locally primitive if and only if Σ is.*

We end this section by introducing the definition of the bidirect product of graphs. Let Σ be a connected bipartite graph with biparts U and W. The bidirect square $\Sigma^{\times_{bi}2}$ is defined to be the graph with vertex set $(U \times U) \cup (W \times W)$ such that $(u_1, u_2) \sim (w_1, w_2)$ if and only if that both $u_1 \sim w_1$ and $u_2 \sim w_2$ in Σ (where $\sim$ denotes adjacency). Clearly, $\Sigma^{\times_{bi}2}$ is a connected component of $\Sigma^{\times 2}$.

3. Basic Graphs

In this section, let Γ be a G-locally primitive and vertex-transitive graph of order $2p^2$, where $G \leq \mathsf{Aut}\Gamma$ and p is an odd prime. The vertex-quasiprimitive case is considered in Section 3.1, and the vertex-biquasiprimitive case is studied in Section 3.2.

3.1. Vertex-Quasiprimitive Case

Suppose that G is quasiprimitive on $V\Gamma$. Then, G is a permutation group of degree $2p^2$. Set $\mathrm{N} := \mathsf{soc}(G)$, which is the product of all minimal normal subgroups of G. By Theorem 2, G is of type AS or PA.

We first give an example satisfying the main theorem.

Example 1. *Let $X = \mathrm{PSU}(3,5).\mathbb{Z}_2$ and $H(\cong S_7)$ be a maximal subgroup of X. By Magma, there exists an involution $g \in X \backslash H$ such that $\langle H, g\rangle = X$ and $H \cap H^g = A_6.\mathbb{Z}_2$. Define a coset graph* $\mathrm{HS}(50) := \mathsf{Cos}(X, H, HgH)$*. A calculation by Magma shows that* $\mathsf{Aut}(\Gamma) = X$*, $HS(50)$ is $(X, 3)$-transitive, which is essentially the Hoffman–Singleton graph of order* 50 *with valency* 7.

Now, we consider the case that G is almost simple.

Lemma 7. *Suppose that G is almost simple and quasiprimitive on $V\Gamma$. Then, Γ is 2-arc transitive, and Γ is one of the following:*

(1) $\Gamma \cong \mathsf{K}_{2p^2}$ *and* $\mathsf{soc}(G) = \mathsf{A}_{2p^2}$ *or* $\mathrm{PSL}(2,q)$*;*

(2) $\Gamma \cong \mathrm{HS}(50)$ *is a Hoffman–Singleton graph, which is a 3-transitive non-Cayley graph.*

Proof. Note that G is quasiprimitive on $V\Gamma$ and $T = \mathsf{soc}(G)$. Then

$$G_\alpha/T_\alpha \cong G_\alpha/T \cap G_\alpha \cong TG_\alpha/T \cong G/T \cong O,$$

where $O \le \mathrm{Out}(T)$. Thus, G is primitive if and only if T is primitive, and $|V\Gamma| = |\alpha^G| = |\alpha^T| = |T : T_\alpha| = 2p^2$ for some $\alpha \in V\Gamma$. Thus, the couple (T, T_α) is listed in Table 1.

For row 1, $T = \mathsf{soc}(G) = \mathsf{A}_{2p^2}$ is primitive on $V\Gamma$, and T is 2-transitive on $[T : T_\alpha]$. Thus, it follows that $\Gamma \cong \mathsf{K}_{2p^2}$ is a complete graph. Since $G = \mathsf{Aut}(\Gamma) = \mathsf{S}_{2p^2}$, Γ is $(G,2)$-arc transitive. Clearly, Γ is G-locally primitive arc-transitive.

For row 2, $T = \mathsf{soc}(G) = \mathrm{PSU}(3,5)$, and $T_\alpha = \mathsf{A}_7$. Since G_α is primitive on $\Gamma(\alpha)$, the arc stabilizer $G_{\alpha\beta}$ is a maximal subgroup of the vertex stabilizer G_α for each $\beta \in \Gamma(\alpha)$. Note that $G/T \cong G_\alpha/T_\alpha \le \mathsf{Out}(T)$. Then, $T_{\alpha\beta}$ is a maximal subgroup of T_α. Thus, $k = \mathrm{val}(\Gamma) = |G_\alpha : G_{\alpha\beta}| = |T_\alpha : T_{\alpha\beta}|$. By Atlas [15], one knows that the possible value of k is $7, 21$ or 35. If $k = 7$, by Example 1, $\Gamma \cong \mathrm{HS}(50)$, which is a 3-transitive non-Cayley graph. If $k = 21$ or 35, by Magma, $T_{\alpha\beta} = \mathrm{PSL}(2,7)$, S_5 or $(\mathsf{A}_4 \times \mathbb{Z}_3) : \mathbb{Z}_2$. In these cases, one has that $\mathrm{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$. Thus, Γ is not connected. Therefore, there is no graphs in these cases.

For row 3, $T = \mathsf{soc}(G) = \mathrm{PSL}(2,q)$, and $T_\alpha = [q] : \mathbb{Z}_{q+1}$, which is a parabolic subgroup of T. Since $\mathrm{PSL}(2,q)$ is 2-transitive on $V\Gamma = [T : T_\alpha]$ with $q = 2p^2 - 1$. Hence, $\Gamma \cong \mathsf{K}_{2p^2}$ with $\mathrm{val}(\Gamma) = q$, and $\mathrm{PSL}(2,q) \le G \le P\Gamma L(2,q)$.

Finally, let $T = \mathrm{PSL}(n,q)$ and $T_\alpha = H_1$. Note that $\frac{q^n-1}{q-1} = p^2$. By Lemma 2, n is a prime. Suppose that $n = 2$. Note that $(p+1)(p-1) = q = 2^e$. It follows that $p = 3$ and $e = 3$. Thus, $T = \mathrm{PSL}(2,8)$ and $P_1 \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7$, and thus $T_\alpha = H_1 \cong \mathrm{D}_{14}$. Now, $|T{:}T_\alpha| = 36 \ne 2p^2$, a contradiction occurred. Suppose that $n \ge 3$. If $(n,q) = (n,2^e) = (3,2)$, then $T = \mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ and $T_\alpha = H_1 \cong \mathrm{D}_6$. Now, $|V\Gamma| = |T{:}T_\alpha| = 28 \ne 2p^2$ for any prime p, which is a contradiction. Assume that $(n,q) = (3,2^e)$ and $e \ge 2$. Let r be an odd prime divisor of $q+1 = 2^e+1$. As $(q-1,q+1) = (2^e-1, 2^e+1) = 1$, we have that $(r, q(q-1)) = (r, 2^e(2^e-1)) = 1$.

Now, it follows that $(r, \frac{|P_1|}{|\mathrm{PSL}(n-1,q)|}) = 1$. Since $(q+1) \mid |H_1|$, $|H_1|$ and $|H_1^{\Gamma(\alpha)}|$ have the same prime divisors, one has that $r \mid |H_1^{\Gamma(\alpha)}|$. It follows that $\mathrm{PSL}(n-1,q) = \mathrm{PSL}(2,2^e)$ is a combinatorial factor of $H_1^{\Gamma(\alpha)}$. As $H_1^{\Gamma(\alpha)} = T_\alpha^{\Gamma(\alpha)}$ is a primitive permutation group, it concludes with Theorem 2 that $T_\alpha^{\Gamma(\alpha)}$ is almost simple and $\mathsf{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(2,2^e)$. By checking the maximal subgroup of $\mathrm{PSL}(2,2^e)$, we have that either $2^e = 11$ or 2^e+1 is a prime. Clearly, the former case is impossible. For the later case that 2^e+1 is a Fermat prime, then $e = 2^f$ for some positive integer f. Now, $q^2+q+1 = 2^{2e}+2^e+1 = (2^e+1)^2 - 2^e = (2^{2^f}+1)^2 - (2^{2^{f-1}})^2 = (2^{2^f}+1+2^{2^{f-1}})(2^{2^f}+1-2^{2^{f-1}})$. By easy calculation, there exists no prime p satisfying $q^2+q+1 = p^2$.

Assume that $n \ge 4$. If $(n,q) = (7,2)$, then $7 \mid |H_1|$, but $7 \nmid \frac{|P_1|}{|\mathrm{PSL}(n-1,q)|} = \frac{|P_1|}{|\mathrm{PSL}(6,2^e)|}$. If $(n,q) \ne (7,2)$, then by Lemma 1 $q^{n-1}-1$ has a primitive prime divisor, say r, and $r \nmid \frac{|P_1|}{|\mathrm{PSL}(n-1,q)|}$. Since $|H_1|$ and $|H_1^{\Gamma(\alpha)}|$ have the same prime divisors, we conclude that $\mathrm{PSL}(n-1,q)$ is a combinatorial factor of $H_1^{\Gamma(\alpha)}$. Hence, the primitive permutation group $H_1^{\Gamma(\alpha)}$ is almost simple with socle $\mathrm{PSL}(n-1,q)$. Thus, $\frac{q^{n-1}-1}{q-1} = \frac{(2^e)^{n-1}-1}{2^e-1}$ is a prime. It follows from Lemma 2 that $n-1$ is a prime, noting that n is a prime. Then, $n = 3$, which is a contradiction with the assumption. $\square$

The following lemma considers the case that G is of type PA.

Lemma 8. *Suppose that G is a quasiprimitive permutation group of product action type on $V\Gamma$. Then, no graphs appear.*

Proof. By assumption, let $N = \mathsf{soc}(G) = T \times \cdots \times T = T^d$. Then, N is not regular and also has no subgroup that is regular on $V\Gamma$. Further, there exists an almost simple group U with socle T satisfying that $G \leq \mathrm{U} \wr \mathrm{S}_d$. If U is a permutation group on a set Δ, then $V\Gamma := \Delta \times \cdots \times \Delta = \Delta^d$. For a vertex $\alpha = (\delta, \cdots, \delta) \in V\Gamma$, then $|N{:}N_\alpha| = |T{:}T_\delta|^d = 2p^2$, it follows that $d = 1$, which is a contradiction with $d \geq 2$. □

3.2. Vertex-Biquasiprimitive Case

Suppose that G acts biquasiprimitively on $V\Gamma$ with biparts $\mathrm{B}_1, \mathrm{B}_2$. Then, $V\Gamma = \mathrm{B}_1 \cup \mathrm{B}_2$ and $|\mathrm{B}_1| = |\mathrm{B}_2| = p^2$. Set $G^+ = G_{\mathrm{B}_1} = \{g \in G | \mathrm{B}_1^g = \mathrm{B}_1\}$. Then, $G^+ = G_{\mathrm{B}_2}$. Clearly, G^+ is the normal subgroup of G with index 2 and quasiprimitive on B_1 and B_2.

If G^+ acts unfaithfully on B_1 or B_2, by reference [16] (Lemma 5.2), $\Gamma \cong \mathsf{K}_{p^2,p^2}$ and $\mathbb{Z}_{p^2}^2 : \mathbb{Z}_2 \leq G \leq \mathsf{Aut}(\mathsf{K}_{p^2,p^2}) = \mathsf{S}_{p^2} \wr \mathbb{Z}_2$. Suppose that G^+ acts faithfully on both B_1 and B_2. By [9] (Theorem 2.2), quasiprimitive permutation groups of prime-power degree is primitive. Then, G^+ is primitive on both B_1 and B_2. Thus, G is a biprimitive permutation group on $V\Gamma$. By Lemma 4 and Theorem 2, the following result is obtained.

Lemma 9. *Suppose that G^+ is faithful on both B_1 and B_2. Then the actions of G^+ on B_i are permutationally isomorphic, and one of the following holds:*

(1) G^+ is affine and $\mathsf{soc}(G^+) = \mathbb{Z}_p^2$;

(2) G^+ is almost simple and $\mathsf{soc}(G^+) = \mathsf{A}_{p^2}$ or $\mathrm{PSL}(n,q)$ with n, q satisfying $\frac{q^n-1}{q-1} = p^2$. In addition, G^+ is 2-transtive on B_i, $i = 1, 2$;

(3) G^+ is of product action type and $\mathsf{soc}(G^+) = T^2$, where $T = \mathsf{A}_p$, M_{11}, M_{23}, $\mathrm{PSL}(2,11)$, or $\mathrm{PSL}(n,q)$ with n, q satisfying $\frac{q^n-1}{q-1} = p$.

The next lemma determines the graph according to the structure of G^+.

Lemma 10. *Let Γ be a G-locally primitive graph, and G be biquasiprimitive on $V\Gamma$ with biparts B_1 and B_2. Assume that $G^+ = G_{\mathrm{B}_1} = G_{\mathrm{B}_2}$ is faithful on both B_1 and B_2. Then, one of the followings holds:*

(1) Γ is a bi-normal Cayley graph of the generalized dihedral group $\mathrm{Dih}(\mathbb{Z}_p^2)$;

(2) $\Gamma \cong \mathsf{K}_{p^2,p^2} - p^2\mathsf{K}_2$, $\mathrm{PH}(n,q)$ or $\overline{\mathrm{PH}}(n,q)$ with $\frac{q^n-1}{q-1} = p^2$;

(3) Γ is the standard double cover of $\Sigma^{\times 2}$, where $\Sigma = \mathsf{K}_p^{\times 2}$; and

(4) $\Gamma = \Sigma^{\times_{bi} 2}$, where $\Sigma = \mathrm{D}_2^1(11,5)$, $\overline{\mathrm{D}}_2^1(11,5)$, $\mathrm{PH}(n,q)$ or $\overline{\mathrm{PH}}(n,q)$ with $\frac{q^n-1}{q-1} = p$.

Proof. By Lemma 9, G^+ is a primitive permutation group of p^2 degree and is of type HA, AS or PA.

Assume that G^+ is of type HA. Then, $N := \mathrm{soc}(G^+) = \mathbb{Z}_p^2$, which is regular on B_i. Since Nchar G^+ and $G^+ \lhd G$, we have that $N \lhd G$. By Lemma 5, Γ is stated as in (1);

Assume that G^+ is of type AS. Let $\mathrm{soc}(G^+) = T$. Then, T is non-abelian simple and transitive on both B_1 and B_2. Let $C = \mathrm{C}_G(T)$ be the centralizer of T in G. Then, $C \lhd G$. Suppose that $C \neq 1$. Since $C \cap G^+ = \mathrm{C}_{G^+}(T) = 1$, it follows that $\langle C, G^+\rangle = C \times G^+ = G$, and thus $C = \langle g\rangle \cong \mathbb{Z}_2$, where g interchanges B_1 and B_2. Clearly, C is semiregular and has p^2 orbits on $V\Gamma$. Now, the quotient graph Γ_C induced by C is a G^+-locally primitive arc-transitive graph of order p^2. By [17] (Lemma 2.2), Γ is the standard double cover of Γ_C. By Lemma 9 (2), we know that G^+ is 2-transitive on $V\Gamma_C$. Thus, $\Gamma_C = \mathsf{K}_{p^2}$. It follows that $\Gamma \cong \mathsf{K}_{p^2,p^2} - p^2\mathsf{K}_2$.

Suppose that $C = 1$. Then, $T \lhd G \leq \mathrm{Aut}(T)$. Thus, G is an almost simple group. Assume that the induced permutations $(G^+)^{\mathrm{B}_1}$ and $(G^+)^{\mathrm{B}_2}$ are permutation equivalent. By Lemma 4, G^+ is 2-transitive on B_i, so the orbital graph of G^+ acting on B_i is K_{p^2}. For $\alpha \in B_1$ and $\beta \in B_2$, because G_α and G_β are conjugate in G, by Lemma 6, Γ is the standard double cover of K_{p^2}. Thus, $\Gamma \cong \mathsf{K}_{p^2,p^2} - p^2\mathsf{K}_2$. Assuming that $(G^+)^{\mathrm{B}_1}$ and $(G^+)^{\mathrm{B}_2}$ are permutation inequivalent, by Lemma 4, $T = \mathrm{PSL}(n,q)$ and n,q satisfy $\frac{q^n-1}{q-1} = p^2$. As G_α and G_β are not conjugate in G, we get $\Gamma \cong \mathrm{PH}(n,q)$ or $\overline{\mathrm{PH}}(n,q)$ from [14] (Example 3.6).

Assume that the action of G^+ on B_i is of type PA. By Lemma 9, $N = \mathrm{soc}(G^+) = T \times T$, where T is stated as in (3) of Lemma 9. Noting that G^+ is transitive on B_i and $G_\alpha = G_\alpha^+$, then the orbital graph Σ of G^+ acting on B_2 is G-locally primitive arc transitive. Let $V\Sigma = B_2 = \Delta \times \Delta$. For a vertex $\alpha = (\delta,\delta) \in V\Sigma$, since $G_\alpha^{\Sigma(\alpha)}$ is primitive, we have that $\Sigma(\alpha)$ is an orbit of G_α on $B_2 \setminus \{\alpha\}$. By [14] (Lemma 2.4), $\Sigma(\alpha) = \Delta(\delta)^2$, where $\Delta(\delta)$ is an orbit of H in $\Delta \setminus \{\delta\}$ and H is almost simple with socle T and primitive on Δ satisfying that $G^+ \leq H \wr \mathrm{S}_2$. It follows that $\Sigma = \Pi \times \Pi = \Pi^{\times 2}$. Since $T = \mathrm{A}_p$, M_{11}, M_{23}, $\mathrm{PSL}(2,11)$, or $\mathrm{PSL}(n,q)$ with n,q satisfying $\frac{q^n-1}{q-1} = p$, which are 2-transitive on Δ. We conclude that $\Pi = \mathsf{K}_p$ is a complete graph. That is $\Sigma = \mathsf{K}_p^{\times 2}$.

Suppose that $(G^+)^{\mathrm{B}_1}$ and $(G^+)^{\mathrm{B}_2}$ are permutation equivalent. By Lemma 6, Γ is the standard double cover of $\Sigma = \mathsf{K}_p^{\times 2}$. Suppose that $(G^+)^{\mathrm{B}_1}$ and $(G^+)^{\mathrm{B}_2}$ are permutation inequivalent. By Lemma 9, $T = \mathrm{PSL}(2,11)$, and $T_{\delta_1} \cong T_{\delta_2} \cong \mathrm{A}_5$ or $T = \mathrm{PSL}(n,q)$ acts on 1-subspace or hyperplane of n-dimensional linear space over the field F_q and n,q satisfying that $\frac{q^n-1}{q-1} = p$. Since Γ is N-edge transitive, N_{b_1} and N_{b_2} are not conjugate in N, by [14] (Lemma 3.9), $\Gamma = \Sigma^{\times_{bi} 2}$, is a bidirect product of Σ, where $\Sigma = \mathrm{D}_2^1(11,5)$, $\overline{\mathrm{D}}_2^1(11,5)$, $\mathrm{PH}(n,q)$, $\overline{\mathrm{PH}}(n,q)$. $\square$

Proof of Theorem 1. Now, we are ready to complete the proof of the main Theorem 1. Let Γ be a G-locally primitive graph of order $2p^2$, where p is an odd prime.

Assume that G is quasiprimitive on $V\Gamma$. By Lemmas 7 and 8, Γ is the complete graph K_{2p^2} or the Hoffman–Singleton graph HS(50). Thus, the graphs in Theorem 1 (1), (2) hold.

Assume that G is biquasiprimitive on $V\Gamma$. Then, $\Gamma = \mathsf{K}_{p^2,p^2}$ or the standard double cover of K_{p^2}. By Lemma 10, either Γ is a bi-normal Cayley graph on the generalized dihedral group $\mathrm{Dih}(\mathbb{Z}_p^2)$, or Γ is the graph as in Theorem 1 (3)–(5). Thus, the proof of Theorem 1 is completed. $\square$

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