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On the Converse Jensen-Type Inequality for Generalized f-Divergences and Zipf–Mandelbrot Law

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Abstract: Motivated by some recent investigations about the sharpness of the Jensen inequality, this paper deals with the sharpness of the converse of the Jensen inequality. These results are then used for deriving new inequalities for different types of generalized f-divergences. As divergences measure the differences between probability distributions, these new inequalities are then applied on the Zipf–Mandelbrot law as a special kind of a probability distribution.

Keywords: Green function; converse Jensen inequality; f-divergence; Zipf–Mandelbrot law

MSC: 26D15; 60E05; 94A17

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1. Introduction

Some of the most famous inequalities in mathematics are surely the Jensen inequality and its converse. The converse Jensen inequality is given by Lah and Ribarič in [1] and separately by Edmundson in [2], so it is sometimes referred by Edmundson–Lah–Ribarič inequality. The Jensen inequality and its converse are closely connected with the Hermite-Hadamard inequality, and these three inequalities have always been the great inspiration for further investigations, generalizations, refinements, improvements and extensions. An interested reader can consult several very new papers, published just in the last few months, in order to obtain a more comprehensive understanding of the recent research progress in this field (see for example [3–10]).

In the recent papers [11,12] the authors investigated the sharpness of the Jensen inequality. However, how sharp is the converse of the Jensen inequality?

Let $[\alpha, \beta]$ be an interval in $\mathbb{R}$. Consider the Green functions $G_k : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$, $(k = 0, 1, 2, 3, 4)$ defined by

$$G_0(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha} & \text{for } \alpha \le s \le t, \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha} & \text{for } t \le s \le \beta, \end{cases}$$

$$G_1(t,s) = \begin{cases} \alpha - s, & \text{for } \alpha \le s \le t, \\ \alpha - t, & \text{for } t \le s \le \beta, \end{cases}$$

$$G_2(t,s) = \begin{cases} t - \beta, & \text{for } \alpha \le s \le t, \\ s - \beta, & \text{for } t \le s \le \beta, \end{cases}$$

$$G_3(t,s) = \begin{cases} t - \alpha, & \text{for } \alpha \le s \le t, \\ s - \alpha, & \text{for } t \le s \le \beta, \end{cases}$$

$$G_4(t,s) = \begin{cases} \beta - s, & \text{for } \alpha \le s \le t, \\ \beta - t, & \text{for } t \le s \le \beta. \end{cases}$$

By means of these functions, the authors in [13,14] gave the uniform treatment of the Jensen type inequalities, allowing the measure also to be negative. In this paper, we continue this investigation, and we concentrate on the converse Jensen inequality.

The paper is organized as follows: after this introduction, in the Section 2 we give our main results. We analyse the sharpness of the converse of the Jensen inequality. Here, instead of the convexity of the function, we use previously mentioned Green functions. After the first theorem, the following corollaries give us some further results and an example with the condition which is easier to verify. In the Section 3, the analogous results in discrete case are presented. As we know, the Jensen inequality is important when obtaining inequalities for divergences. Therefore, in our Section 4 we use our results with the converse Jensen inequality in order to derive new inequalities for different types of generalized f-divergences. According to their definition, divergences measure the differences between probability distributions. So, to conclude the paper, in the Section 5 we apply our results with f-divergences on the special kind of a probability distribution defined as Zipf–Mandelbrot law.

2. Main Results

To simplify the notation, we denote

$$\overline{g} = \frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}.$$

We give our first result.

Theorem 1. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, where $Im(g) \subseteq [\alpha, \beta]$. Let $m, M \in [\alpha, \beta]$ $(m \neq M)$ be such that $m \leq g(t) \leq M$ for all $t \in [a, b]$. Let $\lambda : [a, b] \to \mathbb{R}$ be a function of bounded variation, such that $\lambda(a) \neq \lambda(b)$. Let $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$.*

Then

$$\left| \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \right| \leq Q \cdot \left\| \varphi'' \right\|_p$$

holds, where

$$Q = \begin{cases} \left[\int_\alpha^\beta \left| \frac{M-\overline{g}}{M-m} G_k(m,s) + \frac{\overline{g}-m}{M-m} G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \right|^q ds \right]^{\frac{1}{q}}, & \text{for } q \neq \infty; \\ \sup_{s \in [\alpha,\beta]} \left\{ \left| \frac{M-\overline{g}}{M-m} G_k(m,s) + \frac{\overline{g}-m}{M-m} G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \right| \right\}, & \text{for } q = \infty. \end{cases}$$

Proof. Using the functions G_k $(k = 0, 1, 2, 3, 4)$ we can represent every function $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, as:

$$\varphi(x) = \frac{\beta - x}{\beta - \alpha}\varphi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\varphi(\beta) + \int_\alpha^\beta G_0(x,s)\varphi''(s)ds,$$

$$\varphi(x) = \varphi(\alpha) + (x - \alpha)\varphi'(\beta) + \int_\alpha^\beta G_1(x,s)\varphi''(s)ds,$$

$$\varphi(x) = \varphi(\beta) + (x - \beta)\varphi'(\alpha) + \int_\alpha^\beta G_2(x,s)\varphi''(s)ds,$$

$$\varphi(x) = \varphi(\beta) - (\beta - \alpha)\varphi'(\beta) + (x - \alpha)\varphi'(\alpha) + \int_\alpha^\beta G_3(x,s)\varphi''(s)ds,$$

$$\varphi(x) = \varphi(\alpha) + (\beta - \alpha)\varphi'(\alpha) - (\beta - x)\varphi'(\beta) + \int_\alpha^\beta G_4(x,s)\varphi''(s)ds,$$

which can be easily shown by integrating by parts. For instance, for $k = 0$ we have

$$\begin{aligned}
\int_\alpha^\beta G_0(x,s)\varphi''(s)ds &= \int_\alpha^x G_0(x,s)\varphi''(s)ds + \int_x^\beta G_0(x,s)\varphi''(s)ds \\
&= \int_\alpha^x \frac{(x-\beta)(s-\alpha)}{\beta-\alpha}\varphi''(s)ds + \int_x^\beta \frac{(s-\beta)(x-\alpha)}{\beta-\alpha}\varphi''(s)ds \\
&= \frac{x-\beta}{\beta-\alpha}\int_\alpha^x (s-\alpha)\varphi''(s)ds + \frac{x-\alpha}{\beta-\alpha}\int_x^\beta (s-\beta)\varphi''(s)ds \\
&= \frac{x-\beta}{\beta-\alpha}\left[\left((s-\alpha)\varphi'(s)\right)|_\alpha^x - \varphi(s)|_\alpha^x\right] + \frac{x-\alpha}{\beta-\alpha}\left[\left((s-\beta)\varphi'(s)\right)|_x^\beta - \varphi(s)|_x^\beta\right] \\
&= \varphi(x) - \frac{\beta-x}{\beta-\alpha}\varphi(\alpha) - \frac{x-\alpha}{\beta-\alpha}\varphi(\beta),
\end{aligned}$$

which proves the first identity. For $k = 1$ we have

$$\begin{aligned}
\int_\alpha^\beta G_1(x,s)\varphi''(s)ds &= \int_\alpha^x G_1(x,s)\varphi''(s)ds + \int_x^\beta G_1(x,s)\varphi''(s)ds \\
&= \int_\alpha^x (\alpha-s)\varphi''(s)ds + \int_x^\beta (\alpha-x)\varphi''(s)ds \\
&= \left((\alpha-s)\varphi'(s)\right)|_\alpha^x + \varphi(s)|_\alpha^x + (\alpha-x)\varphi'(s)|_x^\beta \\
&= \varphi(x) - \varphi(\alpha) + (\alpha-x)\varphi'(\beta),
\end{aligned}$$

and this gives us the second identity. The other identities can be proved analogously.

Furthermore, by simple calculation using these identities it can be shown that for every function $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$, and for any $k \in \{0,1,2,3,4\}$ holds

$$\begin{aligned}
&\frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \\
&= \int_\alpha^\beta \left[\frac{M-\overline{g}}{M-m}G_k(m,s) + \frac{\overline{g}-m}{M-m}G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)}\right]\varphi''(s)ds.
\end{aligned}$$

Now, using the triangle inequality for integrals we get

$$\begin{aligned}
&\left|\frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)}\right| \\
&= \left|\int_\alpha^\beta \left[\frac{M-\overline{g}}{M-m}G_k(m,s) + \frac{\overline{g}-m}{M-m}G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)}\right]\varphi''(s)ds\right| \\
&\leq \int_\alpha^\beta \left|\left[\frac{M-\overline{g}}{M-m}G_k(m,s) + \frac{\overline{g}-m}{M-m}G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)}\right]\varphi''(s)\right|ds,
\end{aligned}$$

and then applying the Hölder inequality we get the statement of our theorem. □

Let us see what happens for $q = 1$, $p = \infty$. If the term

$$\frac{M-\overline{g}}{M-m}G_k(m,s) + \frac{\overline{g}-m}{M-m}G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)}$$

has the same positivity for all $s \in [\alpha,\beta]$, then we can calculate Q. The following result holds.

Corollary 1. *Let $g : [a,b] \to \mathbb{R}$ be continuous function and $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$, where $Im(g) \subseteq [\alpha,\beta]$. Let $m, M \in [\alpha,\beta]$ $(m \neq M)$ be such that $m \leq g(t) \leq M$ for all $t \in [a,b]$. Let $\lambda : [a,b] \to \mathbb{R}$ be a function of bounded variation, such that $\lambda(a) \neq \lambda(b)$. If for any $k \in \{0,1,2,3,4\}$ for all $s \in [\alpha,\beta]$ the inequality*

$$\frac{M-\overline{g}}{M-m} G_k(m,s) + \frac{\overline{g}-m}{M-m} G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \geq 0 \tag{1}$$

holds, or if for any $k \in \{0,1,2,3,4\}$ for all $s \in [\alpha,\beta]$ the reverse inequality in (1) holds, then

$$\begin{aligned} &\left| \frac{M-\overline{g}}{M-m} \varphi(m) + \frac{\overline{g}-m}{M-m} \varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \right| \\ &\leq \frac{1}{2} \cdot \|\varphi''\|_\infty \cdot \left| \frac{M-\overline{g}}{M-m} \cdot m^2 + \frac{\overline{g}-m}{M-m} \cdot M^2 - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right|. \end{aligned} \tag{2}$$

Proof. Applying Theorem 1 for $q = 1$, $p = \infty$ we get

$$\begin{aligned} &\left| \frac{M-\overline{g}}{M-m} \varphi(m) + \frac{\overline{g}-m}{M-m} \varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \right| \\ &\leq \|\varphi''\|_\infty \cdot \int_\alpha^\beta \left| \frac{M-\overline{g}}{M-m} G_k(m,s) + \frac{\overline{g}-m}{M-m} G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \right| ds. \end{aligned} \tag{3}$$

If the term $\frac{M-\overline{g}}{M-m} G_k(m,s) + \frac{\overline{g}-m}{M-m} G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)}$ doesn't change it's positivity for all $s \in [\alpha,\beta]$, we can calculate the integral on the right side of (3).

Let us start with the case when $k = 0$. We have

$$\begin{aligned} \int_\alpha^\beta G_0(t,s)ds &= \int_\alpha^t \frac{(t-\beta)(s-\alpha)}{\beta-\alpha} ds + \int_t^\beta \frac{(s-\beta)(t-\alpha)}{\beta-\alpha} ds \\ &= \frac{t-\beta}{\beta-\alpha} \int_\alpha^t (s-\alpha)ds + \frac{t-\alpha}{\beta-\alpha} \int_t^\beta (s-\beta)ds \\ &= \frac{t-\beta}{\beta-\alpha} \left(\frac{1}{2}(t^2-\alpha^2) - \alpha(t-\alpha) \right) + \frac{t-\alpha}{\beta-\alpha} \left(\frac{1}{2}(\beta^2-t^2) - \beta(\beta-t) \right) \\ &= \frac{1}{2}(t-\alpha)(t-\beta), \end{aligned}$$

and therefore it is

$$\begin{aligned} &\int_\alpha^\beta \left(\frac{M-\overline{g}}{M-m} G_0(m,s) + \frac{\overline{g}-m}{M-m} G_0(M,s) - \frac{\int_a^b G_0(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \right) ds \\ &= \frac{M-\overline{g}}{M-m} \int_\alpha^\beta G_0(m,s)ds + \frac{\overline{g}-m}{M-m} \int_\alpha^\beta G_0(M,s)ds - \frac{1}{\int_a^b d\lambda(x)} \int_a^b \left(\int_\alpha^\beta G_0(g(x),s)ds \right) d\lambda(x) \\ &= \frac{1}{2} \frac{M-\overline{g}}{M-m} (m-\alpha)(m-\beta) + \frac{1}{2} \frac{\overline{g}-m}{M-m} (M-\alpha)(M-\beta) \\ &\quad - \frac{1}{\int_a^b d\lambda(x)} \int_a^b \left(\frac{1}{2}(g(x)-\alpha)(g(x)-\beta) \right) d\lambda(x) \\ &= \frac{1}{2} \cdot \left[\frac{M-\overline{g}}{M-m} \cdot m^2 + \frac{\overline{g}-m}{M-m} \cdot M^2 - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right]. \end{aligned}$$

Similarly, when we consider the case when $k = 1$, we have

$$\begin{aligned}
\int_\alpha^\beta G_1(t,s)ds &= \int_\alpha^t (\alpha - s)ds + \int_t^\beta (\alpha - t)ds \\
&= \alpha \int_\alpha^t ds - \int_\alpha^t sds + (\alpha - t)\int_t^\beta ds \\
&= \alpha(t-\alpha) - \frac{1}{2}(t^2 - \alpha^2) + (\alpha - t)(\beta - t) \\
&= \frac{1}{2}(t-\alpha)(t+\alpha-2\beta),
\end{aligned}$$

and we obtain

$$\begin{aligned}
&\int_\alpha^\beta \left(\frac{M-\overline{g}}{M-m} G_1(m,s) + \frac{\overline{g}-m}{M-m} G_1(M,s) - \frac{\int_a^b G_1(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \right) ds \\
&= \frac{M-\overline{g}}{M-m} \int_\alpha^\beta G_1(m,s)ds + \frac{\overline{g}-m}{M-m} \int_\alpha^\beta G_1(M,s)ds - \frac{1}{\int_a^b d\lambda(x)} \int_a^b \left(\int_\alpha^\beta G_1(g(x),s)ds \right) d\lambda(x) \\
&= \frac{1}{2}\frac{M-\overline{g}}{M-m}(m-\alpha)(m+\alpha-2\beta) + \frac{1}{2}\frac{\overline{g}-m}{M-m}(M-\alpha)(M+\alpha-2\beta) \\
&\quad - \frac{1}{\int_a^b d\lambda(x)} \int_a^b \left(\frac{1}{2}(g(x)-\alpha)(g(x)+\alpha-2\beta) \right) d\lambda(x) \\
&= \frac{1}{2} \cdot \left[\frac{M-\overline{g}}{M-m} \cdot m^2 + \frac{\overline{g}-m}{M-m} \cdot M^2 - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right].
\end{aligned}$$

For $k = 2$ we have

$$\begin{aligned}
\int_\alpha^\beta G_2(t,s)ds &= \int_\alpha^t (t-\beta)ds + \int_t^\beta (s-\beta)ds \\
&= \frac{1}{2}(t-\beta)(t+\beta-2\alpha),
\end{aligned}$$

for $k = 3$

$$\begin{aligned}
\int_\alpha^\beta G_3(t,s)ds &= \int_\alpha^t (t-\alpha)ds + \int_t^\beta (s-\alpha)ds \\
&= \frac{1}{2}\left[(t-\alpha)^2 + (\beta-\alpha)^2\right],
\end{aligned}$$

and for $k = 4$

$$\begin{aligned}
\int_\alpha^\beta G_4(t,s)ds &= \int_\alpha^t (\beta-s)ds + \int_t^\beta (\beta-t)ds \\
&= \frac{1}{2}\left[(\beta-t)^2 + (\beta-\alpha)^2\right],
\end{aligned}$$

and using the same procedure, we get the same result. Thus, for $k = 0,1,2,3,4$ we have that

$$\begin{aligned}
&\int_\alpha^\beta \left(\frac{M-\overline{g}}{M-m} G_k(m,s) + \frac{\overline{g}-m}{M-m} G_k(M,s) - \frac{\int_a^b G_k(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \right) ds \\
&= \frac{1}{2} \cdot \left[\frac{M-\overline{g}}{M-m} \cdot m^2 + \frac{\overline{g}-m}{M-m} \cdot M^2 - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right].
\end{aligned}$$

If for all $s \in [\alpha, \beta]$ the inequality (1) holds, then (3) becomes

$$\begin{aligned}&\left| \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \right| \\ &\leq \frac{1}{2} \cdot \|\varphi''\|_\infty \cdot \left[\frac{M-\overline{g}}{M-m} \cdot m^2 + \frac{\overline{g}-m}{M-m} \cdot M^2 - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right],\end{aligned}$$

and if for all $s \in [\alpha, \beta]$ the reverse inequality in (1) holds, then (3) becomes

$$\begin{aligned}&\left| \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \right| \\ &\leq \frac{1}{2} \cdot \|\varphi''\|_\infty \cdot \left[\frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} - \frac{M-\overline{g}}{M-m} \cdot m^2 - \frac{\overline{g}-m}{M-m} \cdot M^2 \right].\end{aligned}$$

So, if for all $s \in [\alpha, \beta]$ the inequality (1) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (1) holds, in both cases (2) is valid. □

Remark 1. *Note that (2) can also be expressed as*

$$\begin{aligned}&\left| \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \right| \\ &\leq \frac{1}{2} \cdot \|\varphi''\|_\infty \cdot \left| \overline{g}(m+M) - m \cdot M - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right|.\end{aligned}$$

Let us consider the case when $k = 0$. If we set that $m = \alpha$ and $M = \beta$, we have that $G_0(\alpha, s) = G_0(\beta, s) = 0$, and the inequality (1) transforms into

$$\frac{\int_a^b G_0(g(x), s)d\lambda(x)}{\int_a^b d\lambda(x)} \leq 0. \tag{4}$$

Therefore, we have the following result.

Corollary 2. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, where $Im(g) \subseteq [\alpha, \beta]$. Let $\lambda : [a, b] \to \mathbb{R}$ be a function of bounded variation, such that $\lambda(a) \neq \lambda(b)$. If for all $s \in [\alpha, \beta]$ the inequality (4) holds, or if for all $s \in [\alpha, \beta]$ the reverse inequality in (4) holds, then*

$$\begin{aligned}&\left| \frac{\beta-\overline{g}}{\beta-\alpha}\varphi(\alpha) + \frac{\overline{g}-\alpha}{\beta-\alpha}\varphi(\beta) - \frac{\int_a^b \varphi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \right| \\ &\leq \frac{1}{2} \cdot \|\varphi''\|_\infty \cdot \left| \frac{\beta-\overline{g}}{\beta-\alpha} \cdot \alpha^2 + \frac{\overline{g}-\alpha}{\beta-\alpha} \cdot \beta^2 - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right| \\ &= \frac{1}{2} \cdot \|\varphi''\|_\infty \cdot \left| \overline{g}(\alpha+\beta) - \alpha \cdot \beta - \frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} \right|.\end{aligned}$$

As we can see, this case looks much simpler, while the condition (4) is easier to verify than the condition (1). Similar results could also be given in cases when $k = 1, 2, 3, 4$, but we are not mentioning them here, because their conditions are not so simple.

3. On the Converse Jensen Type Inequality in Discrete Case

In this section, we give our results in the discrete case. We omit the proofs, as they are similar to those in the integral case from the previous section. We introduce the notation: $U_n = \sum_{i=1}^n u_i$, $\overline{x} = \frac{1}{U_n}\sum_{i=1}^n u_i x_i$.

For $x_i \in [a,b] \subseteq [\alpha,\beta]$, $a \neq b$, $u_i \in \mathbb{R}$ $(i = 1,\dots,n)$ such that $U_n \neq 0$, we have that for every function $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$, holds:

$$\begin{aligned}&\frac{b-\overline{x}}{b-a}\varphi(a) + \frac{\overline{x}-a}{b-a}\varphi(b) - \frac{1}{U_n}\sum_{i=1}^n u_i\varphi(x_i)\\ &\qquad = \int_\alpha^\beta \left(\frac{b-\overline{x}}{b-a}G_k(a,s) + \frac{\overline{x}-a}{b-a}G_k(b,s) - \frac{1}{U_n}\sum_{i=1}^n u_i G_k(x_i,s)\right)\varphi''(s)ds.\end{aligned}$$

Using that fact, we obtain the following result.

Theorem 2. *Let $x_i \in [a,b] \subseteq [\alpha,\beta]$, $a \neq b$, $u_i \in \mathbb{R}$ $(i = 1,\dots,n)$ be such that $U_n \neq 0$, and let $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$. Let $p,q \in \mathbb{R}$, $1 \leq p,q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$.*

Then

$$\left|\frac{b-\overline{x}}{b-a}\varphi(a) + \frac{\overline{x}-a}{b-a}\varphi(b) - \frac{1}{U_n}\sum_{i=1}^n u_i\varphi(x_i)\right| \leq L \cdot \left\|\varphi''\right\|_p$$

holds, where

$$L = \begin{cases} \left[\int_\alpha^\beta \left|\frac{b-\overline{x}}{b-a}G_k(a,s) + \frac{\overline{x}-a}{b-a}G_k(b,s) - \frac{1}{U_n}\sum_{i=1}^n u_i G_k(x_i,s)\right|^q ds\right]^{\frac{1}{q}}, & \textit{for } q \neq \infty;\\ \sup_{s\in[\alpha,\beta]}\left\{\left|\frac{b-\overline{x}}{b-a}G_k(a,s) + \frac{\overline{x}-a}{b-a}G_k(b,s) - \frac{1}{U_n}\sum_{i=1}^n u_i G_k(x_i,s)\right|\right\}, & \textit{for } q = \infty.\end{cases}$$

Let us now see what happens for $q = 1$, $p = \infty$. If the term

$$\frac{b-\overline{x}}{b-a}G_k(a,s) + \frac{\overline{x}-a}{b-a}G_k(b,s) - \frac{1}{U_n}\sum_{i=1}^n u_i G_k(x_i,s)$$

has the same positivity for all $s \in [\alpha,\beta]$, then we can calculate L. The following result holds.

Corollary 3. *Let $x_i \in [a,b] \subseteq [\alpha,\beta]$, $a \neq b$, $u_i \in \mathbb{R}$ $(i = 1,\dots,n)$ be such that $U_n \neq 0$, and let $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$. If for any $k \in \{0,1,2,3,4\}$ for all $s \in [\alpha,\beta]$ the inequality*

$$\frac{b-\overline{x}}{b-a}G_k(a,s) + \frac{\overline{x}-a}{b-a}G_k(b,s) - \frac{1}{U_n}\sum_{i=1}^n u_i G_k(x_i,s) \geq 0 \tag{5}$$

holds, or if for any $k \in \{0,1,2,3,4\}$ for all $s \in [\alpha,\beta]$ the reverse inequality in (5) holds, then

$$\left|\frac{b-\overline{x}}{b-a}\varphi(a) + \frac{\overline{x}-a}{b-a}\varphi(b) - \frac{1}{U_n}\sum_{i=1}^n u_i\varphi(x_i)\right| \leq \frac{1}{2}\cdot\left\|\varphi''\right\|_\infty \cdot \left|(a+b)\overline{x} - ab - \frac{1}{U_n}\sum_{i=1}^n u_i x_i^2\right|.$$

In the case when $k = 0$, $a = \alpha$ and $b = \beta$, the inequality (5) transforms into

$$\frac{1}{U_n}\sum_{i=1}^n u_i G_0(x_i,s) \leq 0 \tag{6}$$

and we obtain the following result.

Corollary 4. *Let $x_i \in [\alpha,\beta]$, $\alpha \neq \beta$, $u_i \in \mathbb{R}$ $(i = 1,...,n)$ be such that $U_n \neq 0$, and let $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$. If for all $s \in [\alpha,\beta]$ the inequality (6) holds, or if for all $s \in [\alpha,\beta]$ the reverse inequality in (6) holds, then*

$$\left|\frac{\beta-\overline{x}}{\beta-\alpha}\varphi(\alpha)+\frac{\overline{x}-\alpha}{\beta-\alpha}\varphi(\beta)-\frac{1}{U_n}\sum_{i=1}^{n}u_i\varphi(x_i)\right| \leq \frac{1}{2}\cdot\|\varphi''\|_\infty\cdot\left|(\alpha+\beta)\overline{x}-\alpha\beta-\frac{1}{U_n}\sum_{i=1}^{n}u_ix_i^2\right|.$$

4. Inequalities for Generalized f-Divergences

I. Csiszár in [15] defined the f-divergence

$$C_f(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} p_i f\left(\frac{q_i}{p_i}\right)$$

for a function $f : \mathbb{R}_+ \to \mathbb{R}$ and two positive probability distributions $\mathbf{p} = (p_1,\ldots,p_n) \in \mathbb{R}_+^n$, $\mathbf{q} = (q_1,\ldots,q_n) \in \mathbb{R}_+^n$. He considered the case when the function f is convex. Although several other authors ([16,17]) also introduced and studied this divergence, it is well known as the Csiszár f-divergence.

There exist various kinds of divergences, and all of them measure the differences between probability distributions. We focus here on the f-differences which are generalized using weights (see [18,19]), and we apply our results from the previous section in order to get new results and inequalities for these generalized f-divergences.

The generalized Csiszár f-divergence is defined by

$$C_f(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i p_i f\left(\frac{q_i}{p_i}\right),$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ and $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$.

To simplify our results, we use the following notations

$$P_r = \sum_{i=1}^{n} r_i p_i,$$

$$\overline{Q}_r = \frac{1}{P_r}\sum_{i=1}^{n} r_i q_i.$$

Theorem 3. *Let $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$ be such that $\frac{q_i}{p_i} \in [a,b] \subseteq [\alpha,\beta]$, $(i = 1,...,n)$, $a \neq b$, and let $p,q \in \mathbb{R}$, $1 \leq p,q \leq \infty$, be such that $\frac{1}{p}+\frac{1}{q} = 1$. If $f : [\alpha,\beta] \to \mathbb{R}$, $f \in C^2([\alpha,\beta])$, then*

$$\left|\frac{b-\overline{Q}_r}{b-a}f(a)+\frac{\overline{Q}_r-a}{b-a}f(b)-\frac{1}{P_r}C_f(\mathbf{q},\mathbf{p};\mathbf{r})\right| \leq L\cdot\|f''\|_p$$

holds, where

$$L = \begin{cases} \left[\int_\alpha^\beta\left|\frac{b-\overline{Q}_r}{b-a}G_k(a,s)+\frac{\overline{Q}_r-a}{b-a}G_k(b,s)-\frac{1}{P_r}\sum_{i=1}^{n}r_ip_iG_k(\frac{q_i}{p_i},s)\right|^q ds\right]^{\frac{1}{q}}, & \text{for } q \neq \infty; \\ \sup_{s\in[\alpha,\beta]}\left\{\left|\frac{b-\overline{Q}_r}{b-a}G_k(a,s)+\frac{\overline{Q}_r-a}{b-a}G_k(b,s)-\frac{1}{P_r}\sum_{i=1}^{n}r_ip_iG_k(\frac{q_i}{p_i},s)\right|\right\}, & \text{for } q = \infty. \end{cases} \tag{7}$$

Proof. Substituting $\varphi := f$,

$$u_i := \frac{r_ip_i}{\sum\limits_{i=1}^{n}r_ip_i}, \quad x_i := \frac{q_i}{p_i}, \quad i = 1,\ldots,n,$$

our result directly follows from Theorem 2. □

The generalized Kullback–Leibler divergence is defined by

$$KL(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i q_i \log \frac{q_i}{p_i},$$

where $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$. For this divergence we have the following result.

Theorem 4. *Let $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$ be such that $\frac{q_i}{p_i} \in [a,b] \subseteq [\alpha,\beta]$, $(i = 1,\dots,n)$, $a \neq b$, $a,b > 0$ and let $p,q \in \mathbb{R}$, $1 \leq p,q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{b - \overline{Q}_r}{b-a} a \log a + \frac{\overline{Q}_r - a}{b-a} b \log b - \frac{1}{P_r} KL(\mathbf{q},\mathbf{p};\mathbf{r}) \right| \leq L \cdot \left\| (id \cdot \log)'' \right\|_p$$

holds, where id is the identity function and L is as defined in (7).

Proof. This result follows directly from Theorem 3 by setting $f(t) = t \log t$, $t > 0$. □

For the generalized Hellinger divergence defined by

$$He(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i (\sqrt{q_i} - \sqrt{p_i})^2$$

the following result holds.

Theorem 5. *Let $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$ be such that $\frac{q_i}{p_i} \in [a,b] \subseteq [\alpha,\beta]$, $(i = 1,\dots,n)$, $a \neq b$, $a,b > 0$ and let $p,q \in \mathbb{R}$, $1 \leq p,q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{b - \overline{Q}_r}{b-a} (1 - \sqrt{a})^2 + \frac{\overline{Q}_r - a}{b-a} (1 - \sqrt{b})^2 - \frac{1}{P_r} He(\mathbf{q},\mathbf{p};\mathbf{r}) \right| \leq L \cdot \left\| f'' \right\|_p$$

holds, where $f(t) = (1 - \sqrt{t})^2$ and L is as defined in (7).

Proof. This result follows directly from Theorem 3 by setting $f(t) = (1 - \sqrt{t})^2$, $t > 0$. □

The generalized Rényi divergence is defined by

$$Re_\gamma(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i q_i^\gamma p_i^{1-\gamma}$$

where $\gamma \in \langle 1, +\infty \rangle$.

Theorem 6. *Let $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$ be such that $\frac{q_i}{p_i} \in [a,b] \subseteq [\alpha,\beta]$, $(i = 1,\dots,n)$, $a \neq b$, $a,b > 0$ and let $p,q \in \mathbb{R}$, $1 \leq p,q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{b - \overline{Q}_r}{b-a} a^\gamma + \frac{\overline{Q}_r - a}{b-a} b^\gamma - \frac{1}{P_r} Re_\gamma(\mathbf{q},\mathbf{p};\mathbf{r}) \right| \leq L \cdot \left\| f'' \right\|_p$$

holds, where $f(t) = t^\gamma$ $(t > 0, \gamma > 1)$ and L is as defined in (7).

Proof. This result follows directly from Theorem 3 by setting $f(t) = t^\gamma$ $(t > 0, \gamma > 1)$. □

The generalized χ^2-divergence is defined by

$$D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i \frac{(q_i - p_i)^2}{p_i}.$$

The following result holds.

Theorem 7. *Let $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}^n_+$ be such that $\frac{q_i}{p_i} \in [a,b] \subseteq [\alpha, \beta]$, $(i = 1, ..., n)$, $a \neq b$, $a, b > 0$ and let $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{b - \overline{Q}_r}{b-a}(a-1)^2 + \frac{\overline{Q}_r - a}{b-a}(b-1)^2 - \frac{1}{P_r} D_{\chi^2}(\mathbf{q}, \mathbf{p}; \mathbf{r}) \right| \leq L \cdot \|f''\|_p$$

holds, where $f(t) = (t-1)^2$, $t > 0$, and L is as defined in (7).

Proof. This result follows directly from Theorem 3 by setting $f(t) = (t-1)^2$, $t > 0$. □

The generalized Shannon entropy of a positive probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ is defined by

$$H(\mathbf{p}; \mathbf{r}) = -\sum_{i=1}^{n} r_i p_i \log(p_i).$$

It is a special case of the generalized Csiszár f-divergence $C_f(\mathbf{q}, \mathbf{p}; \mathbf{r})$ if we set $\mathbf{q} = (1, \ldots, 1)$ and $f(t) = \log t$, $t > 0$. We have the following.

Theorem 8. *Let $\mathbf{p}, \mathbf{r} \in \mathbb{R}^n_+$ be such that $\frac{1}{p_i} \in [a,b] \subseteq [\alpha, \beta]$, $(i = 1, ..., n)$, $a \neq b$, $a, b > 0$ and let $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{b - \frac{1}{P_r}\sum_{i=1}^{n} r_i}{b-a} \log a + \frac{\frac{1}{P_r}\sum_{i=1}^{n} r_i - a}{b-a} \log b - \frac{1}{P_r} H(\mathbf{p}; \mathbf{r}) \right| \leq L \cdot \|\log''\|_p$$

holds, where L is as defined in (7).

5. Applications to Zipf–Mandelbrot Law

Definition 1 ([20]). *Zipf–Mandelbrot law is a discrete probability distribution, depends on three parameters $N \in \{1, 2, \ldots\}$, $t \in [0, \infty\rangle$ and $v > 0$, and it is defined by*

$$\phi(i; N, t, v) := \frac{1}{(i+t)^v H_{N,t,v}}, \quad i = 1, \ldots, N,$$

where

$$H_{N,t,v} := \sum_{j=1}^{N} \frac{1}{(j+t)^v}.$$

When $t = 0$, then Zipf–Mandelbrot law becomes Zipf's law.

The Zipf–Mandelbrot law got its name after the linguist George Kingsley Zipf, who gave its primary form, and after the mathematician Benoit Mandelbrot, who gave its generalization. The Zipf law goes after the frequency of a certain word in the text, and it is used in bibliometric and in information science. It is used in linguistics, but also in economics (as Pareto's law) when analysing the distribution of the wealth. Apart from that, this law can be found also in other disciplines like mathematics, physics, biology, computer science, social sciences, demography, etc. Here we are going to concentrate on its mathematical aspect of course. (More about the Zipf–Mandelbrot law in mathematical context can be found in [21].)

As the Zipf–Mandelbrot law is a probability distribution, and f-divergences measure the differences between two probability distributions, we can apply the results from the previous section on the Zipf–Mandelbrot law.

Suppose $\mathbf{p}, \mathbf{q}$ are two Zipf–Mandelbrot laws with parameters $N \in \{1, 2, \ldots\}$, $t_1, t_2 \geq 0$ and $v_1, v_2 > 0$, respectively. Then

$$p_i = \phi(i; N, t_1, v_1) := \frac{1}{(i+t_1)^{v_1} H_{N,t_1,v_1}}, \quad i = 1, \ldots, N, \tag{8}$$

and

$$q_i = \phi(i; N, t_2, v_2) := \frac{1}{(i+t_2)^{v_2} H_{N,t_2,v_2}}, \quad i = 1, \ldots, N, \tag{9}$$

where

$$H_{N,t_k,v_k} := \sum_{j=1}^{N} \frac{1}{(j+t_k)^{v_k}}, \quad k = 1, 2.$$

The generalized Csiszár divergence for such $\mathbf{p}, \mathbf{q}$, and for $\mathbf{r} \in \mathbb{R}_+^n$ is given by

$$C_f(\mathbf{q}, \mathbf{p}; \mathbf{r}) = \frac{1}{H_{N,t_1,v_1}} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}} f\left(\frac{(i+t_1)^{v_1} H_{N,t_1,v_1}}{(i+t_2)^{v_2} H_{N,t_2,v_2}} \right). \tag{10}$$

Using (8) and (9), we get the following expressions for P_r and $\overline{Q}_r$:

$$P_r = \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1} H_{N,t_1,v_1}} = \frac{1}{H_{N,t_1,v_1}} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}, \tag{11}$$

$$\overline{Q}_r = \frac{\sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2} H_{N,t_2,v_2}}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1} H_{N,t_1,v_1}}} = \frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}} \cdot \frac{\sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}}, \tag{12}$$

and we obtain the following result.

Corollary 5. *Let $\mathbf{p}, \mathbf{q}$ be two Zipf–Mandelbrot laws with parameters $N \in \{1, 2, \ldots\}$, t_1, $t_2 \geq 0$ and $v_1, v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}_+^n$ such that*

$$\frac{q_i}{p_i} := \frac{(i+t_1)^{v_1} H_{N,t_1,v_1}}{(i+t_2)^{v_2} H_{N,t_2,v_2}} \in [a, b] \subseteq [\alpha, \beta] \text{ for } i = 1, \ldots, N, \ (a \neq b).$$

Let $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$, then

$$\left| \frac{b - \overline{Q}_r}{b-a} f(a) + \frac{\overline{Q}_r - a}{b-a} f(b) - \frac{1}{P_r} C_f(\mathbf{q}, \mathbf{p}; \mathbf{r}) \right| \leq L \cdot \|f''\|_p$$

where p_i, q_i, P_r, $\overline{Q}_r$, $C_f(\mathbf{q}, \mathbf{p}; \mathbf{r})$ are as defined in (8)–(12), and L is as defined in (7).

The generalized Kullbach-Leibler divergence of two Zipf–Mandelbrot laws $\mathbf{p}, \mathbf{q}$ with parameters $N \in \{1, 2, \ldots\}$, t_1, $t_2 \geq 0$ and v_1, $v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}_+^n$, is given by:

$$KL(\mathbf{q}, \mathbf{p}; \mathbf{r}) = \frac{1}{H_{N,t_2,v_2}} \sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2}} \log\left(\frac{(i+t_1)^{v_1} H_{N,t_1,v_1}}{(i+t_2)^{v_2} H_{N,t_2,v_2}} \right). \tag{13}$$

The following holds.

Corollary 6. *Let $\mathbf{p}, \mathbf{q}$ be two Zipf–Mandelbrot laws with parameters $N \in \{1, 2, \ldots\}$, t_1, $t_2 \geq 0$ and $v_1, v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}_+^n$ such that*

$$\frac{q_i}{p_i} := \frac{(i+t_1)^{v_1} H_{N,t_1,v_1}}{(i+t_2)^{v_2} H_{N,t_2,v_2}} \in [a, b] \subseteq [\alpha, \beta] \text{ for } i = 1, \ldots, N, \ (a, b > 0, \ a \neq b).$$

Let $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \frac{b - \overline{Q}_r}{b-a} a \log a + \frac{\overline{Q}_r - a}{b-a} b \log b - \frac{1}{P_r} KL(\mathbf{q}, \mathbf{p}; \mathbf{r}) \right| \leq L \cdot \|(id \cdot \log)''\|_p$$

holds, where id is the identity function and p_i, q_i, P_r, $\overline{Q}_r$, $KL(\mathbf{q},\mathbf{p};\mathbf{r})$ are as defined in (8), (9), (11)–(13), and L is as defined in (7).

The generalized Hellinger divergence for two Zipf–Mandelbrot laws $\mathbf{p},\mathbf{q}$ with parameters $N \in \{1,2,\ldots\}$, t_1, $t_2 \geq 0$ and v_1, $v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}^n_+$, has the following representation:

$$He(\mathbf{q},\mathbf{p};\mathbf{r}) = \frac{1}{H_{N,t_1,v_1}H_{N,t_2,v_2}}\sum_{i=1}^{N} r_i \frac{\left(\sqrt{(i+t_1)^{v_1}H_{N,t_1,v_1}} - \sqrt{(i+t_2)^{v_2}H_{N,t_2,v_2}}\right)^2}{(i+t_1)^{v_1}(i+t_2)^{v_2}}. \tag{14}$$

The following result holds.

Corollary 7. *Let $\mathbf{p},\mathbf{q}$ be two Zipf–Mandelbrot laws with parameters $N \in \{1,2,\ldots\}$, t_1, $t_2 \geq 0$ and $v_1, v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}^n_+$ such that*

$$\frac{q_i}{p_i} := \frac{(i+t_1)^{v_1}H_{N,t_1,v_1}}{(i+t_2)^{v_2}H_{N,t_2,v_2}} \in [a,b] \subseteq [\alpha,\beta] \ \text{for} \ i = 1,\ldots,N, \ (a,b > 0, \ a \neq b).$$

Let $p,q \in \mathbb{R}$, $1 \leq p,q \leq \infty$, be such that $\frac{1}{p}+\frac{1}{q} = 1$. Then

$$\left|\frac{b-\overline{Q}_r}{b-a}(1-\sqrt{a})^2 + \frac{\overline{Q}_r-a}{b-a}(1-\sqrt{b})^2 - \frac{1}{P_r}He(\mathbf{q},\mathbf{p};\mathbf{r})\right| \leq L \cdot \|f''\|_p$$

where $f(t) = (1-\sqrt{t})^2$, and p_i, q_i, P_r, $\overline{Q}_r$, $He(\mathbf{q},\mathbf{p};\mathbf{r})$ are as defined in (8), (9), (11), (12) and (14), respectively, and L is as defined in (7).

The generalized Rényi divergence for two Zipf–Mandelbrot laws $\mathbf{p},\mathbf{q}$ with parameters $N \in \{1,2,\ldots\}$, t_1, $t_2 \geq 0$ and v_1, $v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}^n_+$, has the following representation:

$$Re_\gamma(\mathbf{q},\mathbf{p};\mathbf{r}) = \frac{H^{\gamma-1}_{N,t_1,v_1}}{H^{\gamma}_{N,t_2,v_2}}\sum_{i=1}^{N} r_i \frac{(i+t_1)^{(\gamma-1)v_1}}{(i+t_2)^{\gamma v_2}}, \tag{15}$$

where $\gamma \in \langle 1,+\infty\rangle$. The following result holds.

Corollary 8. *Let $\mathbf{p},\mathbf{q}$ be two Zipf–Mandelbrot laws with parameters $N \in \{1,2,\ldots\}$, t_1, $t_2 \geq 0$ and $v_1, v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}^n_+$ such that*

$$\frac{q_i}{p_i} := \frac{(i+t_1)^{v_1}H_{N,t_1,v_1}}{(i+t_2)^{v_2}H_{N,t_2,v_2}} \in [a,b] \subseteq [\alpha,\beta] \ \text{for} \ i = 1,\ldots,N, \ (a,b > 0, \ a \neq b).$$

Let $p,q \in \mathbb{R}$, $1 \leq p,q \leq \infty$, be such that $\frac{1}{p}+\frac{1}{q} = 1$. Then

$$\left|\frac{b-\overline{Q}_r}{b-a}a^\gamma + \frac{\overline{Q}_r-a}{b-a}b^\gamma - \frac{1}{P_r}Re_\gamma(\mathbf{q},\mathbf{p};\mathbf{r})\right| \leq L \cdot \|f''\|_p$$

holds, where $f(t) = t^\gamma$ $(t > 0, \gamma > 1)$, and p_i, q_i, P_r, $\overline{Q}_r$, $Re(\mathbf{q},\mathbf{p};\mathbf{r})$ are as defined in (8), (9), (11), (12) and (15), respectively, and L is as defined in (7).

The generalized χ^2-divergence for two Zipf–Mandelbrot laws $\mathbf{p},\mathbf{q}$ with parameters $N \in \{1,2,\ldots\}$, t_1, $t_2 \geq 0$ and v_1, $v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}^n_+$, has the following representation:

$$D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r}) = H_{N,t_1,v_1} \cdot \sum_{i=1}^{N} r_i (i+t_1)^{v_1}\left(\frac{1}{(i+t_2)^{v_2}H_{N,t_2,v_2}} - \frac{1}{(i+t_1)^{v_1}H_{N,t_1,v_1}}\right)^2. \tag{16}$$

We have the following result.

Corollary 9. *Let* $\mathbf{p}, \mathbf{q}$ *be two Zipf–Mandelbrot laws with parameters* $N \in \{1, 2, \ldots\}$, $t_1, t_2 \geq 0$ *and* $v_1, v_2 > 0$, *respectively, and* $\mathbf{r} \in \mathbb{R}^n_+$ *such that*

$$\frac{q_i}{p_i} := \frac{(i+t_1)^{v_1} H_{N,t_1,v_1}}{(i+t_2)^{v_2} H_{N,t_2,v_2}} \in [a,b] \subseteq [\alpha, \beta] \text{ for } i = 1, \ldots, N, \ (a, b > 0, \ a \neq b).$$

Let $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left| \frac{b - \overline{Q}_r}{b-a}(a-1)^2 + \frac{\overline{Q}_r - a}{b-a}(b-1)^2 - \frac{1}{P_r} D_{\chi^2}(\mathbf{q}, \mathbf{p}; \mathbf{r}) \right| \leq L \cdot \|f''\|_p$$

holds, where $f(t) = (t-1)^2$, $t > 0$, *and* p_i, q_i, P_r, $\overline{Q}_r$, $D_{\chi^2}(\mathbf{q}, \mathbf{p}; \mathbf{r})$ *are as defined in (8), (9), (11), (12) and (16), respectively, and* L *is as defined in (7).*

In addition, at the end, we also give the result for the generalized Shannon entropy of a Zipf–Mandelbrot law $\mathbf{p}$ with parameters $N \in \{1, 2, \ldots\}$, $t_1 \geq 0$, $v_1 > 0$, and $\mathbf{r} \in \mathbb{R}^n_+$, which has the following representation:

$$H(\mathbf{p}; \mathbf{r}) = \frac{1}{H_{N,t_1,v_1}} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}} \log\left[(i+t_1)^{v_1} H_{N,t_1,v_1}\right]. \tag{17}$$

Corollary 10. *Let* $\mathbf{p}$ *be a Zipf–Mandelbrot law with parameters* $N \in \{1, 2, \ldots\}$, $t_1 \geq 0$ *and* $v_1 > 0$, *and* $\mathbf{r} \in \mathbb{R}^n_+$, *such that*

$$\frac{1}{p_i} := (i+t_1)^{v_1} H_{N,t_1,v_1} \in [a,b] \subseteq [\alpha, \beta] \text{ for } i = 1, \ldots, N, \ (a, b > 0, \ a \neq b).$$

Let $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left| \frac{b - \frac{1}{P_r}\sum_{i=1}^n r_i}{b-a} \log a + \frac{\frac{1}{P_r}\sum_{i=1}^n r_i - a}{b-a} \log b - \frac{1}{P_r} H(\mathbf{p}; \mathbf{r}) \right| \leq L \cdot \|\log''\|_p$$

holds, where p_i, P_r, $H(\mathbf{p}; \mathbf{r})$ *are as defined in (8), (11) and (17), respectively, and* L *is as defined in (7).*

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