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Traveling Waves for the Generalized Sinh-Gordon Equation with Variable Coefficients

Lewa' Alzaleq [1,*], Du'a Al-zaleq [2] and Suboh Alkhushayni [2]

[1] Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq 25113, Jordan
[2] Computer Information Science Department, Minnesota State University, Mankato, MN 56001, USA; dua.al-zaleq@mnsu.edu (D.A.-z.); Suboh.Alkhushayni@mnsu.edu (S.A.)
* Correspondence: lewa.alzaleq@aabu.edu.jo

Abstract: The sinh-Gordon equation is simply the classical wave equation with a nonlinear sinh source term. It arises in diverse scientific applications including differential geometry theory, integrable quantum field theory, fluid dynamics, kink dynamics, and statistical mechanics. It can be used to describe generic properties of string dynamics for strings and multi-strings in constant curvature space. In the present paper, we study a generalized sinh-Gordon equation with variable coefficients with the goal of obtaining analytical traveling wave solutions. Our results show that the traveling waves of the variable coefficient sinh-Gordon equation can be derived from the known solutions of the standard sinh-Gordon equation under a specific selection of a choice of the variable coefficients. These solutions include some real single and multi-solitons, periodic waves, breaking kink waves, singular waves, periodic singular waves, and compactons. These solutions might be valuable when scientists model some real-life phenomena using the sinh-Gordon equation where the balance between dispersion and nonlinearity is perturbed.

Keywords: sinh-Gordon equation; space and time dependent coefficients; soliton; periodic waves; traveling wave solution

MSC: 35C07; 35C08; 35B10; 81Q05; 35N10

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1. Introduction

The sinh-Gordon equation in its standard form

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + \sinh(u(x,t)) = 0, \tag{1}$$

is a completely nonlinear integrable partial differential equation that is widely used in physics and sciences [1]. This equation has broad-spectrum scientific applications in integrable quantum field theory, fluid dynamics, kink dynamics, differential geometry theory, and statistical mechanics. Early examples include particular surfaces of constant mean curvature and Josephson junctions between two superconductors [1–4]. The geometrical interpretation of Equation (1) was shown by studying surfaces of constant Gaussian curvature in a three-dimensional pseudo-Riemannian manifold of constant curvature [5,6]. It can be used to describe generic properties of string dynamics for strings and multi-strings in constant curvature space [6,7]. It also arises in models of interacting charged particles in plasma physics, the interaction of neighboring particles of equal mass in a lattice formation with a crystal, and on effects of weak dislocation potential on nonlinear wave propagation in the anharmonic crystal [1,8,9].

Equation (1) involves the d'Alembertian $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ and the hyperbolic function sinh of the function $u(x,t)$. The solution $u(x,t)$ is supposed to be a real-valued function and clearly a purely complex solution $u = i\hat{u}$ of Equation (1) satisfies the sine-Gordon equation

$$\frac{\partial^2}{\partial t^2}\hat{u}(x,t) - \frac{\partial^2}{\partial x^2}\hat{u}(x,t) + \sin(\hat{u}(x,t)) = 0.$$

Equation (1) is a perturbation of the well known linear Klein–Gordon equation [10]

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + u(x,t) = 0.$$

It is rewritten in a system model as

$$\begin{aligned}
&\frac{\partial}{\partial t}u(x,t) = -v(x,t),\\
&\frac{\partial}{\partial t}v(x,t) + \frac{\partial^2}{\partial x^2}u(x,t) - \sinh(u(x,t)) = 0.
\end{aligned}$$

Using the transformation $\mathbf{x} = \frac{1}{2}(x+t), \mathbf{t} = \frac{1}{2}(x-t)$, Equation (1) becomes the famous sinh-Gordon equation

$$\frac{\partial^2}{\partial \mathbf{x}\partial \mathbf{t}}u(\mathbf{x},\mathbf{t}) = \sinh(u(\mathbf{x},\mathbf{t})), \tag{2}$$

and hence obtaining the analytical solutions for Equation (2) is similar to finding the analytic solutions of Equation (1). Equation (2) is an integrable system and has a self-adjoint Lax pair [11]. It is known that Equation (2) has an auto-Backlund transformation [12]

$$\begin{aligned}
&\frac{\partial}{\partial \mathbf{x}}u(\mathbf{x},\mathbf{t}) + \frac{\partial}{\partial \mathbf{x}}v(\mathbf{x},\mathbf{t}) = -4\,\lambda\,\sinh\left(\frac{u(\mathbf{x},\mathbf{t})}{2} - \frac{v(\mathbf{x},\mathbf{t})}{2}\right),\\
&\frac{\partial}{\partial \mathbf{t}}u(\mathbf{x},\mathbf{t}) - \frac{\partial}{\partial \mathbf{t}}v(\mathbf{x},\mathbf{t}) = -\frac{1}{\lambda}\,\sinh\left(\frac{u(\mathbf{x},\mathbf{t})}{2} + \frac{v(\mathbf{x},\mathbf{t})}{2}\right),
\end{aligned} \tag{3}$$

and hence if u is a solution of Equation (2), then v can be determined by the auto-Backlund transformation (3). Thus, it could be said that the function v satisfies the sinh-Gordon equation in the form of

$$\frac{\partial^2}{\partial \mathbf{x}\partial \mathbf{t}}v(\mathbf{x},\mathbf{t}) = \sinh(v(\mathbf{x},\mathbf{t})).$$

Many mathematicians and physicists studied Equations (1) and (2) from different aspects [2,5–7,9–17]. The Painlevé property was used in [2] to investigate the sinh-Gordon equation. Nonlocal symmetries and conservation laws of the Sinh-Gordon equation were obtained in [12]. The sinh-Gordon equations in $(1+1)$, $(2+1)$, and $(3+1)$ dimensions were investigated and the one soliton solution and the two soliton solutions were formally derived for each model [16]. Several analytic solutions were obtained in [15,17] by using the tanh method and in [13] by using the Exp-function method. The bifurcation theory of the dynamical system was used in [14] to obtain more analytic solutions to Equation (1) such as periodic wave solutions, breaking kink wave solutions, and compactons. The authors in [6] found elliptic solutions for Equation (1) and showed that these elliptic solutions are orbitally stable with respect to subharmonic perturbations of the arbitrary periods. The direct and inverse scattering problems were solved in [18] for the elliptic sinh-Gordon equation and it was also shown that the inverse scattering transform might be useful in the analysis of localized singular solutions. Three numerical techniques were proposed in [19] for solving the two-dimensional sinh-Gordon equation using the moving least squares, RBF-PS collocation, and radial basis function meshless methods.

A soliton in nonlinear dispersive systems is a self-reinforcing pulse that maintains its shape during propagation with constant velocity. Its caused by canceling the nonlinear and dispersive effects in the medium. Soliton solutions are well known in physics and engineering fields including fluid dynamics, optics, surface wave propagation, and shallow water waves. Many new studies have focused on finding N soliton solutions for systems of nonlinear partial differential equations, for instance, N solitons and Bäcklund transformations of the Boussinesq–Burgers system have been carried out in [20] for the shallow water waves in a lake or near an ocean beach. In [21], scaling transformations, hetero-Bäcklund transformations, bilinear forms, and N solitons have been carried out for a generalized $(2+1)$-dimensional dispersive long-wave system on the shallow water of an open sea or a wide channel of finite depth. The soliton solutions of the sinh-Gordon Equation (1) are expressed by [16]

$$u(x,t) = 4\,\text{arctanh}\left(\frac{f(x,t)}{g(x,t)}\right), \tag{4}$$

where $f(x,t)$ and $g(x,t)$ are auxiliary functions. The single soliton solution of the sinh-Gordon Equation (1) is given by

$$u(x,t) = 4\,\text{arctanh}\left(\exp\left(\kappa\,x \pm \sqrt{\kappa^2-1}\,t\right)\right), \tag{5}$$

whereas, the two-soliton solution is

$$u(x,t) = 4\,\text{arctanh}\left(\frac{\exp(\kappa_1\,x \pm \omega_1\,t) + \exp(\kappa_2\,x \pm \omega_2\,t)}{1 - \left(\frac{1-\kappa_1\,\kappa_2+\omega_1\,\omega_2}{1+\kappa_1\,\kappa_2-\omega_1\,\omega_2}\right)\,\exp((\kappa_1+\kappa_2)\,x \pm (\omega_1+\omega_2)\,t)}\right), \tag{6}$$

where $\omega_1 = \sqrt{\kappa_1^2 - 1}$ and $\omega_2 = \sqrt{\kappa_2^2 - 1}$.

More analytic traveling wave solutions for Equation (1) can be obtained by using the transformation

$$v(x,t) = \exp(u(x,t)),$$

so that

$$u(x,t) = \ \text{arccosh}\left(\frac{v(x,t)+v^{-1}(x,t)}{2}\right). \tag{7}$$

Thus, the authors in [13,17] obtained the solution

$$u(x,t) = \ \text{arccosh}\left(-\frac{1}{2}\,\frac{\tanh^4\left(\frac{1}{2\sqrt{c^2-1}}(x-ct)\right)+1}{\tanh^2\left(\frac{1}{2\sqrt{c^2-1}}(x-ct)\right)}\right), \tag{8}$$

where $c^2 > 1$ and the solution

$$u(x,t) = \ \text{arccosh}\left(\frac{1}{2}\,\frac{\tanh^4\left(\frac{1}{2\sqrt{1-c^2}}(x-ct)\right)+1}{\tanh^2\left(\frac{1}{2\sqrt{1-c^2}}(x-ct)\right)}\right), \tag{9}$$

where $c^2 < 1$. Another traveling wave solution to Equation (1) is

$$u(x,t) = 2\,\ln\left(\tanh\left(\frac{1}{2}\,\kappa\,x \pm \frac{1}{2}\,\sqrt{\kappa^2-1}\,t\right)\right). \tag{10}$$

Other possible solutions for the Equation (1) can be obtained from Equations (8)–(10) by replacing the tanh function with coth, tan, or cot functions (see [17]).

A number of researchers have studied several extensions of the standard sinh-Gordon equation. For example, analytic traveling wave solutions for the generalized double sinh-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \kappa\,\frac{\partial^2 u}{\partial x^2} + 2\alpha\,\sinh(nu) + \beta\,\sinh(2nu) = 0,$$

where n is a positive integer, obtained in [22] by using a new function approach based on the hyperbolic function cosh, in [23] by using the Exp-function method, and in [24] by using the $(\frac{G'}{G})$-expansion method. In [25–27], various travelling waves, periodic solutions, and Jacobi elliptic function solutions are derived for the combined sinh–cosh-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \kappa \frac{\partial^2 u}{\partial x^2} + \alpha \sinh(nu) + \beta \cosh(nu) = 0.$$

where n is a positive integer. The dynamical behavior and analytic traveling wave solutions are obtained for the generalized double combined sinh–cosh-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \kappa \frac{\partial^2 u}{\partial x^2} + \alpha \sinh(nu) + \alpha \cosh(nu) + \beta \sinh(2nu) + \beta \cosh(2nu) = 0, \quad (11)$$

where n is a positive integer [13,28–30].

In the real-life world, nonlinear systems with variable coefficients can be used to study more complex phenomena, including special cases of nonlinear integrable systems with constant coefficients. Therefore, it is beneficial to study nonlinear systems with variable coefficients. Hence, several nonlinear systems with space- and time-dependent coefficients have been studied and solved; see, for instance, the generalized sine-Gordon equation with variable coefficients [31–33], the generalized sinh-Gordon equation with variable coefficients [4,9,34], the Fisher–KPP equation with a time-dependent Allee effect [35], the generalized Fisher equation with time-dependent coefficients [36–38], the Korteweg–de Vries equation with variable coefficients and (or) nonuniformity terms [39,40], the three-coupled variable-coefficient nonlinear Schrödinger system [41], the $(2+1)$-dimensional generalized variable-coefficient Boiti–Leon–Pempinelli system [42], a population model with time-dependent advection and an autocatalytic-type growth [43], and so on.

The analytical studies of an inhomogeneous sinh-Gordon equation that is space- and/or time-dependent have been limited. A generalized sinh-Gordon with variable coefficient can be expressed as

$$\frac{\partial^2 u}{\partial x \partial t} = \gamma(x,t) \sinh(u(x,t)), \quad (12)$$

where $\gamma(x,t)$ is the variable coefficient. Solitary and extended wave solutions to Equation (12) were found in [4,34]. As opposed to the non-integrable Equation (12), let us consider a different version of the generalized sinh-Gordon equation with variable coefficients expressed as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sinh(\mu(x,t)\, u(x,t) + \nu(x,t)) = 0, \quad (13)$$

where $\mu(x,t)$ and $\nu(x,t)$ are variable coefficients. Clearly, Equation (13) is reduced to the sinh-Gordon Equation (1) in the case of $\mu(x,t) = 1$ and $\nu(x,t) = 0$. If the sinh term in (13) is expanded, then the equation can be expressed as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sinh(\mu(x,t)\, u(x,t)) \cosh \nu(x,t) + \cosh(\mu(x,t)\, u(x,t)) \sinh \nu(x,t) = 0. \quad (14)$$

To our knowledge, the generalized sinh-Gordon equation with variable coefficients (13) has not been solved to date. In this paper, by employing a function transformation in a judicious manner, we construct various analytical traveling wave solutions to the generalized sinh-Gordon equation with variable coefficients (13). These solutions include some real single and multi-solitons, breaking kink waves, periodic waves, singular waves, periodic singular waves, and compactons.

2. Traveling Wave Solutions for the Case $\nu(x,t) = 0$

In this section, we employ the known solutions of Equation (1) in order to find analytic solutions for the generalized sinh-Gordon equation with variable coefficients. Let us consider the case when $\nu(x,t) = 0$. Then Equation (13) becomes

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sinh(\mu(x,t)\, u(x,t)) = 0, \tag{15}$$

where $\mu(x,t)$ is a variable coefficient. We start by introducing the variable transformation

$$u(x,t) = \frac{f(x,t)}{\mu(x,t)}, \tag{16}$$

where $f(x,t)$ is an auxiliary function to be determined later. It should be pointed out that with the transformation (16), the sinh terms of Equations (1) and (15) will be exactly the same. Therefore, the thought is to separately compare the second derivative terms $\partial_{tt}u$ and $\partial_{xx}u$ of both the generalized sinh-Gordon Equation (15) and the standard sinh-Gordon Equation (1) and decide if it is possible to find a suitable variable coefficient function $\mu(x,t)$. Substituting the transformation (16) into the second derivative term $\partial_{tt}u$ in Equation (15), we find the expression

$$\frac{\frac{\partial^2}{\partial t^2} f(x,t)}{\mu(x,t)} - \frac{2\,\frac{\partial}{\partial t} f(x,t)\,\frac{\partial}{\partial t}\mu(x,t)}{\mu^2(x,t)} + \frac{2\,f(x,t)\left(\frac{\partial}{\partial t}\mu(x,t)\right)^2}{\mu^3(x,t)} - \frac{f(x,t)\,\frac{\partial^2}{\partial t^2}\mu(x,t)}{\mu^2(x,t)}.$$

Now, equating the last expression and the linear term $\partial_{tt}u = \partial_{tt}f$ and solving the resulting equation for $\mu(x,t)$, we obtain

$$\begin{aligned}\mu(x,t) &= \frac{f(x,t)}{\left(\frac{\partial}{\partial t} f(x,t)\right)t - \int t\,\frac{\partial^2}{\partial t^2} f(x,t)\,\mathrm{d}t + g_1(x) + g_2(x)\,t} \\ &= \frac{f(x,t)}{f(x,t) + g_1(x) + g_2(x)\,t},\end{aligned} \tag{17}$$

where $g_1(x)$ and $g_2(x)$ are arbitrary functions. Repeating this process for $\partial_{xx}u$, we find

$$\mu(x,t) = \frac{f(x,t)}{f(x,t) + g_3(t) + g_4(t)x}, \tag{18}$$

where $g_3(t)$ and $g_4(t)$ are arbitrary functions.

If we compare (17) and (18), we find

$$\mu(x,t) = \frac{f(x,t)}{f(x,t) + x(A_1 t + A_2) + A_3 t + A_4}, \tag{19}$$

where A_1, A_2, A_3, and A_4 are arbitrary constants. Equation (19) can be used to find analytic traveling wave solutions for (15). These solutions include real solitons, periodic waves, breaking kink waves, singular waves, periodic singular waves, and compactons.

Theorem 1. *The generalized sinh-Gordon equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + \sinh\left(\frac{f(x,t)}{f(x,t) + x(A_1 t + A_2) + A_3 t + A_4}\,u(x,t)\right) = 0, \tag{20}$$

has the analytic traveling wave solution

$$u(x,t) = f(x,t) + x(A_1 t + A_2) + A_3 t + A_4,$$

provided that $f(x,t)$ is a solution for the standard sinh-Gordon Equation (1), where A_1, A_2, A_3, and A_4 are arbitrary constants.

For example, if we use the single soliton solution (5), the generalized sinh-Gordon equation given by

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + \sinh\left(\frac{4\,\mathrm{arctanh}\left(\exp\left(\kappa\,x \pm \sqrt{\kappa^2-1}\,t\right)\right)}{4\,\mathrm{arctanh}\left(\exp\left(\kappa\,x \pm \sqrt{\kappa^2-1}\,t\right)\right) + x(A_1t + A_2) + A_3t + A_4}\,u(x,t)\right) = 0,$$

admits the following traveling wave solution

$$u(x,t) = 4\,\mathrm{arctanh}\left(\exp\left(\kappa\,x \pm \sqrt{\kappa^2-1}\,t\right)\right) + x(A_1t + A_2) + A_3t + A_4. \tag{21}$$

Additionally, using the two-soliton solution (6), the generalized sinh-Gordon equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + \sinh\left(\frac{4\,\mathrm{arctanh}\left(\frac{\exp(\kappa_1\,x \pm \omega_1\,t) + \exp(\kappa_2\,x \pm \omega_2\,t)}{1 - \left(\frac{1-\kappa_1\,\kappa_2+\omega_1\,\omega_2}{1+\kappa_1\,\kappa_2-\omega_1\,\omega_2}\right)\exp((\kappa_1+\kappa_2)\,x \pm (\omega_1+\omega_2)\,t)}\right)}{4\,\mathrm{arctanh}\left(\frac{\exp(\kappa_1\,x \pm \omega_1\,t) + \exp(\kappa_2\,x \pm \omega_2\,t)}{1 - \left(\frac{1-\kappa_1\,\kappa_2+\omega_1\,\omega_2}{1+\kappa_1\,\kappa_2-\omega_1\,\omega_2}\right)\exp((\kappa_1+\kappa_2)\,x \pm (\omega_1+\omega_2)\,t)}\right) + x(A_1t + A_2) + A_3t + A_4}\,u(x,t)\right) = 0,$$

admits the analytic solution

$$u(x,t) = 4\,\mathrm{arctanh}\left(\frac{\exp(\kappa_1\,x \pm \omega_1\,t) + \exp(\kappa_2\,x \pm \omega_2\,t)}{1 - \left(\frac{1-\kappa_1\,\kappa_2+\omega_1\,\omega_2}{1+\kappa_1\,\kappa_2-\omega_1\,\omega_2}\right)\exp((\kappa_1+\kappa_2)\,x \pm (\omega_1+\omega_2)\,t)}\right) + x(A_1t + A_2) + A_3t + A_4,$$

where $\omega_1 = \sqrt{\kappa_1^2 - 1}$ and $\omega_2 = \sqrt{\kappa_2^2 - 1}$. Another analytic traveling wave solution for the generalized sinh-Gordon equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + \sinh\left(\frac{2\,\ln\left(\tanh\left(\frac{1}{2}\,\kappa\,x \pm \frac{1}{2}\,\sqrt{\kappa^2-1}\,t\right)\right)}{2\,\ln\left(\tanh\left(\frac{1}{2}\,\kappa\,x \pm \frac{1}{2}\,\sqrt{\kappa^2-1}\,t\right)\right) + x(A_1t + A2) + A_3t + A_4}\,u(x,t)\right) = 0,$$

can be taken from Equation (10) that is given by

$$u(x,t) = 2\,\ln\left(\tanh\left(\frac{1}{2}\,\kappa\,x \pm \frac{1}{2}\,\sqrt{\kappa^2-1}\,t\right)\right) + x(A_1t + A2) + A_3t + A_4.$$

Further, more analytic solutions to Equation (15) can be obtained using the alternate form for $\mu(x,t)$ that is given by

$$\mu(x,t) = \frac{f(x,t)}{f(x,t) + G_1(x+t) + G_2(x-t) + x(A_1t + A_2) + A_3t + A_4}, \tag{22}$$

where $G_1(x+t)$ and $G_2(x-t)$ are arbitrary differentiable functions and A_1, A_2, A_3, and A_4 are arbitrary constants.

Theorem 2. *The generalized sinh-Gordon equation*

$$\frac{\partial^2}{\partial t^2}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t) + \sinh\left(\frac{f(x,t)}{f(x,t)+G_1(x+t)+G_2(x-t)+x(A_1t+A_2)+A_3t+A_4}u(x,t)\right)=0, \tag{23}$$

has the analytic traveling wave solution

$$u(x,t)=f(x,t)+G_1(x+t)+G_2(x-t)+x(A_1t+A_2)+A_3t+A_4,$$

provided that $f(x,t)$ is a solution for the standard sinh-Gordon Equation (1), where $G_1(x+t)$ and $G_2(x-t)$ are arbitrary differentiable functions and A_1, A_2, A_3, and A_4 are arbitrary constants.

For example, a periodic wave solution for the standard sinh-Gordon Equation (1) [14]

$$u(x,t)= \operatorname{arccosh}\left(\frac{v(x,t)+v^{-1}(x,t)}{2}\right),$$

where

$$v(x,t)=v_1-(v_1-v_2)\mathrm{JacobiSN}^2\left(\frac{1}{2}\sqrt{\frac{v_1}{c^2-1}}(x-ct),\sqrt{\frac{v_1-v_2}{v_1}}\right),$$

$v_1=1+\frac{\kappa}{2}+\frac{1}{2}\sqrt{\kappa^2+4\kappa}$, and $v_2=1+\frac{\kappa}{2}-\frac{1}{2}\sqrt{\kappa^2+4\kappa}$. Then, the traveling wave solution for the generalized sinh-Gordon equation

$$\frac{\partial^2}{\partial t^2}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t) + \sinh\left(\frac{\operatorname{arccosh}\left(\frac{v(x,t)+v^{-1}(x,t)}{2}\right)}{\operatorname{arccosh}\left(\frac{v(x,t)+v^{-1}(x,t)}{2}\right)+G_1(x+t)+G_2(x-t)+x(A_1t+A_2)+A_3t+A_4}u(x,t)\right)=0,$$

is given by

$$u(x,t)= \operatorname{arccosh}\left(\frac{v(x,t)+v^{-1}(x,t)}{2}\right)+G_1(x+t)+G_2(x-t)+x(A_1t+A_2)+A_3t+A_4, \tag{24}$$

which is a bounded periodic wave solution when G_1 and G_2 are bounded functions and $A_1=A_2=A_3=0$. Further, other solutions can be obtained when any of the constants A_1, A_2, and A_3 are selected non-zero. However, the resulting solutions will be unbounded.

3. Traveling Wave Solutions for the Case $\nu(x,t)\neq 0$

In this section, our goal is to look for an expression of $\nu(x,t)$ that can be used to find more analytic traveling waves for the generalized sinh-Gordon Equation (13). Let us initially consider the case when $\mu(x,t)=1$. Then Equation (13) becomes

$$\frac{\partial^2}{\partial t^2}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)+\sinh(u(x,t)+\nu(x,t))=0. \tag{25}$$

If we substitute the variable transformation

$$u(x,t)=f(x,t)-\nu(x,t),$$

into Equation (25), we find that

$$\frac{\partial^2}{\partial t^2}f(x,t)-\frac{\partial^2}{\partial t^2}\nu(x,t)-\frac{\partial^2}{\partial x^2}f(x,t)+\frac{\partial^2}{\partial x^2}\nu(x,t)+\sinh(f(x,t))=0$$

Setting $\nu(x,t) = 0$, we find

$$\frac{\partial^2}{\partial t^2} f(x,t) - \frac{\partial^2}{\partial x^2} f(x,t) + \sinh(f(x,t)) = 0$$

Equating the two equations above and simplifying, by eliminating the like terms, we find that

$$\frac{\partial^2}{\partial t^2} \nu(x,t) - \frac{\partial^2}{\partial x^2} \nu(x,t) = 0$$

Solving the above equation for $\nu(x,t)$, we obtain

$$\nu(x,t) = H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4, \tag{26}$$

where $H_1(x+t)$ and $H_2(x-t)$ are arbitrary differentiable functions and B_1, B_2, B_3, and B_4 are arbitrary constants.

Theorem 3. *The generalized sinh-Gordon equation*

$$\frac{\partial^2}{\partial t^2} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) + \sinh\Big(u(x,t) + H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4\Big) = 0, \tag{27}$$

has the analytic traveling wave solution

$$u(x,t) = f(x,t) - \Big(H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4\Big), \tag{28}$$

provided that $f(x,t)$ is a solution for the standard sinh-Gordon Equation (1), where $H_1(x+t)$ and $H_2(x-t)$ are arbitrary differentiable functions and B_1, B_2, B_3, and B_4 are arbitrary constants.

Now, let us consider the case when $\mu(x,t) \neq 1$. Substituting the transformation

$$u(x,t) = \frac{f(x,t) - \nu(x,t)}{\mu(x,t)}.$$

into Equation (13) gives us

$$\begin{aligned}
&\frac{\frac{\partial^2}{\partial t^2} f(x,t) - \frac{\partial^2}{\partial t^2} \nu(x,t)}{\mu(x,t)} - \frac{2\left(\frac{\partial}{\partial t} f(x,t) - \frac{\partial}{\partial t} \nu(x,t)\right) \frac{\partial}{\partial t}\mu(x,t)}{\mu^2(x,t)} + \frac{2\,(f(x,t) - \nu(x,t))\left(\frac{\partial}{\partial t}\mu(x,t)\right)^2}{\mu^3(x,t)} \\
&- \frac{(f(x,t) - \nu(x,t)) \frac{\partial^2}{\partial t^2}\mu(x,t)}{\mu^2(x,t)} - \frac{\frac{\partial^2}{\partial x^2} f(x,t) - \frac{\partial^2}{\partial x^2} \nu(x,t)}{\mu(x,t)} + \frac{2\left(\frac{\partial}{\partial x} f(x,t) - \frac{\partial}{\partial x} \nu(x,t)\right) \frac{\partial}{\partial x}\mu(x,t)}{\mu^2(x,t)} \\
&- \frac{2\,(f(x,t) - \nu(x,t))\left(\frac{\partial}{\partial x}\mu(x,t)\right)^2}{\mu^3(x,t)} + \frac{(f(x,t) - \nu(x,t)) \frac{\partial^2}{\partial x^2}\mu(x,t)}{\mu^2(x,t)} + \sinh(f(x,t)) = 0.
\end{aligned}$$

Setting $\nu(x,t) = 0$, we find

$$\begin{aligned}
&\frac{\frac{\partial^2}{\partial t^2} f(x,t)}{\mu(x,t)} - \frac{2\left(\frac{\partial}{\partial t} f(x,t)\right) \frac{\partial}{\partial t}\mu(x,t)}{\mu^2(x,t)} + \frac{2\,f(x,t)\left(\frac{\partial}{\partial t}\mu(x,t)\right)^2}{\mu^3(x,t)} - \frac{f(x,t) \frac{\partial^2}{\partial t^2}\mu(x,t)}{\mu^2(x,t)} - \frac{\frac{\partial^2}{\partial x^2} f(x,t)}{\mu(x,t)} \\
&+ \frac{2\,\frac{\partial}{\partial x} f(x,t) \frac{\partial}{\partial x}\mu(x,t)}{\mu^2(x,t)} - \frac{2\,f(x,t)\left(\frac{\partial}{\partial x}\mu(x,t)\right)^2}{\mu^3(x,t)} + \frac{f(x,t) \frac{\partial^2}{\partial x^2}\mu(x,t)}{\mu^2(x,t)} + \sinh(f(x,t)) = 0.
\end{aligned}$$

Equating the two equations above and simplifying, by eliminating the like terms, we find that

$$\frac{\partial^2}{\partial t^2}\left(\frac{\nu(x,t)}{\mu(x,t)}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\nu(x,t)}{\mu(x,t)}\right) = 0. \tag{29}$$

Solving the above equation for $\nu(x,t)$, we obtain

$$\nu(x,t) = \mu(x,t)\Big(H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4\Big), \tag{30}$$

where $H_1(x+t)$ and $H_2(x-t)$ are arbitrary differentiable functions and B_1, B_2, B_3, and B_4 are arbitrary constants.

Theorem 4. *The generalized sinh-Gordon equation given by*

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t)\\&+\sinh\Big(\mu(x,t)\,u(x,t) + \mu(x,t)\Big(H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4\Big)\Big) = 0,\end{aligned} \tag{31}$$

has the analytic traveling wave solution

$$u(x,t) = \frac{f(x,t) - \mu(x,t)\Big(H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4\Big)}{\mu(x,t)}, \tag{32}$$

provided that

$$u(x,t) = \frac{f(x,t)}{\mu(x,t)},$$

is a solution for the generalized sinh-Gordon equation with variable coefficient (15), where $H_1(x+t)$ and $H_2(x-t)$ are arbitrary differentiable functions and B_1, B_2, B_3, and B_4 are arbitrary constants.

Equations (28) and (32) are bounded and unbounded real-valued traveling waves for the generalized sinh-Gordon Equations (27) and (31), respectively. For example, the breaking kink wave solution and breaking anti-kink wave solution for the standard sinh-Gordon Equation (1) is [14]

$$u(x,t) = 2\,\mathrm{arctanh}\left(\mathrm{sech}\left(\frac{1}{\sqrt{1-c^2}}(x - c\,t)\right)\right),$$

where $|c| < 1$, and so, the generalized sinh-Gordon equation given by

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t)\\&+\sinh\left(\left(\frac{2\,\mathrm{arctanh}\left(\mathrm{sech}\left(\frac{1}{\sqrt{1-c^2}}(x-c\,t)\right)\right)}{2\,\mathrm{arctanh}\left(\mathrm{sech}\left(\frac{1}{\sqrt{1-c^2}}(x-c\,t)\right)\right) + G_1(x+t) + G_2(x-t) + x(A_1 t + A_2) + A_3 t + A_4}\right)\right.\\&\left.\Big[u(x,t) + H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4\Big]\right) = 0,\end{aligned} \tag{33}$$

has the following analytic traveling wave

$$\begin{aligned}u(x,t) =&\,2\,\mathrm{arctanh}\left(\mathrm{sech}\left(\frac{1}{\sqrt{1-c^2}}(x-c\,t)\right)\right) + G_1(x+t) + G_2(x-t) + x(A_1 t + A_2) + A_3 t + A_4\\&-\Big(H_1(x+t) + H_2(x-t) + x(B_1 t + B_2) + B_3 t + B_4\Big).\end{aligned} \tag{34}$$

From (34), one can obtain a variety of traveling wave solutions for different choices of $G_1, G_2, H_1, H_2, A_1, A_2, A_3, A_4, B_1, B_2, B_3$, and B_4. On the other hand, other types of solutions for the generalized sinh-Gordon Equations (27) and (31) such as single and multi-solitons, periodic waves, singular waves, periodic singular waves, and compactons can be obtained in a similar manner using Equations (28) and (32) and here, for the sake of brevity, we omit the details.

4. Conclusions

Sinh-Gordon equation is an important soliton equation in the field of soliton. It has applications in various fields, such as differential geometry, integrable quantum field theory, fluid dynamics, and kink dynamics. It can be used to describe surfaces with a constant negative Gaussian curvature. In this paper, we showed that real-valued traveling waves and soliton solutions could be obtained for the generalized sinh-Gordon equation with variable coefficients by utilizing the transformation of variables innovatively and the known solutions of the standard sinh-Gordon equation. The analytic solutions are new and have not been reported elsewhere in the literature. In addition, with the aid of *Maple*, we have verified all the solutions by substituting them back into the generalized sinh-Gordon Equation (13). These solutions can be viewed as more general and extensions of the solutions of the standard sinh-Gordon equation. Additionally, these solutions can be of great value when modeling real-life phenomena using the sinh-Gordon equation where the balance between dispersion and nonlinearity is perturbed.

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