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Modified Mann Subgradient-like Extragradient Rules for Variational Inequalities and Common Fixed Points Involving Asymptotically Nonexpansive Mappings

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Abstract: In a real Hilbert space, we aim to investigate two modified Mann subgradient-like methods to find a solution to pseudo-monotone variational inequalities, which is also a common fixed point of a finite family of nonexpansive mappings and an asymptotically nonexpansive mapping. We obtain strong convergence results for the sequences constructed by these proposed rules. We give some examples to illustrate our analysis.

Keywords: modified Mann subgradient-like extragradient rule; pseudo-monotone variational inequality; common fixed point problem; asymptotically nonexpansive mapping; line-search process

MSC: 90C25; 90C30; 90C60; 68Q25; 49M25; 90C22

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1. Introduction

Let the $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ represent the inner product and induced norm in a real Hilbert space H, respectively. We denote by P_C the nearest point projection from H onto C, where $\emptyset \neq C \subset H$ and C is convex and closed. Given $T : C \to H$ a nonlinear mapping, we denote by $\mathrm{Fix}(T)$ the fixed point set of T, i.e., $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$. Let the $\mathbf{R}$, $\to$ and $\rightharpoonup$ indicate the set of all real numbers, the strong convergence, and the weak convergence, respectively. A self-mapping $T : C \to C$ is referred to as being asymptotically nonexpansive if $\exists \{\psi_n\} \subset [0, +\infty)$ s.t. $\lim_{n\to\infty} \psi_n = 0$ and

$$\|T^n x - T^n y\| \leq \|x - y\| + \psi_n \|x - y\| \quad \forall n \geq 1,\ x, y \in C \tag{1}$$

and T is nonexpansive when $\psi_n = 0$.

Given a continuous mapping $A : H \to H$, a variational inequality problem (denoted by (VIP)) is:

$$\text{find } z^* \in C \text{ such that } \langle Az^*, z - z^* \rangle \geq 0 \ \forall z \in C.$$

Let us denote the set of the solution VIP by $\mathrm{VI}(C, A)$. In 1976, Korpelevich [1] put forth the extragradient method, which has been one of the most effective approaches for solving the VIP:

$$\begin{cases} y_n = P_C(x_n - \zeta A x_n), \\ x_{n+1} = P_C(x_n - \zeta A y_n) \quad \forall n \geq 0, \end{cases} \tag{2}$$

for $\zeta \in (0, \frac{1}{L})$ with L being the Lipschitz constant of A. Weak convergence results of (2) have been obtained in studies [2–22] and references therein.

The extragradient method (2) involves solving a minimization problem over C at each iteration when P_C has no closed-form solution. This could make the extragradient method

(2) computationally expensive. In study [6], Censor et al. modified (2) and introduced the subgradient extragradient:

$$\begin{cases} y_n = P_C(x_n - \zeta A x_n), \\ D_n = \{v \in H : \langle x_n - \zeta A x_n - y_n, v - y_n \rangle \le 0\}, \\ x_{n+1} = P_{D_n}(x_n - \zeta A y_n), \end{cases} \tag{3}$$

for $\zeta \in (0, \frac{1}{L})$ with L being the Lipschitz constant of A. Thong and Hieu [19] added an inertial extrapolation step to (3): $x_0, x_1 \in H$,

$$\begin{cases} v_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(v_n - \zeta A w_n), \\ D_n = \{v \in H : \langle v_n - \zeta A w_n - y_n, v - y_n \rangle \le 0\}, \\ x_{n+1} = P_{C_n}(v_n - \zeta A y_n), \end{cases} \tag{4}$$

for $\zeta \in (0, \frac{1}{L})$ with L being the Lipschitz constant of A, and the weak convergence being obtained. In study [22], Reich et al. suggested the modified projection-type method for solving the VIP with the pseudo-monotone and uniformly continuous mapping A, given a sequence $\{\alpha_n\} \subset (0,1)$ and a contraction $f : C \to C$ with constant $\varrho \in [0,1)$. For any initial $x_1 \in C$, the sequence $\{x_n\}$ is constructed below.

Furthermore, it was proven in study [22] that the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly. Subsequently, Ceng, Yao and Shehu [21] proposed a Mann-type method of (2) to solve pseudo-monotone variational inequalities and the common fixed point problem of many finitely nonexpansive self-mappings $\{T_i\}_{i=1}^N$ on C and an asymptotically nonexpansive self-mapping $T_0 := T$ on C. Given a contraction $f : C \to C$ with constant $\varrho \in [0,1)$, let $\{\sigma_n\} \subset [0,1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1 \; \forall n \ge 1$ and $T_n := T_{n \bmod N}$. For any initial $x_1 \in C$, the sequence $\{x_n\}$ is constructed below (Algorithm 2).

Algorithm 1 (see study [22]). **Initialization:** Given $\mu > 0$, $l \in (0,1)$, $\lambda \in (0, \frac{1}{\mu})$.

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute $y_n = P_C(x_n - \lambda A x_n)$ and $r_\lambda(x_n) := x_n - y_n$. If $r_\lambda(x_n) = 0$, then stop. x_n is a solution of VI(C, A). Otherwise;

Step 2. Compute $w_n = x_n - \zeta_n r_\lambda(x_n)$, where $\zeta_n := l^{j_n}$ and j_n is the smallest nonnegative integer j, satisfying $\langle A x_n - A(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n) \rangle \le \frac{\mu}{2} \|r_\lambda(x_n)\|^2$;

Step 3. Compute $x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_{C_n}(x_n)$, where $C_n := \{x \in C : h_n(x_n) \le 0\}$ and $h_n(x) = \langle A w_n, x - x_n \rangle + \frac{\zeta_n}{2\lambda} \|r_\lambda(x_n)\|^2$.

Under suitable conditions, it was proven in study [21] that the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega = \bigcap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ if and only if $\lim_{n \to \infty}(\|x_n - x_{n+1}\| + \|x_n - y_n\|) = 0$ provided $T^n z_n - T^{n+1} z_n \to 0$, where $x^* = P_\Omega(I - \rho F + f)x^*$.

In a real Hilbert space H, let the VIP and CFPP represent the pseudo-monotone variational inequality problem with uniformly continuous mapping A, the common fixed point problem of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, and an asymptotically nonexpansive mapping $T_0 := T$, respectively. Inspired by the above research works, we propose and analyze two modified Mann subgradient-like extragradient algorithms with the line-search process for solving the VIP and CFPP. The proposed algorithms are based on the Mann iteration method, (3) method with the line-search process, and the viscosity approximation method. Under some conditions, we establish some strong convergence results for the sequences constructed by these proposed rules. Finally, our main results are applied to handle the VIP and CFPP in an illustrated example.

Algorithm 2 (see study [21]). **Initialization:** Given $\mu > 0$, $l \in (0,1)$, $\lambda \in (0, \frac{1}{\mu})$.

Iterative Steps: Given x_n, compute

Step 1. Set $w_n = (1-\sigma_n)x_n + \sigma_n T_n x_n$, and compute $y_n = P_C(w_n - \lambda A w_n)$ and $r_\lambda(w_n) := w_n - y_n$;

Step 2. Compute $t_n = w_n - \zeta_n r_\lambda(w_n)$, where $\zeta_n := l^{j_n}$ and j_n is the smallest nonnegative integer j,
satisfying $\langle Aw_n - A(w_n - l^j r_\lambda(w_n)), w_n - y_n\rangle \le \frac{\mu}{2}\|r_\lambda(w_n)\|^2$;

Step 3. Compute $z_n = P_{C_n}(w_n)$ and $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n z_n$,
where $C_n := \{x \in C : h_n(x) \le 0\}$ and $h_n(x) = \langle At_n, x - w_n\rangle + \frac{\zeta_n}{2\lambda}\|r_\lambda(w_n)\|^2$.

Again set $n := n+1$ and go to Step 1.

The structure of the article is specified below. In Section 2, we first recall some concepts and basic results. Section 3 explores the strong convergence analysis of our proposed methods. Finally, in Section 4, an illustrated example is given. Our results complement related results by Ceng, Yao, and Shehu [21]; Reich et al. [22]; and Ceng and Shang [9]. Indeed, it is worth emphasizing that our problem of finding an element $x^* \in \Omega = \bigcap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A)$ is more general and more interesting than the corresponding problem of finding an element $x^* \in \mathrm{VI}(C,A)$ in study [22]. Moreover, our strong convergence theorems are more advantageous and more clever than the corresponding strong convergence ones in studies [9,21] because the conclusion $x_n \to x^* \in \Omega \Leftrightarrow \|x_n - x_{n+1}\| + \|x_n - y_n\| \to 0\ (n \to \infty)$ in the corresponding strong convergence theorems [9,21] is updated by our conclusion $x_n \to x^* \in \Omega$. Without question, the strong convergence criteria for the sequence $\{x_n\}$ in this paper are more convenient and more beneficial in comparison with those of studies [9,21].

2. Preliminaries

We say that $T : C \to H$ is

(a) L-Lipschitz continuous (or L-Lipschitzian) if $\exists L > 0$ such that $\|Tu - Tv\| \le L\|u - v\|\ \forall u, v \in C$;
(b) Monotone if $\langle Tu - Tv, u - v\rangle \ge 0\ \forall u, v \in C$;
(c) pseudo-monotone if $\langle Tu, v - u\rangle \ge 0 \Rightarrow \langle Tv, v - u\rangle \ge 0\ \forall u, v \in C$;
(d) ϖ-strongly monotone if $\exists \varpi > 0$ such that $\langle Tu - Tv, u - v\rangle \ge \varpi\|u - v\|^2\ \forall u, v \in C$;
(e) Sequentially weakly continuous if $\forall\{u_n\} \subset C$, we have $u_n \rightharpoonup u \Rightarrow Tu_n \rightharpoonup Tu$.

One can see that (b) implies (c) but the converse fails. Given $v \in H$, there exists a unique nearest point in C, denoted by $P_C v$ (P_C is called a metric projection of H onto C), such that $\|v - P_C v\| \le \|v - z\|\ \ \forall z \in C$. According to reference [17], we know that the following hold:

(a) $\langle u - v, P_C u - P_C v\rangle \ge \|P_C u - P_C v\|^2\ \ \forall u, v \in H$;
(b) $\langle u - P_C u, v - P_C u\rangle \le 0\ \ \forall u \in H, v \in C$;
(c) $\|u - v\|^2 \ge \|u - P_C u\|^2 + \|v - P_C u\|^2\ \ \forall u \in H, v \in C$;
(d) $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle\ \ \forall u, v \in H$;
(e) $\|\lambda u + \mu v\|^2 = \lambda\|u\|^2 + \mu\|v\|^2 - \lambda\mu\|u - v\|^2\ \ \forall u, v \in H$, $\forall \lambda, \mu \in \mathbb{R}$ with $\lambda + \mu = 1$.

Lemma 1. *(see reference [4]). Let H_1 and H_2 be two real Hilbert spaces. Suppose that $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

It is easy from the subdifferential inequality of $\frac{1}{2}\|\cdot\|^2$:

$$\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle \quad \forall u, v \in H.$$

Lemma 2. *(see reference [23]). Let h be a real-valued function on H and define $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\psi > 0$, then* $\mathrm{dist}(x, K) \geq \psi^{-1} \max\{h(x), 0\}$ $\forall x \in C$, *where* $\mathrm{dist}(x, K)$ *denotes the distance of x to K.*

Lemma 3. *(see reference [6], Lemma 2.1). Assume that $A : C \to H$ is pseudo-monotone and continuous. Then $u^* \in C$ is a solution to the VIP $\langle Au^*, u - u^*\rangle \geq 0$ $\forall u \in C$, if and only if $\langle Au, u - u^*\rangle \geq 0$ $\forall u \in C$.*

Lemma 4. *(see reference [24]). Let $\{b_n\}$ be a sequence of nonnegative numbers satisfying: $b_{n+1} \leq (1 - \varsigma_n)b_n + \varsigma_n\gamma_n$ $\forall n \geq 1$, where $\{\varsigma_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\varsigma_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \varsigma_n = \infty$, and (ii) $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\varsigma_n\gamma_n| < \infty$. Then $\lim_{n\to\infty} b_n = 0$.*

Lemma 5. *(see reference [25]). Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Then $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, where I is the identity mapping of X.*

Lemma 6. *(see reference [26]). Let $\{\Gamma_n\}$ be a sequence of real numbers such that there exists $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$, satisfying $\Gamma_{n_k} < \Gamma_{n_k+1}$ for each integer $k \geq 1$. Define*

$$\eta(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where integer $n_0 \geq 1$ and $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then (i) $\eta(n_0) \leq \eta(n_0 + 1) \leq \cdots$ and $\eta(n) \to \infty$; (ii) $\Gamma_{\eta(n)} \leq \Gamma_{\eta(n)+1}$ and $\Gamma_n \leq \Gamma_{\eta(n)+1}$ $\forall n \geq n_0$.

3. Our Contributions

Assume that

$T : C \to C$ is an asymptotically nonexpansive mapping and $T_i : C \to C$ is a nonexpansive mapping for $i = 1, ..., N$ such that the sequence $\{T_n\}_{n=1}^{\infty}$ is defined as in Algorithm 1.

$A : H \to H$ is pseudo-monotone and uniformly continuous on C, s.t. $\|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|$ for each $\{x_n\} \subset C$ with $x_n \rightharpoonup z$.

$f : C \to C$ is a contraction with constant $\varrho \in [0, 1)$, and $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \neq \emptyset$ with $T_0 := T$.

$\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $\{\sigma_n\} \subset [0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{\psi_n}{\alpha_n} = 0$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$.

Lemma 7. *The Armijo-type search rule (5) is well defined, and consequently* $\frac{\|r_\lambda(w_n)\|^2}{\lambda} \leq \langle Aw_n, r_\lambda(w_n)\rangle$.

Proof. From $l \in (0, 1)$ and uniform continuity of A on C, one has $\lim_{j\to\infty}\langle Aw_n - A(w_n - l^j r_\lambda(w_n)), r_\lambda(w_n)\rangle = 0$. If $r_\lambda(w_n) = 0$, then $j_n = 0$. If $r_\lambda(w_n) \neq 0$, then $\exists$ (integer) $j_n \geq 0$, satisfying (3.1). By the firm nonexpansivity of P_C, one obtains $\|x - P_C y\|^2 \leq \langle x - y, x - P_C y\rangle$ $\forall x \in C, y \in H$. Putting $y = w_n - \lambda A w_n$ and $x = w_n$, one gets $\|w_n - P_C(w_n - \lambda A w_n)\|^2 \leq \lambda\langle Aw_n, w_n - P_C(w_n - \lambda A w_n)\rangle$, and hence $\frac{\|r_\lambda(w_n)\|^2}{\lambda} \leq \langle Aw_n, r_\lambda(w_n)\rangle$. $\square$

Lemma 8. *Let $p \in \Omega$ and assume the function h_n is formulated as (3.2). Then, $h_n(w_n) = \frac{\zeta_n}{2\lambda}\|r_\lambda(w_n)\|^2$ and $h_n(p) \leq 0$. In addition, if $r_\lambda(w_n) \neq 0$, then $h_n(w_n) > 0$.*

Let $\{u_n\}$ be the sequence constructed in Algorithm 3.

Algorithm 3 Initialization: Given $\mu > 0$, $l \in (0,1)$, $\lambda \in (0, \frac{1}{\mu})$. Pick $u_1 \in C$.

Iterative Steps: Given u_n, compute

Step 1. Set $w_n = (1-\sigma_n)u_n + \sigma_n T^n u_n$, and compute $y_n = P_C(w_n - \lambda A w_n)$ and $r_\lambda(w_n) := w_n - y_n$;

Step 2. Compute $t_n = w_n - \zeta_n r_\lambda(w_n)$, where $\zeta_n := l^{j_n}$ and j_n is the smallest nonnegative integer j,
satisfying

$$\langle Aw_n - A(w_n - l^j r_\lambda(w_n)), w_n - y_n\rangle \leq \tfrac{\mu}{2}\|r_\lambda(w_n)\|^2. \tag{5}$$

Step 3. Compute $z_n = P_{D_n}(w_n)$ and $u_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \gamma_n T_n z_n$,
where $D_n := \{x \in C : h_n(x) \leq 0\}$ and

$$h_n(x) = \langle At_n, x - w_n\rangle + \tfrac{\zeta_n}{2\lambda}\|r_\lambda(w_n)\|^2. \tag{6}$$

Again set $n := n+1$ and go to Step 1.

Proof. The first claim of Lemma 8 is evident. Let us show the second claim. In fact, for $p \in \Omega$, by Lemma 3 one has $\langle At_n, t_n - p\rangle \geq 0$. So, one obtains that

$$h_n(p) = \langle At_n, p - w_n\rangle + \frac{\zeta_n}{2\lambda}\|r_\lambda(w_n)\|^2 \leq -\zeta_n\langle At_n, r_\lambda(w_n)\rangle + \frac{\zeta_n}{2\lambda}\|r_\lambda(w_n)\|^2. \tag{7}$$

Furthermore, from (5) one has $\langle Aw_n - At_n, r_\lambda(w_n)\rangle \leq \frac{\mu}{2}\|r_\lambda(w_n)\|^2$. Thus, by Lemma 7 we get

$$\langle At_n, r_\lambda(w_n)\rangle \geq \langle Aw_n, r_\lambda(w_n)\rangle - \frac{\mu}{2}\|r_\lambda(w_n)\|^2 \geq (\frac{1}{\lambda} - \frac{\mu}{2})\|r_\lambda(w_n)\|^2. \tag{8}$$

Combining (7) and (8) arrives at

$$h_n(p) \leq -\frac{\zeta_n}{2}(\frac{1}{\lambda} - \mu)\|r_\lambda(w_n)\|^2 \leq 0. \tag{9}$$

□

Lemma 9. *Let $\{u_n\}$ be the sequence constructed in Algorithm 3, s.t. $u_n - y_n \to 0$, $u_n - u_{n+1} \to 0$, $u_n - T^n u_n \to 0$, $u_n - T_n u_n \to 0$. Suppose that $T^n u_n - T^{n+1}u_n \to 0$ and $\exists\{u_{n_k}\} \subset \{u_n\}$ s.t. $u_{n_k} \rightharpoonup z$. Then $z \in \Omega$.*

Proof. Using Algorithm 3, one obtains $w_n - u_n = \sigma_n(T^n u_n - u_n)$ $\forall n \geq 1$, and hence $\|w_n - u_n\| \leq \|T^n u_n - u_n\|$. Using the hypothesis $u_n - T^n u_n \to 0$, we have

$$\lim_{n\to\infty}\|w_n - u_n\| = 0, \tag{10}$$

which together with the hypothesis $u_n - y_n \to 0$, implies that

$$\|w_n - y_n\| \leq \|w_n - u_n\| + \|u_n - y_n\| \to 0 \quad (n \to \infty).$$

Besides this, combining $w_n - u_n \to 0$ and $u_{n_k} \rightharpoonup z$ yields $w_{n_k} \rightharpoonup z$. □

Let us show that $\lim_{n\to\infty}\|u_n - T_i u_n\| = 0$ for $i = 1,\ldots,N$. In fact, note that for $i = 1,\ldots,N$,

$$\begin{aligned}\|u_n - T_{n+i}u_n\| &\le \|u_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}u_n\| \\ &\le 2\|u_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\|.\end{aligned}$$

By $u_n - u_{n+1} \to 0$ and $u_n - T_n u_n \to 0$, we get $\lim_{n\to\infty}\|u_n - T_{n+i}u_n\| = 0$ for $i = 1,\ldots,N$. This immediately arrives at

$$\lim_{n\to\infty}\|u_n - T_i u_n\| = 0 \quad \text{for } i = 1,\ldots,N. \tag{11}$$

Moreover, we claim that $u_n - Tu_n \to 0$ is as $n \to \infty$. In fact, combining the hypotheses $u_n - T^n u_n \to 0$ and $T^n u_n - T^{n+1}u_n \to 0$, guarantees that

$$\begin{aligned}\|u_n - Tu_n\| &\le \|u_n - T^n u_n\| + \|T^n u_n - T^{n+1}u_n\| + \|T^{n+1}u_n - Tu_n\| \\ &\le \|u_n - T^n u_n\| + \|T^n u_n - T^{n+1}u_n\| + (1+\psi_1)\|T^n u_n - u_n\| \\ &= (2+\psi_1)\|u_n - T^n u_n\| + \|T^n u_n - T^{n+1}u_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{12}$$

Next, let us show $z \in \mathrm{VI}(C,A)$. In fact, since C is convex and closed, from $\{u_n\} \subset C$ and $u_{n_k} \rightharpoonup z$ we get $z \in C$. In what follows, we consider two cases. If $Az = 0$, then it is clear that $z \in \mathrm{VI}(C,A)$ because $\langle Az, x - z\rangle \ge 0\ \forall x \in C$. Assume that $Az \ne 0$. Then, it follows from $w_n - y_n \to 0$ and $w_{n_k} \rightharpoonup z$ that $y_{n_k} \rightharpoonup z$. Using the assumption on A, instead of the sequentially weak continuity of A, we get $0 < \|Az\| \le \liminf_{k\to\infty}\|Ay_{n_k}\|$. So, we could suppose that $\|Ay_{n_k}\| \ne 0\ \forall k \ge 1$. Furthermore, from $y_n = P_C(w_n - \lambda Aw_n)$, we have $\langle w_n - \lambda Aw_n - y_n, u - y_n\rangle \le 0\ \forall u \in C$. Thus,

$$\frac{1}{\lambda}\langle w_n - y_n, u - y_n\rangle + \langle Aw_n, y_n - w_n\rangle \le \langle Aw_n, u - w_n\rangle \quad \forall u \in C. \tag{13}$$

According to the uniform continuity of A on C, one knows that $\{Aw_{n_k}\}$ is bounded (due to Lemma 1). Note that $\{y_{n_k}\}$ is bounded as well. Then, by (13) we have $\liminf_{k\to\infty}\langle Aw_{n_k}, u - w_{n_k}\rangle \ge 0\ \forall u \in C$. To show that $z \in \mathrm{VI}(C,A)$, we pick $\{\varepsilon_k\} \subset (0,1)$ s.t. $\varepsilon_k \downarrow 0$ as $k \to \infty$. For each $k \ge 1$, let m_k be the smallest natural number, such that

$$\langle Aw_{n_j}, u - w_{n_j}\rangle + \varepsilon_k \ge 0 \quad \forall j \ge m_k. \tag{14}$$

Since $\{\varepsilon_k\}$ is nonincreasing, it can be readily seen that $\{m_k\}$ is increasing. Noticing that $Aw_{m_k} \ne 0\ \forall k \ge 1$ (due to $\{Aw_{m_k}\} \subset \{Aw_{n_k}\}$), we set $\mu_{m_k} = \frac{Aw_{m_k}}{\|Aw_{m_k}\|^2}$, and we get $\langle Aw_{m_k}, \mu_{m_k}\rangle = 1$. By (14), it gives $\langle Aw_{m_k}, u + \varepsilon_k\mu_{m_k} - w_{m_k}\rangle \ge 0$. The pseudomonotonicity of A then gives $\langle A(u + \varepsilon_k\mu_{m_k}), u + \varepsilon_k\mu_{m_k} - w_{m_k}\rangle \ge 0$, i.e.,

$$\langle Au, u - w_{m_k}\rangle \ge \langle Au - A(u + \varepsilon_k\mu_{m_k}), x + \varepsilon_k\mu_{m_k} - w_{m_k}\rangle - \varepsilon_k\langle Au, \mu_{m_k}\rangle. \tag{15}$$

Observe that $w_{n_k} \rightharpoonup z$, $\{w_{m_k}\} \subset \{w_{n_k}\}$ and $\varepsilon_k \downarrow 0$ as $k \to \infty$. So it follows that $0 \le \limsup_{k\to\infty}\|\varepsilon_k\mu_{m_k}\| = \limsup_{k\to\infty}\frac{\varepsilon_k}{\|Aw_{m_k}\|} \le \frac{\limsup_{k\to\infty}\varepsilon_k}{\liminf_{k\to\infty}\|Aw_{n_k}\|} = 0$. Hence, we get $\varepsilon_k\mu_{m_k} \to 0$ as $k \to \infty$.

Next, we show that $z \in \Omega$. Passing to the limit as $k \to \infty$ in (15), we have $\langle Au, u - z\rangle = \liminf_{k\to\infty}\langle Au, u - w_{m_k}\rangle \ge 0\ \forall u \in C$. Using Lemma 3, $z \in \mathrm{VI}(C,A)$. Furthermore, for $i = 1,\ldots,N$, since Lemma 5 guarantees the demiclosedness of $I - T_i$ at zero, from $u_{n_k} \rightharpoonup z$ and $u_{n_k} - T_i u_{n_k} \to 0$ (due to (11)) we deduce that $z \in \mathrm{Fix}(T_i)$. Thus, $z \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Hence, from $u_{n_k} \rightharpoonup z$ and $u_{n_k} - Tu_{n_k} \to 0$ (due to (12)), we obtain that $z \in \mathrm{Fix}(T)$. Therefore, $z \in \Omega$.

Lemma 10. *Assume $\{w_n\}$ in Algorithm 3 is such that $\zeta_n\|r_\lambda(w_n)\|^2 \to 0$ as $n \to \infty$. Then, $w_n - y_n \to 0$.*

Proof. On the contrary, suppose that $\limsup_{n\to\infty} \|w_n - y_n\| = a > 0$. Then, $\exists\{n_k\} \subset \{n\}$ s.t.

$$\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = a > 0. \tag{16}$$

Note that $\lim_{k\to\infty} \zeta_{n_k}\|r_\lambda(w_{n_k})\|^2 = 0$. In what follows, we consider two cases. □

Case 1. $\liminf_{k\to\infty} \zeta_{n_k} > 0$. In this case, we might assume that $\exists\zeta > 0$ s.t. $\zeta_{n_k} \geq \zeta > 0\ \forall k \geq 1$. Then, it follows that $\|w_{n_k} - y_{n_k}\|^2 = \frac{1}{\zeta_{n_k}}\zeta_{n_k}\|w_{n_k} - y_{n_k}\|^2 \leq \frac{1}{\zeta} \cdot \zeta_{n_k}\|r_\lambda(w_{n_k})\|^2$, which hence yields

$$0 < a^2 = \lim_{k\to\infty} \|w_{n_k} - y_{n_k}\|^2 \leq \lim_{k\to\infty} \left[\frac{1}{\zeta} \cdot \zeta_{n_k}\|r_\lambda(w_{n_k})\|^2\right] = 0. \tag{17}$$

This reaches a contradiction.

Case 2. $\liminf_{k\to\infty} \zeta_{n_k} = 0$. In this case, there exists a subsequence of $\{\zeta_{n_k}\}$, still denoted by $\{\zeta_{n_k}\}$, such that $\lim_{k\to\infty} \zeta_{n_k} = 0$. Putting $v_{n_k} = \frac{1}{l}\zeta_{n_k}y_{n_k} + (1 - \frac{1}{l}\zeta_{n_k})w_{n_k}$, we get $v_{n_k} = w_{n_k} - \frac{1}{l}\zeta_{n_k}(w_{n_k} - y_{n_k})$. Since $\lim_{n\to\infty} \zeta_n\|r_\lambda(w_n)\|^2 = 0$, we have

$$\lim_{k\to\infty} \|v_{n_k} - w_{n_k}\|^2 = \lim_{k\to\infty} \frac{1}{l^2}\zeta_{n_k} \cdot \zeta_{n_k}\|w_{n_k} - y_{n_k}\|^2 = 0. \tag{18}$$

From the step size rule (5) and the definition of v_{n_k}, it follows that

$$\langle Aw_{n_k} - Av_{n_k}, w_{n_k} - y_{n_k}\rangle > \frac{\mu}{2}\|w_{n_k} - y_{n_k}\|^2. \tag{19}$$

Using the uniform the continuity of A on C, from (18) we deduce that $\lim_{k\to\infty} \|Aw_{n_k} - Av_{n_k}\| = 0$, which together with (19) leads to $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$. Thus, this contradicts with (16). Consequently, $\lim_{n\to\infty} \|w_n - y_n\| = 0$.

Theorem 1. *Suppose that the sequence $\{u_n\}$ is constructed by Algorithm 3. Then, $u_n \to u^* \in \Omega$ provided $T^n u_n - T^{n+1}u_n \to 0$, where $u^* \in \Omega$ is the unique solution to the VIP: $\langle(I - f)u^*, p - u^*\rangle \geq 0\ \ \forall p \in \Omega$.*

Proof. First of all, since $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ and $\lim_{n\to\infty} \frac{\psi_n}{\alpha_n} = 0$, we may assume, without loss of generality, that $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ and $\psi_n \leq \frac{\alpha_n(1-\varrho)}{2}\ \forall n \geq 1$. Clearly, $P_\Omega \circ f : C \to C$ is a contraction. Hence, there exists $u^* \in C$, such that $u^* = P_\Omega f(u^*)$. Therefore, $u^* \in \Omega = \bigcap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ of the VIP

$$\langle(I - f)u^*, p - u^*\rangle \geq 0 \quad \forall p \in \Omega. \tag{20}$$

Next, we show the conclusion of the theorem. With this aim, we consider the following steps.

Step 1. We claim that the following inequality holds:

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \mathrm{dist}^2(w_n, D_n) \quad \forall p \in \Omega. \tag{21}$$

Indeed, one has

$$\begin{aligned}\|z_n - p\|^2 = \|P_{D_n}w_n - p\|^2 &\leq \|w_n - p\|^2 - \|P_{D_n}w_n - w_n\|^2 \\ &= \|w_n - p\|^2 - \mathrm{dist}^2(w_n, D_n),\end{aligned}$$

which immediately yields

$$\|z_n - p\| \leq \|w_n - p\| \quad \forall n \geq 1. \tag{22}$$

Thus

$$\begin{aligned}\|w_n - p\| &\le (1-\sigma_n)\|u_n - p\| + \sigma_n\|T^n u_n - p\| \\ &\le (1-\sigma_n)\|u_n - p\| + \sigma_n(1+\psi_n)\|u_n - p\| \\ &\le (1+\psi_n)\|u_n - p\|,\end{aligned}$$

which together with (22), yields

$$\|z_n - p\| \le \|w_n - p\| \le (1+\psi_n)\|u_n - p\| \quad \forall n \ge 1. \tag{23}$$

Thus, from (23) and $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \ge 1$ it follows that

$$\begin{aligned}&\|u_{n+1} - p\| \le \alpha_n\|f(u_n) - p\| + \beta_n\|u_n - p\| + \gamma_n\|T_n z_n - p\| \\ &\le \alpha_n(\|f(u_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|u_n - p\| + \gamma_n(1+\psi_n)\|u_n - p\| \\ &\le \alpha_n(\varrho\|u_n - p\| + \|f(p) - p\|) + \beta_n\|u_n - p\| + \gamma_n\|u_n - p\| + \tfrac{\alpha_n(1-\varrho)}{2}\|u_n - p\| \\ &= [1 - \tfrac{\alpha_n(1-\varrho)}{2}]\|u_n - p\| + \alpha_n\|f(p) - p\| \le \max\{\|u_n - p\|, \tfrac{2\|f(p)-p\|}{1-\varrho}\}.\end{aligned}$$

Thus, $\|u_n - p\| \le \max\{\|u_1 - p\|, \frac{2\|f(p)-p\|}{1-\varrho}\}\ \forall n \ge 1$. This $\{u_n\}$ is bounded, and so are $\{w_n\}, \{y_n\}, \{z_n\}, \{f(u_n)\}, \{At_n\}, \{T^n u_n\}, \{T_n z_n\}$.

Step 2. Let us obtain

$$\begin{aligned}&\gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 + \|z_n - w_n\|^2\Big] + \beta_n\gamma_n\|u_n - T_n z_n\|^2 \\ &\le \|u_n - p\|^2 - \|u_{n+1} - p\|^2 + \psi_n K + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle\end{aligned}$$

for some $K > 0$. To prove this, we first note that

$$\begin{aligned}&\|u_{n+1} - p\|^2 = \|\alpha_n(f(u_n) - p) + \beta_n(u_n - p) + \gamma_n(T_n z_n - p)\|^2 \\ &\le \|\beta_n(u_n - p) + \gamma_n(T_n z_n - p)\|^2 + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle \\ &\le \beta_n\|u_n - p\|^2 + \gamma_n\|z_n - p\|^2 - \beta_n\gamma_n\|u_n - T_n z_n\|^2 + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle.\end{aligned} \tag{24}$$

On the other hand, by Algorithm 3 one has

$$\begin{aligned}&\|z_n - p\|^2 = \|P_{D_n} w_n - p\|^2 \le \|w_n - p\|^2 - \|z_n - w_n\|^2 \\ &= (1-\sigma_n)\|u_n - p\|^2 + \sigma_n\|T^n u_n - p\|^2 - (1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 - \|z_n - w_n\|^2 \\ &\le (1+\psi_n)^2\|u_n - p\|^2 - (1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 - \|z_n - w_n\|^2.\end{aligned} \tag{25}$$

Substituting (25) into (24), one gets

$$\begin{aligned}&\|u_{n+1} - p\|^2 \\ &\le \beta_n\|u_n - p\|^2 + \gamma_n\Big[(1+\psi_n)^2\|u_n - p\|^2 - (1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 - \|z_n - w_n\|^2\Big] \\ &\quad - \beta_n\gamma_n\|u_n - T_n z_n\|^2 + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle \\ &\le (1-\alpha_n)\|u_n - p\|^2 - \gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 + \|z_n - w_n\|^2\Big] \\ &\quad + \psi_n(2+\psi_n)\|u_n - p\|^2 - \beta_n\gamma_n\|u_n - T_n z_n\|^2 + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle \\ &\le \|u_n - p\|^2 - \gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 + \|z_n - w_n\|^2\Big] - \beta_n\gamma_n\|u_n - T_n z_n\|^2 \\ &\quad + \psi_n K + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle,\end{aligned}$$

where $\sup_{n\ge1}(2+\psi_n)\|u_n - p\|^2 \le K$ for some $K > 0$. This immediately implies that

$$\begin{aligned}&\gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 + \|z_n - w_n\|^2\Big] + \beta_n\gamma_n\|u_n - T_n z_n\|^2 \\ &\le \|u_n - p\|^2 - \|u_{n+1} - p\|^2 + \psi_n K + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle.\end{aligned}$$

Step 3. We show that

$$\gamma_n\Big[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2 \le \|u_n - p\|^2 - \|u_{n+1} - p\|^2 + \alpha_n\|f(u_n) - p\|^2 + \psi_n K.$$

Indeed, we claim that for some $L > 0$,

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \Big[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2. \tag{26}$$

Thanks to the boundedness of $\{At_n\}$, we know that $\exists L > 0$ s.t. $\|At_n\| \leq L \; \forall n \geq 1$, which arrives at

$$|h_n(u) - h_n(v)| = |\langle At_n, u - v\rangle| \leq \|At_n\|\|u - v\| \leq L\|u - v\| \quad \forall u, v \in D_n.$$

This hence ensures that $h_n(\cdot)$ is L-Lipschitz continuous on D_n. By Lemmas 2 and 8, one obtains

$$\mathrm{dist}(w_n, D_n) \geq \frac{1}{L}h_n(w_n) = \frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2. \tag{27}$$

Combining (21) and (27) immediately yields

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \Big[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2.$$

From Algorithm 3, (23), and (26) it follows that

$$\begin{aligned}
&\|u_{n+1} - p\|^2 \leq \alpha_n\|f(u_n) - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|T_n z_n - p\|^2 \\
&\leq \alpha_n\|f(u_n) - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\{(1 + \psi_n)^2\|u_n - p\|^2 - \Big[\tfrac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2\} \\
&\leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 + \psi_n(2 + \psi_n)\|u_n - p\|^2 - \gamma_n\Big[\tfrac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2 \\
&\leq \alpha_n\|f(u_n) - p\|^2 + \psi_n K + \|u_n - p\|^2 - \gamma_n\Big[\tfrac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2.
\end{aligned}$$

This immediately yields

$$\gamma_n\Big[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2 \leq \|u_n - p\|^2 - \|u_{n+1} - p\|^2 + \alpha_n\|f(u_n) - p\|^2 + \psi_n K.$$

Step 4. We show that

$$\|u_{n+1} - p\|^2 \leq (1 - \alpha_n(1 - \varrho))\|u_n - p\|^2 + \alpha_n(1 - \varrho)\Big[\frac{2\langle f(p) - p, u_{n+1} - p\rangle}{1 - \varrho} + \frac{\psi_n}{\alpha_n} \cdot \frac{K}{1 - \varrho}\Big]. \tag{28}$$

Indeed, from Algorithm 3 and (23), one has

$$\begin{aligned}
&\|u_{n+1} - p\|^2 = \|\alpha_n(f(u_n) - f(p)) + \beta_n(u_n - p) + \gamma_n(T_n z_n - p) + \alpha_n(f(p) - p)\|^2 \\
&\leq \|\alpha_n(f(u_n) - f(p)) + \beta_n(u_n - p) + \gamma_n(T_n z_n - p)\|^2 + 2\alpha_n\langle f(p) - p, u_{n+1} - p\rangle \\
&\leq \alpha_n\|f(u_n) - f(p)\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|z_n - p\|^2 + 2\alpha_n\langle f(p) - p, u_{n+1} - p\rangle \\
&\leq \alpha_n\|f(u_n) - f(p)\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n(1 + \psi_n)^2\|u_n - p\|^2 + 2\alpha_n\langle f(p) - p, u_{n+1} - p\rangle \\
&\leq \varrho\alpha_n\|u_n - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|u_n - p\|^2 + \psi_n(2 + \psi_n)\|u_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(p) - p, u_{n+1} - p\rangle \\
&\leq \varrho\alpha_n\|u_n - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|u_n - p\|^2 + \psi_n K + 2\alpha_n\langle f(p) - p, u_{n+1} - p\rangle \\
&= [1 - \alpha_n(1 - \varrho)]\|u_n - p\|^2 + \psi_n K + 2\alpha_n\langle f(p) - p, u_{n+1} - p\rangle \\
&= (1 - \alpha_n(1 - \varrho))\|u_n - p\|^2 + \alpha_n(1 - \varrho)\Big[\tfrac{2\langle f(p) - p, u_{n+1} - p\rangle}{1 - \varrho} + \tfrac{\psi_n}{\alpha_n} \cdot \tfrac{K}{1 - \varrho}\Big].
\end{aligned}$$

Step 5. We obtain the strong convergence to $u^* \in \Omega$, satisfying (20).
Indeed, putting $p = u^*$, we deduce from (28) that

$$\|u_{n+1} - u^*\|^2 \leq (1 - \alpha_n(1 - \varrho))\|u_n - u^*\|^2 + \alpha_n(1 - \varrho)\Big[\frac{2\langle f(u^*) - u^*, u_{n+1} - u^*\rangle}{1 - \varrho} + \frac{\psi_n}{\alpha_n} \cdot \frac{K}{1 - \varrho}\Big]. \tag{29}$$

Setting $\Gamma_n = \|u_n - u^*\|^2$ we show $\Gamma_n \to 0$ $(n \to \infty)$. □

Case 1. Assume there exists $n_0 \geq 1$ such that $\{\Gamma_n\}$ is nonincreasing. Thus, $\lim_{n\to\infty} \Gamma_n = \hbar < +\infty$ and $\lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$. Putting $p = u^*$, from Step 2 and $\{\gamma_n\} \subset [a,b] \subset (0,1)$, we obtain

$$\begin{aligned}
&a[(1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 + \|z_n - w_n\|^2] + (1-\alpha_n - b)a\|u_n - T_n z_n\|^2 \\
&\leq \gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 + \|z_n - w_n\|^2\Big] + \beta_n\gamma_n\|u_n - T_n z_n\|^2 \\
&\leq \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2 + \psi_n K + 2\alpha_n\langle f(u_n) - u^*, u_{n+1} - u^*\rangle \\
&\leq \Gamma_n - \Gamma_{n+1} + \psi_n K + 2\alpha_n\|f(u_n) - u^*\|\|u_{n+1} - u^*\|.
\end{aligned}$$

Since $0 < \liminf_{n\to\infty}\sigma_n \leq \limsup_{n\to\infty}\sigma_n < 1$, $\psi_n \to 0$, $\alpha_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$, from the boundedness of $\{u_n\}$ one has

$$\lim_{n\to\infty}\|u_n - T^n u_n\| = \lim_{n\to\infty}\|u_n - T_n z_n\| = \lim_{n\to\infty}\|w_n - z_n\| = 0. \tag{30}$$

So, it follows from Algorithm 3 and (30) that

$$\|w_n - u_n\| = \sigma_n\|T^n u_n - u_n\| \leq \|T^n u_n - u_n\| \to 0 \quad (n\to\infty),$$

and

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \alpha_n\|f(u_n) - u_n\| + \gamma_n\|T_n z_n - u_n\| \\
&\leq \alpha_n\|f(u_n) - u_n\| + \|T_n z_n - u_n\| \to 0 \quad (n\to\infty).
\end{aligned} \tag{31}$$

Putting $p = u^*$, from Step 3 we obtain

$$\begin{aligned}
\gamma_n\left[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2 &\leq \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2 + \alpha_n\|f(u_n) - u^*\|^2 + \psi_n K \\
&= \Gamma_n - \Gamma_{n+1} + \psi_n K + \alpha_n\|f(u_n) - u^*\|^2.
\end{aligned}$$

Since $0 < \liminf_{n\to\infty}\gamma_n$, $\psi_n \to 0$, $\alpha_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$, from the boundedness of $\{u_n\}$ one gets

$$\lim_{n\to\infty}\left[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2 = 0.$$

Hence, by Lemma 10 we deduce that

$$\lim_{n\to\infty}\|w_n - y_n\| = 0,$$

which immediately yields

$$\|u_n - y_n\| \leq \|u_n - w_n\| + \|w_n - y_n\| \to 0 \quad (n\to\infty) \tag{32}$$

From the boundedness of $\{u_n\}$, it follows that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle f(u^*) - u^*, u_n - u^*\rangle = \lim_{k\to\infty}\langle f(u^*) - u^*, u_{n_k} - u^*\rangle. \tag{33}$$

Since H is reflexive and $\{u_n\}$ is bounded, we may assume, without loss of generality, that $u_{n_k} \rightharpoonup \widetilde{x}$. Thus, from (33) one gets

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(u^*) - u^*, u_n - u^*\rangle &= \lim_{k\to\infty}\langle f(u^*) - u^*, u_{n_k} - u^*\rangle \\
&= \langle f(u^*) - u^*, \widetilde{x} - u^*\rangle.
\end{aligned} \tag{34}$$

Furthermore, by Algorithm 3 we get $u_{n+1} - z_n = \alpha_n(f(u_n) - z_n) + \beta_n(u_n - z_n) + \gamma_n(T_n z_n - z_n)$, which immediately yields

$$\begin{aligned}
\gamma_n\|T_n z_n - z_n\| &\leq \|u_{n+1} - z_n\| + \alpha_n(\|f(u_n)\| + \|z_n\|) + \beta_n\|u_n - z_n\| \\
&\leq \|u_{n+1} - u_n\| + 2(\|u_n - w_n\| + \|w_n - z_n\|) + \alpha_n(\|f(u_n)\| + \|z_n\|).
\end{aligned}$$

Since $u_n - u_{n+1} \to 0$, $w_n - u_n \to 0$, $w_n - z_n \to 0$, $\alpha_n \to 0$, $\liminf_{n\to\infty} \gamma_n > 0$ and $\{u_n\}, \{z_n\}$ are bounded, we obtain $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$, which together with the nonexpansivity of each T_n, arrives at

$$\begin{aligned}\|u_n - T_n u_n\| &\le \|u_n - z_n\| + \|z_n - T_n z_n\| + \|T_n z_n - T_n u_n\| \\ &\le 2\|u_n - z_n\| + \|z_n - T_n z_n\| \\ &\le 2(\|u_n - w_n\| + \|w_n - z_n\|) + \|z_n - T_n z_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Since $u_n - y_n \to 0$, $u_n - u_{n+1} \to 0$, $u_n - T^n u_n \to 0$, $u_n - T_n u_n \to 0$ and $u_{n_k} \rightharpoonup \widetilde{x}$, by Lemma 9 we infer that $\widetilde{x} \in \Omega$. Hence from (20) and (34) one gets

$$\limsup_{n\to\infty} \langle f(u^*) - u^*, u_n - u^* \rangle = \langle f(u^*) - u^*, \widetilde{x} - u^* \rangle \le 0, \tag{35}$$

which immediately leads to

$$\begin{aligned}&\limsup_{n\to\infty} \langle f(u^*) - u^*, u_{n+1} - u^* \rangle \\ &= \limsup_{n\to\infty} \Big[\langle f(u^*) - u^*, u_{n+1} - u_n \rangle + \langle f(u^*) - u^*, u_n - u^* \rangle \Big] \\ &\le \limsup_{n\to\infty} \Big[\|f(u^*) - u^*\| \|u_{n+1} - u_n\| + \langle f(u^*) - u^*, u_n - u^* \rangle \Big] \le 0.\end{aligned} \tag{36}$$

Note that $\{\alpha_n(1-\varrho)\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n(1-\varrho) = \infty$, and

$$\limsup_{n\to\infty} \Big[\frac{2\langle f(u^*) - u^*, u_{n+1} - u^* \rangle}{1-\varrho} + \frac{\psi_n}{\alpha_n} \cdot \frac{K}{1-\varrho} \Big] \le 0.$$

Consequently, applying Lemma 4 to (29), one has $\lim_{n\to\infty} \|u_n - u^*\|^2 = 0$.

Case 2. Suppose that $\exists \{\Gamma_{n_k}\} \subset \{\Gamma_n\}$ s.t. $\Gamma_{n_k} < \Gamma_{n_k+1}$ $\forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\eta : \mathcal{N} \to \mathcal{N}$ by

$$\eta(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

By Lemma 6, we get

$$\Gamma_{\eta(n)} \le \Gamma_{\eta(n)+1} \quad \text{and} \quad \Gamma_n \le \Gamma_{\eta(n)+1}.$$

Putting $p = u^*$, from Step 2 we have

$$\begin{aligned}&a\Big[(1-\sigma_{\eta(n)})\sigma_{\eta(n)} \|u_{\eta(n)} - T^{\eta(n)} u_{\eta(n)}\|^2 + \|z_{\eta(n)} - w_{\eta(n)}\|^2\Big] \\ &\quad + (1 - \alpha_{\eta(n)} - b)a\|u_{\eta(n)} - T_{\eta(n)} z_{\eta(n)}\|^2 \\ &\le \gamma_{\eta(n)} \Big[(1-\sigma_{\eta(n)})\sigma_{\eta(n)} \|u_{\eta(n)} - T^{\eta(n)} u_{\eta(n)}\|^2 + \|z_{\eta(n)} - w_{\eta(n)}\|^2\Big] \\ &\quad + \beta_{\eta(n)}\gamma_{\eta(n)} \|u_{\eta(n)} - T_{\eta(n)} z_{\eta(n)}\|^2 \\ &\le \Gamma_{\eta(n)} - \Gamma_{\eta(n)+1} + \psi_{\eta(n)} K + 2\alpha_{\eta(n)} \langle f(u_{\eta(n)}) - u^*, x_{\eta(n)+1} - u^* \rangle \\ &\le \psi_{\eta(n)} K + 2\alpha_{\eta(n)} \|f(u_{\eta(n)}) - u^*\| \|x_{\eta(n)+1} - u^*\|,\end{aligned}$$

which immediately yields

$$\lim_{n\to\infty} \|u_{\eta(n)} - T^{\eta(n)} u_{\eta(n)}\| = \lim_{n\to\infty} \|u_{\eta(n)} - T_{\eta(n)} z_{\eta(n)}\| = \lim_{n\to\infty} \|w_{\eta(n)} - z_{\eta(n)}\| = 0.$$

Putting $p = u^*$, from Step 3 we get

$$\begin{aligned}\gamma_{\eta(n)} [\tfrac{\zeta_{\eta(n)}}{2\lambda L} \|r_\lambda(w_{\eta(n)})\|^2]^2 &\le \Gamma_{\eta(n)} - \Gamma_{\eta(n)+1} + \alpha_{\eta(n)} \|f(u_{\eta(n)}) - u^*\|^2 + \psi_{\eta(n)} K \\ &\le \psi_{\eta(n)} K + \alpha_{\eta(n)} \|f(u_{\eta(n)}) - u^*\|^2,\end{aligned}$$

which hence leads to

$$\lim_{n\to\infty} \Big[\frac{\zeta_{\eta(n)}}{2\lambda L} \|r_\lambda(w_{\eta(n)})\|^2 \Big]^2 = 0.$$

Utilizing the same inferences as in the proof of Case 1, we deduce that

$$\lim_{n\to\infty}\|w_{\eta(n)}-y_{\eta(n)}\|=\lim_{n\to\infty}\|w_{\eta(n)}-u_{\eta(n)}\|=\lim_{n\to\infty}\|u_{\eta(n)+1}-u_{\eta(n)}\|=0,$$

and

$$\limsup_{n\to\infty}\langle f(u^*)-u^*,u_{\eta(n)+1}-u^*\rangle\le 0.$$

On the other hand, from (29) we obtain

$$\begin{aligned}\alpha_{\eta(n)}(1-\varrho)\Gamma_{\eta(n)} &\le \Gamma_{\eta(n)}-\Gamma_{\eta(n)+1}+\alpha_{\eta(n)}(1-\varrho)\Big[\tfrac{2\langle f(u^*)-u^*,u_{\eta(n)+1}-u^*\rangle}{1-\varrho}+\tfrac{\psi_{\eta(n)}}{\alpha_{\eta(n)}}\cdot\tfrac{K}{1-\varrho}\Big]\\ &\le \alpha_{\eta(n)}(1-\varrho)\Big[\tfrac{2\langle f(u^*)-u^*,u_{\eta(n)+1}-u^*\rangle}{1-\varrho}+\tfrac{\psi_{\eta(n)}}{\alpha_{\eta(n)}}\cdot\tfrac{K}{1-\varrho}\Big].\end{aligned}$$

which hence arrives at

$$\limsup_{n\to\infty}\Gamma_{\eta(n)}\le\limsup_{n\to\infty}\Big[\frac{2\langle f(u^*)-u^*,u_{\eta(n)+1}-u^*\rangle}{1-\varrho}+\frac{\psi_{\eta(n)}}{\alpha_{\eta(n)}}\cdot\frac{K}{1-\varrho}\Big]\le 0.$$

Thus, $\lim_{n\to\infty}\|u_{\eta(n)}-u^*\|^2=0$. Furthermore, note that

$$\begin{aligned}&\|u_{\eta(n)+1}-u^*\|^2-\|u_{\eta(n)}-u^*\|^2\\ &=2\langle u_{\eta(n)+1}-u_{\eta(n)},u_{\eta(n)}-u^*\rangle+\|u_{\eta(n)+1}-u_{\eta(n)}\|^2\\ &\le 2\|u_{\eta(n)+1}-u_{\eta(n)}\|\|u_{\eta(n)}-u^*\|+\|u_{\eta(n)+1}-u_{\eta(n)}\|^2\to 0\quad(n\to\infty).\end{aligned}$$

Thanks to $\Gamma_n\le\Gamma_{\eta(n)+1}$, we get

$$\begin{aligned}&\|u_n-u^*\|^2\le\|u_{\eta(n)+1}-u^*\|^2\\ &\le\|u_{\eta(n)}-u^*\|^2+2\|u_{\eta(n)+1}-u_{\eta(n)}\|\|u_{\eta(n)}-u^*\|+\|u_{\eta(n)+1}-u_{\eta(n)}\|^2\to 0\quad(n\to\infty).\end{aligned}$$

That is, $u_n\to u^*$ as $n\to\infty$.

Theorem 2. *Suppose $T:C\to C$ is nonexpansive and $\{u_n\}$ is constructed by: $u_1\in C$,*

$$\begin{cases} w_n=(1-\sigma_n)u_n+\sigma_n Tu_n,\\ y_n=P_C(w_n-\lambda Aw_n),\\ t_n=(1-\zeta_n)w_n+\zeta_n y_n,\\ z_n=P_{D_n}(w_n),\\ u_{n+1}=\alpha_n f(u_n)+\beta_n u_n+\gamma_n T_n z_n,\end{cases}$$

where for each $n\ge 1$, D_n and ζ_n that are chosen as in Algorithm 3, then $u_n\to u^*\in\Omega$, where $u^*\in\Omega$ is the unique solution to the VIP: $\langle(I-f)u^*,p-u^*\rangle\ge 0\ \forall p\in\Omega$.

Proof. Step 1. $\{u_n\}$ is bounded. Indeed, using the same arguments as in Step 1 of the proof of Theorem 1, we obtain the desired assertion.

Step 2.

$$\begin{aligned}&\gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n-Tu_n\|^2+\|z_n-w_n\|^2\Big]+\beta_n\gamma_n\|u_n-T_n z_n\|^2\\ &\le\|u_n-p\|^2-\|u_{n+1}-p\|^2+2\alpha_n\langle f(u_n)-p,u_{n+1}-p\rangle.\end{aligned}$$

Indeed, using the same arguments as in Step 2 of the proof of Theorem 1, we have the result.

Step 3.

$$\gamma_n\Big[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2\le\|u_n-p\|^2-\|u_{n+1}-p\|^2+\alpha_n\|f(u_n)-p\|^2.$$

The same arguments in Step 3 of the proof of Theorem 1 give the conclusion.

Step 4.

$$\|u_{n+1}-p\|^2 \le (1-\alpha_n(1-\varrho))\|u_n-p\|^2 + \alpha_n(1-\varrho)\cdot\frac{2\langle f(p)-p, u_{n+1}-p\rangle}{1-\varrho}.$$

The results follow from the same arguments as in Step 4 of the proof of Theorem 1.

Step 5. $\{u_n\}$ converges strongly to $u^* \in \Omega$, which satisfies (20), with $T_0 = T$ as a nonexpansive mapping. Letting $p = u^*$, we deduce from Step 4 that

$$\|u_{n+1}-u^*\|^2 \le (1-\alpha_n(1-\varrho))\|u_n-u^*\|^2 + \alpha_n(1-\varrho)\cdot\frac{2\langle f(u^*)-u^*, u_{n+1}-u^*\rangle}{1-\varrho}. \quad (37)$$

Setting $\Gamma_n = \|u_n-u^*\|^2$, we show $\Gamma_n \to 0$ $(n\to\infty)$ by considering the two cases below. □

Case 1. If there exists an integer $n_0 \ge 1$ such that $\{\Gamma_n\}$ is nonincreasing, then $\lim_{n\to\infty}\Gamma_n = \hbar < +\infty$ and $\lim_{n\to\infty}(\Gamma_n-\Gamma_{n+1}) = 0$. Putting $p = u^*$, from Step 2 and $\{\gamma_n\} \subset [a,b] \subset (0,1)$ we obtain

$$\begin{aligned}
&a[(1-\sigma_n)\sigma_n\|u_n-Tu_n\|^2 + \|z_n-w_n\|^2] + (1-\alpha_n-b)a\|u_n-T_nz_n\|^2\\
&\le \gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n-Tu_n\|^2 + \|z_n-w_n\|^2\Big] + \beta_n\gamma_n\|u_n-T_nz_n\|^2\\
&\le \Gamma_n-\Gamma_{n+1} + 2\alpha_n\langle f(u_n)-u^*, u_{n+1}-u^*\rangle\\
&\le \Gamma_n-\Gamma_{n+1} + 2\alpha_n\|f(u_n)-u^*\|\|u_{n+1}-u^*\|,
\end{aligned}$$

which hence yields

$$\lim_{n\to\infty}\|u_n-Tu_n\| = \lim_{n\to\infty}\|u_n-T_nz_n\| = \lim_{n\to\infty}\|w_n-z_n\| = 0. \quad (38)$$

Putting $p = u^*$, from Step 3 we obtain

$$\gamma_n\left[\tfrac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2 \le \Gamma_n-\Gamma_{n+1} + \alpha_n\|f(u_n)-u^*\|^2,$$

which immediately leads to

$$\lim_{n\to\infty}\left[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2 = 0.$$

By inference, as in Case 1, of the proof of Theorem 1, we deduce

$$\lim_{n\to\infty}\|w_n-y_n\| = \lim_{n\to\infty}\|w_n-u_n\| = \lim_{n\to\infty}\|u_{n+1}-u_n\| = 0, \quad (39)$$

and

$$\limsup_{n\to\infty}\langle f(u^*)-u^*, u_{n+1}-u^*\rangle \le 0. \quad (40)$$

Consequently, applying Lemma 4 to (37), one has $\lim_{n\to\infty}\|u_n-u^*\|^2 = 0$.

Case 2. Suppose that $\exists\{\Gamma_{n_k}\} \subset \{\Gamma_n\}$ s.t. $\Gamma_{n_k} < \Gamma_{n_k+1}$ $\forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\eta : \mathcal{N} \to \mathcal{N}$ by

$$\eta(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

By Lemma 6, we get

$$\Gamma_{\eta(n)} \le \Gamma_{\eta(n)+1} \quad \text{and} \quad \Gamma_n \le \Gamma_{\eta(n)+1}.$$

The conclusion follows using same arguments as in Case 2 of the proof of Theorem 1.

We introduce a viscosity extragradient-like iterative method.

We point out that Lemmas 7–10 still hold for Algorithm 4.

Algorithm 4 Initialization: Given $\mu > 0,\ l \in (0,1),\ \lambda \in (0, \frac{1}{\mu})$. Let $u_1 \in C$ be arbitrary.

Iterative Steps: Given u_n, calculate

Step 1. Set $w_n = (1-\sigma_n)u_n + \sigma_n T^n u_n$, and compute $y_n = P_C(w_n - \lambda A w_n)$ and $r_\lambda(w_n) := w_n - y_n$.

Step 2. Compute $t_n = w_n - \zeta_n r_\lambda(w_n)$, where $\zeta_n := l^{j_n}$ and j_n is the smallest nonnegative integer j,
satisfying

$$\langle Aw_n - A(w_n - l^j r_\lambda(w_n)), w_n - y_n\rangle \le \frac{\mu}{2}\|r_\lambda(w_n)\|^2. \tag{41}$$

Step 3. Compute $z_n = P_{D_n}(w_n)$ and $u_{n+1} = \alpha_n f(u_n) + \beta_n w_n + \gamma_n T_n z_n$,
where $D_n := \{x \in C : h_n(x) \le 0\}$ and

$$h_n(x) = \langle At_n, x - w_n\rangle + \frac{\zeta_n}{2\lambda}\|r_\lambda(w_n)\|^2. \tag{42}$$

Theorem 3. *Suppose $\{u_n\}$ is constructed by Algorithm 4. Then, $u_n \to u^* \in \Omega$ provided $T^n u_n - T^{n+1}u_n \to 0$, where $u^* \in \Omega$ is the unique solution to the VIP: $\langle (I-f)u^*, p - u^*\rangle \ge 0\ \forall p \in \Omega$.*

Proof. By $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$ and $\lim_{n\to\infty} \frac{\psi_n}{\alpha_n} = 0$, we have, without loss of generality, that $\{\gamma_n\} \subset [a,b] \subset (0,1)$ and $\psi_n \le \frac{\alpha_n(1-\varrho)}{2}\ \forall n \ge 1$. By the same arguments as in the proof of Theorem 3.1, we have $u^* \in \Omega = \bigcap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A)$.

Next, we show the conclusion of the theorem. With this aim, we divide the rest of the proof into several steps.

Step 1. $\{u_n\}$ is bounded. Using the same arguments as in Step 1 of the proof of Theorem 3.1, we have inequalities (21)–(23). Thus, from (23) and $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \ge 1$, it follows that

$$\begin{aligned}
&\|u_{n+1} - p\| \le \alpha_n(\|f(u_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|w_n - p\| + \gamma_n\|z_n - p\| \\
&\le \alpha_n(\varrho\|u_n - p\| + \|f(p) - p\|) + \beta_n\|w_n - p\| + \gamma_n\|w_n - p\| \\
&\le \alpha_n(\varrho\|u_n - p\| + \|f(p) - p\|) + (\beta_n + \gamma_n)(1+\psi_n)\|u_n - p\| \\
&\le \alpha_n(\varrho\|u_n - p\| + \|f(p) - p\|) + (\beta_n + \gamma_n)\|u_n - p\| + \frac{\alpha_n(1-\varrho)}{2}\|u_n - p\| \\
&= [1 - \frac{\alpha_n(1-\varrho)}{2}]\|u_n - p\| + \frac{\alpha_n(1-\varrho)}{2}\cdot\frac{2\|f(p)-p\|}{1-\varrho} \\
&\le \max\{\|u_n - p\|, \frac{2\|f(p)-p\|}{1-\varrho}\}.
\end{aligned}$$

Inducting, we obtain $\|u_n - p\| \le \max\{\|u_1 - p\|, \frac{2\|f(p)-p\|}{1-\varrho}\}\ \forall n \ge 1$. Thus, $\{u_n\}$ is bounded, and so are the sequences $\{w_n\}, \{y_n\}, \{z_n\}, \{f(u_n)\}, \{At_n\}, \{T^n u_n\}, \{T_n z_n\}$.

Step 2. We show that

$$\begin{aligned}
&\gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 + \|z_n - w_n\|^2\Big] + \beta_n\gamma_n\|w_n - T_n z_n\|^2 \\
&\le \|u_n - p\|^2 - \|u_{n+1} - p\|^2 + \psi_n K + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle
\end{aligned}$$

for some $K > 0$. To prove this, we first note that

$$\begin{aligned}
&\|u_{n+1} - p\|^2 = \|\alpha_n(f(u_n) - p) + \beta_n(w_n - p) + \gamma_n(T_n z_n - p)\|^2 \\
&\le \|\beta_n(w_n - p) + \gamma_n(T_n z_n - p)\|^2 + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle \\
&\le \beta_n\|w_n - p\|^2 + \gamma_n\|z_n - p\|^2 - \beta_n\gamma_n\|w_n - T_n z_n\|^2 + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle.
\end{aligned} \tag{43}$$

On the other hand, using the same inferences as in (25) one has

$$\|z_n - p\|^2 \le (1+\psi_n)^2\|u_n - p\|^2 - (1-\sigma_n)\sigma_n\|u_n - T^n u_n\|^2 - \|z_n - w_n\|^2. \tag{44}$$

Substituting (44) into (43), one gets

$$\begin{aligned}
&\|u_{n+1}-p\|^2 \\
&\le \beta_n(1+\psi_n)^2\|u_n-p\|^2+\gamma_n[(1+\psi_n)^2\|u_n-p\|^2-(1-\sigma_n)\sigma_n\|u_n-T^nu_n\|^2-\|z_n-w_n\|^2] \\
&\quad -\beta_n\gamma_n\|w_n-T_nz_n\|^2+2\alpha_n\langle f(u_n)-p,u_{n+1}-p\rangle \\
&\le (1-\alpha_n)\|u_n-p\|^2-\gamma_n[(1-\sigma_n)\sigma_n\|u_n-T^nu_n\|^2+\|z_n-w_n\|^2] \\
&\quad +\psi_n(2+\psi_n)\|u_n-p\|^2-\beta_n\gamma_n\|w_n-T_nz_n\|^2+2\alpha_n\langle f(u_n)-p,u_{n+1}-p\rangle \\
&\le \|u_n-p\|^2-\gamma_n\Big[(1-\sigma_n)\sigma_n\|u_n-T^nu_n\|^2+\|z_n-w_n\|^2\Big]-\beta_n\gamma_n\|w_n-T_nz_n\|^2 \\
&\quad +\psi_nK+2\alpha_n\langle f(u_n)-p,u_{n+1}-p\rangle,
\end{aligned}$$

where $\sup_{n\ge1}(2+\psi_n)\|u_n-p\|^2\le K$ for some $K>0$. This immediately implies that

$$\begin{aligned}
&\gamma_n[(1-\sigma_n)\sigma_n\|u_n-T^nu_n\|^2+\|z_n-w_n\|^2]+\beta_n\gamma_n\|w_n-T_nz_n\|^2 \\
&\le \|u_n-p\|^2-\|u_{n+1}-p\|^2+\psi_nK+2\alpha_n\langle f(u_n)-p,u_{n+1}-p\rangle.
\end{aligned}$$

Step 3. We show that

$$\gamma_n\Big[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2\le\|u_n-p\|^2-\|u_{n+1}-p\|^2+\alpha_n\|f(u_n)-p\|^2+\psi_nK.$$

Indeed, using the same argument as that of (26), we obtain that for some $L>0$,

$$\|z_n-p\|^2\le\|w_n-p\|^2-\Big[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2. \tag{45}$$

From Algorithm 4, (23), and (45) it follows that

$$\begin{aligned}
&\|u_{n+1}-p\|^2\le\alpha_n\|f(u_n)-p\|^2+\beta_n\|w_n-p\|^2+\gamma_n\|z_n-p\|^2 \\
&\le\alpha_n\|f(u_n)-p\|^2+\beta_n\|w_n-p\|^2+\gamma_n[\|w_n-p\|^2-[\tfrac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2] \\
&\le\alpha_n\|f(u_n)-p\|^2+(1+\psi_n)^2\|u_n-p\|^2-\gamma_n[\tfrac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2 \\
&\le\alpha_n\|f(u_n)-p\|^2+\|u_n-p\|^2+\psi_nK-\gamma_n\Big[\tfrac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\Big]^2,
\end{aligned}$$

which hence yields the desired assertion.

Step 4. We show that

$$\|u_{n+1}-p\|^2\le(1-\alpha_n(1-\varrho))\|u_n-p\|^2+\alpha_n(1-\varrho)\Big[\frac{2\langle f(p)-p,u_{n+1}-p\rangle}{1-\varrho}+\frac{\psi_n}{\alpha_n}\cdot\frac{K}{1-\varrho}\Big].$$

Indeed, from Algorithm 4 and (3.19), one has

$$\begin{aligned}
&\|u_{n+1}-p\|^2 \\
&\le\|\alpha_n(f(u_n)-f(p))+\beta_n(w_n-p)+\gamma_n(T_nz_n-p)\|^2+2\alpha_n\langle f(p)-p,u_{n+1}-p\rangle \\
&\le\varrho\alpha_n\|u_n-p\|^2+\beta_n\|w_n-p\|^2+\gamma_n\|z_n-p\|^2+2\alpha_n\langle f(p)-p,u_{n+1}-p\rangle \\
&\le\varrho\alpha_n\|u_n-p\|^2+(1-\alpha_n)\|w_n-p\|^2+2\alpha_n\langle f(p)-p,u_{n+1}-p\rangle \\
&\le\varrho\alpha_n\|u_n-p\|^2+(1-\alpha_n)\|u_n-p\|^2+\psi_n(2+\psi_n)\|u_n-p\|^2+2\alpha_n\langle f(p)-p,u_{n+1}-p\rangle \\
&\le[1-\alpha_n(1-\varrho)]\|u_n-p\|^2+\psi_nK+2\alpha_n\langle f(p)-p,u_{n+1}-p\rangle,
\end{aligned}$$

which hence leads to the desired assertion.

Step 5. $\{u_n\}$ converges strongly to the unique solution $u^*\in\Omega$, which satisfies (20). This follows the argument in Step 5 of the proof of Theorem 1. □

Theorem 4. *Suppose $T : C \to C$ is nonexpansive and $\{u_n\}$ is constructed by: $u_1 \in C$,*

$$\begin{cases} w_n = (1 - \sigma_n)u_n + \sigma_n T u_n, \\ y_n = P_C(w_n - \lambda A w_n), \\ t_n = (1 - \zeta_n)w_n + \zeta_n y_n, \\ z_n = P_{D_n}(w_n), \\ u_{n+1} = \alpha_n f(u_n) + \beta_n w_n + \gamma_n T_n z_n, \end{cases}$$

where for each $n \geq 1$, D_n and ζ_n are chosen as in Algorithm 4, then $u_n \to u^* \in \Omega$, where $u^* \in \Omega$ is the unique solution to the VIP: $\langle (I - f)u^*, p - u^* \rangle \geq 0 \ \forall p \in \Omega$.

Proof. **Step 1.** By Step 1 of the proof of Theorem 2, we see that $\{u_n\}$ is bounded.

Step 2. By the same arguments as in Step 2 of the proof of Theorem 2, we have

$$\gamma_n[(1 - \sigma_n)\sigma_n\|u_n - Tu_n\|^2 + \|z_n - w_n\|^2] + \beta_n\gamma_n\|u_n - T_n z_n\|^2 \\ \leq \|u_n - p\|^2 - \|u_{n+1} - p\|^2 + 2\alpha_n\langle f(u_n) - p, u_{n+1} - p\rangle,$$

for some $K > 0$.

Step 3. Step 3 of the proof of Theorem 2 gives

$$\gamma_n\left[\frac{\zeta_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2 \leq \|u_n - p\|^2 - \|u_{n+1} - p\|^2 + \alpha_n\|f(u_n) - p\|^2.$$

Step 4. Step 4 of the proof of Theorem 2 gives

$$\|u_{n+1} - p\|^2 \leq (1 - \alpha_n(1 - \varrho))\|u_n - p\|^2 + \alpha_n(1 - \varrho) \cdot \frac{2\langle f(p) - p, u_{n+1} - p\rangle}{1 - \varrho}.$$

Step 5. By arguments as in Step 5 of the proof of Theorem 2, we have that $\{u_n\}$ converges strongly to the unique solution $u^* \in \Omega$, satisfying (20). □

Remark 1. *Compared with the corresponding results in Ceng et al. [21], Reich et al. [22], and Ceng and Shang [9], our results improve and extend them in the following aspects.*

(i) Although the same problem of finding an element of $\bigcap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ as considered in this paper was studied in reference [21], our strong convergence theorems are more advantageous and more subtle than the corresponding strong convergence ones in reference [21] because the conclusion $u_n \to u^* \in \Omega \Leftrightarrow \|u_n - u_{n+1}\| + \|u_n - y_n\| \to 0$ $(n \to \infty)$ in the corresponding strong convergence theorems [21] is updated by our conclusion $u_n \to u^* \in \Omega$. Without doubt, the strong convergence criteria for the sequence $\{u_n\}$ in this paper are more convenient and more beneficial in comparison with those of reference [21]. In addition, to overcome the weakness of the strong convergence criteria in reference [21] (i.e., $\lim_{n\to\infty}(\|u_n - u_{n+1}\| + \|u_n - y_n\|) = 0$), we make use of Maingé's technique (i.e., Lemma 6) to derive successfully the conclusion $u_n \to u^* \in \Omega$.

(ii) Our results reduce to the results in reference [22] when $T_i = I$, where I is the identity mapping for $i = 0, 1, ..., N$.

(iii) The operator A in reference [9] is extended from being Lipschitz continuous and sequentially weak in continuity mapping to A being uniformly continuous with $\|Az\| \leq \liminf_{n\to\infty}\|Au_n\|$ for each $\{u_n\} \subset C$ with $u_n \rightharpoonup z \in C$. Furthermore, the hybrid inertial subgradient extragradient method with the line-search process in reference [9] is extended in this paper. For example, the original inertial technique $w_n = T_n u_n + \alpha_n(T_n u_n - T_n x_{n-1})$ is replaced by our Mann iteration approach $w_n = (1 - \sigma_n)u_n + \sigma_n T^n u_n$, and the original iterative step $u_{n+1} = \beta_n f(u_n) + \gamma_n u_n + ((1 - \gamma_n)I - \beta_n\rho F)T^n z_n$ is replaced by our simpler iterative one $u_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \gamma_n T_n z_n$. It is worth mentioning that the definition of z_n in the former formulation of u_{n+1} is very different from the definition of z_n in the latter formulation of u_{n+1}.

(iv) We intend to apply the SP-iteration studied in reference [27] to the problem of finding an element of $\bigcap_{i=0}^{N}\mathrm{Fix}(T_i)\cap \mathrm{VI}(C,A)$ considered in this paper in our next project. As part of our future project, we will apply our results to the appearance of fractals using ideas given in reference [28].

4. Applications

In what follows, we give the following illustrated example. Put $\mu = l = \lambda = \frac{1}{2}$, $\sigma_n = \frac{1}{3}$, $\alpha_n = \frac{1}{3(n+1)}$, $\beta_n = \frac{n}{3(n+1)}$ and $\gamma_n = \frac{2}{3}$.

We first provide an example of Lipschitz continuous and pseudo-monotone mapping A, asymptotically nonexpansive mapping T and nonexpansive mapping T_1 with $\Omega = \mathrm{Fix}(T_1)\cap\mathrm{Fix}(T)\cap\mathrm{VI}(C,A)\neq\emptyset$. Let $C=[-3,3]$ and $H=\mathbf{R}$ with the inner product $\langle a,b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The initial point u_1 is randomly chosen in C. Take $f(u) = \frac{1}{3}u\ \forall u\in C$ with $\varrho = \frac{1}{3}$. Let $A: H\to H$ and $T, T_1: C\to C$ be defined as $Au := \frac{1}{1+|\sin u|} - \frac{1}{1+|u|}$, $Tu := \frac{2}{5}\sin u$, and $T_1 u := \sin u$ for all $u\in C$. We now claim that A is pseudo-monotone and Lipschitz continuous. Indeed, for all $u,v\in H$ we have

$$\begin{aligned}\|Au-Av\| &\le \left|\tfrac{\|v\|-\|u\|}{(1+\|u\|)(1+\|v\|)}\right| + \left|\tfrac{\|\sin v\|-\|\sin u\|}{(1+\|\sin u\|)(1+\|\sin v\|)}\right| \\ &\le \tfrac{\|v-u\|}{(1+\|u\|)(1+\|v\|)} + \tfrac{\|\sin v-\sin u\|}{(1+\|\sin u\|)(1+\|\sin v\|)} \\ &\le \|u-v\| + \|\sin u-\sin v\| \le 2\|u-v\|.\end{aligned}$$

This implies that A is Lipschitz continuous. Next, we show that A is pseudo-monotone. For each $u,v\in H$, it is easy to see that

$$\langle Au, v-u\rangle = \left(\frac{1}{1+|\sin u|} - \frac{1}{1+|u|}\right)(v-u)\ge 0 \Rightarrow \langle Av, v-u\rangle = \left(\frac{1}{1+|\sin v|} - \frac{1}{1+|v|}\right)(v-u)\ge 0.$$

Besides, it is easy to verify that T is asymptotically nonexpansive with $\psi_n = (\frac{2}{5})^n\ \forall n\ge 1$, such that $\|T^{n+1}z_n - T^n z_n\|\to 0$ as $n\to\infty$. Indeed, we observe that

$$\|T^n u - T^n v\| \le \frac{2}{5}\|T^{n-1}u - T^{n-1}v\| \le \cdots \le \left(\frac{2}{5}\right)^n\|u-v\| \le (1+\psi_n)\|u-v\|,$$

and

$$\|T^{n+1}u_n - T^n u_n\| \le \left(\frac{2}{5}\right)^{n-1}\|T^2u_n - Tu_n\| = \left(\frac{2}{5}\right)^{n-1}\left\|\frac{2}{5}\sin(Tu_n) - \frac{2}{5}\sin u_n\right\| \le 2\left(\frac{2}{5}\right)^n \to 0.$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and

$$\lim_{n\to\infty}\frac{\psi_n}{\alpha_n} = \lim_{n\to\infty}\frac{(2/5)^n}{1/3(n+1)} = 0.$$

In addition, it is clear that T_1 is nonexpansive and $\mathrm{Fix}(T_1) = \{0\}$. Therefore, $\Omega = \mathrm{Fix}(T_1)\cap\mathrm{Fix}(T)\cap\mathrm{VI}(C,A) = \{0\}\neq\emptyset$. In this case, Algorithm 3 can be rewritten as follows:

$$\begin{cases} w_n = \frac{2}{3}u_n + \frac{1}{3}T^n u_n, \\ y_n = P_C(w_n - \frac{1}{2}Aw_n), \\ t_n = (1-\zeta_n)w_n + \zeta_n y_n, \\ z_n = P_{D_n}(w_n), \\ u_{n+1} = \frac{1}{3(n+1)}\cdot\frac{1}{3}u_n + \frac{n}{3(n+1)}u_n + \frac{2}{3}T_1 z_n \quad \forall n\ge 1, \end{cases} \tag{46}$$

where for each $n\ge 1$, D_n and ζ_n are chosen as in Algorithm 3. Then, by Theorem 1, we know that $\{u_n\}$ converges to $0\in\Omega = \mathrm{Fix}(T_1)\cap\mathrm{Fix}(T)\cap\mathrm{VI}(C,A)$.

More so, since $Tu := \frac{2}{5}\sin u$ is also nonexpansive, we consider the modified version of Algorithm 3, that is,

$$\begin{cases} w_n = \frac{2}{3}u_n + \frac{1}{3}Tu_n, \\ y_n = P_C(w_n - \frac{1}{2}Aw_n), \\ t_n = (1-\zeta_n)w_n + \zeta_n y_n, \\ z_n = P_{D_n}(w_n), \\ u_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{3}u_n + \frac{n}{3(n+1)}u_n + \frac{2}{3}T_1 z_n \quad \forall n \geq 1, \end{cases} \tag{47}$$

where for each $n \geq 1$, D_n and ζ_n are chosen as above. Then, by Theorem 2, we know that $\{u_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. In particular, we compare the performance of the new algorithm (4.2) with the Reich et al. [22] method using similar parameters as above. We choose the following initial input and take $\|u_{n+1} - u_n\| < 5E^{-5}$ as the stopping criterion:

Case I: $u_1 = 2$; Case II: $u_1 = \exp(\frac{150}{77})$; Case III: $u_1 = \frac{3}{4}\pi$; Case IV: $u_1 = 7$.

The numerical results are shown in Table 1 and Figure 1. One can observe from the table and figures that our proposed Algorithm 1 outperforms the method proposed by Reich et al. [22] based on our test example.

Table 1. Numerical results showing performance of our new method and the Reich et al. [22] method.

	New Algorithm		**Riech et al. [22] alg.**	
	Iter.	Time	Iter.	Time
Case I	17	0.0082	72	0.0176
Case II	24	0.0079	168	0.0717
Case III	58	0.0131	71	0.0166
Case IV	119	0.0298	164	0.0455

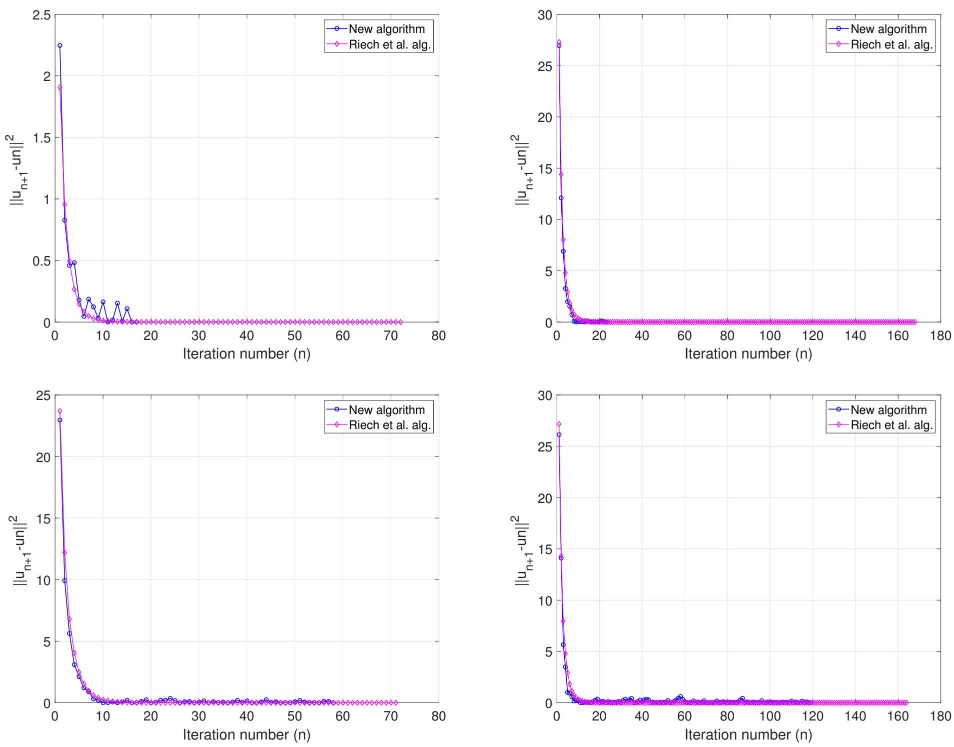

Figure 1. Computation result showing performance of our new method and the Reich et al. [22] method: **Top Left**: Case I; **Top Right**: Case II; **Bottom Left**: Case III; **Bottom Right**: Case IV.

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