mathematics

Article

Mild Solutions of Second-Order Semilinear Impulsive Differential Inclusions in Banach Spaces

Martina Pavlačková [1,*] and Valentina Taddei [2]

[1] Department of Informatics and Mathematics, Moravian Business College Olomouc, tř. Kosmonautů 1288/1, 77900 Olomouc, Czech Republic
[2] Department of Sciences and Methods for Engineering, University of Modena and Reggio Emilia, Via G. Amendola, 2-pad. Morselli, 42122 Reggio Emilia, Italy; valentina.taddei@unimore.it
* Correspondence: pavlackovam@centrum.cz

Abstract: In this paper, the existence of a mild solution to the Cauchy problem for impulsive semilinear second-order differential inclusion in a Banach space is investigated in the case when the nonlinear term also depends on the first derivative. This purpose is achieved by combining the Kakutani fixed point theorem with the approximation solvability method and the weak topology. This combination enables obtaining the result under easily verifiable and not restrictive conditions on the impulsive terms, the cosine family generated by the linear operator and the right-hand side while avoiding any requirement for compactness. Firstly, the problems without impulses are investigated, and then their solutions are glued together to construct the solution to the impulsive problem step by step. The paper concludes with an application of the obtained results to the generalized telegraph equation with a Balakrishnan–Taylor-type damping term.

Keywords: second-order Cauchy problem; Banach spaces; cosine family; approximation solvability method; mild solution

MSC: 34A60; 34G25

check for updates

Citation: Pavlačková, M.; Taddei, V. Mild Solutions of Second-Order Semilinear Impulsive Differential Inclusions in Banach Spaces. *Mathematics* **2022**, *10*, 672. https://doi.org/10.3390/math10040672

Academic Editor: Leonid Piterbarg

Received: 14 January 2022
Accepted: 17 February 2022
Published: 21 February 2022
Corrected: 31 August 2023

1. Introduction

The theory of impulsive differential equations has undergone considerable development in recent years since it serves as realistic mathematical descriptions of real situations containing abrupt changes, as well as other phenomena, such as harvesting or treatment of diseases. The study of the multivalued case of differential inclusions takes into account the presence of discontinuous right-hand sides, and it is related to control theory problems, arising from practical applications concerning population genetics, power-law fluids, and many other branches.

For the basic theory on impulsive differential equations and inclusions in finite-dimensional spaces, the reader is referred to the literature [1–3]. The recent results about impulsive differential equations and inclusions can be found, e.g., in [4–7] or [8] and see also the references therein.

The investigation of impulsive differential equations and inclusions in infinite-dimensional Banach spaces has been undertaken by a lot of authors starting from the end of the last century—see, e.g., [9–12] and the references therein for the problems governed by first-order impulsive equations.

The theory for the first-order problems in Banach spaces has been quite deeply studied in comparison to the higher-order problems that still possess many unsolved tasks. In [13,14], the existence of strong solutions to Cauchy problems for implicit linear equations in Banach spaces with irreversible operators in the main part, i.e., Sobolev-type equations, was studied by reducing the problem into a system of the two first-order semilinear equations and introducing the concept of a degenerate semigroup, which extends the classical definition of semigroup. In [14], the authors also applied Lyapunov–Schmidt's method to singular linear differential equations, which, in the case when the main part consists of a Fredholm operator, allowed studying the existence of classical solutions by reducing infinite-dimensional equations into finite-dimensional ones. Moreover, on the basis of the formulas obtained in [14], it was possible to describe the set of both the initial conditions and the right-hand sides under which the problem turns out to be solvable in the class of classical solutions. One of the main advantages of the theorems obtained in [13,14] lies in the fact that the results have a constructive character and can be easily implemented on computers. In [15,16], the existence of a classical solution for an implicit evolutionary inclusion in a separable Hilbert space was obtained by introducing the concept of a resolvent and Yosida approximation of a maximal monotone operator with respect to a strongly linear monotone operator.

Several authors have been also studying the semilinear second-order problems in Banach spaces—see, e.g., [17–20] or [21]. In these papers, the existence of mild solutions to the second-order (impulsive) initial value problems in the case when the right-hand side (r.h.s.) does not depend on the first derivative of a solution was investigated.

Only a few authors have dealt with the second-order problem in a Banach space whose r.h.s. also depends on the first derivative (sometimes solving the second-order problem by reducing it to a first-order equation)—see, e.g., [22–31] and the references therein. In [22,23], the operator A (in Equation (1) below) was supposed to be bounded, and impulses were not present; however, boundary value problems were studied there instead of the Cauchy problem. In [24,25], a mild solution to a not impulsive problem with a nonlocal boundary condition was obtained, and in the second paper, the nonlinearity was allowed to also depend on an integral term, but in both cases, the r.h.s. must satisfy the Lipschitz condition. In [27], Lipschitz assumptions were used too—the Lipschitz continuity of the r.h.s. was required, and the Lipschitz continuity of the impulse conditions and a Lipschitz-type condition on the derivative of the cosine family generated by the linear part along the impulses were proposed. Furthermore, in [28], the local Lipschitz continuity put on the r.h.s. played the key role in the proof of the main result. In [29], the linear operator A depends on t, and the existence of a mild solution for the Cauchy problem has been studied in the case of a C^1 nonlinear term. In [30,31], the authors, respectively, studied the Cauchy problem for a functional semilinear integro-differential equation and the Cauchy problem for an impulsive problem, assuming in both cases that the cosine family generated by the linear part is compact.

Some of the too strong assumptions put on the r.h.s. and/or the impulsive conditions (mainly the Lipschitz continuity put on the r.h.s. and on impulse conditions) were removed in [26], where the second-order impulsive integro-differential evolution equations in Banach spaces were studied. Conversely, some quite strict or not easily verified conditions, such as uniform continuity of the r.h.s. or employment of the measure of non-compactness within the assumptions, remained in the paper.

The aim of this paper is to study the existence of mild solutions to second-order multivalued impulsive problems in infinite-dimensional Banach spaces without reducing the second-order problem to a first-order one. The results are obtained under easily verifiable conditions and without assuming any compactness on the impulsive terms and/or to the r.h.s., which becomes the main advantage of the paper.

More concretely, the paper deals with the Cauchy problem for impulsive semilinear second-order differential inclusion in a Banach space of the form

$$\left.\begin{array}{c}\ddot{x}(t) \in Ax(t) + F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0,T] \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta x|_{t=t_k} = I_k(x(t_k), \dot{x}(t_k)),\ k = 1,2,\ldots,m, \\ \Delta \dot{x}|_{t=t_k} = \overline{I}_k(x(t_k), \dot{x}(t_k)),\ k = 1,2,\ldots,m, \\ x(0) = x_0,\ \dot{x}(0) = \overline{x}_0. \end{array}\right\} \tag{1}$$

Throughout the paper, we assume:

(i) E is a reflexive Banach space that has a Schauder basis;
(ii) $A : D(A) \subset E \to E$, $D(A)$ dense in E, is a closed, linear, and densely defined operator generating a cosine family $\{C(t)\}_{t\in\mathbb{R}}$;
(iii) $F\colon [0,T] \times E \times E \multimap E$ is multivalued mapping with nonempty, bounded, closed and convex values;
(iv) $x_0, \overline{x}_0 \in E$;
(v) $I_k,\ \overline{I}_k \in C(E \times E,\ E)$, for all $k = 1, 2, \ldots, m$.

The proof of our main results will be based on the Kakutani fixed point theorem for multivalued mappings. The direct use of the fixed point theorem requires strong compactness conditions, which are usually guaranteed by requiring the compactness of the cosine family generated by the linear term or assuming some conditions based on employing the measure of noncompactness of the nonlinear term.

In this paper, the approximation solvability method will be applied, which consists of introducing a sequence of approximating problems with values in finite-dimensional spaces, combined with the usage of the weak topology, which will allow avoiding any requirement of compactness. A limiting argument will then lead to a solution to the original problem. As a consequence of the used techniques, the obtained solution will be localized in a suitable bounded set.

The existence of a mild solution for the Cauchy impulsive problem (1) will be obtained by a step-by-step reduction of the impulsive problem to problems without impulses. The results will be proven without the transformation of the second-order problem to the first-order one because such a transformation may lack important information about the original problem.

The approximation solvability method that will be used was introduced in [32] to study fully nonlinear first-order problems in Hilbert spaces. Its application was then extended to first-order semilinear problems in Banach spaces in [33] and to fully nonlinear second-order problems in Hilbert spaces in [34]. To the best of our knowledge, our paper is the first paper that applies the approximation solvability method to semilinear second-order problems.

As mentioned before, weak topology will also be employed. It was first exploited to prove existence results in [35]. A lot of papers then appeared, where the same technique was applied to study first- and second-order equations and inclusions (of functional and neutral types), fractional equations, controllability problems, and so on. In particular, it was used in [36] to obtain the existence of a solution for a semilinear second-order equation in a Banach space with the r.h.s. not depending on the first derivative. In this paper, we will extend the results in [36] to differential inclusions with the r.h.s. also depending on the first derivative.

The paper is organized as follows. In Section 2, the necessary preliminaries about cosine families and multivalued analysis are mentioned. This section also contains lemmas and propositions that are used in the proof of our main results. The main theorems for problems without impulses are contained in Section 3. Subsequently, making use of the results for non-impulsive problems, the impulsive ones are studied in Section 4. Finally, the application of the proved theory to a generalized telegraph equation with a Balakrishnan–Taylor-type damping term is shown in Section 5.

2. Preliminaries

Let E be an infinite-dimensional real Banach space with norm $\|\cdot\|$, and let us denote the Banach space dual to E by E^*. The notation $\mathcal{L}(E)$ stands for the Banach space of linear and bounded operators from E into itself. For every $x \in E$ and $r > 0$, $B_r(x)$ is the open ball centered in x with radius r. Throughout this paper, by E^ω, we denote the space E endowed with the weak topology. Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C,\varepsilon)$ will denote, as usual, the set $C + \varepsilon B$, where B is the open unit ball in E, i.e., $B = B_1(0) = \{x \in E \mid ||x|| \le 1\}$.

We denote the $C([0,T],E)$-norm and the $C^1([0,T],E)$-norm, respectively, by $||\cdot||_C$ and $||\cdot||_{C^1}$, defined by

$$||x||_C = \max_{t\in[0,T]} ||x(t)||, \text{ for all } x \in C([0,T],E),$$

$$||x||_{C^1} = \max\{||x||_C, ||\dot{x}||_C\}, \text{ for all } x \in C^1([0,T],E)$$

and the $L^1([0,T],E)$-norm by $||\cdot||_{L^1}$ defined for all $x \in L^1([0,T],E)$ by

$$||x||_{L^1} = \int_0^T ||x(t)||\, dt.$$

Let $PC^1([0,T],E)$ be the space of all functions $x : [0,T] \to E$ such that

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0,t_1], \\ x_{[1]}(t), & \text{for } t \in (t_1,t_2], \\ . & \\ . & \\ . & \\ x_{[m]}(t), & \text{for } t \in (t_m,T], \end{cases} \tag{2}$$

where $x_{[0]} \in C^1([0,t_1],E), x_{[i]} \in C^1((t_i,t_{i+1}],E)$, $x(t_i^+)$, and $\dot{x}(t_i^+)$ exist in E for every $i = 1,\ldots,m$. For $x \in PC^1([0,T],E), \Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limit of $x(t)$ at $t = t_k$, respectively, and $\Delta\dot{x}|_{t=t_k}$ has a similar meaning. In a similar way, we can define the space $PC([0,T],E)$ as the space of functions $x : [0,T] \to E$, satisfying Definition (2), with $x_{[0]} \in C([0,t_1],E), x_{[i]} \in C((t_i,t_{i+1}],E)$, for every $i = 1,\ldots,m$. The space $PC([0,T],E)$ is a normed space endowed with the norm

$$||x||_{PC} := \sup_{t\in[0,T]} ||x(t)|| \tag{3}$$

and the space $PC^1([0,T],E)$ is a normed space with the norm

$$||x||_{PC^1} := \max\{||x||_{PC}, ||\dot{x}||_{PC}\}. \tag{4}$$

Definition 1. *A sequence $\{e_n\}_n$ of vectors in E is a Schauder basis for E if for every $x \in E$, there exists a unique sequence of real numbers $\alpha_n = \alpha_n(x)$, $n \in \mathbb{N}$, such that*

$$\left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| \to 0, \quad \text{as } n \to \infty.$$

Given a Schauder basis $\{e_n\}_n$ for E, let $E_n = \text{span}\{e_1,\ldots,e_n\}$ denote the n-dimensional Banach space generated by the first n vectors of the basis, and let $\mathbb{P}_n : E \to E_n$ be the natural projection of E onto E_n, i.e.,

$$\mathbb{P}_n\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k e_k.$$

It holds that $\alpha_n \in E^*$ for every $n \in \mathbb{N}$ (see [37], pp. 18–20) and that the sequence $\{\|\mathbb{P}_n\|\}_n$ is bounded, i.e., there exists $K \geq 1$, such that

$$\|\mathbb{P}_n(x)\| \leq K\|x\| \quad \forall n \in \mathbb{N},\ \forall x \in E \tag{5}$$

(see [38], (Proposition 1.a.2)). The Schauder basis $\{e_n\}_n$ is said to be monotone if $K = 1$, i.e., if $\|\mathbb{P}_n\| = 1$, for every $n \in \mathbb{N}$.

Remark 1. *Trivially, if E is a separable Hilbert space, every orthonormal system of E is a monotone Schauder basis. Moreover, if a Banach space admits a Schauder basis, it is separable. On the other hand, it was proven in [39] that there exists a separable Banach space without a Schauder basis. However, for every* $1 < p < \infty$ *and for each bounded subset* $\Omega \subset \mathbb{R}^n$, $L^p(\Omega, \mathbb{R})$ *has a monotone Schauder basis (see, e.g., [40], (Chap. 1.3 and 1.4)).*

Some of the main properties of the projection $\mathbb{P}_n$ are contained in the following lemma (see [33] (Lemma 2.2), [34] (Lemma 6), and [41] (Proposition 2.4)).

Lemma 1. *The projection* $\mathbb{P}_n : E \to E_n$ *satisfies the following properties:*
(a) $\mathbb{P}_n : E^\omega \to E_n$ *is continuous;*
(b) *If* $x_n \to x$, *then* $\mathbb{P}_n(x_n) \to x$;
(c) *If* $x_n \rightharpoonup x$, *then* $\mathbb{P}_n(x_n) \rightharpoonup x$;
(d) *If* $f_n \rightharpoonup f$ *in* $L^1([0,T], E)$, *then* $\mathbb{P}_n f_n \rightharpoonup f$ *in* $L^1([0,T], E)$,
(e) *For every* $x \in E$, $||P_n(x) - x|| \to 0$.

In the following, the cosine family generated by the operator A will be employed. For this purpose, its definition and main properties will be discussed now.

A one-parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the space E into itself is called a strongly continuous cosine family if:

- $C(t+s) + C(s-t) = 2C(s)C(t)$, for all t, $s \in \mathbb{R}$;
- $C(0) = I$;
- The map $t \to C(t)x$ is continuous in $\mathbb{R}$ for each fixed $x \in E$.

If $\{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family, then there exist $M \geq 1$ and $\omega \geq 0$ such that, for all $t \in \mathbb{R}$,

$$\|C(t)\| \leq Me^{\omega|t|} \tag{6}$$

and moreover, the set

$$D(A) = \left\{ x \in E : \exists \lim_{t\to 0^+} \frac{C(t)x - x}{t^2} \right\}$$

is dense in E.

The linear closed operator $A : D(A) \subset E \to E$ defined by

$$Ax = \frac{d^2}{dt^2}\Big[C(t)x\Big]_{t=0} = 2\lim_{t\to 0^+} \frac{C(t)x - x}{t^2}$$

is called the infinitesimal generator of the cosine family.

In what follows, we shall also make use of the following set:

$$X = \Big\{ x \in E \,|\, C(\cdot)x \text{ is continuously differentiable} \Big\}.$$

The one-parameter family $\{S(t)\}_{t\in\mathbb{R}}$ of the bounded linear operators mapping the space E into itself defined, for all $t \in \mathbb{R}$ and $x \in E$, by

$$S(t)x = \int_0^t C(s)x\, ds \tag{7}$$

is called the strongly continuous sine family associated with the cosine family.

The families $\{S(t)\}_{t\in\mathbb{R}}$ and $\{C(t)\}_{t\in\mathbb{R}}$ possess several important properties; the most crucial are summarized in the following lemma.

Lemma 2. *(see, e.g., [42] (Propositions 2.1, 2.2)) The families* $\{C(t)\}_{t\in\mathbb{R}}$ *and* $\{S(t)\}_{t\in\mathbb{R}}$ *satisfy the following properties:*
(a) $C(t) = C(-t)$, *for all* $t \in \mathbb{R}$;

(b) $S(t) = -S(-t)$, *for all* $t \in \mathbb{R}$;
(c) *The map* $t \to S(t)x$ *is continuous, for each fixed* $x \in E$;
(d) $S(s+t) + S(s-t) = 2S(s)C(t)$, *for all* t, $s \in \mathbb{R}$;
(e) $S(s+t) = S(s)C(t) + S(t)C(s)$, *for all* t, $s \in \mathbb{R}$;
(f) *For all* $t, \hat{t} \in \mathbb{R}$,

$$\left|S(t) - S(\hat{t})\right| \leq M\left|\int_{\hat{t}}^{t} \mathrm{e}^{\omega|s|}\, ds.\right| \tag{8}$$

(g) $C(s)$, $S(s)$, $C(t)$, *and* $S(t)$ *commute, for all* t, $s \in \mathbb{R}$;
(h) $S(t)x \in X$, *for every* $t \in \mathbb{R}, x \in E$;
(i) $S(t)x \in D(A), \lim_{t\to 0} AS(t)x = 0, \frac{d}{dt}C(t)x = AS(t)x$ *and* $\frac{d^2}{dt^2}S(t)x = AS(t)x$, *for every* $x \in X, t \in \mathbb{R}$;
(j) $C(t)x \in D(A), \frac{d^2}{dt}C(t)x = AC(t)x = C(t)Ax$, *and* $AS(t)x = S(t)Ax$, *for every* $x \in D(A), t \in \mathbb{R}$;
(k) $C(t+s) - C(t-s) = 2AS(t)S(s)$, *for all* $s, t \in \mathbb{R}$.

It follows from the definition of S that $S(0) = 0$. Therefore, using Equation (8), it is easy to prove that

$$\|S(t)\| \leq \begin{cases} M\frac{|e^{\omega|t|}-1|}{\omega} & \text{if } \omega \neq 0 \\ M|t| & \text{if } \omega = 0. \end{cases} \tag{9}$$

In the following, we will use the next lemma.

Lemma 3. *Let* $[a,b] \subset \mathbb{R}$ *be a compact interval and* E *a Banach space. Then, the map* $c : [a,b] \times E \to E$ *defined as*

$$c(t,x) = C(t)x$$

is continuous.

Proof. Fix $(t_0, x_a), (t,x) \in [a,b] \times E$. Then, it follows from Equation (6) that

$$\begin{aligned} \|c(t,x) - c(t_0,x_a)\| = & \quad \|C(t)x - C(t_0)x_a\| \leq \|C(t)x - C(t)x_a\| + \|C(t)x_a - C(t_0)x_a\| \\ \leq & \quad Me^{\omega|t|}\|x - x_a\| + \|C(t)x_a - C(t_0)x_a\|. \end{aligned}$$

Since $t \to C(t)x_a$ is continuous, it is possible to find the constant $\delta > 0$ for every $\epsilon > 0$, such that, if $|t - t_0| \leq \delta$, then $\|C(t)x_a - C(t_0)x_a\| \leq \epsilon$. Thus, assuming without a loss of generality that $\delta \leq \epsilon$ and taking $(t,x) \in [a,b] \times E$ with $|t - t_0| \leq \delta$ and $\|x - x_a\| \leq \delta$, we obtain that

$$\|c(t,x) - c(t_0,x_a)\| \leq Me^{\omega \max\{|t_0-\delta|,|t_0+\delta|\}}\delta + \epsilon \leq Me^{\omega(|t_0|+\epsilon)}\epsilon + \epsilon$$

which yields the thesis. □

It was proven in [43] that A is the generator of the cosine family $\{C(t)\}_{t\in\mathbb{R}}$ if and only if set X endowed with the norm

$$||x||_X = ||x||_E + \max_{t\in[0,1]} ||AS(t)x||_E \tag{10}$$

is a Banach space, where the maximum is achieved according the compactness of $[0,1]$, and the operator valued function

$$G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators in $X \times E$ generated by the operator

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

defined on $D(A) \times X$. Therefore, $AS(t) : X \to E$ is a bounded linear operator, and, similar to Lemma 3 (and recalling Lemma 2 (i)), it easily follows that the map $c_1 : [a, b] \times X \to E$ defined by

$$c_1(t, x) = C'(t)x \tag{11}$$

is continuous as well.

The next result establishes a relationship between the generator of a cosine family and the generator of a C_0 semigroup.

Lemma 4. *(see [44] (Theorems 2.5 and 4.9)) Assume that A is the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then, A generates a C_0 semigroup $\{T(t)\}_{t\geq 0}$. Moreover, $\{C(t)\}_{t\in\mathbb{R}}$ satisfies Equation (6) if and only if $\|T(t)\| \leq 2Me^{\omega^2 t}$ for every $t \geq 0$.*

We shall now introduce the definitions and notions from the multivalued analysis that we will need in the sequel. Let X, Y be two metric spaces. We say that H is a multivalued mapping from X to Y (written $H\colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $H(x)$ of Y is given.

A multivalued mapping $H\colon X \multimap Y$ is called upper semicontinuous (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid H(x) \subset U\}$ is open in X. It is called completely continuous if $H(C)$ is relatively compact for every bounded set $C \subset X$. If H is u.s.c. with convex values, then H has a closed graph (see [45] (Theorem 1.1.4)). If H is u.s.c. and completely continuous with compact values, then it has a closed graph ([45] (Theorem 1.1.5)). Conversely, if H is a completely continuous multivalued mapping with compact values and has a closed graph, then H is u.s.c. (see [46] (Theorem 1.1.5)).

Let $J \subset \mathbb{R}$ be a compact interval. A mapping $H : J \multimap Y$ with closed values, where Y is a separable metric space, is called measurable if, for each open subset $U \subset Y$, the set $\{t \in J \mid H(t) \subset U\}$ belongs to a σ-algebra of subsets of J. If Y is separable, the measurability is indifferent strong and weak measurability (see [47] (Chap. II)).

In the proof of the main result, the measure of non-compactness will be used. For this purpose, some basic facts concerning this notion will be mentioned now.

Definition 2. *Let N be a partially ordered set, E be a Banach space, and $P(E)$ denote the family of all subsets of E. A function $\beta\colon P(E) \to N$ is called a measure of non-compactness (m.n.c.) in E if $\beta(\overline{co\ \Omega}) = \beta(\Omega)$ for all $\Omega \in P(E)$, where $\overline{co\ \Omega}$ denotes the closed convex hull of Ω.*

An m.n.c. β is called:

(i) Monotone if $\beta(\Omega_1) \leq \beta(\Omega_2)$, for all $\Omega_1 \subset \Omega_2 \subset E$;
(ii) Nonsingular if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for all $x \in E$ and $\Omega \subset E$;
(iii) Regular when $\beta(\Omega) = 0$ if and only if Ω is relatively compact.

The typical example of an m.n.c. is the Hausdorff measure of noncompactness γ defined, for all $\Omega \subset E$, by

$$\gamma(\Omega) := \inf\{\varepsilon > 0 \mid \exists x_1, \ldots, x_n \in E\colon \Omega \subset \cup_{i=1}^n B_\varepsilon(x_i)\}.$$

The Hausdorff measure of noncompactness is monotone, nonsingular, and regular.

The notion of a solution will be understood in a mild sense. Namely, given $x_0 \in X$, by a mild solution of the non-impulsive problem

$$\left.\begin{array}{r} \ddot{x}(t) \in Ax(t) + F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T], \\ x(0) = x_0,\ \dot{x}(0) = \overline{x}_0 \end{array}\right\} \tag{12}$$

we mean a C^1-function $x\colon [0,T] \to E$ such that, for all $t \in [0,T]$,

$$x(t) = C(t)\,x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s)f(s)\,ds, \tag{13}$$

where

$$f \in S^1_{F,x} = \{f \in L^1([0,T],E) : f(t) \in F(t,x(t),x'(t)),\ \text{for a.a. } t \in [0,T]\}. \tag{14}$$

Notice that, for every $f \in L^1([0,T],E)$, the function defined in Equation (13) is continuously differentiable (according to condition (i) in Lemma 2, to [48] (Lemma II.4.1), and since $x_0 \in X$), and, for all $t \in [0,T]$, it holds that

$$\dot{x}(t) = AS(t)\,x_0 + C(t)\overline{x}_0 + \int_0^t C(t-s)f(s)\,ds.$$

In order to ensure that the $S^1_{F,x} \neq \emptyset$, an appropriate selection result is needed.

Proposition 1. *(see, e.g., [35] [Proposition 2.2]) Let $[a,b] \subset \mathbb{R}$ be a compact interval, E_1, E_2 be Banach spaces and $G\colon [a,b] \times E_1 \multimap E_2$ be a multivalued mapping satisfying:*

(A1) $G(t,x)$ is nonempty, convex, and weakly compact, for every $t \in [a,b]$ and $x \in E_1$;
(A2) For every $x \in E_1$, $G(\cdot,x)$ has a measurable selection;
(A3) For a.a. $t \in [a,b]$, $G(t,\cdot) : E_1^w \multimap E_2^w$ is weakly sequentially closed;
(A4) For each bounded $\Omega \subset E_1$, there exists $\eta_\Omega \in L^1([a,b],\mathbb{R})$ such that, for a.a. $t \in [a,b]$,

$$\sup_{x \in \Omega} ||G(t,x)|| \le \eta_\Omega(t).$$

Then, for every $q \in C([a,b],E_1)$, there exists $f \in L^1([a,b],E_2)$ such that $f(t) \in G(t,q(t))$ for a.a. $t \in [a,b]$.

Let us note that the previous result was proven in [35] for Banach spaces $E_1 = E_2$, but it is also valid in this more general case.

In the study of the impulsive problem in the sequel, we shall consider the initial problem starting from $t_k > 0$. In the following proposition, we obtain the mild solution formula for this case.

Proposition 2. *Let $x_a \in X, \overline{x}_a \in E, a \in (0,T), b \in (a,T]$. Then, x is a mild solution of the problem*

$$\left.\begin{array}{c} \ddot{x}(t) \in Ax(t) + F(t,x(t),\dot{x}(t)), \quad t \in [a,b] \\ x(a) = x_a,\ \dot{x}(a) = \overline{x}_a \end{array}\right\} \tag{15}$$

if and only if there exists $f \in S^1_{F,x}$, such that

$$x(t) = C(t-a)\,x_a + S(t-a)\overline{x}_a + \int_a^t S(t-s)f(s)\,ds. \tag{16}$$

Proof. Observe that $x : [a,b] \to E$ is a mild solution of Equation (15) if and only if the function defined as $y(r) = x(r+a)$ is a mild solution of

$$\left.\begin{array}{c} \ddot{y}(r) \in Ay(r) + G(r,y(r),\dot{y}(r)) \quad r \in [0,b-a] \\ y(0) = x_a,\ \dot{y}(0) = \overline{x}_a \end{array}\right\}$$

where $G(r,y(r)\dot{y}(r)) = F(r+a,x(r+a),\dot{x}(r+a))$. According to the definition, there exists $g \in S^1_{G,y}$ such that Equation (13) holds for every $r \in [0,b-a]$. Thus,

$$y(r) = C(r)y(0) + S(r)\dot{y}(0) + \int_0^r S(r-p)g(p)dp.$$

Define $f(s) = g(s-a)$ for every $s \in [a,b]$. Trivially, $f \in S^1_{F,x}$ and

$$\begin{aligned} x(t) = y(t-a) &= C(t-a)x_a + S(t-a)\overline{x}_a + \int_0^{t-a} S(t-a-p)g(p)dp \\ &= C(t-a)x_a + S(t-a)\overline{x}_a + \int_0^{t-a} S(t-a-p)f(p+a)dp \\ &= C(t-a)x_a + S(t-a)\overline{x}_a + \int_a^{t} S(t-s)f(s)ds, \end{aligned}$$

which proves the thesis. □

Remark 2. *Notice that the explicit formula for the mild solution of Equation (15) can be obtained also as restriction to $[a,b]$ of a mild solution defined in $[0,T]$. In fact, assume that $x : [0,T] \to E$ is a mild solution of Equation (12). Then, according to the definition, there exists $f \in S^1_{F,x}$ such that Equation (13) holds for every $t \in [0,T]$. In particular,*

$$x_a = C(a)x_0 + S(a)\overline{x}_0 + \int_0^a S(a-s)f(s)ds$$

and

$$\overline{x}_a = AS(a)\,x_0 + C(a)\overline{x}_0 + \int_0^a C(a-s)f(s)\,ds.$$

Now, exploiting the definition of the cosine family and Lemma 2.2 (e), (j), and (k), we obtain

$$C(t-a)C(a) + S(t-a)AS(a) = C(t-a)C(a) + AS(t-a)S(a) = C(t-a+a) = C(t),$$

$$C(t-a)S(a) + S(t-a)C(a) = S(t-a+a) = S(t),$$

and

$$C(t-a)S(a-s) + S(t-a)C(a-s) = S(t-a+a-s) = S(t-s)$$

which imply

$$\begin{aligned} x(t) = \;& C(t)x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s)f(s)ds \\ =\;& C(t)x_0 + S(t)\overline{x}_0 + \int_0^a S(t-s)f(s)ds + \int_a^t S(t-s)f(s)ds \\ =\;& [C(t-a)C(a) + S(t-a)AS(a)]x_0 + [C(t-a)S(a) + S(t-a)C(a)]\overline{x}_0 + \\ & \int_0^a [C(t-a)S(a-s) + S(t-a)C(a-s)]f(s)\,ds + \int_a^t S(t-s)f(s)ds \\ =\;& C(t-a)[C(a)x_0 + S(a)\overline{x}_0 + \int_0^a S(a-s)f(s)ds] + \\ & S(t-a)[AS(a)\,x_0 + C(a)\overline{x}_0 + \int_0^a C(a-s)f(s)\,ds] + \int_a^t S(t-s)f(s)ds \\ =\;& C(t-a)x_a + S(t-a)\overline{x}_a + \int_a^t S(t-s)f(s)ds. \end{aligned}$$

3. Existence of a Mild Solution for the Cauchy Problem without Impulses

In this section, with fixed $a, b \in [0,T]$ with $b > a$, the existence of a mild solution to the Cauchy problem in $[a,b]$ will be discussed.At first, we will use the natural projections of the space E onto the finite-dimensional spaces generated by the first n vectors of the Schauder basis to introduce a sequence of finite-dimensional approximating problems. Then, we will apply the Kakutani fixed point theorem, obtaining a solution for each finite-dimensional problem. Our technique will allow obtaining the localization of each solution in a given bounded set of C^1 functions. Finally, we will apply a limiting procedure based on the usage of the weak topology to obtain a solution to the original problem.

Since both the finite-dimensional spaces and the weak topology in a reflexive Banach space enjoy natural compactness properties, we will avoid any requirements for compactness within the assumptions.

In the next section, the obtained results for non-impulsive problems will be used to study the impulse problem on the interval $[0,T]$.

Theorem 1. *Consider the Cauchy problem (15), where $x_a \in X$, and assume that $F : [a,b] \times E \times E \multimap E$ satisfies the following assumptions:*

(F1) *$F(t,x,y)$ is nonempty, convex, closed, and bounded for every $t \in [a,b]$ and $x, y \in E$;*

$(F2)$ *For every* $(x,y) \in E \times E$, $F(\cdot,x,y)$ *has a measurable selection;*
$(F3)$ *For a.a.* $t \in [a,b]$, $F(t,\cdot,\cdot) : E^w \times E^w \multimap E^w$ *is weakly u.s.c.;*
$(F4)$ *For every* $n \in \mathbb{N}$, *there exists* $\varphi_n \in L^1([a,b],\mathbb{R})$ *with*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_{L^1}}{n} = 0$$

and such that

$$||z|| \le \varphi_n(t),$$

for a.a. $t \in [a,b]$, *every* $(x,y) \in nB \times nB$ *and every* $z \in F(t,x,y)$, *where* $B = \{x \in E \mid ||x|| \le 1\}$.

Then, the Cauchy problem (15) *has a solution.*

Proof. For the sake of simplicity, we will assume all along with the proof that space E has a monotone Schauder basis, i.e., that $\|\mathbb{P}_m\| \le 1$, for every $m \in \mathbb{N}$. Let us note that the proof also works in the general case with little changes.

Since the proof consists of several parts, it will be split into the relevant steps from now on.

Step 1. *Introduction of a sequence of approximating operators*

To prove the existence of a solution to the problem (15), we will use the approximation solvability method. Thus, for each $m \in \mathbb{N}$, consider the multimap $G_m : [a,b] \times E \times E \to E_m$ defined as $G_m = \mathbb{P}_m \circ F$ and the operator $\Sigma_m : C^1([a,b],E_m) \multimap C^1([a,b],E_m)$ defined as

$$\Sigma_m(q)(t) = \left\{ \mathbb{P}_m C(t)\, x_a + \mathbb{P}_m S(t)\overline{x}_a + \int_a^t \mathbb{P}_m S(t-s) f(s)\, ds : f \in S^1_{G_m,q} \right\}. \tag{17}$$

Let us note that, since $\mathbb{P}_m$ is a bounded and linear operator with $\|P_m\| \le 1$, the mapping G_m satisfies properties $(F1)-(F4)$ as well. Therefore, for every $q \in C^1([a,b],E_m)$, the existence of a selection $f \in S^1_{G_m,q}$ is guaranteed by Proposition 1 taking $\eta_\Omega = \varphi_n$ with $\Omega \subset nB \times nB$. We stress that, since the natural projection is bounded and since the function

$$y(t) = C(t-a)\, x_a + S(t-a)\overline{x}_a + \int_a^t S(t-s) f(s)\, ds$$

is continuously differentiable and

$$\dot{y}(t) = AS(t-a)x_a + C(t-a)\overline{x}_a + \int_a^t C(t-s)\, f(s) ds,$$

Σ_m is well defined and

$$\dot{h}(t) = \mathbb{P}_m AS(t-a)x_a + \mathbb{P}_m C(t-a)\overline{x}_a + \int_a^t \mathbb{P}_m C(t-s) f(s)\, ds,$$

for every $q \in C^1([a,b],E_m)$ and every $h \in \Sigma_m(q)$.

In order to show that Σ_m has a fixed point, we will prove that it satisfies all assumptions of the Kakutani fixed point theorem ([49] (Theorem 1)). For this purpose, given $n \in \mathbb{N}$, we use the following notation

$$nB_m = \{q \in C^1([a,b];E_m) : \|q(t)\|, \|\dot{q}(t)\| \le n, \text{ for every } t \in [a,b]\}.$$

(a) *Proving that the solution mapping* Σ_m *has convex values.*

Let $q \in C^1([a,b],E_m)$ and let $h_1, h_2 \in \Sigma_m(q)$. Then, there exist $f_1, f_2 \in S^1_{G_m,q}$ such that

$$h_i(t) = \mathbb{P}_m C(t-a)\, x_a + \mathbb{P}_m S(t-a)\overline{x}_a + \int_a^t \mathbb{P}_m S(t-s) f_i(s)\, ds,\ i = 1,2.$$

Let $\alpha \in [0,1]$. Then, for each $t \in [a,b]$, we obtain that

$$\begin{aligned}(\alpha h_1 + (1-\alpha)h_2)(t) = \;& \mathbb{P}_m C(t-a)\, x_a + \mathbb{P}_m S(t-a)\overline{x}_a \\ & + \int_a^t \mathbb{P}_m S(t-s)(\alpha f_1(s) + (1-\alpha) f_2(s))\, ds.\end{aligned}$$

Since G_m has convex values and $\mathbb{P}_m$ is linear, it holds that

$$\alpha h_1 + (1-\alpha)h_2 \in \Sigma_m(q).$$

(b) *Proving that Σ_m has a closed graph.*

Assume that $(q_k,\, h_k) \to (q,\, h)$ in $C^1([a,b],E_m) \times C^1([a,b],E_m)$, where $h_k \in \Sigma_m(q_k)$, for all $k \in \mathbb{N}$, and let us prove that $h \in \Sigma_m(q)$.

Since, for all $k \in \mathbb{N}$, $h_k \in \Sigma_m(q_k)$, there exists, for all $k \in \mathbb{N}$, $f_k \in S^1_{G_m,\,q_k}$ such that

$$h_k(t) = \mathbb{P}_m C(t-a)x_a + \mathbb{P}_m S(t-a)\overline{x}_a + \int_a^t \mathbb{P}_m S(t-s) f_k(s)\, ds, \quad \text{for a.a. } t \in [a,b].$$

Since every converging sequence is bounded, there exists $n \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and every $s \in [a,b]$, $\|q_k(s)\| \le n$, $\|\dot{q}_k(s)\| \le n$. Then, $(F4)$ and the monotonicity of the Schauder basis yield that the sequence $\{f_k\}_k \subset L^1([a,b],E_m)$ is bounded and uniformly integrable, and, for a.a. $s \in [a,b]$, the sequence $\{f_k(s)\}_k$ is bounded in E_m. Since E_m is finite-dimensional, according to the Dunford–Pettis Theorem (see [50], p. 294), we have the existence of a subsequence, denoted as the sequence, and a function f such that $f_k \rightharpoonup f$ in $L^1([a,b],E_m)$.

Let us now prove that $f \in S^1_{G_m,q}$. Due to Mazur's convexity theorem, for each $k \in \mathrm{N}$, there exists $p_k \in \mathrm{N}$ and positive numbers $\beta_{k,i},\ i = 0,\ \ldots,\ p_k$, such that $\sum_{i=0}^{p_k} \beta_{k,i} = 1$ and $r_k := \sum_{i=0}^{p_k} \beta_{k,i}\, f_{k+i} \to f$ in $L^1([a,b],E)$. From the sequence $\{r_k\}_k$, we extract a subsequence, denoted as the sequence as usual, such that $r_k(t) \to f(t)$, for all $t \in [a,b]\backslash N_1$ with $\lambda(N_1) = 0$. Moreover, for all $t \in [a,b]\backslash N_2$ with $\lambda(N_2) = 0$, $G_m(t,\,\cdot,\,\cdot)$ is weakly u.s.c.

Put $N = N_1 \cup N_2$ and consider $\mathrm{t}_0 \in [a,b]\backslash N$. Then, for every weak neighborhood V of $G_m\big(t_0,\, q(t_0),\, \dot{q}(t_0)\big)$, there exists a weak neighborhood W of $\big(q(t_0),\, \dot{q}(t_0)\big)$ such that $G_m(t,\, x, y) \subset V$ when $(x,y) \in W$. Since the uniform convergence implies the weak pointwise convergence, it follows that $q_k(t_0) \rightharpoonup q(t_0)$ and $\dot{q}_k(t_0) \rightharpoonup \dot{q}(t_0)$. Thus, there exists $\overline{k}$ such that, for all $k \ge \overline{k}$, $(q_k(t_0),\, \dot{q}_k(t_0)) \in W$, yielding that $f_k(t_0) \in G_m\big(t_0,\, q_k(t_0), \dot{q}_k(t_0)\big) \subset V$, i.e., that $r_k(t_0) \in V$, because G_m is convex valued. Since $r_k(t_0) \to f(t_0)$, it follows that $f(t_0) \in \overline{V}$, for every weak neighborhood V of $G_m\big(t_0,\, q(t_0),\, \dot{q}(t_0)\big)$. Since G_m is closed valued, the proof is complete.

Given $\mathit{\Phi} \in E^*$ and $t \in [a,\ b]$, consider the operator Φ: $L^1([a,t],E) \to \mathrm{R}$ defined by

$$\Phi(p) := \mathit{\Phi}\bigg(\int_a^t S(t-s)p(s)\, ds\bigg).$$

Since $S(t-s)$ is bounded and linear, for every $t,\ s$, Φ is clearly linear and bounded. Moreover, $f_k \rightharpoonup f$ also in $L^1([a,t],E)$, and hence, we have that

$$\mathit{\Phi}\bigg(\int_a^t S(t-s) f_k(s)\, ds\bigg) = \Phi(f_k) \to \Phi(f) = \mathit{\Phi}\bigg(\int_a^t S(t-s) f(s)\, ds\bigg).$$

By the arbitrariness of $\mathit{\Phi}$, we conclude that

$$\int_a^t S(t-s) f_k(s)\, ds \rightharpoonup \int_a^t S(t-s) f(s)\, ds.$$

Hence, since P_m is a linear and bounded operator taking values in the finite-dimensional space E_m, it holds that

$$\int_a^t P_m\, S(t-s) f_k(s)\, ds \to \int_a^t P_m\, S(t-s) f(s)\, ds.$$

The uniqueness of the limit then imply that

$$h(t) = \mathbb{P}_m C(t-a) x_a + \mathbb{P}_m S(t-a)\overline{x}_a + \int_a^t \mathbb{P}_m S(t-s) f(s)\, ds, \quad \text{for a.a. } t \in [a,b].$$

(c) *Showing that Σ_m maps bounded sets into bounded sets.*

Let $C \subset C^1([a,b], E_m)$ be bounded, $q \in C$, and $h \in \Sigma_m(q)$. Then, there exists $f \in S^1_{G_m,q}$ such that

$$h(t) = \mathbb{P}_m C(t-a)\, x_a + \mathbb{P}_m S(t-a)\overline{x}_a + \int_a^t \mathbb{P}_m S(t-s) f(s)\, ds \tag{18}$$

and

$$\dot{h}(t) = \mathbb{P}_m A S(t-a)\, x_a + \mathbb{P}_m C(t-a)\overline{x}_a + \int_a^t \mathbb{P}_m C(t-s) f(s)\, ds. \tag{19}$$

According to Equations (6) and (9), there exists

$$K_0 = \begin{cases} M\frac{e^{\omega b}-1}{\omega} & \text{if } \omega \neq 0 \\ Mb & \text{if } \omega = 0 \end{cases} \tag{20}$$

such that for every $t \in [a,b]$,

$$||C(t)|| \leq Me^{\omega b}$$

and

$$\|S(t)\| \leq K_0.$$

Since $x_a \in X$, denoted by

$$L = \max_{t\in[a,b]} \left\| \frac{d}{dt} C(t) x_a \right\| \tag{21}$$

from Lemma 2 *(i)*, we obtain, for every $t \in [0,T]$, that

$$||AS(t)x_a|| \leq L.$$

Now, since C is bounded, there exists $n \in \mathbb{N}$ such that $C \subset nB_m$. Thus, the monotonicity of the Schauder basis and $(F4)$ yield

$$\begin{aligned} ||h(t)|| \leq\ & \|C(t-a)\|\|x_a\| + \|S(t-a)\|\|\overline{x}_a\| + \textstyle\int_0^T \|S(t-s)\|\|f(s)\|\, ds \\ \leq\ & Me^{\omega b}||x_a|| + K_0(||\overline{x}_a|| + ||\varphi_n||_{L^1}) \end{aligned} \tag{22}$$

and

$$\begin{aligned} ||\dot{h}(t)|| \leq\ & \|AS(t-a)\, x_a\| + \|C(t-a)\|\|\overline{x}_a\| + \textstyle\int_0^t \|C(t-s)\|\|f(s)\|\, ds \\ \leq\ & L + Me^{\omega b}(||\overline{x}_a|| + ||\varphi_n||_{L^1}). \end{aligned} \tag{23}$$

Therefore, Σ_m maps bounded sets into bounded sets.

(d) *Showing that Σ_m maps bounded sets into relatively compact sets.*

Let $C \subset C^1([a,b], E_m)$ be bounded, $q \in C$, and $h \in \Sigma_m(q)$. Then, h fulfills Equations (18) and (19) for some $f \in S^1_{G_m,q}$.

Fix $t_0 \in [a,b]$. According to point (c), Equation (23), and (F4), for all $t \in [a,b]$, we obtain that

$$\|h(t) - h(t_0)\| \leq \left| \int_{t_0}^t \|\dot{h}(s)\|\, ds \right| \leq \left(L + Me^{\omega b}(||\overline{x}_a|| + ||\varphi_n||_{L^1}) \right) |t - t_0|$$

which implies the equicontinuity of $\Sigma_m(C)$.

Moreover, for all $t \in [a,b]$, applying Lemma 2 *(i)* and Equation (6) and introducing the constant n as in point (c), we obtain that

$$\begin{aligned}
\|\dot{h}(t-a)-\dot{h}(t_0-a)\| \le\ & \|\mathbb{P}_m(AS(t-a)-AS(t_0-a))x_a\| \\
&+\|\mathbb{P}_m(C(t-a)-C(t_0-a))\overline{x}_a\| \\
&+\|\textstyle\int_a^{t_0}\mathbb{P}_m(C(t-s)-C(t_0-s))f(s)\,ds\| \\
&+\|\textstyle\int_{t_0}^{t}\mathbb{P}_m C(t-s)f(s)\,ds\| \\
\le\ & \|(C'(t-a)-C'(t_0-a))x_a\| \\
&+\|(C(t-a)-C(t_0-a))\overline{x}_a\| \\
&+\|\textstyle\int_a^{t_0}\mathbb{P}_m[C(t-s)-C(t_0-s)]f(s)\,ds\| \\
&+Me^{\omega T}|\textstyle\int_{t_0}^{t}\varphi_n(s)\,ds|.
\end{aligned} \tag{24}$$

Let us now prove that all four summands in Equation (24) are uniformly continuous.

Notice that $t_0,t \in [a,b] \subset [0,T]$ imply that $t_0-a, t-a \in [0,T]$. Since $x_a \in X$, the map $t \to C'(t)x_a$ is continuous and hence, uniformly continuous in the compact set $[0,T]$. Similarly, the map $t \to C(t)\overline{x}_a$ is uniformly continuous in $[0,T]$ as well. Thus, the first two summands are uniformly continuous.

Applying the properties listed in Lemma 2, we obtain that

$$\begin{aligned}
C(t-s)-C(t_0-s) =\ & C\left(\frac{t+t_0}{2}-s+\frac{t-t_0}{2}\right)-C\left(\frac{t+t_0}{2}-s-\frac{t-t_0}{2}\right) \\
=\ & 2AS\left(\frac{t+t_0}{2}-s\right)S\left(\frac{t-t_0}{2}\right) \\
=\ & 2AS\left(\frac{t-t_0}{2}\right)S\left(\frac{t+t_0}{2}-s\right) \\
=\ & 2AS\left(\frac{t-t_0}{2}\right)\left[S\left(\frac{t+t_0}{2}\right)C(-s)+S(-s)C\left(\frac{t+t_0}{2}\right)\right] \\
=\ & 2AS\left(\frac{t-t_0}{2}\right)S\left(\frac{t+t_0}{2}\right)C(s)-2AS\left(\frac{t-t_0}{2}\right)C\left(\frac{t+t_0}{2}\right)S(s) \\
=\ & \left[C\left(\frac{t-t_0}{2}+\frac{t+t_0}{2}\right)-C\left(\frac{t-t_0}{2}-\frac{t+t_0}{2}\right)\right]C(s) \\
& -A\left[S\left(\frac{t-t_0}{2}+\frac{t+t_0}{2}\right)+S\left(\frac{t-t_0}{2}-\frac{t+t_0}{2}\right)\right]S(s) \\
=\ & [C(t)-C(t_0)]C(s)-A[S(t)-S(t_0)]S(s) \\
=\ & [C(t)-C(t_0)]C(s)-[C'(t)-C'(t_0)]S(s).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\int_a^{t_0}[C(t-s)-C(t_0-s)]f(s)\,ds\right\| \le\ & \|[C(t)-C(t_0)]\textstyle\int_a^{t_0}C(s)f(s)\,ds\| \\
&+\|[C'(t)-C'(t_0)]\textstyle\int_a^{t_0}S(s)f(s)\,ds\|.
\end{aligned}$$

Consider now the set

$$I_1=\left\{\int_a^{t_0}C(s)f(s)ds : q \in nB_m, f \in S^1_{G_m,q}\right\}. \tag{25}$$

According to $(F4)$, and since E_m is finite-dimensional, we obtain that, for every $s \in [a,b]$, the set

$$\left\{f(s) : q \in nB_m, f \in S^1_{G_m,q}\right\} \subset E_m \subset E$$

is bounded and hence relatively compact. Since $C(s)$ is linear and bounded, for every $s \in [a,b]$, the set

$$D(s)=\left\{C(s)f(s) : q \in nB_m, f \in S^1_{G_m,q}\right\}$$

is relatively compact as well. Therefore, the separability of E (see Remark 1) and [46] (Theorem 4.2.3) imply that

$$\gamma(I_1) \le \int_a^{t_0} \gamma(D(s))ds = 0.$$

The regularity of the Hausdorff measure of noncompactness γ then implies that I_1 is relatively compact in E.

According to Lemma 3, $(t, x) \to C(t)x$ is continuous in $[a,\ b] \times E$ and, hence, uniformly continuous in the pre-compact set $[a,\ b] \times I_1$.

According to Lemma 2 (*h*), since $S(s)$ is also linear and bounded, for every $s \in [a,\ b]$, we obtain by the same reasoning that

$$\left\{S(s)f(s) : q \in\ nB_{m,}\ ,\ f \in S^1_{G_m,q}\right\} \subset E_m \cap\ X\ \subset X$$

is relatively compact, hence

$$I_2 = \left\{\int_a^{t_0} S(s)f(s)\ ds : q \in\ nB_{m,}\ ,\ f \in S^1_{G_m,q}\right\}$$

is relatively compact in X.

According to (11), $(t, x) \to C'(t)x$ is continuous in $[a,\ b] \times X$ and, hence, uniformly continuous in the pre-compact set $[a,\ b] \times I_2$. Thus, we also obtained the uniform continuity of the third term in Equation (24). The equicontinuity of the set $\{h' : h \in \Sigma_m(q), q \in C\}$ then follows from the absolute continuity of the integral contained in the last term of Equation (24).

Subsequently, it follows from the Arzela–Ascoli theorem that $\Sigma_m(C)$ is relatively compact in $C^1([a, b], E_m)$ for every bounded $C \subset C^1([a, b], E_m)$.

(e) *Showing that there exists $N \in \mathbb{N}$ independent of m such that $\Sigma_m(NB_m) \subset NB_m$*

By Equations (22) and (23), we have that there exist two constants

$$K_1 = \max\left\{Me^{\omega b}||x_a|| + K_0||\overline{x}_a||, L + Me^{\omega b}||\overline{x}_a||\right\}$$

and

$$K_2 = \max\left\{K_0, Me^{\omega b}\right\} \tag{26}$$

such that, for every $n, m \in \mathbb{N}, q \in nB_m$, and every $h \in \Sigma_m(q)$,

$$\|h\|_{C^1} \le K_1 + K_2\|\varphi_n\|_{L^1}. \tag{27}$$

According to (F4), there exists a subsequence, still denoted as the sequence, such that

$$\lim_{n\to\infty} \frac{K_1 + K_2\|\varphi_n\|_{L^1}}{n} = 0.$$

Therefore, there exists $N > 0$ such that

$$\frac{K_1 + K_2\|\varphi_N\|_{L^1}}{N} < 1,$$

which, combined with Equation (27), implies that

$$\frac{1}{N}\|h\|_{C^1} < 1,$$

i.e., that $h \in NB_m$, for every $t \in [a, b], m \in \mathbb{N}, q \in NB_m, h \in \Sigma_m(q)$, and the claim is proven.

Since Σ_m is closed and maps bounded sets into relatively compact sets, it has compact values; hence, it is u.s.c. Thus, $\Sigma_m : NB_m \multimap NB_m$ is a u.s.c. compact map with convex and closed values. Applying the Kakutani fixed point theorem, we obtain that, for all $m \in \mathbb{N}$, the operator Σ_m has a fixed point q_m. Because of the technique used, we are also able to localize the fixed point in the set

$$NB = \left\{ q \in C^1([a,b], E) : \|q(t)\|, \|\dot{q}(t)\| \leq N, \text{ for every } t \in [a,b] \right\}.$$

Step 2. *Limiting procedure.*

Let us now prove that the sequence $\{q_m\}_m$ found in Step 1 admits a subsequence pointwise weakly converging to a solution q of Problem (15).

The sequence $\{q_m\}_m$ satisfies, for all $m \in \mathbb{N}$ and $t \in [a,b]$,

$$q_m(t) = \mathbb{P}_m C(t-a)\, x_a + \mathbb{P}_m S(t-a)\overline{x}_a + \int_a^t \mathbb{P}_m S(t-s) f_m(s)\, ds,$$

where $f_m \in S^1_{G_m, q_m}$, for every $m \in \mathbb{N}$. Thus, there exists $g_m \in S^1_{F, q_m}$ such that $f_m = \mathbb{P}_m g_m$. Since $q_m \in NB$ for every m, we then obtain from (F4) that

$$\|g_m(s)\| \leq \varphi_N(s),$$

for a.e. $s \in [a,b]$. Therefore, $\{g_m\}_m$ is bounded and uniformly integrable and $\{g_m(s)\}_m$ is bounded for a.a $s \in [a,b]$. Since E is reflexive, according the Dunford–Pettis Theorem, we obtain the existence of a subsequence, denoted as the sequence, and of a function f such that $g_m \rightharpoonup f$ in $L^1([a,b], E)$. From Lemma 1 *(d)*, we then also obtain that $f_m \rightharpoonup f$ in $L^1([a,b], E)$.

Given $\phi \in E^*$ and $t \in [a,b]$, consider the operator $\Phi : L^1([a,t], E) \to \mathbb{R}$ defined by

$$\Phi(p) := \phi\left(\int_a^t S(t-s) p(s)\, ds \right).$$

According to Equation (9), Φ is clearly linear and bounded. Moreover, $f_m \rightharpoonup f$ also in $L^1([a,t], E)$, and hence, we have that

$$\Phi(f_m) = \phi\left(\int_a^t S(t-s) f_m(s)\, ds \right) \to \phi\left(\int_a^t S(t-s) f(s)\, ds \right) = \Phi(f).$$

By the arbitrariness of ϕ, we conclude that $S(t-\cdot) f_m \rightharpoonup S(t-\cdot) f$ in $L^1([a,t], E)$ for every $t \in [a,b]$ and, by applying Lemma 1 *(d)* again, that $\mathbb{P}_m S(t-\cdot) f_m \rightharpoonup S(t-\cdot) f$ in $L^1([a,t], E)$. In particular,

$$\int_a^t \mathbb{P}_m S(t-s) f_m(s)\, ds \rightharpoonup \int_a^t S(t-s) f(s)\, ds,$$

and therefore,

$$q_m(t) \rightharpoonup q(t) = C(t-a)\, x_a + S(t-a)\overline{x}_a + \int_a^t S(t-s) f(s)\, ds.$$

Similarly, it is possible to prove that $\dot{q}_m(t) \rightharpoonup \dot{q}(t)$ for every $t \in [a,b]$.

It remains to be proven that $f \in S^1_{F,q}$. Due to Mazur's convexity theorem, for each $m \in \mathbb{N}$, there exist $p_m \in \mathbb{N}$ and positive numbers $\beta_{m,i}$, $i = 0, \ldots, p_m$, such that $\sum_{i=0}^{p_m} \beta_{m,i} = 1$ and $r_m := \sum_{i=0}^{p_m} \beta_{m,i} g_{m+i} \to f$ in $L^1([a,b], E)$. From the sequence $\{r_m\}_m$, we extract a subsequence, denoted as the sequence as usual, such that $r_m(t) \to f(t)$, for all $t \in [a,b] \setminus N_1$, with $\lambda(N_1) = 0$. Since F is convex valued, $r_m(t) \in F(t, q_m(t), \dot{q}_m(t))$ for every $t \in [a,b]$.

Moreover, for all $t \in [a,b] \setminus N_2$, with $\lambda(N_2) = 0$, $F(t,\cdot,\cdot)$ is weakly u.s.c. Put $N = N_1 \cup N_2$ and consider $t_0 \in [a,b] \setminus N$. Then, for every weak neighborhood V of $F(t_0, q(t_0), \dot{q}(t_0))$, there exists a weak neighbourhood W of $(q(t_0), \dot{q}(t_0))$ such that $F(t,x,y) \subset V$ when

$(x, y) \in W$. Since $q_m(t_0) \rightharpoonup q(t_0)$ and $\dot{q}_m(t_0) \rightharpoonup \dot{q}(t_0)$, there exists $\overline{m}$ such that, for all $m \geq \overline{m}$, $(q_m(t_0), \dot{q}_m(t_0)) \in W$, yielding that $g_m(t_0) \in F(t_0, q_m(t_0), \dot{q}_m(t_0)) \subset V$. Since $g_m(t_0) \to f(t_0)$, it follows that $f(t_0) \in \overline{V}$ for every weak neighborhood V of $F(t_0, q(t_0), \dot{q}(t_0))$, and since F is closed valued, the proof is complete. □

The following theorem shows that it is possible to prove the result when assuming the growth condition $(F4^{'})$ instead of $(F4)$ in Theorem 1. For a comparison between these conditions, we refer to [41]. The sketch of the proof is a generalization of the technique used in [51] for second-order inclusions, where the nonlinear term does not depend on the first derivative.

Theorem 2. *Consider the Cauchy problem* (15)*, where* $x_a \in X$*, and* $F : [a, b] \times E \times E \multimap E$ *satisfies conditions* $(F1) - (F3)$*. Moreover, let the following assumption hold:*

$(F4^{'})$ *There exist* $\alpha, \beta \in L^1([a, b], E)$ *such that, for a.a.* $t \in [a, b]$ *and all* x*,* $y \in E$*,*

$$||F(t, x, y)|| \leq \alpha(t) \max\{||x||, ||y||\} + \beta(t). \tag{28}$$

Then, the Cauchy problem (15) *has a solution.*

Proof. The result can be proven similarly to Theorem 1 when replacing $\varphi_n(t)$ by $n\alpha(t) + \beta(t)$ and consequently modifying the proof. More concretely, the most difficult point is showing that there exists a bounded and convex set H_m, such that $\Sigma_m(H_m) \subset H_m$ for all $m \in \mathbb{N}$.

For this purpose, for every fixed $j \in \mathbb{N}$, define

$$q_j = \max_{t\in[a,b]} \int_a^b e^{-j(t-s)} \chi_{[a,t]}(s)\alpha(s)\, ds,$$

whose existence is guaranteed by the continuity. For every $j \in \mathbb{N}$, let t_j be the point where the maximum is achieved. Since $\{t_j\}_j \subset [a, b]$, there exists $\bar{t}$ such that (eventually passing to a subsequence) $t_j \to \bar{t}$. Thus, the sequence $\{\phi_j\}_j \subset L^1([a, b], E)$ defined as $\phi_j(s) = e^{-j(t_j-s)}\chi_{[a,t_j]}(s)\alpha(s)$ converges pointwise to 0. The convergence is dominated, which implies that $\phi_j \to 0$ in $L^1([a, b], E)$. In particular, there exists a subsequence, still denoted as the sequence, such that $q_j \to 0$. Take $\bar{j} \in \mathbb{N}$ and $R \in \mathbb{R}$ such that $1 - K_2 q_{\bar{j}} > 0$, and

$$R > \frac{e^{-\bar{j}a}\left[\max\{Me^{\omega b}\|x_a\|, L\} + K_2(\|\overline{x}_a\| + \|\beta\|_1)\right]}{1 - K_2 q_{\bar{j}}},$$

where K_2 is the constant introduced in Equation (26). Moreover, let us consider the bounded and convex set

$$H_m = \left\{x \in C^1([a, b], E_m) : \max_{t\in[a,b]} (e^{-\bar{j}t} \max\{\|x(t)\|, \|\dot{x}(t)\|\}) \leq R\right\}.$$

Now, with reasoning like in Equations (22) and (23), we have that, for every $q \in H_m, h \in \Sigma_m(q), t \in [a,b]$,

$$\begin{aligned} e^{-\bar{j}t}\|h(t)\| &\le e^{-\bar{j}t}Me^{\omega b}\|x_a\| \\ &+e^{-\bar{j}t}K_0\|\overline{x}_a\| + e^{-\bar{j}t}K_0\int_a^t(\alpha(s)\max\{\|x(s)\|,\|\dot{x}(s)\|\}+\beta(s))ds \\ &\le e^{-\bar{j}t}Me^{\omega b}\|x_a\| + e^{-\bar{j}t}K_0\|\overline{x}_a\| + e^{-\bar{j}t}K_0\|\beta\|_1 \\ &+e^{-\bar{j}t}K_0\int_a^t e^{\bar{j}s}\alpha(s)e^{-\bar{j}s}\max\{\|x(s)\|,\|\dot{x}(s)\|\}ds \\ &\le e^{-\bar{j}t}[Me^{\omega b}\|x_a\| + K_0(\|\overline{x}_a\| + \|\beta\|_1)] + K_0R\int_a^t e^{-\bar{j}(t-s)}\alpha(s)ds \\ &= e^{-\bar{j}t}[Me^{\omega b}\|x_a\| + K_0(\|\overline{x}_a\| + \|\beta\|_1)] + K_0R\int_a^b e^{-\bar{j}(t-s)}\chi_{[a,t]}(s)\alpha(s)ds \\ &\le e^{-\bar{j}a}[Me^{\omega b}\|x_a\| + K_0(\|\overline{x}_a\| + \|\beta\|_1)] + K_0Rq_{\bar{j}} \\ &< R(1-K_2q_{\bar{j}}) + K_0Rq_{\bar{j}} \\ &< R(1-K_2q_{\bar{j}}) + K_2Rq_{\bar{j}} < R \end{aligned}$$

and similarly

$$\begin{aligned} e^{-\bar{j}t}\|\dot{h}(t)\| &\le e^{-\bar{j}t}L + e^{-\bar{j}t}Me^{\omega b}\|\overline{x}_a\| + e^{-\bar{j}t}Me^{\omega b}\int_a^t(\alpha(s)\max\{\|x(s)\|,\|\dot{x}(s)\|\}+\beta(s))ds \\ &\le e^{-\bar{j}a}\Big[L + Me^{\omega b}(\|\overline{x}_a\| + \|\beta\|_1)\Big] + Me^{\omega b}Rq_{\bar{j}} < R. \end{aligned}$$

Therefore, $h \in H_m$. Since H_m is a subset of the bounded set

$$H = \{x \in C^1([a,b],E) : \max_{t\in[a,b]}(e^{-\bar{j}t}\max\{\|x(t)\|,\|\dot{x}(t)\|\}) \le R\},$$

the conclusion then follows like in the proof of Theorem 1. □

4. Existence of a Mild Solution for the Impulsive Problem

In this section, the existence of a mild solution to the impulsive problem (1) will be discussed. The solution will be found in the class of piecewise continuously differentiable functions. We will firstly apply, in each interval of continuous differentiability, the theorems that were proven in the previous section for the problems without impulses. Then, we will glue the particular solutions, exploiting Formula (2). The gluing technique will enable obtaining the conclusions under no compactness requirements on the impulsive terms.

Theorem 3. *Consider the Cauchy impulsive problem (1), where $x_0 \in X$, $I_k(y,e) \in X$ for every $y \in X, e \in E$ and $F : [0,T] \times E \times E \multimap E$ satisfies conditions $(F1)-(F4)$ or $(F1)-(F4')$ for $[a,b] = [0,T]$. Then, the Cauchy impulsive problem (1) has a mild solution $x \in PC^1([0,T],E)$ satisfying, for all $t \in [0,T]$,*

$$\begin{aligned} x(t) = \; & C(t)\,x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s)f(s)\,ds \\ & + \sum_{0<t_i<t}\big[C(t-t_i)I_i(x(t_i),\,\dot{x}(t_i)) + S(t-t_i)\overline{I}_i(x(t_i),\,\dot{x}(t_i))\big], \end{aligned} \tag{29}$$

where $f \in S^1_{F,x}$.

Proof. The proof will be given in three steps.
Step 1. Let us consider the problem on the interval $[0,t_1]$:

$$\left.\begin{aligned} \ddot{x}(t) \in Ax(t) + F(t,x(t),\dot{x}(t)), \quad & \text{for a.a. } t \in [0,t_1], \\ x(0) = x_0,\ \dot{x}(0) = \overline{x}_0. & \end{aligned}\right\} \tag{30}$$

According to Theorems 1 or 2 with $[a,b] = [0,t_1]$, this problem has a solution $x_{[0]}(t) \in C^1([0,t_1],E)$ verifying

$$x_{[0]}(t) = C(t)\,x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s)f_{[0]}(s)\,ds, \tag{31}$$

and

$$\dot{x}_{[0]}(t) = AS(t)\,x_0 + C(t)\overline{x}_0 + \int_0^t C(t-s)f_{[0]}(s)\,ds, \tag{32}$$

where $f_{[0]} \in L^1([0,t_1],E)$ and $f_{[0]}(t) \in F(t, x_{[0]}(t), \dot{x}_{[0]}(t))$, for every $t \in [0,t_1]$.
Step 2. Let us consider now the following problem on $[t_1,t_2]$:

$$\left.\begin{array}{r} \ddot{x}(t) \in Ax(t) + F(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \in [t_1,t_2], \\ x(t_1) = I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + x_{[0]}(t_1), \\ \dot{x}(t_1) = \overline{I}_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + \dot{x}_{[0]}(t_1). \end{array}\right\} \tag{33}$$

By $x_0 \in X$, we obtain that $x_{[0]}(t) \in X$ for every $t \in [0,t_1]$. In particular, $x_{[0]}(t_1) \in X$, yielding that $I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) \in X$, because $I(y,e) \in X$ for every $y \in X$. Since X is a linear subspace of E, we then obtain that $I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + x_{[0]}(t_1) \in X$ and, analogously to Step 1, it is possible to obtain that there exists a mild solution $x_{[1]}(t) \in C^1([t_1,t_2],E)$ of Problem (33) such that

$$\begin{aligned} x_{[1]}(t) = \;& C(t-t_1)\,[I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + x_{[0]}(t_1)] \\ & + S(t-t_1)[\overline{I}_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + \dot{x}_{[0]}(t_1)] + \textstyle\int_{t_1}^t S(t-s)f_{[1]}(s)\,ds, \end{aligned}$$

where $f_{[1]} \in L^1([t_1,t_2],E)$ and $f_{[1]}(t) \in F(t, x_{[1]}(t), \dot{x}_{[1]}(t))$, for every $t \in [t_1,t_2]$. Since, according to Equations (31) and (32),

$$x_{[0]}(t_1) = C(t_1)\,x_0 + S(t_1)\overline{x}_0 + \int_0^{t_1} S(t_1-s)f_{[0]}(s)\,ds,$$

$$\dot{x}_{[0]}(t_1) = AS(t_1)\,x_0 + C(t_1)\overline{x}_0 + \int_0^{t_1} C(t_1-s)f_{[0]}(s)\,ds,$$

reasoning like in Proposition 2, and denoted

$$f(t) = \begin{cases} f_{[0]}(t), & \text{for } t \in [0,t_1], \\ f_{[1]}(t), & \text{for } t \in (t_1,t_2], \end{cases}$$

we obtain that

$$\begin{aligned} x_{[1]}(t) = \;& C(t-t_1)\,[I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + x_{[0]}(t_1)] \\ & + S(t-t_1)[\overline{I}_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + \dot{x}_{[0]}(t_1)] + \textstyle\int_{t_1}^t S(t-s)f(s)\,ds, \\ = \;& C(t-t_1)[I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + C(t_1)\,x_0 + S(t_1)\overline{x}_0 + \textstyle\int_0^{t_1} S(t_1-s)f(s)\,ds] \\ & + S(t-t_1)[\overline{I}_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + AS(t_1)\,x_0 + C(t_1)\overline{x}_0 + \textstyle\int_0^{t_1} C(t_1-s)f(s)\,ds] \\ & + \textstyle\int_{t_1}^t S(t-s)f(s)\,ds \\ = \;& [C(t-t_1)C(t_1) + S(t-t_1)AS(t_1)]x_0 + [C(t-t_1)S(t_1) + S(t-t_1)C(t_1)]\overline{x}_0 \\ & + C(t-t_1)\,I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + S(t-t_1)\overline{I}_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) \\ & + \textstyle\int_0^{t_1}[C(t-t_1)S(t_1-s) + S(t-t_1)C(t_1-s)]f(s)\,ds + \int_{t_1}^t S(t-s)f(s)\,ds \\ = \;& C(t)x_0 + S(t)\overline{x}_0 + C(t-t_1)\,I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + S(t-t_1)\overline{I}_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) \\ & + \textstyle\int_0^{t_1} S(t-s)f(s)\,ds + \int_{t_1}^t S(t-s)f(s)\,ds \\ = \;& C(t)x_0 + S(t)\overline{x}_0 + C(t-t_1)\,I_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) + S(t-t_1)\overline{I}_1(x_{[0]}(t_1),\, \dot{x}_{[0]}(t_1)) \\ & + \textstyle\int_0^t S(t-s)f(s)\,ds. \end{aligned}$$

Step 3. Repeating this procedure for all $k = 1, ..., m$, we obtain that x defined by

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0, t_1], \\ x_{[1]}(t), & \text{for } t \in (t_1, t_2], \\ . \\ . \\ . \\ x_{[m]}(t), & \text{for } t \in (t_m, T], \end{cases}$$

is a mild solution of Problem (1) satisfying Equation (29) because

$$f(t) = \begin{cases} f_{[0]}(t), & \text{for } t \in [0, t_1], \\ f_{[1]}(t), & \text{for } t \in (t_1, t_2], \\ . \\ . \\ . \\ f_{[m]}(t), & \text{for } t \in (t_m, T] \end{cases}$$

belongs to $S^1_{F,x}$. □

Remark 3. *We stress that, since X is a linear subspace, the condition $I_k(y,e) \in X$, for every $y \in X, e \in E$, is satisfied whenever I_k is linear in the first variable.*

5. An Application

In this section, we apply the theoretical result obtained in the previous section to the following problem

$$\left.\begin{array}{l} z''_{tt} = \sum_{k,l=1}^{n} a_{kl}(x)\frac{\partial^2 z}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x)\frac{\partial z}{\partial x_j} + c(x)z + d(t,x)f(\int_\Omega k(x,s)z_t(t,s)ds), \\ \quad t \in [0,T], x \in \Omega \\ z(t_k^+, x) = z(t_k^-, x) + p_k z(t_k, x), \quad k = 1,2,\dots,m, \quad x \in \Omega \\ \dot{z}(t_k^+, x) = z(t_k^-, x) + q_k z(t_k, x), \quad k = 1,2,\dots,m, \quad x \in \Omega \\ z(0,x) = z_0(x), z_t(0,x) = \bar{z}_0(x), \quad x \in \Omega \\ z(t,0) = 0, \nabla z(t,0) = 0, \quad t \in [0,T], \end{array}\right\} \tag{34}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $z_0 \in W^{1,p}(\Omega) \cap C_0(\Omega), \bar{z}_0 \in L^p(\Omega), 1 < p < \infty$. The linear operator in Equation (34) appears frequently in many equations modeling phenomena, coming from electrostatics, continuum mechanics, and also other branches. In particular, when the linear operator reduces to the Laplacian and the integral term is replaced by the first-order time derivative, Equation (34) represents the multidimensional telegraph equation, which governs both the voltage and current of electrical transmission. Here, we consider a nonlinear Balakrishnan–Taylor-type damping term like in [52–54], generalizing the results to the case of a strongly elliptic linear part in $\mathbb{R}^n$.

In order to apply previous results to the problem (34), let us consider the following assumptions on functions $a_{kl} : \overline{\Omega} \to \mathbb{R}$, $b_j, c : \Omega \to \mathbb{R}$, $d : [0,T] \times \Omega \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $k : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$:

(a_1) $a_{kl} \in C(\overline{\Omega}), b_j, c \in L^\infty(\Omega), a_{kl}(x) = a_{lk}(x)$ for every $x \in \Omega$;

(a_2) There exists $C_0 \geq 0$ such that $\sum_{k,l=1}^{n} a_{kl}(x)s_k s_l \geq C_0|s|^2$ for every $s \in \mathbb{R}^n$, and a.a. $x \in \overline{\Omega}$;

(k) $k \in C(\overline{\Omega} \times \overline{\Omega})$;

(f) $f \in C(\mathbb{R})$;

(d_1) d is measurable;

(d_2) There exists $\alpha \in L^1((0,T),\mathbb{R}), \gamma \in L^p(\Omega), \lambda : [0,+\infty) \to [0,+\infty)$ increasing, with

$$\lim_{c \to +\infty} \frac{\lambda(c)}{c} = 0,$$

such that $|d(t,x)| \leq \alpha(t)\gamma(x)$, for a.a. $t \in [0,T]$, $x \in \Omega$ and $|f(c)| \leq \lambda(|c|)$, for every $c \in \mathbb{R}$;

(pq) $p_k, q_k \in \mathbb{R}$, for every $k = 1, ..., m$.

In order to rewrite the problem (34) into the abstract form, as in Problem (1), it is necessary to define the Banach space E, the operator A, and the nonlinear term F. The Banach space E is defined as the space $L^p(\Omega)$. Therefore, solution x of the studied problem will belong to the space $PC^1([0,T], L^p(\Omega))$. Moreover, we identify z and d, respectively, with functions $t \to z(t,\cdot)$ and $t \to d(t,\cdot)$.

The mapping $F\colon [0,T] \times L^p(\Omega) \times L^p(\Omega) \to L^p(\Omega)$ is defined by

$$F(t,z,u)(x) := d(t,x) f\left(\int_\Omega k(x,s)u(s)ds\right).$$

If we denote

$$\bar{k} := \max\{|k(x,s)| : x, s \in \overline{\Omega}\},$$

then, according to (d_2) and the Holder inequality, it holds that

$$\begin{aligned} \left| d(t,x) f\left(\int_\Omega k(x,s)u(s)ds\right)\right| &\leq \alpha(t)\gamma(x)\lambda\left(\left|\int_\Omega k(x,s)u(s)ds\right|\right) \\ &\leq \alpha(t)\gamma(x)\lambda\left(\int_\Omega |k(x,s)u(s)|ds\right) \\ &\leq \alpha(t)\gamma(x)\lambda\left(\bar{k}|\Omega|^{1-\frac{1}{p}}\|u\|_p\right). \end{aligned} \tag{35}$$

Thus, $F(t,z,u) \in L^p(\Omega)$ for every $t \in [0,T]$ and $z, u \in L^p(\Omega)$.

Moreover, by (d_1), we obtain that $F(\cdot,z,u)$ is measurable for every $z, u \in L^p(\Omega)$. Consider now a sequence $\{u_n\}$ weakly converging to u in $L^p(\Omega)$. Notice that, for every $x \in \Omega, k(x,\cdot) \in L^q(\Omega)$, where $q = \frac{p}{p-1}$. Thus,

$$\int_\Omega k(x,s)u_n(s)ds \to \int_\Omega k(x,s)u(s)ds.$$

Condition (f) then implies that

$$f\left(\int_\Omega k(x,s)u_n(s)ds\right) \to f\left(\int_\Omega k(x,s)u(s)ds\right).$$

Since every weakly converging sequence is bounded, we obtain the existence of a positive constant L such that, for every $n \in \mathbb{N}$, $\|u_n\|_p \leq L$. Hence, recalling Equation (35), we obtain that

$$\left| d(t,x) f\left(\int_\Omega k(x,s)u_n(s)ds\right)\right| \leq \alpha(t)\gamma(x)\lambda\left(\bar{k}|\Omega|^{1-\frac{1}{p}}L\right).$$

The convergence is also dominated, which implies that $F(t,\cdot,\cdot)$ is weakly continuous, for a.e. $t \in [0,T]$.

Recalling Equation (35) again, when $\|u\|_p \leq n$, we have that

$$\begin{aligned} \|F(t,z,u)\|_p^p &= \int_\Omega \left| d(t,x) f\left(\int_\Omega k(x,s)u(s)ds\right)\right|^p dx \leq \int_\Omega \left[\alpha(t)\gamma(x)\lambda\left(\bar{k}|\Omega|^{1-\frac{1}{p}}n\right)\right]^p dx \\ &= \alpha(t)^p \lambda\left(\bar{k}|\Omega|^{1-\frac{1}{p}}n\right)^p \|\gamma\|_p^p. \end{aligned}$$

Thus, (d_2) implies $(F4)$, with $\varphi_n(t) = \alpha(t)\lambda\left(\bar{k}|\Omega|^{1-\frac{1}{p}}n\right)\|\gamma\|_p$.

The symmetric strongly elliptic linear operator

$$A : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$$

is defined as

$$Au(x) := \sum_{k,l=1}^{n} a_{kl}(x)\frac{\partial^2 u}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u.$$

Since there exists a constant $D \geq 0$ such that

$$\|u\|_{2,p} \leq D(\|Au|_{0,p} + \|u\|_{0,p}), \tag{36}$$

for every $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [55] (Theorem 7.3.1)), operator A generates an analytic semigroup $\{T(t)\}_{t\geq 0}$, with

$$\|T(t)\| \leq D,$$

for every $t \geq 0$ (see [55] (Theorem 7.3.5)), and moreover, it generates a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ (see [48] (Section IV.8)). Lemma 4 then yields that

$$\|C(t)\| \leq \frac{D}{2}. \tag{37}$$

Clearly, $X = W^{1,p}(\Omega) \cap C_0(\Omega)$. Notice finally that the impulse functions $I_k(y,e) = p_k y$ depend linearly on y.

The conclusions from previous paragraphs can be summarized in the following result, which deals with the existence of a mild solution of Problem (34) and which is the direct consequence of Theorem 3.

Theorem 4. *Consider the Cauchy impulsive problem (34), where assumptions $(a_1) - (pq)$ are verified. Assume $z_0 \in W^{1,p}(\Omega)$ with compact support. Then, the Cauchy impulsive problem (34) has a mild solution $u \in PC^1([0,T], L^p(\Omega))$.*

Remark 4. *It is possible to generalize the previous result in the case when the condition $f \in C(\mathbb{R})$ is replaced by the following one:*

(f_2) *There exist $r_1 < r_2 < \ldots < r_k$ such that $f(\cdot)$ is continuous, for $r \neq r_i$, and $f(\cdot)$ has discontinuities at r_i, for $i = 1, \ldots, k$, with $f(r_i^{\mp}) := \lim_{r\to r_i^{\mp}} f(r) \in \mathbb{R}$ and $|f(r)| \leq \lambda(|r|)$, for every $r \neq r_i$, where λ is as in (d_2).*

In this case, it is appropriate to define the multivalued mapping $\tilde{f} : \mathbb{R} \multimap \mathbb{R}$ by the formula

$$\tilde{f}(r) := \begin{cases} f(r) & \text{if } r \neq r_i, \\ [\min\{f(r_i), f(r_i^-), f(r_i^+)\},\ \max\{f(r_i), f(r_i^-), f(r_i^+)\}] & \text{if } r = r_i, \\ & i = 1,2,\ldots,k. \end{cases}$$

In the abstract form $F\colon [0,T] \times L^p(\Omega) \times L^p(\Omega) \multimap L^p(\Omega)$ is defined as the closed, bounded, and convex valued multimap

$$F(t,z,u)(x) := d(t,x)\tilde{f}\left(\int_\Omega k(x,s)u(s)ds\right).$$

If we fix $i = 1, \ldots k$, then for every $\epsilon > 0$, it is possible to find $\delta > 0$ such that for every r with $0 < |r - r_i| \leq \delta$, it follows that

$$\tilde{f}(r_i) - \epsilon \leq f(r_i^-) - \epsilon < f(r) < f(r_i^+) + \epsilon \leq \tilde{f}(r_i) + \epsilon$$

and $f(r) = \tilde{f}(r)$, i.e., the function $r \to \min\{f(r), f(r^-), f(r^+)\}$ is lower semicontinuous and the function $r \to \max\{f(r), f(r^-), f(r^+)\}$ is upper semicontinuous. Therefore, $\tilde{f}$ is an upper semicontinuous multimap. Moreover, for every $r > r_k$, $\tilde{f}(r) = f(r)$. Thus, $\tilde{f}(r)$ satisfies

$$|c| \leq \lambda(|r|),$$

for every $c \in \tilde{f}(r)$. Using the same reasoning as in the paragraphs before Theorem 4, it is possible to prove that F verifies (F3) and (F4). Since all other assumptions hold as well, Theorem 3 yields the existence of a mild solution for Problem (34).

Remark 5. *It is possible to extend the previous result to a nonlinear term satisfying a linear growth condition. More precisely, instead of the equation contained in Equation (34), consider the equation*

$$z''_{tt} = \sum_{k,l=1}^{n} a_{kl}(x)\frac{\partial^2 z}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x)\frac{\partial z}{\partial x_j} + c(x)z + p(t)z_t + d(t,x)f\left(\int_\Omega k(x,s)z_t(t,s)ds\right). \quad (38)$$

In this case, the nonlinear term F reads as

$$F(t,z,u)(x) = p(t)u + d(t,x)f\left(\int_\Omega k(x,s)u(t,s)ds\right).$$

Assume that condition (d_2) is replaced by:

(d_2') *There exist $\delta \in L^1((0,T),\mathbb{R}), \gamma \in L^p(\Omega), l \geq 0$, such that $|p(t)| \leq \delta(t), |d(t,x)| \leq \delta(t)\gamma(x)$, for a.a. $t \in [0,T]$, $x \in \Omega$, and $|f(c)| \leq l|c|$, for every $c \in \mathbb{R}$.*

The result can be proven by similarly modifying the proof. More concretely, it is sufficient to take into account that Equation (35) is replaced by

$$\left| p(t)u(x) + d(t,x)f\left(\int_\Omega k(x,s)u(s)ds\right)\right| \leq \delta(t)[|u(x)| + \gamma(x)l\bar{k}|\Omega|^{1-\frac{1}{p}}\|u\|_p].$$

Hence, recalling that for every $a, b \geq 0, p > 1$, the estimate $(a+b)^p \leq 2^p(a^p + b^p)$ holds, we obtain

$$\|F(t,z,u)\|_p \leq 2\delta(t)[1 + (\|\gamma\|_p l\bar{k}|\Omega|^{1-\frac{1}{p}})^p]\|u\|_p.$$

Thus, (F_4') holds with $\beta(t) = 0$ and $\alpha(t) = 2\delta(t)[1 + (\|\gamma\|_p l\bar{k}|\Omega|^{1-\frac{1}{p}})^p]$. The existence of a mild solution for Equation (38) satisfying the Cauchy and the impulsive conditions from Equation (34) then follows by applying Theorem 3 again.

Similar equations as Equation (38) were considered in [36,53,56], but all of them were related to a one-dimensional variable x only.

In [53], the authors considered a constant coefficient p in Equation (38) and a kernel k only depending on s. The nonlinear term also included a Carathéodory function F, independent of z_t, which was asked to satisfy, at most, a linear growth with an additional constraint. Thus, their nonlinear term cannot completely be compared with our map, but we stress that we can allow it to have linear growth without the need for any constraint. Moreover, the linear term A in [53] was equal to z''_{xx}, and the proof of the existence of a mild solution made use of the compactness of the cosine family generated by A, while our method does not need any compactness and hence can take into account the possible presence of first-order derivatives.

In [56], the coefficient p was continuous, as well as the kernel k, which again only depended on s. The function f did not depend on x, and it was allowed to have a linear growth, but again with an additional constraint.

In [36], the coefficient $p \in L^\infty((0,T),\mathbb{R})$. The nonlinear term consisted of a weakly closed function F only depending on t and z, which, moreover, is assumed to be strictly sublinear.

6. Conclusions and Future Studies

In this paper, the existence of a mild solution to the Cauchy problem for an impulsive semilinear second-order differential inclusion in a Banach space is investigated. The results are obtained by the combination of the Kakutani fixed point theorem with the approximation solvability method and weak topology. The applied method enables obtaining the conclusions without any requirement of the compactness of the r.h.s. and/or the cosine family generated by the linear term and without the transformation of the studied second-order problem to the corresponding first-order one. The proved theoretical results are applied to the generalized telegraph equation with a Balakrishnan–Taylor-type damping term. The conclusions of the paper generalize several previous results dealing with the Cauchy problem for semilinear second-order differential equations or inclusions in Banach spaces since the r.h.s. considered also depends on the first derivative and since the conclusions are proven under less restrictive conditions.

Some directions for possible future research related to the studied topics are the following:

1. Our theoretical result does not only establish the existence of a solution but also its localization in a bounded set; from the applications point of view, it could be very interesting to investigate the existence of a bounded solution in an unbounded interval, which could be obtained thanks to the localization property;
2. Our theoretical result is based on the application of a fixed point result. Thus, at most, a linear growth condition on the nonlinear term is allowed; the study of the abstract inclusion by means of a continuation principle would allow relaxing the growth condition, thereby enlarging the class of models to which it can be applied;
3. As stressed in the Introduction, the interest in studying an inclusion mainly comes from the possibility of dealing with control problems; we think that it would be interesting to implement our technique for studying the controllability of the system due to its importance in real-life problems;
4. We investigated the Cauchy problem associated with the inclusion; several more general boundary conditions could be introduced, such as, e.g., periodic, antiperiodic, Dirichlet, multipoint, integral, and so on;
5. As the linear term, we considered the generator A of a cosine family; many applications involve generators $A(t)$ of fundamental systems possibly perturbed by suitable linear operators; thus, it is worth applying our techniques to more general problems.

Author Contributions: Writing—original draft preparation, M.P. and V.T.; writing—review and editing, M.P. and V.T.; funding acquisition, M.P. and V.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the European Structural and Investment Funds (Operational Programme Research, Development and Education) and by the Ministry of Education, Youth and Sports of the Czech Republic under the Grant No. CZ.02.2.69/0.0/0.0/18_054/0014592 The Advancement of Capacities for Research and Development at Moravian Business College Olomouc. This research was performed within the framework of the grant MIUR-PRIN 2020F3NCPX "Mathematics for industry 4.0 (Math4I4)".

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Taddei is the member of Gruppo Nazionale di Analisi Matematica, Probabilitá e le loro Applicazioni of Istituto Nazionale di Alta Matematica.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Benchohra, M.; Henderson, J.; Ntouyas, S.K. *Impulsive Differential Equations and Inclusions 2*; Hindawi: New York, NY, USA, 2006.
2. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific: Singapore; London, UK, 1989.
3. Yang, T. *Impulsive Control Theory*; Springer: Berlin/Heidelberg, Germany, 2001.
4. Liu, X.; Zeng, Y.M. Analytic and numerical stability of delay differential equations with variable impulses. *Appl. Math. Comput.* **2019**, *358*, 293–304. [CrossRef]
5. Pavlačková, M.; Taddei, V.A. Bounding function approach to impulsive Dirichlet problem with an upper-Carathéodory right-hand side. *Electron. J. Diff. Eq.* **2019**, *12*, 1–18.
6. Pavlačková, M.; Taddei, V. On the impulsive Dirichlet problem for second-order differential inclusions. *J. Qual. Theory Differ. Equ.* **2020**, *13*, 1–22. [CrossRef]
7. Santra, S.S.; Ghosh, A.; Bazighifan, O.; Khedher, K.M.; Nofal, T.A. Second-order impulsive differential systems with mixed and several delays. *Adv. Differ. Equ.* **2021**, *2021*, 318. [CrossRef]
8. Wang, Z.; Cao, J.; Cai, Z. Finite-time stability of impulsive differential inclusion: Applications to discontinuous impulsive neural networks. *Discret. Contin. Dyn. Syst. B* **2021**, *26*, 2677–2692. [CrossRef]
9. Ahmed, N.U. Optimal impulsive control for impulsive systems in Banach spaces. *Int. J. Differ. Equ. Appl.* **2000**, *1*, 37–52.
10. Ahmed, N.U.; Teo, K.L.; Hou, S.H. Nonlinear impulsive systems on infinite dimensional spaces. *Nonlinear Anal.* **2003**, *54*, 907–925. [CrossRef]
11. Fan, Z.; Li, G. Existence results for semilinear differential equations with nonlocal and impulsive conditions. *J. Funct. Anal.* **2010**, *258*, 1709–1727. [CrossRef]
12. Luo, Y. Existence results for impulsive semilinear differential inclusions with nonlinear boundary conditions. *Bound. Value Probl.* **2018**, *2018*, 165. [CrossRef]
13. Sviridyuk, G.A.; Fedorov V.E. *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*; VSP: Utrecht, The Netherlands, 2003.
14. Sidorov, N.; Loginov, B.; Sinitsyn, A.; Falaleev, M. *Lyapunov-Schmidt Methods in Nonlinear Analysis and Applications*; Mathematics and Its Applications, No. 550; Springer Science+Business Media: Dordrecht, The Netherlands, 2002.
15. Tolstonogov, A.A. A Multivalued History-Dependent Operator and Implicit Evolution Inclusions I. *Sib. Math. J.* **2021**, *62*, 545–553. [CrossRef]
16. Tolstonogov, A.A. A Multivalued History-Dependent Operator and Implicit Evolution Inclusions II. *Sib. Math. J.* **2021**, *62*, 747–762. [CrossRef]
17. Benchohra, M.; Rezoug, N.; Samet, B.; Zhou, Y. Second Order Semilinear Volterra-Type Integro-Differential Equations with Non-Instantaneous Impulses. *Mathematics* **2019**, *7*, 1134. [CrossRef]
18. Cernea A. A note on the solutions of a second-order evolution in non separable Banach spaces. *Comment. Math. Univ. Carol.* **2017**, *58*, 307–314.
19. Henriquez, H.R.; Pierri, M.; Rolnik, V. Pseudo S-asymptotically periodic solutions of second-order abstract Cauchy problems. *Appl. Math. Comput.* **2016**, *274*, 590–603. [CrossRef]
20. Vijayakumar, V.; Sivasankaran, S.; Mani Mallika, A. Existence of global solutions for second order impulsive abstract functional integrodifferential equations. *Dyn. Contin. Discret. Impuls. Syst. Ser. A Math. Anal.* **2011**, *18*, 747–766.
21. Xiao, J.-Z.; Zhu, X.-H.; Cheng, R. The solution sets for second order semilinear impulsive multivalued boundary value problems. *Comput. Math. Appl.* **2012**, *64*, 147–160. [CrossRef]
22. Andres, J.; Malaguti, L.; Pavlačková, M. A Scorza-Dragoni approach to second-order boundary value problems in abstract spaces. *Appl. Math. Inf. Sci.* **2012**, *6*, 29–44.
23. Andres, J.; Malaguti, L.; Pavlačková, M. On second-order boundary value problems in Banach spaces: A bound sets approach. *Topol. Meth. Nonlinear Anal.* **2011**, *37*, 303–341.
24. Byszewski, L.; Winiarska, T. An abstract nonlocal second order evolution problem. *Opuscula Math.* **2012**, *32*, 75–82. [CrossRef]
25. Tidke, H.L.; Dhakne, M.B. Global existence of mild solutions of second order nonlinear Volterra integrodifferential equations in Banach spaces. *Differ. Equ. Dyn. Syst.* **2009**, *17*, 331–342. [CrossRef]
26. Hao, X.; Liu, L. Mild Solution of Second-Order Impulsive Integro-Differential Evolution Equations of Volterra Type in Banach Spaces. *Qual. Theory Dyn. Syst.* **2020**, *19*, 18. [CrossRef]
27. Hernández, E. A second-order impulsive Cauchy problem. *Int. J. Math. Math. Sci.* **2002**, *31*, 451–461.
28. Peng, Y.; Xiang, X. Second order nonlinear impulsive time-variant systems with unbounded perturbation and optimal controls. *J. Ind. Manag. Optim.* **2008**, *4*, 17–32. [CrossRef]
29. Winiarska, T. Extrapolation Banach spaces and abstract semilinear second order differential equations. *Int. J. Differ. Equ. Appl.* **2002**, *6*, 449–465.
30. Benchohra, M.; Gatsori, E.; Ntouyas, S.K. Existence results for functional semilinear damped integrodifferential equations. *Libertas Math.* **2006**, *26*, 97–108.
31. Benchohra, M.; Henderson, J.; Ntouyas, S.K.; Ouahab, A. Existence results for impulsive semilinear damped differential inclusions. *Electron. J. Qual. Theory Differ. Equ.* **2003**, *11*, 19. [CrossRef]

32. Benedetti, I.; Loi, N.V.; Malaguti, L.; Obukhovskii, V. An approximation solvability method for nonlocal differential problems in Hilbert spaces. *Commun. Contemp. Math.* **2017**, *19*, 1650002. [CrossRef]
33. Benedetti, I.; Loi, N.V.; Taddei, V. An approximation solvability method for nonlocal semilinear differential problems in Banach spaces. *Discr. Cont. Dyn. Syst. Ser. A* **2017**, *37*, 2977–2998. [CrossRef]
34. Benedetti, I.; Loi, N.V.; Malaguti, L.; Taddei, V. Nonlocal diffusion second order partial differential equations. *J. Diff. Equ.* **2017**, *262*, 1499–1523. [CrossRef]
35. Benedetti, I.; Malaguti, L.; Taddei, V. Semilinear evolution equations in abstract spaces and applications. *Rend. Ist. Mat. Univ. Trieste* **2012**, *44*, 371–388.
36. Vijayakumar, V.; Murugesu, R. Controllability for a class of second-order evolution differential inclusions without compactness. *Appl. Anal.* **2019**, *98*, 1367–1385. [CrossRef]
37. Singer, I. *Bases in Banach Spaces I*; Springer: Berlin/Heildelberg, Germany; New York, NY, USA, 1970.
38. Lindenstrauss, J.; Tzafriri, L. *Classical Banach Spaces I: Sequence Spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete*; No. 92; Springer: Berlin, Germany; New York, NY, USA, 1977.
39. Enflo, P. A counterexample to the approximation problem in Banach spaces. *Acta Math.* **1973**, *130*, 309–317. [CrossRef]
40. Johnson, W.B.; Lindenstrauss, J. *Handbook of the Geometry of Banach Spaces Vol. I*; North-Holland Publishing Co.: Amsterdam, The Netherlands, 2001.
41. Malaguti, L.; Perrotta, S.; Taddei, V. L^p exact controllability of partial differential equations with nonlocal terms. *Evol. Equ. Control Theory* **2021**, 1–32. *in press*.
42. Travis, C.C.; Webb, G.F. Cosine families and abstract nonlinear second order differential equations. *Acta Math. Acad. Sci. Hung.* **1978**, *32*, 75–96. [CrossRef]
43. Kisyński, J. On cosine operator functions and one parameter group of operators. *Studia Math.* **1972**, *49*, 93–105. [CrossRef]
44. Sova, M. Cosine operator functions. *Rozpr. Mat*. **1966**, *49*, 1–47.
45. Kamenskii, M.; Obukhovskii, V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*; de Gruyter: Berlin, Germany, 2001.
46. Kozak, M. A fundamental solution of a second-order differential equation in Banach space. *Univ. Iagel. Acta Math.* **1995**, *32*, 275–289.
47. Diestel, J.; Uhl, J.J., Jr. *Vector Measures*; Math. Surveys, No. 15; AMS: Providence, RI, USA, 1977.
48. Fattorini, H.O. *Second Order Linear Differential Equations in Banach Spaces*; North-Holland: New York, NY, USA, 1985.
49. Kakutani, S.A. Generalization of Brouwer's Fixed Point Theorem. *Duke Math. J.* **1941**, *8*, 457–459. [CrossRef]
50. Dunford, N.; Schwartz, J.T. *Linear Operators*; A Wiley-Interscience Publication, John Wiley and Sons, Inc.: New York, NY, USA, 1988.
51. Cardinali, T.; Gentili, S. An existence theorem for a non-autonomous second order nonlocal multivalued problem. *Stud. Univ. Babeş-Bolyai Math.* **2017**, *62*, 101–117. [CrossRef]
52. Arthi G.; Balachandran, K. Controllability of damped second-order impulsive neutral functional differential systems with infinite delay. *J. Optim. Theory Appl.* **2012**, *152*, 799–813. [CrossRef]
53. Hernandez E.; Balachandran, K.; Annapoorani N. Existence results for a damped second order abstract functional differential equation with impulses. *Math. Comput. Model.* **2009**, *50*, 1583–1594. [CrossRef]
54. Mallika Arjunan, M.; Nadaf, N.Y. Existence and controllability results for damped second order impulsive functional differential systems with state-dependent delay. *Opusc. Math.* **2014**, *34*, 503–522. [CrossRef]
55. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Applied Mathematical Sciences, No. 44; Springer: New York, NY, USA, 1983.
56. Henriquez, H.R.; Poblete, V.; Pozo, J.C. Mild solutions of nonautonomous second order problems with nonlocal initial conditions. *J. Math. Anal. Appl.* **2014**, *412*, 1064–1083. [CrossRef]