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Hardy Inequalities and Interrelations of Fractional Triebel–Lizorkin Spaces in a Bounded Uniform Domain

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Abstract: The interrelations of Triebel–Lizorkin spaces on smooth domains of Euclidean space $\mathbb{R}^n$ are well-established, whereas only partial results are known for the non-smooth domains. In this paper, Ω is a non-smooth domain of $\mathbb{R}^n$ that is bounded and uniform. Suppose $p, q \in [1, \infty)$ and $s \in (n(\frac{1}{p} - \frac{1}{q})_+, 1)$ with $n(\frac{1}{p} - \frac{1}{q})_+ := \max\{n(\frac{1}{p} - \frac{1}{q}), 0\}$. The authors show that three typical types of fractional Triebel–Lizorkin spaces, on Ω: $F^s_{p,q}(\Omega)$, $\mathring{F}^s_{p,q}(\Omega)$ and $\widetilde{F}^s_{p,q}(\Omega)$, defined via the restriction, completion and supporting conditions, respectively, are identical if Ω is E-thick and supports some Hardy inequalities. Moreover, the authors show the condition that Ω is E-thick can be removed when considering only the density property $F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega)$, and the condition that Ω supports Hardy inequalities can be characterized by some Triebel–Lizorkin capacities in the special case of $1 \le p \le q < \infty$.

Keywords: Triebel–Lizorkin space; Hardy inequality; uniform domain; fractional Laplacian

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1. Introduction

The Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ on the Euclidean space $\mathbb{R}^n$, with parameters $s \in \mathbb{R}$ and $p, q \in (0, \infty]$, were introduced in 1970s (see [1–3]). They provide a unified treatment of various kinds of classical concrete function spaces, such as Sobolev spaces, Hölder-Zygmund spaces, Bessel-potential spaces, Hardy spaces and BMO spaces. Nowadays, the theory of $F^s_{p,q}(\mathbb{R}^n)$ is well-established in the literature as has numerous applications (see [4–10] and their references).

When trying to extend the theory of Triebel–Lizorkin space from $\mathbb{R}^n$ to a domain Ω of $\mathbb{R}^n$, one usually meets the fundamental problem of identifying the interrelations among a number of related spaces that are defined from distinct perspectives. In particular, there are three typical ways of defining Triebel–Lizorkin spaces on Ω (see, e.g., [10]). To be precise, let $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ be the collection of all infinitely differentiable functions in $\mathbb{R}^n$ with compact supports in Ω and $\mathcal{D}'(\Omega)$ the dual space of $\mathcal{D}(\Omega)$. For any $s \in \mathbb{R}$ and $p, q \in (0, \infty]$, recall that

(I) $F^s_{p,q}(\Omega) := \{f \in \mathcal{D}'(\Omega) : \text{there is a } g \in F^s_{p,q}(\mathbb{R}^n) \text{ with } g|_\Omega = f\}$ being the *restriction Triebel–Lizorkin space* endowed with the quasi-norm

$$\|f\|_{F^s_{p,q}(\Omega)} := \inf \|g\|_{F^s_{p,q}(\mathbb{R}^n)}, \tag{1}$$

where the infimum is taken over all $g \in F^s_{p,q}(\mathbb{R}^n)$ satisfying $g|_\Omega = f$. Here, for any $g \in \mathcal{S}'(\mathbb{R}^n)$, $g|_\Omega$ is the *restriction* of g to Ω, defined as a distribution in Ω such that for any $\varphi \in \mathcal{D}(\Omega)$,

$$(g|_\Omega)(\varphi) := g(\varphi);$$

(II) $\mathring{F}^s_{p,q}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{F^s_{p,q}(\Omega)}}$ is the *completion Triebel–Lizorkin space* that is defined as the completion of $\mathcal{D}(\Omega)$ in $F^s_{p,q}(\Omega)$ with respect to the quasi-norm $\|\cdot\|_{F^s_{p,q}(\Omega)}$, as in (1);

(III) $\widetilde{F}^s_{p,q}(\Omega) := \{f \in \mathcal{D}'(\Omega) : \text{there is a } g \in F^s_{p,q}(\mathbb{R}^n) \text{ with } g|_\Omega = f \text{ and } \operatorname{supp} g \subset \overline{\Omega}\}$ being the *supporting Triebel–Lizorkin space* endowed with the quasi-norm

$$\|f\|_{\widetilde{F}^s_{p,q}(\Omega)} = \inf \|g\|_{F^s_{p,q}(\mathbb{R}^n)},$$

where the infimum is taken over all $g \in F^s_{p,q}(\mathbb{R}^n)$ satisfying $g|_\Omega = f$ and $\operatorname{supp} g \subset \overline{\Omega}$.

Note that if $\Omega = \mathbb{R}^n$ is the Euclidean space, it follows easily from their definitions and the density property of $F^s_{p,q}(\mathbb{R}^n)$ that the aforementioned three kinds of Triebel–Lizorkin spaces are identical (see, e.g., [4]). However, if $\Omega \neq \mathbb{R}^n$, the situation becomes much more complex, since in this case the above density property and many other important properties, including the availability of restriction, trace and extension operators may fail (see, e.g., [6,8]). Indeed, it turns out that the interrelations of the aforementioned three kinds of Triebel–Lizorkin spaces depend heavily on the geometry of domain Ω and parameters s, p and q. Let us review some of the known results on this subject.

If Ω is a bounded C^∞-domain, it is known that the following results are almost sharp (see ([8], Chapter 5)).

(A) $F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega)$, if and only if, one of the following two conditions is satisfied:

(a1) $0 < p < \infty$, $-\infty < s < \frac{1}{p}$ and $0 < q < \infty$;

(a2) $1 < p < \infty$, $s = \frac{1}{p}$ and $0 < q < \infty$.

(B) $\mathring{F}^s_{p,q}(\Omega) = \widetilde{F}^s_{p,q}(\Omega)$, if $0 < p < \infty$, $0 < q < \infty$, $s > \sigma_p := n(\frac{1}{p} - 1)_+$ and $s - \frac{1}{p} \notin \mathbb{Z}_+$.

(C) $F^s_{p,q}(\Omega) = \widetilde{F}^s_{p,q}(\Omega)$, if $0 < p \leq \infty$, $0 < q \leq \infty$ and $\max\{\frac{1}{p} - 1, n(\frac{1}{p} - 1)\} < s < \frac{1}{p}$.

A combination of **(A)**, **(B)** and **(C)** immediately implies the following identities.

$$F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = \widetilde{F}^s_{p,q}(\Omega), \tag{2}$$

if $0 < p < \infty$, $0 < q < \infty$ and $\max\{\frac{1}{p} - 1, n(\frac{1}{p} - 1)\} < s < \frac{1}{p}$.

Note the restriction that $s < \frac{1}{p}$ in the above identities can be relaxed if Ω supports some Hardy inequalities. In particular, it is known that

$$\widetilde{F}^s_{p,q}(\Omega) = F^s_{p,q}(\Omega) \cap L^p(\Omega, d(\cdot, \partial\Omega)^{-s}), \tag{3}$$

if $0 < p < \infty$, $0 < q < \infty$ and

$$s > \sigma_{p,q} := n\left(\frac{1}{\min\{p,q\}} - 1\right)_+,$$

where for any $x \in \Omega$, $d(x, \partial\Omega)$ denotes the distance from x to the boundary $\partial\Omega$ of Ω and

$$L^p(\Omega, d(\cdot, \partial\Omega)^{-s}) := \left\{f : \|f\|_{L^p(\Omega, d(\cdot,\partial\Omega)^{-s})} = \left(\int_\Omega \frac{|f(x)|^p}{d(x, \partial\Omega)^{sp}}\, dx\right)^{1/p} < \infty\right\}$$

denotes the weighted Lebesgue space on Ω. The identity (3) together with **(A)** and **(B)** shows that if Ω supports the Hardy condition $F^s_{p,q}(\Omega) \subset L^p(\Omega, d(\cdot, \partial\Omega)^{-s})$, then identities (2) hold for all

$$1 < p < \infty, 1 \leq q < \infty \quad \text{and} \quad 0 < s < \infty. \tag{4}$$

Recall that on the smooth domain, the Hardy inequalities

$$\|f\|_{L^p(\Omega, d(\cdot,\partial\Omega)^{-s})} \leq C\|f\|_{\widetilde{F}^s_{p,q}(\Omega)}$$

hold for any $f \in \widetilde{F}^s_{p,q}(\Omega)$ with $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > \sigma_p$ with σ_p as in **(B)**.

If Ω is a non-smooth domain, there is no comprehensive treatment compared with what is available for smooth domains. Moreover, in the former case we meet much more complicated situations influenced by the geometry of Ω. Let us mention some of the related results.

(i) If $\Omega \subset \mathbb{R}^n$ is a bounded domain such that its boundary $\partial\Omega$ is porous and has upper Minkowski dimension $D \in (0, n]$, Caetano ([11], Proposition 2.5) proved the following identity.

(A′) $F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega)$, if $0 < p < \infty$, $0 < q < \infty$ and $-\infty < s < (n - D)/p$.

Note that for an arbitrary bounded domain Ω, it holds that $D \in [n-1, n]$, and if $D = n - 1$, then the range of s in **(A′)** equal to that in (a1).

(ii) If $\Omega \subset \mathbb{R}^n$ is a domain whose closure $\overline{\Omega}$ is a n-set, and $\partial\Omega$ is a d-set with $n - 1 < d < n$, Ihnatsyeva et al. ([12], Theorem 4.3) obtained the following inclusion.

(B′) $\mathring{F}^s_{p,q}(\Omega) \subset \widetilde{F}^s_{p,q}(\Omega)$, if $1 < p < \infty$, $1 \le q < \infty$ and $(n - d)/p < s < \infty$.

Note that if $\partial\Omega$ is a d-set with $d < n$, then $\partial\Omega$ is porous (see ([10], Chapter 3)) and has upper Minkowski dimension d (see ([7], Chapter 1)).

(iii) If $\Omega \subset \mathbb{R}^n$ is an arbitrary domain, Triebel ([10], Chapter 2) proved the following identity.

(C′) $F^0_{p,2}(\Omega) = \widetilde{F}^0_{p,2}(\Omega)$, if $1 < p < \infty$.

Moreover, if Ω is a bounded Lipschitz domain, then it is proved in ([9], Proposition 3.1) that identity (2) holds true for all

$$0 < p < \infty,\ \min\{p, 1\} < q < \infty \ \text{ and } \ \max\left\{\frac{1}{p} - 1, n\left(\frac{1}{p} - 1\right)\right\} < s < \frac{1}{p}. \tag{5}$$

Motivated by the aforementioned results, it is natural to ask the following.

Main question: Let Ω be a bounded non-smooth domain. Is it possible to extend identity (2) for parameters from (5) to the general fractional case $s \in (0, 1)$?

In this paper, we give an affirmative answer to the above question in the setting that Ω is a bounded uniform domain, which contains a bounded Lipschitz domain as a special case. Recall that a domain $\Omega \subset \mathbb{R}^n$ is called a *uniform domain* (see [13,14]), if there exist constants c_1 and $c_2 > 0$ such that each pair of points $x, y \in \Omega$ can be connected by a rectifiable curve $\Gamma \subset \Omega$ for which

$$\begin{cases} L(\Gamma) \le c_1|x - y|, \\ \min\{|x - z|, |y - z|\} \le c_2 d(z, \partial\Omega), \quad \text{for any } z \in \Gamma, \end{cases}$$

where $L(\Gamma)$ denotes the length of Γ.

A closely related notion of uniform domain is the so-called E-thick domain. Recall in [10] that a domain $\Omega \subset \mathbb{R}^n$ is said to be *E-thick*, if there exists $j_0 \in \mathbb{N}$ such that for any interior cube $Q^i \subset Q$ satisfying

$$l(Q^i) \sim 2^{-j} \quad \text{and} \quad d(Q^i, \partial\Omega) \sim 2^{-j} \qquad \text{for some } j \ge j_0 \in \mathbb{N},$$

one finds a complementary exterior cube $Q^e \subset \Omega^c = \mathbb{R}^n \setminus \Omega$ satisfying

$$l(Q^e) \sim 2^{-j} \quad \text{and} \quad d(Q^e, \partial\Omega) \sim d(Q^i, Q^e) \sim 2^{-j},$$

where the implicit constants are independent of Q^i, Q^e and j. It is known that any bounded Lipschitz domain is E-thick and uniform; and if a domain Ω is uniform, then $\overline{\Omega}^c$ is E-thick. Moreover, there exists domain in $\mathbb{R}^n$ that is E-thick but not uniform (see ([10], Remark 3.7)). Note that if Ω is E-thick, then $\partial\Omega$ is a d-set with $d \in [n-1, n)$ (see ([10], Proposition 3.18)).

We also need the following Hardy condition.

$(H)_{s,p,q}$*-condition*. Let $1 \le p,\ q < \infty$, $s \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be a domain satisfying $\Omega \neq \mathbb{R}^n$. Ω is said to satisfy the $(H)_{s,p,q}$*-condition* if

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{d(x,\partial\Omega)^s} \right|^p dx < \infty$$

holds for all $f \in F^s_{p,q}(\Omega)$ as in **(I)**.

The main result of the paper is as follows.

Theorem 1. *Let $p,\ q \in [1,\infty)$ and $s \in (n(\frac{1}{p} - \frac{1}{q})_+, 1)$. Assume that Ω is a bounded E-thick uniform domain satisfying the $(H)_{s,p,q}$-condition. Then it holds that*

$$F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = \tilde{F}^s_{p,q}(\Omega) \tag{6}$$

with equivalent norms.

We make some remarks on Theorem 1.

Remark 1. *(i) Theorem 1 gives an affirmative answer to the main question. It extends by necessity the identities (2) for parameter s from the range $s \in (\max\{\frac{1}{p} - 1, n(\frac{1}{p} - 1)\}, \frac{1}{p})$ as in (5) to $s \in (n(\frac{1}{p} - \frac{1}{q})_+, 1)$ and for domain Ω from bounded Lipschitz to bounded uniform, E-thick and supporting the $(H)_{s,p,q}$-condition. Moreover, in the proof of Theorem 1, we establish the following two identities:*

(A″) *$F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega)$, if $1 \le p,\ q < \infty$, $n(\frac{1}{p} - \frac{1}{q})_+ < s < 1$ and Ω is bounded uniform;*
(C″) *$F^s_{p,q}(\Omega) = \widetilde{F}^s_{p,q}(\Omega)$, if $1 \le p,\ q < \infty$, $n(\frac{1}{p} - \frac{1}{q})_+ < s < 1$ and Ω is bounded E-thick,*

which extends by necessity the corresponding identities **(A′)** *and* **(C′)***.*

(ii) As in the Sobolev case (see, e.g., [15,16]), the proof of Theorem 1 relies on an intrinsic norm characterization of the restriction space $F^s_{p,q}(\Omega)$ as in **(I)***. This characterization is established in [17] under the condition $s \in (n(\frac{1}{p} - \frac{1}{q})_+, 1)$, which is shown to be sharp therein. It seems a new method is needed if one considers the case $s \le n(\frac{1}{p} - \frac{1}{q})_+$; see Proposition 1, where a density property is established for a variant of Triebel–Lizorkin space in the full range $s \in (0,1)$. Note that if $1 \le q \le p < \infty$, then $n(\frac{1}{p} - \frac{1}{q})_+ = 0$. In this case, Theorem 1 gives identities (2) for the full range $s \in (0,1)$. We also point out that it is possible to consider the case $s \ge 1$ by using higher order difference. We do not pursue this in the present paper.*

We point out that the most technical part of the proof of Theorem 1 is to prove the first identity

$$F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega), \tag{7}$$

which is also called the density property of $F^s_{p,q}(\Omega)$ and has close relations with other properties, such as zero trace characterization and regularity of the Dirichlet energy integral minimizer (see [18]). As far as we know, if Ω is a non-smooth domain, this density property is only known for some Sobolev spaces, or the case when s is small (see [9,11,15,16,19]). In this paper, we show that the density property (7) holds for bounded uniform domains without the assumption of E-thickness. More precisely, the following result is true.

Theorem 2. *Let $p,\ q \in [1,\infty)$ and $s \in \left(n(\frac{1}{p} - \frac{1}{q})_+, 1\right)$. Assume Ω is a bounded uniform domain satisfying the $(H)_{s,p,q}$-condition. Then the density property (7) holds.*

A few remarks on Theorem 2 are in order.

Remark 2. *(i) Theorem 2 extends by necessity the corresponding density property of $F^s_{p,q}(\Omega)$ by relaxing the restriction $s < (n-D)/p$ as in* **(A′)**. *In particular, if $1 \le p = q < \infty$ and $s \in (0,1)$, since in this case $F^s_{p,p} = W^{s,p}$ becomes the fractional Sobolev space, Theorem 2 implies the the following zero trace characterization of fractional Sobolev space: for any $p \in [1,\infty)$ and $s \in (0,1)$, if Ω is a bounded uniform domain supporting the $(H)_{s,p,p}$-condition, then*

$$W^{s,p}(\Omega) = \mathring{W}^{s,p}(\Omega).$$

Recall that the corresponding characterization at the endpoint case $s = 1$ is a well-known result (see, e.g., [15,16]; see also [19] for a very recent result on the fractional case reached using a different method).

(ii) The proofs of Theorems 1 and 2 are based on a localization technique of Whitney decomposition (see Section 2 below). Since this technique has been extended to the more general setting of volume doubling metric measure space (see, e.g., [20]), it is straightforward to establish our results to this setting, once the corresponding intrinsic norm characterization of the restriction space $F^s_{p,q}(\Omega)$ is established.

Finally, we present further discussion on the Hardy $(H)_{s,p,q}$-condition appearing in Theorems 1 and 2. As announced earlier, we prove Theorems 1 and 2 by using a localization technique of Whitney decomposition, together with a smooth partition of unity. This allows us to decompose each $f \in F^s_{p,q}(\Omega)$ into two parts: the interior part v_ε and boundary part w_ε. It is the estimates of the latter part that need the Hardy $(H)_{s,p,q}$-condition. Note that the $(H)_{s,p,q}$-condition is satisfied once we prove the following Hardy's inequality:

$$\left\| \frac{f}{d(\cdot,\partial\Omega)^s} \right\|_{L^p(\Omega)} \lesssim \|f\|_{F^s_{p,q}(\Omega)}, \tag{8}$$

for any $f \in C_c(\Omega)$. Unfortunately, it is known that (8) may not hold in the uniform domains (see [21]). Thus, a characterization of (8) in this setting is necessary. In this paper, we establish a characterization of (8) in terms of capacities, under the additional condition $1 \le q \le p < \infty$. To be precise, for any $1 \le q \le p < \infty$ and $s \in (0,1)$, let Ω be a uniform domain on $\mathbb{R}^n$ and $K \subset \Omega$ be its compact subset. Define the *capacity* $\mathrm{cap}_{s,p,q}(K,\Omega)$ *of* K by setting

$$\mathrm{cap}_{s,p,q}(K,\Omega) := \inf |f|^p_{\mathcal{F}^s_{p,q}(\Omega)}, \tag{9}$$

where the infimum is taken over all real-valued functions $f \in C_c(\Omega)$ such that $f \ge 1$ on K and

$$|f|_{\mathcal{F}^s_{p,q}(\Omega)} := \left[\int_\Omega \left(\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\, dy \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}. \tag{10}$$

The following result gives the capacity characterization of (8) in the setting of a uniform domain.

Theorem 3. *Let $1 \le q \le p < \infty$ and $s \in (0,1)$. Assume that Ω is a uniform domain. The following are equiavalent.*

(i) *There is a constant $C_1 > 0$ such that*

$$\left\| \frac{f}{d(\cdot,\partial\Omega)^s} \right\|_{L^p(\Omega)} \le C_1 |f|_{\mathcal{F}^s_{p,q}(\Omega)},$$

for any $f \in C_c(\Omega)$.

(ii) *There is a constant $C_2 > 0$ such that*

$$\int_K \frac{1}{d(x,\partial\Omega)^{sp}}\,dx \le C_2 \mathrm{cap}_{s,p,q}(K,\Omega), \qquad (\mathrm{Cap})_{s,p,q}$$

for every compact $K \subset \Omega$.

Based on Theorems 1–3, we immediately obtain the following corollary.

Corollary 1. *Let $1 \le p \le q < \infty$ and $s \in (n(\frac{1}{p} - \frac{1}{q})_+, 1)$. Assume that Ω is a bounded uniform domain satisfying the capacity condition $(\mathrm{Cap})_{s,p,q}$. Then the following two assertions hold.*

(i) $F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega)$ *with equivalent norms.*
(ii) *If, in addition, Ω is E-thick, then $F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = \tilde{F}^s_{p,q}(\Omega)$ with equivalent norms.*

We now make some remarks on Theorem 3 and Corollary 1.

Remark 3.
(i) *Theorem 3 is the extension of the corresponding result in [22], where the authors considered the capacity characterization of Hardy's inequalities in the fractional order Sobolev space. Recall that if Ω is domain with $\partial\Omega$ being a d-set satisfying $n-1 < d < n$, then it is proved in [12] that Hardy's inequalities (8) hold for any $f \in C_c(\Omega)$ with $p \in [1,\infty)$, $q \in [1,\infty]$ and $s > (\frac{n-d}{p}, 1)$. Note that the proof of [12] uses the technique of restriction-extension, whereas the proof of Theorem 3 depends only on the intrinsic norm characterization of $F^s_{p,q}(\Omega)$ defined as in (10).*
(ii) *The restriction $p \le q$ seems technical, which is needed in the proof of Theorem 3 in order to give a dual representation of the capacity in (9). Moreover, since the capacity condition $(\mathrm{Cap})_{s,p,q}$ is difficult to verify, it would be interesting to characterize it in terms of some geometric conditions, which is left for a further study.*

This paper is organized as follows. In Section 2, we collect some necessary technical properties of the Whitney decomposition of the domain Ω that are used out throughout this paper. Section 3.1 is devoted to the proof of Theorem 2. We prove Theorems 1 and 3 in Sections 3.2 and 3.3, respectively.

Notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any $s \in \mathbb{R}$, let $s_+ := \max\{s, 0\}$. For any subset $E \subset \mathbb{R}^n$, $\mathbf{1}_E$ denotes its *characteristic function*. We use C to denote a *positive constant* that is independent of the main parameters involved, whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. For any qualities f, g and h, if $f \le Cg$, we write $f \lesssim g$, and if $f \lesssim g \lesssim f$, we then write $f \sim g$. We also use the following convention: if $f \le Cg$ and $g = h$ or $g \le h$, we write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. Throughout this article, we denote $Q = Q(x, l)$ be the cube with center x and sidelength l whose side parallel to coordinate axes.

2. Preliminaries on Whitney Decomposition

In this section, we collect some basic properties of the Whitney decomposition of domain Ω, with emphasis on those Whitney cubes that are close to the boundary. These properties play an important role in the proofs of our main results. To begin with, we recall the classical form of Whitney decomposition from [23].

Lemma 1 ([23])**.** *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain. There exists a family of cubes $\{Q_j\}_{j=1}^\infty$ with sides parallel to the coordinate axes and satisfying*

(i) $Q_j^o \cap Q_k^o = \emptyset$, *if $j \ne k$, where Q_j^o denotes the interior of Q_j;*

(ii) *For any* $j \in \mathbb{N}$, $\operatorname{diam} Q_j \le d(Q_j, \partial\Omega) \le 4 \operatorname{diam} Q_j$, *where* $\operatorname{diam} Q_j$ *denotes the diameter of* Q_j*;*

(iii) $\Omega = \bigcup_{j=1}^{\infty} Q_j^*$, *where* $Q_j^* = (1+\mu)Q_j$ *is the concentric cube of* Q_j *with sidelength* $(1+\mu)l_j$ *and* $\mu \in [0, \frac{1}{4})$*;*

(iv) *Each* $x \in \Omega$ *is contained in at most* 12^n *cubes* Q_j^**;*

(v) *If* Q_i *and* Q_j *touch, namely,* $\overline{Q}_i \cap \overline{Q}_j \neq \emptyset$ *and* $Q_i^o \bigcap Q_j^o = \emptyset$*, then*

$$\frac{1}{4} \operatorname{diam} Q_i \le \operatorname{diam} Q_j \le 4 \operatorname{diam} Q_i.$$

Throughout this section, for any $\delta > 0$, let Ω_δ be the boundary layer in Ω with length δ defined by setting

$$\Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) < \delta\}. \tag{11}$$

Let $\varepsilon > 0$ and $\{Q_j\}_{j=1}^{\infty}$ be the Whitney decomposition of Ω as in Lemma 1. The following classes of index sets represent three subgroups of $\{Q_j\}_{j=1}^{\infty}$ that are closely related to the boundary layer in Ω.

$$\begin{aligned} &\Lambda_1 := \{j \in \mathbb{N} : d(Q_j, \partial\Omega) \ge \varepsilon\},\ \Lambda_2 := \{j \in \mathbb{N} : d(Q_j, \partial\Omega) < \varepsilon\} \quad \text{and} \\ &\Lambda_3 := \{j \in \mathbb{N} : Q_j \cap (\Omega \setminus \Omega_{14\varepsilon}) \neq \emptyset\} \end{aligned} \tag{12}$$

with $\Omega_{14\varepsilon}$ as in (11).

The following lemma says that a small dilation of the first subgroup $\{Q_j\}_{j\in\Lambda_1}$ of Whitney cubes is contained in the interior of Ω with a positive distance to the boundary $\partial\Omega$.

Lemma 2. *Let* $\varepsilon > 0$ *and* Λ_1 *be the index set as in* (12)*. For any* $j \in \Lambda_1$*, let* $Q_j^* := (1+\tilde{\mu})Q_j$ *be the concentric cube of* Q_j *with sidelength* $(1+\tilde{\mu})l_j$ *and* $\tilde{\mu} \in (0, \frac{1}{16})$*. Then it holds that*

$$\bigcup_{j\in\Lambda_1} Q_j^* \subseteq \left\{x \in \Omega : d(x, \partial\Omega) > \frac{3\tilde{\mu}\varepsilon}{8\sqrt{n}}\right\}. \tag{13}$$

Proof. For any $x \in \bigcup_{j\in\Lambda_1} Q_j^*$, there exists $j \in \Lambda_1$ such that $x \in Q_j^* \subseteq \Omega$. By Lemma 1(iii) and the assumption $0 < \tilde{\mu} < \frac{1}{16}$, we obtain $Q_j^* \subseteq (1+4\tilde{\mu})Q_j \subseteq \Omega$. This, together with Lemma 1(ii) and the definition of Λ_1, implies

$$d(x, \partial\Omega) \ge d\left(Q_j^*, \partial\Omega\right) \ge d\left(Q_j^*, (1+4\tilde{\mu})Q_j\right) = \frac{3}{2}\tilde{\mu} l_j \ge \frac{3\tilde{\mu}\varepsilon}{8\sqrt{n}},$$

which proves (13). □

Our next lemma shows that a small dilation of the second subgroup $\{Q_j\}_{j\in\Lambda_2}$ of Whitney cubes is contained in a boundary layer of Ω.

Lemma 3. *Let* $\varepsilon > 0$ *and* Λ_2 *be the index set as in* (12)*. For any* $j \in \Lambda_2$*, let* $Q_j^{**} := (1+2\tilde{\mu})Q_j$ *be the concentric cube of* Q_j *with sidelength* $(1+2\tilde{\mu})l_j$ *and* $\tilde{\mu} \in (0, \frac{1}{16})$*. Then it holds that*

$$\bigcup_{j\in\Lambda_2} Q_j^{**} \subseteq \Omega_{3\varepsilon} \tag{14}$$

with $\Omega_{3\varepsilon}$ *as in* (11).

Proof. Let $x \in \bigcup_{j\in\Lambda_2} Q_j^{**}$. By (12), Lemma 1(ii), the assumption $0 < \tilde{\mu} < \frac{1}{16}$ and the definition of Λ_2, we have

$$d(x, \partial\Omega) \leq d(Q_j, \partial\Omega) + (1+2\tilde{\mu})l_j\sqrt{n} \leq (2+2\tilde{\mu})d(Q_j, \partial\Omega) < 3\varepsilon,$$

which implies (14). □

The following lemma gives a few interesting properties of the third subgroup $\{Q_j\}_{j\in\Lambda_3}$ of Whitney cubes.

Lemma 4. *Let $\varepsilon > 0$ and Λ_3 be the index set as in (12). Then the following assertions hold.*

(i) $\Omega \setminus \Omega_{14\varepsilon} \subseteq \bigcup_{j\in\Lambda_3} Q_j^*$;

(ii) *For any $j \in \Lambda_3$, it holds $Q_j \cap \Omega_{7\varepsilon} = \varnothing$ and $d(Q_j, \partial\Omega) \geq 7\varepsilon$;*

(iii) *For any $k \in \Lambda_2$, let $Q_k^{**} := (1+2\tilde{\mu})Q_k$ be the concentric cube of Q_k with sidelength $(1+2\tilde{\mu})l_k$ and $\tilde{\mu} \in (0, \frac{1}{16})$. Then for any $j \in \Lambda_3$ and any $x \in Q_k^{**}$, and $y \in Q_j$, it holds that*

$$|x-y| \sim D(Q_j, Q_k), \tag{15}$$

where $D(Q_j, Q_k) := d(Q_j, Q_k) + l_j + l_k$ and the implicit constants are independent of ε, j, k, x and y.

Proof. The assertion (i) follows immediately from the definition of the index set Λ_3. To prove (ii), we first show $Q_j \cap \Omega_{7\varepsilon} = \varnothing$ for any $j \in \Lambda_3$. If not, namely, $Q_j \cap \Omega_{7\varepsilon} \neq \varnothing$, then by Lemma 1(ii), we have

$$\operatorname{diam} Q_j \leq d(Q_j, \partial\Omega) < 7\varepsilon.$$

This implies $Q_j \cap (\Omega \setminus \Omega_{14\varepsilon}) = \varnothing$, which contradicts the definition of Λ_3. Thus, for any $j \in \Lambda_3$, $Q_j \cap \Omega_{7\varepsilon} = \varnothing$, namely, $d(Q_j, \partial\Omega) \geq 7\varepsilon$, which implies (ii).

We now prove (iii). For any $k \in \Lambda_2$, by Lemma 3, we have $Q_k^{**} \subseteq \Omega_{3\varepsilon}$. Let

$$\Gamma_{3\varepsilon} := \{x \in \Omega : \ d(x, \partial\Omega) = 3\varepsilon\}.$$

From (ii), it follows that for each $j \in \Lambda_3$, it holds that $Q_j \cap \Gamma_{3\varepsilon} = \emptyset$ and

$$d(Q_j, Q_k^{**}) \geq 4\varepsilon.$$

Now let $x_j \in \overline{Q}_j$ and $x_k \in \overline{Q}_k^{**}$ such that

$$d(Q_j, Q_k^{**}) = d(x_j, x_k).$$

Let $x_{\tilde{k}}$ be the intersection point of the segment $\overline{x_j x_k}$ and $\Gamma_{3\varepsilon}$. Denote by $Q_{\tilde{k}}$ the Whitney cube that contains $x_{\tilde{k}}$. It is easy to see that

$$d(Q_j, Q_k^{**}) > d(Q_j, Q_{\tilde{k}}). \tag{16}$$

By the definitions of Λ_1, Λ_2 and Lemma 1(iii), it is clear that $\tilde{k} \in \Lambda_1$. This, together with Lemma 1(iii) implies that

$$\frac{\varepsilon}{4} \leq \operatorname{diam} Q_{\tilde{k}} \leq 3\varepsilon. \tag{17}$$

Moreover, since $Q_{\tilde{k}} \cap \Omega_{3\varepsilon} \neq \emptyset$, by Lemma 1(ii) again, it follows that $Q_{\tilde{k}} \subseteq \Omega_{6\varepsilon}$; from (ii), it follows that $Q_j \cap \Omega_{7\varepsilon} = \emptyset$. This means that Q_j and $Q_{\tilde{k}}$ are not touched, and by Lemma 1(v), it holds that

$$d(Q_j, Q_{\tilde{k}}) \geq \frac{1}{4} l_j. \tag{18}$$

Thus, for any $x \in Q_k^{**}$ and $y \in Q_j$, we have

$$\begin{aligned} |x-y| &\leq \operatorname{diam}(Q_k^{**}) + d(Q_j, Q_k) + \operatorname{diam} Q_j \\ &\lesssim l_k + d(Q_j, Q_k) + l_j \sim D(Q_j, Q_k). \end{aligned} \tag{19}$$

On the other hand, by $l_k \leq \varepsilon \lesssim l_j$, (16) and (18), it follows that

$$|x-y| \geq d(Q_j, Q_k^{**}) \geq d(Q_j, Q_{\tilde{k}}) \geq \frac{1}{4} l_j \gtrsim l_j + l_k$$

and by (17), we know that

$$d(Q_j, Q_k) \leq d(Q_j, Q_k^{**}) + \operatorname{diam}(Q_k^{**}) \lesssim |x-y|. \tag{20}$$

By combing (19) and (20), we obtain (iii), which completes the proof of Lemma 4. □

The following lemma on the summation of D as in (15) needs the assumption that Ω is bounded and uniform.

Lemma 5 ([17]). *Let Ω be a bounded uniform domain and $\{Q_j\}_{j=1}^{\infty}$ be the Whitney decomposition of Ω as in Lemma 1. Then there exists a positive constant C such that for any $\eta > 0$ and $j_0 \in \mathbb{N}$, it holds that*

$$\sum_{j=1}^{\infty} \frac{l(Q_j)^n}{D(Q_j, Q_{j_0})^{n+\eta}} \leq \frac{C}{l(Q_{j_0})^{\eta}}$$

We end this section by giving properties of two subgroups of Whitney cubes from Λ_2 as in (12). To this end, for any $i \in \Lambda_2$, we make a subdivision of Λ_2 by setting

$$\Lambda_{21}(i) := \{k \in \Lambda_2 : Q_k^{**} \cap Q_i^* \neq \emptyset\} \quad \text{and} \quad \Lambda_{22}(i) := \{k \in \Lambda_2 : Q_k^{**} \cap Q_i^* = \emptyset\}, \tag{21}$$

where $Q_i^* = (1+\tilde{\mu})Q_i$ and $Q_k^{**} = (1+2\tilde{\mu})Q_k$ with $\tilde{\mu} \in (0, 1/16)$. For any $i \in \Lambda_2$ and $k \in \Lambda_{21}(i)$, let

$$\Lambda_{23}(i,k) := \left\{ j \in \Lambda_2 : Q_j^* \cap Q_i^* \neq \emptyset \text{ or } Q_j^* \cap Q_k^{**} \neq \emptyset \right\}. \tag{22}$$

Lemma 6. *Let Ω be a bounded domain, $\varepsilon > 0$ and Λ_2 be as in (12). Then the following two assertions hold.*

(i) For any $i \in \Lambda_2$, let $\Lambda_{21}(i)$ be the index set as in (21). Then it holds that for any $x \in Q_i^*$,

$$\bigcup_{k \in \Lambda_{21}(i)} Q_k^{**} \subseteq B(x, 7\varepsilon),$$

where $Q_k^{**} = (1+2\tilde{\mu})Q_k$ with $\tilde{\mu} \in (0, 1/(16\sqrt{n}))$;

(ii) For any $i \in \Lambda_2$ and $k \in \Lambda_{21}(i)$, let $\Lambda_{23}(i,k)$ be the index set as in (22). It holds that there exists a number $N \in \mathbb{N}$, independs of i and k, such that

$$\operatorname{Card}(\Lambda_{23}(i,k)) \leq N. \tag{23}$$

Moreover, for any $j \in \Lambda_{23}(i,k)$, the sidelengths l_j and l_i of Q_j and Q_i are comparable, namely,

$$l_i \sim l_j \tag{24}$$

with implicit constants are independent on i and j.

Proof. We first prove (i). For any $x \in Q_i^*$ and $y \in \bigcup_{k\in\Lambda_{21}(i)} Q_k^{**}$, there exists $k \in \Lambda_{21}(i)$ such that $y \in Q_k^{**}$ and

$$|x-y| \le (1+\tilde{\mu})\sqrt{n}l_i + (1+2\tilde{\mu})\sqrt{n}l_k.$$

By Lemma 3, it holds that $d(x,\partial\Omega) < 3\varepsilon$ and $d(y,\partial\Omega) < 3\varepsilon$, which combined with Lemma 1(ii) show that $l_i, l_k < \frac{3\varepsilon}{\sqrt{n}}$. Thus, using the assumption $0 < \tilde{\mu} < \frac{1}{16\sqrt{n}}$, we know

$$|x-y| \le 6(1+2\tilde{\mu})\varepsilon \le 7\varepsilon.$$

This implies $\bigcup_{k\in\Lambda_{21}(i)} Q_k^{**} \subseteq B(x,7\varepsilon)$ and hence verifies (i).

We now prove (ii). To this end, we first claim that for any two Whitney cubes Q_j and Q_k, $Q_j^{**} \cap Q_k^{**} \ne \emptyset$ if and only if Q_j and Q_k touch. Indeed, it suffices to show that Q_j and Q_k touch when $Q_j^{**} \cap Q_k^{**} \ne \emptyset$. Otherwise, if $Q_j^{**} \cap Q_k^{**} \ne \emptyset$ and Q_j and Q_k do not touch, then by Lemma 1(v), we have

$$d(Q_j,Q_k) \ge \frac{1}{4}\max\{l_j,l_k\}.$$

This, together with the assumption $\widetilde{\mu} \in (0, 1/(16\sqrt{n}))$, implies that

$$d(Q_j^{**},Q_k^{**}) \ge d(Q_j,Q_k) - \widetilde{\mu}\sqrt{n}(l_j+l_k) \ge \frac{1}{8}\max\{l_j,l_k\} > 0,$$

which contradicts the assumption $Q_j^{**} \cap Q_k^{**} \ne \emptyset$ and hence verifies the claim. By this and Lemma 1(iv), we know (23) holds with $N = 2(12)^n$. Moreover, the above claim implies that for each $i \in \Lambda_2$, $k \in \Lambda_{21}(i)$ and $j \in \Lambda_{23}(i,k)$, it holds that either Q_j and Q_i touch; or Q_j and Q_k, and Q_i and Q_k, touch. By Lemma 1(v), we conclude that (24) holds true, which completes the proof of (ii) and hence Lemma 6. □

3. Proofs of Main Results

This section is devoted to the proofs of main results of this paper. We first prove Theorem 2 in Section 3.1; then we prove Theorem 1 in Section 3.2. Finally, Section 3.3 is devoted to the proof of Theorem 3.

3.1. Proof of Theorem 2

In this subsection, we prove the density property of Triebel–Lizorkin space $F^s_{p,q}(\Omega)$ (see Theorem 2) via the intrinsic characterization of $F^s_{p,q}(\Omega)$. To this end, we recall the following definitions of intrinsic Triebel–Lizorkin space $\mathcal{F}^s_{p,q}(\Omega)$ from [17].

Definition 1. *Let Ω be a bounded domain in $\mathbb{R}^n$. For any p, $q \in [1,\infty)$ and $s \in (0,1)$. The intrinsic Triebel–Lizorkin space is defined by*

$$\mathcal{F}^s_{p,q}(\Omega) := \{f \in L^p(\Omega) : \|f\|_{\mathcal{F}^s_{p,q}(\Omega)} < \infty\},$$

where

$$\|f\|_{\mathcal{F}^s_{p,q}(\Omega)} := \|f\|_{L^p(\Omega)} + \left[\int_\Omega \left(\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \tag{25}$$
$$=: \|f\|_{L^p(\Omega)} + |f|_{\mathcal{F}^s_{p,q}(\Omega)} < \infty.$$

Let $\mathring{\mathcal{F}}^s_{p,q}(\Omega)$ be the completion of $\mathcal{D}(\Omega)$ in $\mathcal{F}^s_{p,q}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{F}^s_{p,q}(\Omega)}$ as in (25).

Remark 4. *For any $p,\ q \in [1,\infty)$ and $s \in (n(\frac{1}{p}-\frac{1}{q})_+, 1)$, let $F^s_{p,q}(\Omega)$ be the Triebel–Lizorkin space defined as in* **(I)** *of Introduction. If, in addition, Ω is a bounded uniform domain, then it is proved in ([17], Corollary 3.11) that*

$$\mathcal{F}^s_{p,q}(\Omega) = F^s_{p,q}(\Omega) \tag{26}$$

with equivalent norms.

On the other hand, let $\mathring{F}^s_{p,q}(\Omega)$ be the Triebel–Lizorkin space defined as in **(II)** *of Introduction. By (26), we know that for any $p,\ q \in [1,\infty)$ and $s \in (n(\frac{1}{p}-\frac{1}{q})_+, 1)$, it holds that*

$$\mathring{\mathcal{F}}^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega)$$

with equivalent norms.

Note that Theorem 2 is an immediate consequence of Remark 4 and the following density property fo intrinsic Triebel–Lizorkin spaces $\mathcal{F}^s_{p,q}(\Omega)$.

Proposition 1. *Let $p,\ q \in [1,\infty)$ and $s \in (0,1)$. Assume Ω is a bounded uniform domain satisfying the $(H)_{s,p,q}$-condition for all $f \in \mathcal{F}^s_{p,q}(\Omega)$. Then it holds that*

$$\mathcal{F}^s_{p,q}(\Omega) = \mathring{\mathcal{F}}^s_{p,q}(\Omega)$$

with equivalent norms, where $\mathcal{F}^s_{p,q}(\Omega)$ and $\mathring{\mathcal{F}}^s_{p,q}(\Omega)$ are defined as in Definition 1.

Proof. Since Ω is bounded, by an elementary calculation, we know $D(\Omega) \subseteq \mathcal{F}^s_{p,q}(\Omega)$. This immediately implies $\mathring{\mathcal{F}}^s_{p,q}(\Omega) \subset \mathcal{F}^s_{p,q}(\Omega)$. Thus, we only need to prove the converse inclusion $\mathcal{F}^s_{p,q}(\Omega) \subset \mathring{\mathcal{F}}^s_{p,q}(\Omega)$. Since the proof is quite long, we divide it into several steps.

Step 1. Let $\{Q_j\}_{j=1}^\infty$ be the Whitney decomposition of Ω as in Lemma 1 and $\{\psi_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ the corresponding partition of unity satisfying the following properties:

(i) $\psi_j \equiv 1$ on Q_j and $\operatorname{supp}\psi_j \subset Q_j^*$, where $Q_j^* := (1+2\widetilde{\mu})Q$ is the concentric cube of Q_j with sidelength $(1+2\widetilde{\mu})l_j$ and $\widetilde{\mu} \in (0, 1/(16n))$;

(ii) For any $x \in \Omega$, it holds that

$$\sum_{j=1}^\infty \psi_j(x) = 1 \tag{27}$$

(iii) There exists a positive constant C such that for all $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$,

$$|\nabla\psi_j(x)| \le \frac{C}{\operatorname{diam} Q_j}. \tag{28}$$

Now let $f \in \mathcal{F}^s_{p,q}(\Omega)$. For any $\varepsilon > 0$ and $x \in \Omega$, by (27) and the definitions of the index sets Λ_1, Λ_2 as in (12), we write

$$f(x) = \sum_{d(Q_j,\partial\Omega)\geq\varepsilon} \psi_j(x)f(x) + \sum_{d(Q_j,\partial\Omega)<\varepsilon} \psi_j(x)f(x) = \sum_{j\in\Lambda_1} \psi_j(x)f(x) + \sum_{j\in\Lambda_2} \psi_j(x)f(x) \tag{29}$$
$$=: v_\varepsilon(x) + w_\varepsilon(x)$$

with v_ε and w_ε being the interior and boundary parts, respectively.

Step 2. We first consider the interior part v_ε by claiming

$$v_\varepsilon \in \mathcal{F}^s_{p,q}(\Omega). \tag{30}$$

Indeed, let $\psi := \sum_{j\in\Lambda_1}\psi_j(x)$. By the property (i) and (13), it holds that $\psi \in C_0^\infty(\Omega)$, which together with the fact that $v_\varepsilon = \psi f$ implies

$$\|v_\epsilon\|_{L^p(\Omega)} = \|\psi f\|_{L^p(\Omega)} \leq \|\psi\|_{L^\infty(\Omega)}\|f\|_{L^p(\Omega)} < \infty, \tag{31}$$

which implies $v_\varepsilon \in L^p(\Omega)$. On the other hand, by (25), we have

$$|\psi f|_{\mathcal{F}^s_{p,q}(\Omega)} = \left[\int_\Omega\left(\int_\Omega \frac{|\psi(x)f(x)-\psi(y)f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} =: A.$$

Write

$$A \leq \left[\int_\Omega\left(\int_\Omega \frac{|f(x)|^q|\psi(x)-\psi(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} + \left[\int_\Omega\left(\int_\Omega \frac{|f(x)-f(y)|^q|\psi(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$
$$=: A_1 + A_2.$$

We first estimate A_1. Since $\psi \in C_0^\infty(\Omega)$, it follows that

$$A_1 \leq \left[\int_\Omega |f(x)|^p\left(\int_\Omega \|\nabla\psi(x)\|^q_{L^\infty(\Omega)}|x-y|^{q(1-s)-n}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}.$$

Moreover, by the assumption that Ω is bounded, we have

$$A_1 \lesssim \left[\int_\Omega |f(x)|^p\left(\int_0^{\operatorname{diam}\Omega} \rho^{q(1-s)-1}\,d\rho\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$
$$\lesssim \|f\|_{L^p(\Omega)} < \infty.$$

To bound A_2, it is easy to see that

$$A_2 \leq \left[\int_\Omega\left(\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \|\psi\|_{L^\infty(\Omega)} \lesssim |f|_{\mathcal{F}^s_{p,q}(\Omega)} < \infty.$$

Combining the estimates of A_1 and A_2, we conclude that $A < \infty$. This, together with $v\varepsilon \in L^p(\Omega)$, implies $v\varepsilon \in \mathcal{F}^s_{p,q}(\Omega)$, and hence verifies the claim (30).

Step 3. Next we prove $v_\varepsilon \in \mathring{\mathcal{F}}^s_{p,q}(\Omega)$. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\eta \geq 0$ in $\mathbb{R}^n$, $\operatorname{supp}\eta \subseteq B(0,1)$ and $\int_{B(0,1)}\eta(x)\,dx = 1$. Let $0 < \delta < \frac{3\tilde{\mu}\varepsilon}{16\sqrt{n}}$ and $\eta^{(\delta)}$ be the mollifier defined by

$$\eta^{(\delta)}(x) := \delta^{-n}\eta(x/\delta)$$

for any $x \in \mathbb{R}^n$. It is easy to see $\eta^{(\delta)} * v_\varepsilon \in \mathcal{D}(\Omega)$, and by the property of the approximations of identity, we have

$$\left\| \left(\eta^{(\delta)} * v_\varepsilon\right) - v_\varepsilon \right\|_{L^p(\Omega)} \to 0$$

as $\delta \to 0$. Then to prove $v_\varepsilon \in \mathring{\mathcal{F}}^s_{p,q}(\Omega)$, it suffices to show $\left| \left(\eta^{(\delta)} * v_\varepsilon\right) - v_\varepsilon \right|_{\mathcal{F}^s_{p,q}(\Omega)} \to 0$ as $\delta \to 0$. From (25), we deduce

$$\begin{aligned}
&\left| \left(\eta^{(\delta)} * v_\varepsilon\right) - v_\varepsilon \right|_{\mathcal{F}^s_{p,q}(\Omega)} \\
&= \left[\int_\Omega \left(\int_\Omega \frac{\left| \left(\eta^{(\delta)} * v_\varepsilon\right)(x) - v_\varepsilon(x) - \left(\eta^{(\delta)} * v_\varepsilon\right)(y) + v_\varepsilon(y) \right|^q}{|x-y|^{n+sq}} \, dy \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \\
&= \left[\int_\Omega \left(\int_\Omega \frac{\left| \delta^{-n} \int_{B(0,\delta)} [v_\varepsilon(x-z) - v_\varepsilon(y-z)] \eta(z/\delta) \, dz - v_\varepsilon(x) + v_\varepsilon(y) \right|^q}{|x-y|^{n+sq}} \, dy \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \\
&= \left[\int_\Omega \left(\int_\Omega \frac{\int_{B(0,1)} | [v_\varepsilon(x-\delta\tilde{z}) - v_\varepsilon(y-\delta\tilde{z}) - v_\varepsilon(x) + v_\varepsilon(y)] \eta(\tilde{z}) \, d\tilde{z} |^q}{|x-y|^{n+sq}} \, dy \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \\
&\leq \int_{B(0,1)} \left[\int_\Omega \left(\int_\Omega \frac{|v_\varepsilon(x-\delta\tilde{z}) - v_\varepsilon(y-\delta\tilde{z}) - v_\varepsilon(x) + v_\varepsilon(y)|^q}{|x-y|^{n+sq}} \, dy \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \eta(\tilde{z}) \, d\tilde{z}.
\end{aligned} \tag{32}$$

Now, let

$$G(x,y) := \frac{v_\varepsilon(x) - v_\varepsilon(y)}{|x-y|^{\frac{n}{q}+s}}.$$

It is easy to see

$$G(x-\delta\tilde{z}, y-\delta\tilde{z}) - G(x,y) = \frac{v_\varepsilon(x-\delta\tilde{z}) - v_\varepsilon(y-\delta\tilde{z}) - v_\varepsilon(x) + v_\varepsilon(y)}{|x-y|^{\frac{n}{q}+s}}.$$

Since

$$\|G(x,y)\|_{L^p_x(L^q_y)(\Omega\times\Omega)} := \left[\int_\Omega \left(\int_\Omega |G(x,y)|^q \, dy \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}$$

is a mixed Lebegue norm. By the continuity of translation (see ([24], Theorem 2)), we get

$$\lim_{\delta\to 0} \|G(x-\delta\tilde{z}, y-\delta\tilde{z}) - G(x,y)\|_{L^p_x(L^q_y)(\Omega\times\Omega)} = 0 \tag{33}$$

for any $\tilde{z} \in B(0,1)$. Now let

$$\psi_{\varepsilon,\delta}(\tilde{z}) := \left[\int_\Omega \left(\int_\Omega \frac{|v_\varepsilon(x-\delta\tilde{z}) - v_\varepsilon(y-\delta\tilde{z}) - v_\varepsilon(x) + v_\varepsilon(y)|^q}{|x-y|^{n+sq}} \, dy \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \eta(\tilde{z}),$$

for any $\tilde{z} \in B(0,1)$. By (13), the assumption $0 < \delta < \frac{3\tilde{\mu}\varepsilon}{16\sqrt{n}}$ and the change of variables, we obtain

$$\begin{aligned}
\psi_{\varepsilon,\delta}(\tilde{z}) &\lesssim \left[\int_\Omega \left(\int_\Omega \frac{|v_\varepsilon(x-\delta\tilde{z}) - v_\varepsilon(y-\delta\tilde{z})|^q}{|x-y|^{n+sq}}\,dy + \int_\Omega \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \eta(\tilde{z}) \\
&\lesssim \left[\left(\int_\Omega \left(\int_\Omega \frac{|v_\varepsilon(x-\delta\tilde{z}) - v_\varepsilon(y-\delta\tilde{z})|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} + \left(\int_\Omega \left(\int_\Omega \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}\right] \eta(\tilde{z}) \\
&\lesssim \left(\int_\Omega \left(\int_\Omega \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \eta(\tilde{z}).
\end{aligned}$$

This, together with (25) and (30), shows $\psi_{\varepsilon,\delta} \in L^\infty(B(0,1))$. Now, using (32), (33) and the dominated convergence theorem, we get

$$\begin{aligned}
\lim_{\delta\to 0}\left|(\eta^{(\delta)} * v_\varepsilon) - v_\varepsilon\right|_{\mathcal{F}^s_{p,q}(\Omega)} &= \lim_{\delta\to 0}\int_{B(0,1)} \|G(x-\delta\tilde{z}, y-\delta\tilde{z}) - G(x,y)\|_{L^p_x(L^q_y)(\Omega\times\Omega)}\eta(\tilde{z})\,d\tilde{z} \\
&= \int_{B(0,1)} \lim_{\delta\to 0}\psi_{\varepsilon,\delta}(\tilde{z})\,d\tilde{z} = 0,
\end{aligned} \tag{34}$$

which implies $v_\varepsilon \in \mathring{\mathcal{F}}^s_{p,q}(\Omega)$.

Step 4. We still need to verify the boundary part $w_\varepsilon \in \mathring{\mathcal{F}}^s_{p,q}(\Omega)$. To this end, it suffices to prove that

$$\lim_{\varepsilon\to 0}\|w_\varepsilon\|_{L^p(\Omega)} = 0 \tag{35}$$

and

$$\lim_{\varepsilon\to 0}|w_\varepsilon|_{\mathcal{F}^s_{p,q}(\Omega)} = 0. \tag{36}$$

By Lemma 3, we obtain

$$\begin{aligned}
\int_\Omega |w_\varepsilon(x)|^p\,dx &= \int_{\Omega_{3\varepsilon}} |w_\varepsilon(x)|^p\,dx \\
&\le \int_{\Omega_{3\varepsilon}} \left|f(x)\sum_{j=1}^\infty \psi_j(x)\right|^p dx = \int_{\Omega_{3\varepsilon}} |f(x)|^p\,dx,
\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$ and hence implies (35).

Step 5. We now prove (36). By (29) and the fact that $\operatorname{supp}\psi_j \subseteq Q_j^*$, we write

$$
\begin{aligned}
|w_\varepsilon|_{\mathcal{F}^s_{p,q}(\Omega)} &= \left[\int_\Omega\left(\int_\Omega \frac{\left|f(x)\sum_{j\in\Lambda_2}\psi_j(x)-f(y)\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}} \quad (37)\\
&=\left\{\left[\int_{\bigcup\limits_{i\in\Lambda_2}Q_i^*}+\int_{\Omega_{14\varepsilon}\setminus\bigcup\limits_{i\in\Lambda_2}Q_i^*}+\int_{\Omega\setminus\Omega_{14\varepsilon}}\right]\left(\int_\Omega\dots dy\right)^{\frac{p}{q}}dx\right\}^{\frac{1}{p}}\\
&=\left\{\int_{\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left[\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}+\int_{\Omega\setminus\Omega_{14\varepsilon}}\right)\dots dy\right]^{\frac{p}{q}}dx+\int_{\Omega_{14\varepsilon}\setminus\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^*}\dots dy\right)^{\frac{p}{q}}dx\right.\\
&\quad\left.+\int_{\Omega\setminus\Omega_{14\varepsilon}}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^*}\dots dy\right)^{\frac{p}{q}}dx\right\}^{\frac{1}{p}}\\
&\lesssim\left[\int_{\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}\frac{\left|f(x)\sum_{j\in\Lambda_2}\psi_j(x)-f(y)\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\\
&\quad+\left[\int_{\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left(\int_{\Omega\setminus\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}\frac{\left|f(x)\sum_{j\in\Lambda_2}\psi_j(x)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\\
&\quad+\left[\int_{\Omega_{14\varepsilon}\setminus\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^*}\frac{\left|f(x)\sum_{j\in\Lambda_2}\psi_j(x)-f(y)\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\\
&\quad+\left[\int_{\Omega\setminus\Omega_{14\varepsilon}}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^*}\frac{\left|f(y)\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\\
&=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{II}+\mathrm{III}.
\end{aligned}
$$

Step 6. We estimate the above terms in the order of I_2, III, I_1 and II. To estimate I_2, we first write

$$\mathrm{I}_2\le\left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p\left(\int_{\Omega\setminus\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}\frac{1}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}.$$

From the definitions of Q_i^* and Q_i^{**}, it follows that for any $x\in Q_i^*$ and $y\in\Omega\setminus\bigcup\limits_{i\in\Lambda_2}Q_i^{**}$, $|x-y|\ge\frac{\tilde{\mu}l_i}{2}$, where l_i denotes the sidelength of Q_i. Thus, we have

$$\mathrm{I}_2\lesssim\left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p\left(\int_{\frac{\tilde{\mu}l_i}{2}}^\infty\rho^{-sq-1}\,d\rho\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\lesssim\left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p\left(\frac{\tilde{\mu}l_i}{2}\right)^{-sp}dx\right]^{\frac{1}{p}}.$$

Using the properties (ii) and (iv) of Lemma 1, (14) and the $(H)_{s,p,q}$-condition, we obtain

$$\mathrm{I}_2 \lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}\frac{|f(x)|^p}{d(x,\partial\Omega)^{sp}}\,dx\right]^{\frac{1}{p}} \lesssim \left[\int_{\bigcup\limits_{i\in\Lambda_2} Q_i^*}\frac{|f(x)|^p}{d(x,\partial\Omega)^{sp}}\,dx\right]^{\frac{1}{p}} \lesssim \left\|\frac{f}{d(\cdot,\partial\Omega)^s}\right\|_{L^p(\Omega_{3\varepsilon})} \to 0 \tag{38}$$

as $\varepsilon \to 0$, which is desired. That is

$$\lim_{\varepsilon\to 0}\mathrm{I}_2 = 0. \tag{39}$$

Step 7. To bound III, it is easy to see that

$$\begin{aligned}\mathrm{III} &\lesssim \left[\int_{\Omega\backslash\Omega_{14\varepsilon}}\left(\int_{\bigcup\limits_{k\in\Lambda_2} Q_k^{**}}\frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}} \\ &\quad+ \left[\int_{\Omega\backslash\Omega_{14\varepsilon}}\left(\int_{\bigcup\limits_{k\in\Lambda_2} Q_k^{**}}\frac{|f(x)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}} \\ &=: \mathrm{III}_1 + \mathrm{III}_2.\end{aligned}$$

Using the fact that $f \in \mathcal{F}^s_{p,q}(\Omega)$, (14) and (25), we have

$$\mathrm{III}_1 \le \left[\int_\Omega\left(\int_{\Omega_{14\varepsilon}}\frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}} \to 0 \tag{40}$$

as $\varepsilon \to 0$.

Now we estimate III_2. For any $x, y \in \Omega$, let

$$F(x,y) := \frac{f(x)}{|x-y|^{\frac{n}{q}+s}}\mathbf{1}_{\bigcup\limits_{k\in\Lambda_2} Q_k^{**}}(y)\mathbf{1}_{\Omega\backslash\Omega_{14\varepsilon}}(x). \tag{41}$$

It is obvious that

$$\begin{aligned}\mathrm{III}_2 &= \left[\int_\Omega\left(\int_\Omega |F(x,y)|^q\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}} = \|F(x,y)\|_{L^p_x(L^q_y)(\Omega\times\Omega)} \\ &\le \sup_{\|V\|_{L_x^{p'}(L_y^{q'})(\Omega\times\Omega)}\le 1}\left[\int_\Omega\left(\int_\Omega F(x,y)V(x,y)\,dy\right)dx\right].\end{aligned}$$

Let

$$B(F,V) := \left[\int_\Omega\left(\int_\Omega F(x,y)V(x,y)\,dy\right)dx\right].$$

By the definition of F in (41), it holds

$$B(F,V) \le \left[\int_{\Omega\backslash\Omega_{14\varepsilon}}|f(x)|\left(\int_{\bigcup\limits_{k\in\Lambda_2} Q_k^{**}}\frac{|V(x,y)|}{|x-y|^{\frac{n}{q}+s}}\,dy\right)dx\right]. \tag{42}$$

Moreover, since $\|V\|_{L_x^{p'}(L_y^{q'})(\Omega\times\Omega)} \le 1$, we deduce

$$\lim_{\varepsilon\to 0}\left[\int_\Omega\left(\int_{\Omega_{3\varepsilon}}|V(x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}}dx\right]^{\frac{1}{p'}} = 0. \tag{43}$$

Using (42), Lemma 3 and Hölder's inequality, we obtain

$$\begin{aligned}
B(F,V) &\lesssim \sum_{i\in\Lambda_3}\int_{Q_i}|f(x)|\left(\sum_{k\in\Lambda_2}\int_{Q_k^*}|V(x,y)|^{q'}\,dy\right)^{\frac{1}{q'}}\left(\sum_{k\in\Lambda_2}\int_{Q_k^*}\left(\frac{1}{|x-y|^{n+sq}}\,dy\right)^{\frac{1}{q}}\right)dx\\
&\lesssim \sum_{i\in\Lambda_3}\int_{Q_i}|f(x)|\left(\sum_{k\in\Lambda_2}\int_{Q_k^*}|V(x,y)|^{q'}\,dy\right)^{\frac{1}{q'}}\left(\sum_{k\in\Lambda_2}\frac{l(Q_k)^n}{D(Q_i,Q_k)^{n+sq}}\right)^{\frac{1}{q}}dx,
\end{aligned}$$

which, together with Lemmas 4(iv) and 5, implies

$$\begin{aligned}
B(F,V) &\lesssim \sum_{i\in\Lambda_3}\int_{Q_i}|f(x)|\left(\sum_{k\in\Lambda_2}\int_{Q_k^*}|V(x,y)|^{q'}\,dy\right)^{\frac{1}{q'}}\frac{1}{l(Q_i)^s}\,dx\\
&\lesssim \sum_{i\in\Lambda_3}\int_{Q_i}\frac{|f(x)|}{d(x,\partial\Omega)^s}\left(\int_{\Omega_{3\varepsilon}}|V(x,y)|^{q'}\,dy\right)^{\frac{1}{q'}}dx\\
&\sim \int_\Omega\frac{|f(x)|}{d(x,\partial\Omega)^s}\left(\int_{\Omega_{3\varepsilon}}|V(x,y)|^{q'}\,dy\right)^{\frac{1}{q'}}dx\\
&\lesssim \left(\int_\Omega\frac{|f(x)|^p}{d(x,\partial\Omega)^{sp}}\right)^{\frac{1}{p}}\left[\int_\Omega\left(\int_{\Omega_{3\varepsilon}}|V(x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}}dx\right]^{\frac{1}{p'}}.
\end{aligned}$$

Combining the former with (43), we get

$$B(F,V)\to 0$$

as $\varepsilon\to 0$. By this and (40), we conclude that

$$\lim_{\varepsilon\to 0}\mathrm{III} = 0. \tag{44}$$

Step 8. Next we consider I_1. Next,

$$\begin{aligned}
\mathrm{I}_1 \le{}& \left[\int_{\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}\frac{|f(x)-f(y)|^q\left|\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right.\right.\\
&\left.\left.+\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}\frac{|f(x)|^q\left|\sum_{j\in\Lambda_2}\psi_j(x)-\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\\
\lesssim{}& \left[\int_{\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}\frac{|f(x)-f(y)|^q\left|\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\\
&+\left[\int_{\bigcup\limits_{i\in\Lambda_2}Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2}Q_k^{**}}\frac{|f(x)|^q\left|\sum_{j\in\Lambda_2}\psi_j(x)-\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\\
=:{}& \mathrm{I}_{11}+\mathrm{I}_{12}.
\end{aligned}$$

For I_{11}, by (25) and Lemma 3, we know that

$$\mathrm{I}_{12} \leq \left[\int_{\bigcup\limits_{i\in\Lambda_2} Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2} Q_k^{**}} \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \tag{45}$$
$$\leq \left[\int_{\Omega_{3\varepsilon}}\left(\int_{\Omega_{3\varepsilon}} \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}},$$

which turns to 0 as $\varepsilon \to 0$.

To bound I_{12}, by the definitions of the index sets $\Lambda_{21}(i)$ and $\Lambda_{22}(i)$ as in (21), we have

$$\mathrm{I}_{12} \leq \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_2} Q_k^{**}} \frac{|f(x)|^q\left|\sum_{j\in\Lambda_2}\psi_j(x)-\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \tag{46}$$
$$\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_{21}} Q_k^{**}} \frac{|f(x)|^q\left|\sum_{j\in\Lambda_2}\psi_j(x)-\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$
$$+\left[\sum_{i\in\Lambda_2}\int_{Q_i^*}\left(\int_{\bigcup\limits_{k\in\Lambda_{22}} Q_k^{**}} \frac{|f(x)|^q\left|\sum_{j\in\Lambda_2}\psi_j(x)-\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$
$$=: \mathrm{I}_{12}^1+\mathrm{I}_{12}^2.$$

By (28); the definition of the index set Λ_{23} as in (22); Lemmas 5 and 6; and an argument similar to that used in the proof of (38), we obtain

$$\mathrm{I}_{12}^1 \leq \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p\left(\sum_{k\in\Lambda_{21}}\int_{Q_k^{**}} \frac{\left|\sum_{j\in\Lambda_{23}}\psi_j(x)-\sum_{j\in\Lambda_{23}}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \tag{47}$$
$$\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p\left(\sum_{k\in\Lambda_{21}}\int_{Q_k^{**}}\left(\sup_{j\in\Lambda_{23}}\|\nabla\psi_j\|_{L^\infty}\right)^q|x-y|^{q(1-s)-n}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$
$$\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p\left(\int_0^{7\varepsilon} l_i^{-q}\rho^{q(1-s)-1}\,d\rho\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$
$$\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p\left(\int_0^{7\varepsilon} d(x,\partial\Omega)^{-q}\rho^{q(1-s)-1}\,d\rho\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$
$$\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}|f(x)|^p d(x,\partial\Omega)^{-p}\varepsilon^{p(1-s)}\,dx\right]^{\frac{1}{p}}$$
$$\lesssim \left[\int_{\bigcup\limits_{i\in\Lambda_2} Q_i^*}\frac{|f(x)|^p}{d(x,\partial\Omega)^{sp}}\,dx\right]^{\frac{1}{p}} \leq \left\|\frac{f}{d(\cdot,\partial\Omega)^s}\right\|_{L^p(\Omega_{3\varepsilon})} \to 0$$

as $\varepsilon \to 0$.

On the other hand, by (21), we know that if $k \in \Lambda_{22}(i)$, then $Q_k^{**} \bigcap Q_i^* = \emptyset$. For any $x \in Q_i^*$ and $y \in Q_k^{**}$, there exists a positive constant c such that $|x-y| \geq cl_i$—that is, $\bigcup_{k\in\Lambda_{22}} Q_k^{**} \subseteq [B(x, Cl_i)]^{\complement}$. This yields that

$$\begin{aligned}
\mathrm{I}_{12}^2 &\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*}\left(\int_{\bigcup_{k\in\Lambda_{22}} Q_k^{**}} \frac{|f(x)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \\
&\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*} |f(x)|^p \left(\int_{cl_i}^{+\infty} \rho^{-sq-1}\,d\rho\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \\
&\lesssim \left[\sum_{i\in\Lambda_2}\int_{Q_i^*} |f(x)|^p l_i^{-sp}\,dx\right]^{\frac{1}{p}} \\
&\lesssim \left[\int_{\bigcup_{i\in\Lambda_2} Q_i^*} \frac{|f(x)|^p}{d(x,\partial\Omega)^{sp}}\,dx\right]^{\frac{1}{p}} \lesssim \left\|\frac{f}{d(\cdot,\partial\Omega)^s}\right\|_{L^p(\Omega_{3\varepsilon})} \to 0
\end{aligned} \tag{48}$$

as $\varepsilon \to 0$.

Combing (45), (47) and (48), we conclude that

$$\lim_{\varepsilon\to 0} \mathrm{I}_1 = 0. \tag{49}$$

Step 9. Finally, we estimate II. Write

$$\begin{aligned}
\mathrm{II} &\leq \left[\int_{\Omega_{14\varepsilon}}\left(\int_{\bigcup_{k\in\Lambda_2} Q_k^*} \frac{\left|f(x)\sum_{j\in\Lambda_2}\psi_j(x) - f(y)\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \\
&\lesssim \left[\int_{\Omega_{14\varepsilon}}\left(\int_{\bigcup_{k\in\Lambda_2} Q_k^*} \frac{|f(x)|^q\left|\sum_{j\in\Lambda_2}\psi_j(x) - \sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \\
&\quad + \left[\int_{\Omega_{14\varepsilon}}\left(\int_{\bigcup_{k\in\Lambda_2} Q_k^*} \frac{|f(x)-f(y)|^q\left|\sum_{j\in\Lambda_2}\psi_j(y)\right|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \\
&=: \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}$$

By an argument similar to that of I_1, it is easy to see that

$$\lim_{\varepsilon\to 0} \mathrm{II} = 0. \tag{50}$$

Combining (37), (39), (44), (49) and (50), we obtain $\lim_{\varepsilon\to 0} |w_\varepsilon|_{\mathcal{F}^s_{p,q}(\Omega)} = 0$, which proves (36). This, together with (34) and (35) shows $f \in \mathring{\mathcal{F}}^s_{p,q}(\Omega)$ and hence finishes the proof of Proposition 1. □

3.2. Proof of Theorem 1

In this subsection, we prove Theorem 1. To this end, we first recall the following definition of refined localisation Triebel–Lizorkin spaces $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ from ([10], Definition 2.14).

Definition 2 ([10]). *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\{Q_j\}_{j=1}^{\infty}$ be the Whitney decomposition of Ω as in Lemma 1, and $\{\psi_j\}_{j=1}^{\infty}$ be the corresponding partition of unity as in (27) and (28). For any $p, q \in [1, \infty]$ and $s \in (0, \infty)$, the refined localisation Triebel–Lizorkin space $F_{p,q}^{s,\mathrm{rloc}}(\Omega)$ is defined by setting*

$$F_{p,q}^{s,\mathrm{rloc}}(\Omega) := \left\{ f \in D'(\Omega) : \|f\|_{F_{p,q}^{s,\mathrm{rloc}}(\Omega)} := \left(\sum_{j=0}^{\infty} \|\psi_j f\|_{F_{p,q}^s(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} < \infty \right\},$$

where $\|\cdot\|_{F_{p,q}^s(\mathbb{R}^n)}$ denotes the classical Triebel–Lizorkin norm on $\mathbb{R}^n$.

Remark 5.

(i) *Let Ω be a bounded domain. For any $p, q \in [1, \infty]$ and $s \in (0, \infty)$, it is well-known that the space $F_{p,q}^{s,\mathrm{rloc}}(\Omega)$ is independent of the choice of the partition of unity $\{\psi\}_{j=1}^{\infty}$ (see ([10], Theorem 2.16)).*

(ii) *Let Ω be a bounded domain. For any $p, q \in [1, \infty]$ and $s \in (0, 1)$, it is proved in ([10], Theorem 2.18) (see also ([8], Corollary 5.15)) that $F_{p,q}^{s,\mathrm{rloc}}(\Omega)$ can be characterized by the following intrinsic norm:*

$$\left\| \frac{f}{d(\cdot, \partial\Omega)} \right\|_{L^p(\Omega)} + \left\| \left[\int_0^{cd(\cdot,\partial\Omega)} t^{-sq} (d_{t,u} f)^q \, \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\Omega)} \tag{51}$$

for some $c \in (0, 1)$, where for any $u \in (0, 1)$, $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$d_{t,u} f(x) := \left[\frac{1}{t^n} \int_{|h| \le t} |f(x+h) - f(x)|^u \, dh \right]^{1/u}.$$

(iii) *Suppose that Ω is a bounded E-thick domain. Let $\widetilde{F}_{p,q}^s(\Omega)$ be the Triebel–Lizorkin space defined as in* **(III)** *of Introduction. It is known (see ([10], Proposition 3.10)) that for any $p, q \in [1, \infty]$ and $s \in (0, \infty)$,*

$$\widetilde{F}_{p,q}^s(\Omega) = F_{p,q}^{s,\mathrm{rloc}}(\Omega)$$

with equivalent norms.

With the help of Remark 5 and Theorem 2, we now turn to the proof of Theorem 1.

Proof of Theorem 1. Let $p,\ q \in [1, \infty)$ and $s \in \left(n(\frac{1}{p} - \frac{1}{q})_+, 1 \right)$. Since Ω is bounded and uniform, it follows from Remark 4 that

$$\mathcal{F}_{p,q}^s(\Omega) = F_{p,q}^s(\Omega) \quad \text{and} \quad \mathring{\mathcal{F}}_{p,q}^s(\Omega) = \mathring{F}_{p,q}^s(\Omega) \tag{52}$$

with equivalent norms. Moreover, by $(H)_{s,p,q}$-condition and Proposition 1, we know

$$\mathcal{F}_{p,q}^s(\Omega) = \mathring{\mathcal{F}}_{p,q}^s(\Omega).$$

This together with (52) implies that

$$F_{p,q}^s(\Omega) = \mathring{F}_{p,q}^s(\Omega) \tag{53}$$

holds for any $p,\ q \in [1, \infty)$ and $s \in \left(n(\frac{1}{p} - \frac{1}{q})_+, 1 \right)$.

On the other hand, since Ω is an E-thick domain, we deduce from Remark 5(iii) that for any $p, q \in [1, \infty]$ and $s \in (0, 1)$,

$$\widetilde{F}^s_{p,q}(\Omega) = F^{s,\mathrm{rloc}}_{p,q}(\Omega). \tag{54}$$

Moreover, it is proved in ([25], Theorem 3) that the Triebel–Lizorkin space $F^s_{p,q}(\Omega)$, as in **(I)** of Introduction, can also be characterized by the same intrinsic norm of (51). This, combined with Remark 5(ii), implies that for any $p, q \in [1, \infty]$ and $s \in (0, 1)$,

$$F^{s,\mathrm{rloc}}_{p,q}(\Omega) = F^s_{p,q}(\Omega). \tag{55}$$

Taking (53)–(55) together, we conclude that for any $p,\ q \in [1, \infty)$ and $s \in \left(n(\frac{1}{p} - \frac{1}{q})_+, 1\right)$, it holds

$$F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = \widetilde{F}^s_{p,q}(\Omega),$$

which completes the proof of Theorem 1. $\square$

3.3. Proof of Theorem 3

In this subsection, we prove Theorem 3.

Proof of Theorem 3. We first prove the implication (i) $\Rightarrow$ (ii). Assume (i) holds. Let $f \in C_c(\Omega)$ satisfy $f(x) \geq 1$ for any $x \in K$. By (i), we know

$$\int_K \frac{1}{d(x, \partial\Omega)^{sp}}\, dx \leq \int_\Omega \frac{|f(x)|^p}{d(x, \partial\Omega)^{sp}}\, dx \leq C_1^p \left[\int_\Omega \left(\int_\Omega \frac{|f(x) - f(y)|^q}{|x - y|^{n+sq}}\, dy \right)^{\frac{p}{q}} dx \right].$$

Taking the infinum over all such functions f and using (9), we obtain

$$\int_K \frac{1}{d(x, \partial\Omega)^{sp}}\, dx \leq C_1^p \mathrm{cap}_{s,p,q}(K, \Omega),$$

which implies (ii) with $C_2 = C_1^p$.

Now we prove the converse implication that (ii) $\Rightarrow$ (i). Suppose (ii) holds. Then, for any $k \in \mathbb{Z}$, let

$$E_k := \{x \in \Omega : |f(x)| > 2^k\} \quad \text{and} \quad A_k := E_k \setminus E_{k+1}.$$

Observe

$$\Omega = \{x \in \Omega : 0 \leq |f(x)| < \infty\} = F \cup \bigcup_{k \in \mathbb{Z}} A_k \tag{56}$$

with

$$F := \{x \in \Omega : f(x) = 0\}. \tag{57}$$

Hence, by (ii) we obtain

$$\int_\Omega \frac{|f(x)|^p}{d(x, \partial\Omega)^{sp}}\, dx \leq \sum_{k \in \mathbb{Z}} 2^{(k+2)p} \int_{A_{k+1}} \frac{1}{d(x, \partial\Omega)^{sp}}\, dx \leq C_2 2^{2p} \sum_{k \in \mathbb{Z}} 2^{kp} \mathrm{cap}_{s,p,q}(\bar{A}_{k+1}, \Omega). \tag{58}$$

Define the function $f_k : \Omega \to [0,1]$ by

$$f_k(x) := \begin{cases} 1, & |f(x)| \geq 2^{k+1}, \\ \frac{|f(x)|}{2^k} - 1, & 2^k < |f(x)| < 2^{k+1}, \\ 0, & |f(x)| \leq 2^k. \end{cases} \tag{59}$$

It is easy to see $f_k \in C_c(\Omega)$, and it satisfies $f_k = 1$ on $\bar{E}_{k+1} \supset \bar{A}_{k+1}$. Hence, we can take f_k as a test function for the capacity. By (9), we have

$$\begin{aligned} \mathrm{cap}_{s,p,q}(\bar{A}_{k+1}, \Omega) &\leq \int_\Omega \left(\int_\Omega \frac{|f_k(x) - f_k(y)|^q}{|x-y|^{n+sq}} \, dy \right)^{\frac{p}{q}} dx \\ &\leq \sup_{\|h\|_{L^{(p/q)'}(\Omega)} \leq 1} \left[\int_\Omega \left(\int_\Omega \frac{|f_k(x) - f_k(y)|^q}{|x-y|^{n+sq}} \, dy \right) h(x) \, dx \right]^{\frac{p}{q}}. \end{aligned} \tag{60}$$

Using (56) and (57), we get

$$\begin{aligned} &\int_\Omega \left(\int_\Omega \frac{|f_k(x) - f_k(y)|^q}{|x-y|^{n+sq}} \, dy \right) h(x) \, dx \\ &= \int_{F \cup \bigcup\limits_{i \in \mathbb{Z}} A_i} \int_{F \cup \bigcup\limits_{j \in \mathbb{Z}} A_j} \frac{|f_k(x) - f_k(y)|^q}{|x-y|^{n+sq}} h(x) \, dy \, dx \\ &= \left(\int_{\bigcup\limits_{i \in \mathbb{Z}} A_i} \int_{\bigcup\limits_{j \in \mathbb{Z}} A_j} + \int_F \int_{\bigcup\limits_{j \in \mathbb{Z}} A_j} + \int_{\bigcup\limits_{i \in \mathbb{Z}} A_i} \int_F \right) \frac{|f_k(x) - f_k(y)|^q}{|x-y|^{n+sq}} h(x) \, dy \, dx \\ &= \Bigg(\sum_{i \geq k} \sum_{j \geq k} \int_{A_i} \int_{A_j} + \sum_{i \geq k} \sum_{j < k} \int_{A_i} \int_{A_j} + \sum_{i < k} \sum_{j \geq k} \int_{A_i} \int_{A_j} + \sum_{i < k} \sum_{j < k} \int_{A_i} \int_{A_j} \\ &\quad + \sum_{j < k} \int_F \int_{A_j} + \sum_{i < k} \int_{A_i} \int_F + \sum_{j \geq k} \int_F \int_{A_j} + \sum_{i \geq k} \int_{A_i} \int_F \Bigg) \frac{|f_k(x) - f_k(y)|^q}{|x-y|^{n+sq}} h(x) \, dy \, dx. \end{aligned} \tag{61}$$

Now for any $x \in A_i = E_i \setminus E_{i+1}$, by the fact that $2^i < |f(x)| \leq 2^{i+1}$ and the definition of f_k as in (59), we claim that the following assertions hold true.

(i) If $i < k$, then $|f(x)| \leq 2^{i+1} \leq 2^k$, this implies $f_k(x) = 0$;
(ii) If $i = k$, then $f_k(x) = \frac{|f(x)|}{2^k} - 1$;
(iii) If $i > k$, then $|f(x)| > 2^i \geq 2^{k+1}$, which implies $f_k(x) = 1$;
(iv) If $i \leq k \leq j$, for any $x \in A_i$ and $y \in A_j$, it holds that

$$|f_k(x) - f_k(y)| \leq 2 \cdot 2^{-j} |f(x) - f(y)|. \tag{62}$$

We only need to verify (iv). Indeed, let $i \leq k \leq j$, $x \in A_i$ and $y \in A_j$. We consider four cases based on the sizes of i, j and k.

If $i = j$, then by (ii), it is easy to see that

$$|f_k(x) - f_k(y)| = 0. \tag{63}$$

If $j = i + 1$ and $k = i$, then by (ii), (iii) and the assumptions $x \in A_i$, $y \in A_j$, we have

$$|f_k(x) - f_k(y)| = \left| 1 - \left(\frac{|f(x)|}{2^k} - 1 \right) \right| = \left| 2 - \frac{|f(x)|}{2^k} \right| = 2^{-k} \Big| 2^{k+1} - |f(x)| \Big|.$$

Moreover, by the assumption $y \in A_{k+1}$, we know $|f(y)| > 2^{k+1}$. This implies that the above term is bound by

$$2^{-k}\big||f(y)| - |f(x)|\big| \leq 2^{-k}|f(x) - f(y)| = 2 \cdot 2^{-j}|f(x) - f(y)| \tag{64}$$

If $j = i + 1$ and $k = j$, then by a similar argument, we know

$$|f_k(x) - f_k(y)| = \left|\frac{|f(y)|}{2^k} - 1\right| = 2^{-k}\Big||f(y)| - 2^k\Big|.$$

By the assumption $y \in A_{k-1}$, it holds that $|f(y)| \leq 2^k$, so we obtain

$$2^{-k}\big||f(y)| - |f(x)|\big| \leq 2^{-k}|f(x) - f(y)| = 2^{-j}|f(x) - f(y)|. \tag{65}$$

Finally, if $j \geq i + 2$, it holds that

$$|f(x) - f(y)| \geq |f(x)| - |f(y)| \geq 2^{j-1}.$$

By the definition of f_k, we have

$$|f_k(x) - f_k(y)| \leq 1 \leq 2 \cdot 2^{-j}|f(x) - f(y)|. \tag{66}$$

Combining the estimates (63)–(66), we conclude that (62) holds true and hence verifies the claim (iv).

Now by and (i) through (iv), we know that some of the sums in (61) vanish. This, together with (60), implies that

$$\begin{aligned}
\mathrm{cap}_{s,p,q}(\bar{A}_{k+1}, \Omega) &\leq \sup_{\|h\|_{L^{(p/q)'}} \leq 1} \Bigg[\Bigg(\sum_{i \leq k}\sum_{j \geq k}\int_{A_i}\int_{A_j} + \sum_{i \geq k}\sum_{j \leq k}\int_{A_i}\int_{A_j} \\
&\quad + \sum_{j \geq k}\int_F\int_{A_j} + \sum_{i \geq k}\int_{A_i}\int_F\Bigg)\frac{|f_k(x) - f_k(y)|^q}{|x-y|^{n+sq}}h(x)\,dy\,dx\Bigg]^{\frac{p}{q}} \\
&=: \sup_{\|h\|_{L^{(p/q)'}} \leq 1} (\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4)^{\frac{p}{q}}.
\end{aligned}$$

By this and (58), we know that

$$\begin{aligned}
\int_\Omega \frac{|f(x)|^p}{d(x, \partial\Omega)^{sp}}\,dx &\leq CC_2 \sum_{k \in \mathbb{Z}} 2^{kp}\mathrm{cap}_{s,p,q}(\bar{A}_{k+1}, \Omega) \\
&\leq CC_2 \sum_{k \in \mathbb{Z}} 2^{kp} \sup_{\|h\|_{L^{(p/q)'}} \leq 1} \Big(\mathrm{I}_1^{\frac{p}{q}} + \mathrm{I}_2^{\frac{p}{q}} + \mathrm{I}_3^{\frac{p}{q}} + \mathrm{I}_4^{\frac{p}{q}}\Big).
\end{aligned}$$

We first estimate the sum corresponding to I_1. By the properties (i)–(iv) again, we can show that

$$
\begin{aligned}
&CC_2 \sum_{k\in\mathbb{Z}} 2^{kp} \sup_{\|h\|_{L^{(p/q)'}}\le 1} \mathrm{I}_1^{\frac{p}{q}} \\
&\le CC_2 \sum_{k\in\mathbb{Z}} 2^{kp} \sup_{\|h\|_{L^{(p/q)'}}\le 1} \left[2^q \sum_{i\le k}\sum_{j\ge k} \int_{A_i}\int_{A_j} 2^{-jq} \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}} h(x)\,dy\,dx\right]^{\frac{p}{q}} \\
&= CC_2 \sup_{\|h\|_{L^{(p/q)'}}\le 1} \left[\sum_{k\in\mathbb{Z}} 2^{kq} \sum_{i\le k}\sum_{j\ge k} \int_{A_i}\int_{A_j} 2^{-jq} \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}} h(x)\,dy\,dx\right]^{\frac{p}{q}} \\
&= CC_2 \sup_{\|h\|_{L^{(p/q)'}}\le 1} \left[\sum_{i\in\mathbb{Z}}\sum_{j\ge i}\sum_{k=i}^{j} \int_{A_i}\int_{A_j} 2^{(k-j)q} \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}} h(x)\,dy\,dx\right]^{\frac{p}{q}}
\end{aligned}
$$

Since $\sum\limits_{k=i}^{j} 2^{(k-j)q} < \sum\limits_{k=-\infty}^{j} 2^{(k-j)q} \le \frac{1}{1-2^{-q}}$ and by $q \ge 1$, it is obvious that $\frac{1}{1-2^{-q}} \le 2$. Thus,

$$
\begin{aligned}
&CC_2 \sum_{k\in\mathbb{Z}} 2^{kp} \sup_{\|h\|_{L^{(p/q)'}}\le 1} \mathrm{I}_1^{\frac{p}{q}} \\
&\le CC_2 \left(\frac{1}{1-2^{-q}}\right)^{\frac{p}{q}} \sup_{\|h\|_{L^{(p/q)'}}\le 1} \left[\int_\Omega \left(\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right) h(x)\,dx\right]^{\frac{p}{q}} \\
&\le CC_2 \sup_{\|h\|_{L^{(p/q)'}}\le 1} \left[\int_\Omega \left(\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}\,dy\right)^{\frac{p}{q}} dx\right] \left[\int_\Omega h(x)^{(p/q)'}\,dx\right]^{\frac{p/q}{(p/q)'}} \\
&\le CC_2 |f|^p_{\mathcal{F}^s_{p,q}}(\Omega),
\end{aligned}
$$

which is desired. The estimates corresponding to I_2, I_3 and I_4 are similar, the details being omitted. Thus, we conclude that

$$
\int_\Omega \frac{|f(x)|^p}{d(x,\partial\Omega)^{sp}}\,dx \le CC_2 |f|^p_{\mathcal{F}^s_{p,q}}(\Omega),
$$

which implies (i) by letting $C_1 = CC_2$ and hence completes the proof of Theorem 3. □

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