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Hermite-Hadamard-Fejér Type Inequalities with Generalized $\mathcal{K}$-Fractional Conformable Integrals and Their Applications

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Abstract: In this work, we introduce new definitions of left and right-sides generalized conformable $\mathcal{K}$-fractional derivatives and integrals. We also prove new identities associated with the left and right-sides of the Hermite-Hadamard-Fejér type inequality for ϕ-preinvex functions. Moreover, we use these new identities to prove some bounds for the Hermite-Hadamard-Fejér type inequality for generalized conformable $\mathcal{K}$-fractional integrals regarding ϕ-preinvex functions. Finally, we also present some applications of the generalized definitions for higher moments of continuous random variables, special means, and solutions of the homogeneous linear Cauchy-Euler and homogeneous linear $\mathcal{K}$-fractional differential equations to show our new approach.

Keywords: Hermite-Hadamard; ϕ-preinvex function; generalized conformable $\mathcal{K}$-fractional derivative; generalized conformable $\mathcal{K}$-fractional integral; Hölder's inequality; power mean inequality

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1. Introduction

The field of fractional calculus is the generalization of classical differential and integral calculus. There are vast applications of fractional calculus in pure and applied mathematics. There is a significant role of inequalities in different areas of mathematics, and it is active and exciting for researchers. Recently, it has been found that convexity plays a significant part in pure mathematics. A function $\mathcal{F} : \mathcal{I} \subseteq \Re \to \Re$ is named as convex, if the inequality

$$\mathcal{F}\big(\tau\kappa_1 + (1-\tau)\kappa_2\big) \le \tau\mathcal{F}(\kappa_1) + (1-\tau)\mathcal{F}(\kappa_2)$$

holds for all $\kappa_1, \kappa_2 \in \mathcal{I}$ and $\tau \in [0,1]$.

On using classical convexity, a lot of research has been performed on integral inequalities. However, one of the most essential and well-known inequalities is the Hermite-Hadamard inequality.

In [1], the remarkable inequality is stated as: Let $\mathcal{F} : \mathcal{I} \subseteq \Re \to \Re$ be a convex function on the interval $\mathcal{I}$ of real numbers and $\kappa_1, \kappa_2 \in \mathcal{I}$ with $\kappa_1 < \kappa_2$. Then,

$$\mathcal{F}\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\tau)d\tau \le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{1}$$

Suppose $\mathcal{F}$ is a concave function. In that case, the above inequalities will be reversed.

In the area of mathematical inequalities, many mathematicians have paid attention to the inequalities of Hermit-Hadamard due to their importance and applications. Many researchers have generalized the Hermit-Hadamard inequality using the classical convex function.

In [2], Fejér gave a weighted generalization of the inequality (1) for a convex function $\mathcal{F} : \mathcal{I} \subseteq \Re \to \Re$ as follows:

$$\mathcal{F}\Big(\frac{\kappa_1+\kappa_2}{2}\Big)\int_{\kappa_1}^{\kappa_2}\omega(\tau)d\tau \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\omega(\tau)\mathcal{F}(\tau)d\tau \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2}\omega(\tau)d\tau,$$

where $\omega : [\kappa_1;\kappa_2] \to \Re$ is non-negative, integrable, and symmetric about $\tau = \frac{\kappa_1+\kappa_2}{2}$.

First, we mention some preliminary concepts and results that will be helpful in the sequel, for more details see [3–5].

Let $\mathcal{F} : \Omega \to \Re$ and $\vartheta : \Omega \times \Omega \to \Re^n$, where Ω is a nonempty closed set in $\Re^n$, be continuous functions, and let $\phi : \Omega \to \Re$ be a continuous function.
In [6], Noor et al. introduced the concept of ϕ-invex set and ϕ-preinvex functions and related all properties, as follows:

Definition 1. *Let $\kappa_1 \in \Omega$. The set Ω is called ϑ-invex at κ_1 with respect to ϑ and ϕ if*

$$\kappa_1 + \tau e^{i\phi}\vartheta(\kappa_2,\kappa_1) \in \Omega$$

for all $\kappa_1, \kappa_2 \in \Omega$ and $\tau \in [0,1]$.

Remark 1. *Some special cases of Definition 1 are as follows.*
(1) *If $\phi = 0$, then Ω is called an invex set, see [7] and the references therein.*
(2) *If $\vartheta(\kappa_2,\kappa_1) = \kappa_2 - \kappa_1$, then Ω is called a ϕ-convex set, see [8] and the references therein.*
(3) *If $\phi = 0$ and $\vartheta(\kappa_2,\kappa_1) = \kappa_2 - \kappa_1$, then Ω is called a convex set.*

Definition 2. *A function $\mathcal{F}$ on the ϕ-invex set Ω is said to be ϕ-preinvex with respect to ϕ and ϑ, if*

$$\mathcal{F}\big(\kappa_1 + \tau e^{i\phi}\vartheta(\kappa_2,\kappa_1)\big) \leq (1-\tau)\mathcal{F}(\kappa_1) + \tau\mathcal{F}(\kappa_2) \tag{2}$$

holds for all $\kappa_1, \kappa_2 \in \Omega$ and $\tau \in [0,1]$. The function $\mathcal{F}$ is said to be ϕ-preconcave if and only if $-\mathcal{F}$ is ϕ-preinvex.

Remark 2. *Every convex function is a ϕ-preinvex function but not conversely. For example, the function $\mathcal{F}(\mu) = -|\mu|$ is not a convex function, but it is a ϕ-preinvex function with respect to ϑ and $\phi = 0$, where*

$$\vartheta(\kappa_2,\kappa_1) = \begin{cases} \kappa_2 - \kappa_1, & if\ either\ \kappa_1,\kappa_2 \leq\ 0\ or\ \kappa_1,\kappa_2 \geq\ 0, \\ \kappa_1 - \kappa_2, & otherwise. \end{cases} \tag{3}$$

Recently, researchers have expanded their work on Hermite-Hadamard-Fejér type inequalities in the fractional domain by using a wide application of fractional calculus. Hermite-Hadamard-Fejér type inequalities for various classes of function have been identified using fractional integrals.

2. Fractional Calculus

Fractional calculus is a branch of mathematics that deals with studying and applying arbitrary order integrals and derivatives. Fractional calculus is a topic that is both ancient and new at the same time. It is an old issue since, beginning with the assumptions of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed and studied up to the present day. There has been a significant increase in interest in fractional calculus in recent years. The applications have fueled that this calculus are found in numerical analysis and several areas of physics and engineering, including, presumably, fractal phenomena. The Hadamard inequality, which is well known in the field of fractional integrals, is the most celebrated inequality that has been studied for fractional integrals. Now, we give the definition of the conformable fractional derivative with its important properties, which are useful in order to obtain our main results, we suggest [9–17] for articles that deal with fractional integral inequalities using various forms of fractional integral operators to solve them.

In this section, we demonstrate some basic definitions related to fractional calculus.

Definition 3. *Let $\rho > 0$ and $\mathcal{F} \in L([\kappa_1, \kappa_2])$. Then the left and right-sided fractional integrals of Riemann–Liouville, we have*

$$\mathcal{J}_{\rho}^{\kappa_1}\mathcal{F}(\sigma) = \frac{1}{\Gamma(\rho)}\int_{\kappa_1}^{\sigma}(\sigma-\tau)^{\rho-1}\mathcal{F}(\tau)d\tau, \quad \sigma > \kappa_1$$

and

$$^{\kappa_2}\mathcal{J}_{\rho}\mathcal{F}(\sigma) = \frac{1}{\Gamma(\rho)}\int_{\sigma}^{\kappa_2}(\tau-\sigma)^{\rho-1}\mathcal{F}(\tau)d\tau, \quad \sigma < \kappa_2,$$

where $\Gamma(.)$ is the Gamma function.

In the case $\rho = 1$, the fractional integral reduces to the classical integral.

We now give the definition of $\mathcal{K}$-fractional integral which is mainly due to [8].

Definition 4. *Let $\mathcal{F} \in L([\kappa_1, \kappa_2])$. Then for the left-sided and right-sided $\mathcal{K}$-fractional integrals of order ρ, $\mathcal{K} > 0$ are defined as:*

$$\mathcal{J}_{\rho,\mathcal{K}}^{\kappa_1}\mathcal{F}(\sigma) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_{\kappa_1}^{\sigma}(\sigma-\tau)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{F}(\tau)d\tau, \quad \sigma > \kappa_1$$

and

$$^{\kappa_2}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{F}(\sigma) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_{\sigma}^{\kappa_2}(\tau-\sigma)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{F}(\tau)d\tau, \quad \sigma < \kappa_2,$$

where $\mathcal{K}$-gamma function is defined as $\Gamma_{\mathcal{K}}(\rho) = \int_0^{\infty}\tau^{\rho}e^{\frac{\tau^{\mathcal{K}}}{\mathcal{K}}}d\tau$.

A description of a conformable fractional derivative was suggested by Khalil et al. [18]. Consider $\mathcal{F} : [0, \infty) \to \Re$ is called the fractional derivative for $0 < \rho \leq 1$ at $\sigma > 0$,

$$\mathcal{D}_{\rho}(\mathcal{F})(\sigma) = \lim_{\varepsilon \to 0}\frac{\mathcal{F}(\sigma + \varepsilon\sigma^{1-\rho}) - \mathcal{F}(\sigma)}{\varepsilon}.$$

$\mathcal{D}_{\rho}$ satisfies the following properties

Theorem 1. *Let $\rho \in (0,1]$ and $\mathcal{F}, \mathcal{G}$ be ρ differentiable at point $\tau > 0$. Then*

(i) $\mathcal{D}_{\rho}(\kappa_1\mathcal{F} \pm \kappa_2\mathcal{G}) = \kappa_1\mathcal{D}_{\rho}(\mathcal{F}) \pm \kappa_2\mathcal{D}_{\rho}(\mathcal{G})$ *for all* $\kappa_1, \kappa_2 \in \Re$;

(ii) $\mathcal{D}_{\rho}(\mathcal{F}\mathcal{G}) = \mathcal{F}\mathcal{D}_{\rho}(\mathcal{G}) + \mathcal{G}\mathcal{D}_{\rho}(\mathcal{F})$;

(iii) $\mathcal{D}_{\rho}\left(\frac{\mathcal{F}}{\mathcal{G}}\right) = \frac{\mathcal{G}\mathcal{D}_{\rho}(\mathcal{F}) - \mathcal{F}\mathcal{D}_{\rho}(\mathcal{G})}{\mathcal{G}^2}$;

(iv) $\mathcal{D}_{\rho}(\kappa) = 0$, *such that κ is constant; and*

(v) $\mathcal{D}_{\rho}(\tau^s) = s\tau^{s-\rho}$, *such that $s \in \Re$.*

(vi) *Suppose that a function $\mathcal{F}$ is differentiable, then* $\mathcal{D}_\rho(\mathcal{F})(\tau) = \tau^{1-\rho}\frac{d\mathcal{F}}{d\tau}(\tau)$;
(vii) $\mathcal{D}_\rho(e^{\kappa\tau}) = \kappa\tau^{1-\rho}e^{\kappa\tau}, \kappa \in \Re$;
(viii) $\mathcal{D}_\rho(\sin\kappa\tau) = \kappa\tau^{1-\rho}\cos\kappa\tau, \kappa \in \Re$;
(ix) $\mathcal{D}_\rho\left(\frac{1}{\rho}\tau^\rho\right) = 1$;
(x) $\mathcal{D}_\rho\left(\sin\frac{1}{\rho}\tau^\rho\right) = \cos\frac{1}{\rho}\tau^\rho$;
(xi) $\mathcal{D}_\rho\left(\cos\frac{1}{\rho}\tau^\rho\right) = -\sin\frac{1}{\rho}\tau^\rho$; *and*
(xii) $\mathcal{D}_\rho\left(e^{\frac{1}{\rho}\tau^\rho}\right) = e^{\frac{1}{\rho}\tau^\rho}$.

In addition, a function $\mathcal{F} : [\kappa_1, \kappa_2] \to \Re$ is called the conformable integral of order ρ, which is proved in [18],

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\sigma)d_\rho\sigma = \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\sigma)\sigma^{\rho-1}d\sigma, \ \ \rho \ \in (0,1],$$

if the above integral exists and has a finite value.

Remark 3.
$$\mathcal{J}_\rho^{\kappa_1}(\mathcal{F})(s) = \mathcal{J}_1^{\kappa_1}\left(s^{\rho-1}\mathcal{F}\right) = \int_{\kappa_1}^{s} \frac{\mathcal{F}(\sigma)}{\sigma^{1-\rho}}d\sigma,$$
where the integral is the usual Riemann improper integral and $\rho \in (0,1]$.

The Hermite-Hadamard inequalities formed by Anderson [19] for conformable fractional integrals are as follows.

Suppose that $\mathcal{F} : [\kappa_1, \kappa_2] \to \Re$ is ρ-fractional differential function with $0 < \rho \leq 1$ and $\mathcal{F}$ is increasing, then we have

$$\frac{\rho}{\kappa_2^\rho - \kappa_1^\rho} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\sigma)\ d_\rho\sigma \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \tag{4}$$

If $\mathcal{F}$ is decreasing on $[\kappa_1, \kappa_2]$, then we have

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{\rho}{\kappa_2^\rho - \kappa_1^\rho} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\sigma)\ d_\rho\sigma. \tag{5}$$

Remark 4. *It is obvious that if we choose* $\rho = 1$, *then inequalities* (4) *and* (5) *reduce to inequality* (1).

The Hermite-Hadamard-Fejér type inequality for conformable integrals was suggested by Khurshid et al. in [20], which is as shown as:

Theorem 2. *Suppose* $\kappa_1, \kappa_2 \in \mathcal{I}$ *such that* $\vartheta(\kappa_2, \kappa_1) > 0$, $\mathcal{F} : \mathcal{I} = [\kappa_1, \kappa_1 + \vartheta(\kappa_2, \kappa_1)] \to [0, \infty)$ *is a preinvex function; it is symmetric for* $\frac{2\kappa_1 + \vartheta(\kappa_2,\kappa_1)}{2}$, *and function* $\mathcal{G} : [\kappa_1, \kappa_2] \to \Re$ *is a non-negative integrable . Moreover* ϑ *meets the condition C; then, we have:*

$$\mathcal{F}\left(\frac{2\kappa_1 + \vartheta(\kappa_2, \kappa_1)}{2}\right) \int_{\kappa_1}^{\kappa_1+\vartheta(\kappa_2,\kappa_1)} \mathcal{G}(\sigma)\ d_\rho\sigma \leq \int_{\kappa_1}^{\kappa_1+\vartheta(\kappa_2,\kappa_1)} \mathcal{F}(\sigma)\mathcal{G}(\sigma)\ d_\rho\sigma$$
$$\leq \left(\frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_1 + \vartheta(\kappa_2, \kappa_1))}{2}\right) \int_{\kappa_1}^{\kappa_1+\vartheta(\kappa_2,\kappa_1)} \mathcal{G}(\sigma)\ d_\rho\sigma,$$

retained for any $0 < \rho \leq 1$.

The following inequality correlated with the right part of Hermite-Hadamard inequality for preinvex functions is derived.

Theorem 3. *Let $\kappa_1, \kappa_2 \in \mathcal{I}$ such that $\vartheta(\kappa_2, \kappa_1) > 0$ and $\mathcal{F} : \mathcal{I} = [\kappa_1, \kappa_1 + \vartheta(\kappa_2, \kappa_1)] \to [0, \infty)$ be an $\rho-$differentiable function on $(\kappa_1, \kappa_1 + \vartheta(\kappa_2, \kappa_1))$ for $\rho \in (0, 1]$ such that $\mathcal{D}_\rho(\mathcal{F}) \in L_\rho([\kappa_1, \kappa_1 + \vartheta(\kappa_2, \kappa_1)])$. If $|\mathcal{F}'|$ is preinvex function, then :*

$$\left|\frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_1 + \vartheta(\kappa_2, \kappa_1))}{2} - \frac{\rho}{(\kappa_1 + \vartheta(\kappa_2, \kappa_1))^\rho - \kappa_1^\rho} \int_{\kappa_1}^{\kappa_1 + \vartheta(\kappa_2, \kappa_1)} \mathcal{F}(\sigma) d_\rho \sigma\right|$$
$$\leq \frac{\vartheta(\kappa_2, \kappa_1)}{4} \left[|\mathcal{F}'(\kappa_1)| + |\mathcal{F}'(\kappa_2)|\right].$$

This paper is organized as follows: in the next Section 3, the authors introduce the left and right generalized conformable $\mathcal{K}$-fractional derivatives and integrals, which are the $\mathcal{K}$-analogues of the recently introduced fractional conformable derivatives and integrals. In Section 4, we establish two new identities of Hermite-Hadamard-Fejér type inequalities by using ϕ-preinvex functions. In Section 5, we acquire two new weighted approximation versions which are associated with the left and right parts of the Hermite-Hadamard type inequalities for the generalized conformable $\mathcal{K}$-fractional by using ϕ-preinvex functions. In Section 6, we also present some applications to higher moments of continuous random variables, special means, and solutions of the homogeneous linear Cauchy-Euler and $\mathcal{K}$-fractional differential equations to show our new approach. The conclusion is given at the end of this work in Section 7.

3. Generalized Conformable $\mathcal{K}$-Fractional Derivatives and Integrals

Here, we introduce a new definition of a (left and right) generalized conformable $\mathcal{K}$-fractional derivative, which is defined as follows:

Definition 5. (Generalized left conformable $\mathcal{K}$-fractional derivative) *Let $0 < \rho \leq 1$ and the (left) $\mathcal{K}$-fractional derivative starting from κ_1 a function $\mathcal{F} : [\kappa_1, \infty] \to \Re$ with $\mathcal{K} > 0$ is follows as*

$$\mathcal{D}_{\rho,\mathcal{K}}^{\kappa_1}(\mathcal{F})(\tau) = \lim_{\varepsilon \to 0} \frac{\mathcal{F}(\tau + \varepsilon(\tau - \kappa_1)^{1-\frac{\rho}{\mathcal{K}}}) - \mathcal{F}(\tau)}{\varepsilon}. \tag{6}$$

When $\kappa_1 = 0$, we write $\mathcal{D}_{\mathcal{K},\rho}$. If $\left(\mathcal{D}_{\mathcal{K},\rho}\mathcal{F}\right)(\tau)$ exists on (κ_1, κ_2), then $\left(\mathcal{D}_{\mathcal{K},\rho}^{\kappa_1}\mathcal{F}\right)(\kappa_1) = \lim_{\tau \to \kappa_1^+}\left(\mathcal{D}_{\mathcal{K},\rho}^{\kappa}\mathcal{F}\right)(\tau)$.

(Generalized right conformable $\mathcal{K}$-fractional derivative) *Let $0 < \rho \leq 1$, and the (right) $\mathcal{K}$-fractional derivative terminating at κ_2 of a function $\mathcal{F} : [-\infty, \kappa_2] \to \Re$ with $\mathcal{K} > 0$ is as follows*

$${}^{\kappa_2}\mathcal{D}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) = -\lim_{\varepsilon \to 0} \frac{\mathcal{F}(\tau + \varepsilon(\kappa_2 - \tau)^{1-\frac{\rho}{\mathcal{K}}}) - \mathcal{F}(\tau)}{\varepsilon}. \tag{7}$$

If $\left({}^{\kappa_2}\mathcal{D}_{\mathcal{K},\rho}\mathcal{F}\right)(\tau)$ exists on (κ_1, κ_2), then $\left({}^{\kappa_2}\mathcal{D}_{\mathcal{K},\rho}\mathcal{F}\right)(\kappa_2) = \lim_{\tau \to \kappa_2^-}\left({}^{\kappa_2}\mathcal{D}_{\mathcal{K},\rho}\mathcal{F}\right)(\tau)$.

Note that if $\mathcal{F}$ is differentiable then $\mathcal{D}_{\rho,\mathcal{K}}^{\kappa_1}(\mathcal{F})(\tau) = (\tau - \kappa_1)^{1-\frac{\rho}{\mathcal{K}}}\mathcal{F}'(\tau)$ and ${}^{\kappa_2}\mathcal{D}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) = -(\kappa_2 - \tau)^{1-\frac{\rho}{\mathcal{K}}}\mathcal{F}'(\tau)$. It is clear that the conformable $\mathcal{K}$-fractional derivative of the constant function is zero.

Remark 5. *If we choose $\mathcal{K} = 1$, then (6) and (7) reduces the notion of left and right fractional conformable derivatives for a differentiable function $\mathcal{F}$, introduced by Abdeljawad [21], which is defined as*

$$\mathcal{D}_\rho^{\kappa_1}(\mathcal{F})(\tau) = \lim_{\varepsilon \to 0} \frac{\mathcal{F}(\tau + \varepsilon(\tau - \kappa_1)^{1-\rho}) - \mathcal{F}(\tau)}{\varepsilon},$$

and

$${}^{\kappa_2}\mathcal{D}_\rho(\mathcal{F})(\tau) = -\lim_{\varepsilon \to 0} \frac{\mathcal{F}(\tau + \varepsilon(\kappa_2 - \tau)^{1-\rho}) - \mathcal{F}(\tau)}{\varepsilon}.$$

Correspondingly, (left and right) generalized conformable $\mathcal{K}$-fractional integrals for $0 < \rho \leq 1$ can be represented by

Definition 6. (Generalized left conformable $\mathcal{K}$-fractional integral). *For a function $\mathcal{F} : [\kappa_1, \kappa_2] \to \Re$, $0 \leq \kappa_1 < \kappa_2$, then, the generalized left conformable $\mathcal{K}$-fractional integral of $\mathcal{F}$ of order $0 < \rho \leq 1$ with $\mathcal{K} > 0$ is defined as*

$$\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) = \int_{\kappa_1}^{\tau} \mathcal{F}(\sigma)\mathcal{D}_\rho(\sigma, \kappa_1)d\sigma = \int_{\kappa_1}^{\tau} \frac{\mathcal{F}(\sigma)}{(\sigma - \kappa_1)^{1-\frac{\rho}{\mathcal{K}}}} d\sigma, \quad \tau > \kappa_1. \tag{8}$$

(Generalized right $\mathcal{K}$-fractional conformable integral). *For a function $\mathcal{F} : [\kappa_1, \kappa_2] \to \Re$, $0 \leq \kappa_1 < \kappa_2$, then, the generalized right conformable $\mathcal{K}$-fractional integral of $\mathcal{F}$ of order $0 < \rho \leq 1$ with $\mathcal{K} > 0$ is defined as*

$$^{\kappa_2}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) = \int_{\tau}^{\kappa_2} \mathcal{F}(\sigma)\mathcal{D}(\kappa_2, \sigma)d\sigma = \int_{\tau}^{\kappa_2} \frac{\mathcal{F}(\sigma)}{(\kappa_2 - \sigma)^{1-\frac{\rho}{\mathcal{K}}}} d\sigma, \quad \tau < \kappa_2. \tag{9}$$

Remark 6. *If we choose $\mathcal{K} = 1$, then (8) and (9) reduces the notion of left and right fractional conformable integrals for a function $\mathcal{F}$, introduced by Abdeljawad [21], which is defined as*

$$\mathcal{J}^{\kappa_1}_{\rho}(\mathcal{F})(\tau) = \int_{\kappa_1}^{\tau} \frac{\mathcal{F}(\sigma)}{(\sigma - \kappa_1)^{1-\rho}} d\sigma, \ \ \tau > \kappa_1,$$

and

$$^{\kappa_2}\mathcal{J}_{\rho}(\mathcal{F})(\tau) = \int_{\tau}^{\kappa_2} \frac{\mathcal{F}(\sigma)}{(\kappa_2 - \sigma)^{1-\rho}} d\sigma, \ \ \tau < \kappa_2.$$

Lemma 1. *Suppose that $\mathcal{F} : [\kappa_1, \infty) \to \Re$ is continuous, and $0 < \rho \leq 1$ with $\mathcal{K} > 0$, then for all $\tau > \kappa_1$ we have*

$$\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{F}(\tau) = \mathcal{F}(\tau). \tag{10}$$

Proof. Since $\mathcal{F}$ is continuous, then $\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{F}(\tau)$ is clearly differentiable. Hence,

$$\begin{aligned} \mathcal{D}_\rho\left(\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{F}\right)(\tau) &= \tau^{1-\rho}\frac{d}{d\tau}\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) \\ &= \tau^{1-\rho}\frac{d}{d\tau}\int_{\kappa_1}^{\tau}\frac{\mathcal{F}(x)}{x^{1-\rho}}dx \\ &= \tau^{1-\rho}\frac{\mathcal{F}(\tau)}{\tau^{1-\rho}} \\ &= \mathcal{F}(\tau). \end{aligned}$$

□

Lemma 2. *Suppose that $\mathcal{F} : (-\infty, \kappa_2] \to \Re$ is continuous, and $0 < \rho \leq 1$ with $\mathcal{K} > 0$, then, for all $\tau < \kappa_2$ we have*

$$^{\kappa_2}\mathcal{D}_{\rho,\mathcal{K}}\, ^{\kappa_2}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{F}(\tau) = \mathcal{F}(\tau). \tag{11}$$

Proof. Similarly we can proved Lemma 2. So we omit the proof. □

Lemma 3. *Suppose that $\mathcal{F} : (\kappa_1, \kappa_2) \to \Re$ is differentiable, and $0 < \rho \leq 1$ with $\mathcal{K} > 0$, then, for all $\tau > \kappa_1$ we have*

$$\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{F}(\tau) = \mathcal{F}(\tau) - \mathcal{F}(\kappa_1). \tag{12}$$

Proof. Since $\mathcal{F}$ is a differentiable function, so

$$\begin{aligned}\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) &= \int_{\kappa_1}^{\tau}(\sigma-\kappa_1)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{D}_{\rho,\mathcal{K}}\mathcal{F}(\sigma)d\sigma \\ &= \int_{\kappa_1}^{\tau}(\sigma-\kappa_1)^{\frac{\rho}{\mathcal{K}}-1}(\sigma-\kappa_1)^{1-\frac{\rho}{\mathcal{K}}}\mathcal{F}'(\sigma)d\sigma = \mathcal{F}(\tau)-\mathcal{F}(\kappa_1). \quad (13)\end{aligned}$$

□

Lemma 3 can be generalized for the higher order as follows.

Proposition 1. *Let $\rho \in (n, n+1]$ and $\mathcal{F} : [\kappa_1, \infty) \to \Re$ be $(n+1)$ times differentiable for $\tau > \kappa_1$. Then, for all $\tau > \kappa_1$, we have*

$$\mathcal{J}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) = \mathcal{F}(\tau) - \sum_{l=0}^{n}\frac{\mathcal{F}^{(l)}(\kappa_1)(\tau-\kappa_1)^l}{l!}.$$

Proof. By using definition and Theorem 2.1 in [18], we have proved Proposition 1. □

Proposition 2. *Let $\rho \in (n, n+1]$ and $\mathcal{F} : (-\infty, \kappa_2] \to \Re$ be $(n+1)$ times differentiable for $\tau < \kappa_2$. Then, for all $\tau < \kappa_2$ we have*

$${}^{\kappa_2}\mathcal{J}_{\rho,\mathcal{K}}\, {}^{\kappa_2}\mathcal{D}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) = \mathcal{F}(\tau) - \sum_{l=0}^{n}\frac{(-1)^l\mathcal{F}^{(l)}(\kappa_2)(\kappa_2-\tau)^l}{l!}.$$

Proof. By using definition and Theorem 2.1 in [18], we have proved Proposition 2. □

Remark 7. *If $n = 0$ or $0 < \rho \leq 1$ with $\mathcal{K} > 0$ in proposition 2, then, ${}^{\kappa_2}\mathcal{J}_{\rho,\mathcal{K}}\, {}^{\kappa_2}\mathcal{D}_{\rho,\mathcal{K}}(\mathcal{F})(\tau) = \mathcal{F}(\tau) - \mathcal{F}(\kappa_2)$.*

Theorem 4. *(Chain Rule). Suppose that $\mathcal{F}, \mathcal{G} : (\kappa_1, \infty) \to \Re$ are (left) $\mathcal{K}$-differentiable functions, where $0 < \rho \leq 1$ with $\mathcal{K} > 0$. Let $\mathcal{H}(\tau) = \mathcal{F}(\mathcal{G}(\tau))$. Then $\mathcal{H}(\tau)$ is (left) $\mathcal{K}$-differentiable, and for all τ with $\tau \neq \kappa_1$ and $\mathcal{G}(\tau) \neq 0$, we have*

$$\left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{H}\right)(\tau) = \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}(\mathcal{F})\right)(\mathcal{G}(\tau)) \times \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{G}\right)(\tau) \times \mathcal{G}(\tau)^{\frac{\rho}{\mathcal{K}}-1}.$$

If $\tau = \kappa_1$, we have

$$\left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{H}\right)(\kappa_1) = \lim_{\tau\to\kappa_1^+}\left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}(\mathcal{F})\right)(\mathcal{G}(\tau)) \times \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{G}\right)(\tau) \times \mathcal{G}(\tau)^{\frac{\rho}{\mathcal{K}}-1}.$$

Proof. By setting $\mu = \tau + \epsilon(\tau-\kappa_1)^{1-\frac{\rho}{\mathcal{K}}}$ in the definition and using continuity of $\mathcal{G}$, then

$$\begin{aligned}\left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{H}\right)(\tau) &= \lim_{\mu\to\tau}\frac{\mathcal{F}(\mathcal{G}(\mu))-\mathcal{F}(\mathcal{G}(\tau))}{\mu-\tau} \times \tau^{1-\frac{\rho}{\mathcal{K}}} \\ \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{H}\right)(\tau) &= \lim_{\mu\to\tau}\frac{\mathcal{F}(\mathcal{G}(\mu))-\mathcal{F}(\mathcal{G}(\tau))}{\mathcal{G}(\mu)-\mathcal{G}(\tau)} \times \lim_{\mu\to\tau}\frac{\mathcal{G}(\mu)-\mathcal{G}(\tau)}{\mu-\tau} \times \tau^{1-\frac{\rho}{\mathcal{K}}} \\ \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{H}\right)(\tau) &= \lim_{\mathcal{G}(\mu)\to\mathcal{G}(\tau)}\frac{\mathcal{F}(\mathcal{G}(\mu))-\mathcal{F}(\mathcal{G}(\tau))}{\mathcal{G}(\mu)-\mathcal{G}(\tau)} \times \mathcal{G}(\tau)^{1-\frac{\rho}{\mathcal{K}}} \times \mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{G}(\tau) \times \mathcal{G}(\tau)^{\frac{\rho}{\mathcal{K}}-1} \\ \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{H}\right)(\tau) &= \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}(\mathcal{F})\right)(\mathcal{G}(\tau)) \times \left(\mathcal{D}^{\kappa_1}_{\rho,\mathcal{K}}\mathcal{G}\right)(\tau) \times \mathcal{G}(\tau)^{\frac{\rho}{\mathcal{K}}-1}.\end{aligned}$$

□

In this section, we proved the key Lemmas important to prove our main results.

4. Key Lemmas

Throughout this section, we will let $\|\mathcal{W}\|_\infty = \sup_{\tau\in[\kappa_2,\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)]}|\mathcal{W}(\tau)|$, where $\mathcal{W} : [\kappa_2, \kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)] \to \Re$ is a continuous function, $\mathcal{F}'$ is the derivative of $\mathcal{F}$ with respect to variable τ, and $L[\kappa_2, \kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)]$ is the collection of all real-valued Riemann integrable functions defined on $[\kappa_2, \kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)]$ and $0 < \rho \leq 1$.

Lemma 4. *Suppose that an open invex set $\Omega \subseteq \Re$ and ϑ is a continuous bifunction, and $\vartheta : \Omega\times\Omega \to \Re$. Let a function $\mathcal{F} : \Omega \to \Re$ be differentiable and $\mathcal{F}' \in L[\kappa_2, \kappa_2 + e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)]$ with $\kappa_1,\kappa_2 \in \Omega$ and $e^{i\phi}\ \vartheta(\kappa_1,\kappa_2) > 0$. If $\mathcal{W} : [\kappa_2, \kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)] \to \Re$ be an integrable function it is usually symmetric for $\kappa_2 + \frac{1}{2}e^{i\phi}\vartheta(\kappa_1,\kappa_2)$, then*

$$
\begin{aligned}
&\frac{\mathcal{F}\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}}\left[\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\,\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
&-\frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}}\left[\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
&= \int_0^1 \mathcal{U}(\tau,v)\mathcal{F}'\left(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau,
\end{aligned} \tag{14}
$$

where

$$
\mathcal{U}(\tau,v) = \begin{cases} \int_0^\tau v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv, & \tau \in \left[0,\frac{1}{2}\right); \\ \int_1^\tau (1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv, & \tau \in \left[\frac{1}{2},1\right], \end{cases} \tag{15}
$$

and $0 \leq \phi \leq \frac{\pi}{2}$ with $\mathcal{K} > 0$.

Proof. Let

$$
\begin{aligned}
\mathcal{A} &= \int_0^1 \mathcal{U}(\tau,v)\mathcal{F}'\left(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau \\
&= \int_0^{\frac{1}{2}}\left(\int_0^\tau v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv\right)\mathcal{F}'\left(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau \\
&+ \int_{\frac{1}{2}}^1\left(\int_1^\tau (1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv\right)\mathcal{F}'\left(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau \\
&:= \mathcal{A}_1 + \mathcal{A}_2,
\end{aligned}
$$

considering the first integral

$$
\begin{aligned}
\mathcal{A}_1 &= \int_0^{\frac{1}{2}}\left(\int_0^\tau v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv\right)\mathcal{F}'\left(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau \\
&= \frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\left(\int_0^\tau v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv\right)\mathcal{F}\left(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\Big|_0^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)}\int\limits_0^{\frac{1}{2}}\tau^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\mathcal{F}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau \\
&=\frac{\mathcal{F}\Big(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\Big)}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int\limits_0^{\frac{1}{2}}\tau^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau \\
&\quad-\frac{1}{e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)}\int\limits_0^{\frac{1}{2}}\tau^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\mathcal{F}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau.
\end{aligned} \tag{16}
$$

Making the change of variable $z=\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)$ in the above inequality (16), we have

$$
\begin{aligned}
\mathcal{A}_1 \quad &=\frac{\mathcal{F}\Big(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\Big)}{\big(e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\ \vartheta(\kappa_1,\kappa_2)}(z-\kappa_2)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}(z)dz \\
&\quad-\frac{1}{\big(e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\ \vartheta(\kappa_1,\kappa_2)}(z-\kappa_2)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{F}(z)\mathcal{W}(z)dz \\
&=\frac{\mathcal{F}\big(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\big)\Gamma(\rho)}{\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\big(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\big) \\
&\quad-\frac{1}{\big(e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\big(\mathcal{F}\mathcal{W}\big)\big(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\big).
\end{aligned} \tag{17}
$$

Now,

$$
\begin{aligned}
\mathcal{A}_2 \;&=\int\limits_{\frac{1}{2}}^{1}\left(\int\limits_{1}^{\tau}(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\right)\mathcal{F}'\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau \\
&=\frac{1}{e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)}\left(\int\limits_{1}^{\tau}(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\Big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)dv\right)\mathcal{F}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\Big|_{\frac{1}{2}}^{1} \\
&\quad-\frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int\limits_{\frac{1}{2}}^{1}(1-\tau)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\mathcal{F}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau \\
&=-\frac{\mathcal{F}\Big(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\Big)}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int\limits_{1}^{\frac{1}{2}}(1-\tau)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau \\
&\quad-\frac{1}{e^{i\phi}\ \vartheta(\kappa_1,\kappa_2))}\int\limits_{\frac{1}{2}}^{1}(1-\tau)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{F}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\mathcal{W}\Big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau.
\end{aligned} \tag{18}
$$

Making the change of variable $z=\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)$ in the above inequality (18), we have

$$
\mathcal{A}_2 \quad =\frac{\mathcal{F}\Big(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\Big)}{\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\int_{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)-z\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}(z)dz
$$

$$
\begin{aligned}
&-\frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}}\int_{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)-z\right)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{F}(z)\mathcal{W}(z)dz \\
=&\frac{\mathcal{F}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}}\,{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right) \\
&-\frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}}\,{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right).
\end{aligned}
\tag{19}
$$

By adding (17) and (19), we obtain the result that we needed. □

Lemma 5. *If $\mathcal{W}:[\kappa_2,\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)]\to\Re$ is an integrable function, it is usually symmetric for $\kappa_2+\frac{1}{2}e^{i\phi}\vartheta(\kappa_1,\kappa_2)$ with $e^{i\phi}\vartheta(\kappa_1,\kappa_2)>0,\ \ 0\le\phi\le\frac{\pi}{2}$ and $\mathcal{K}>0$. Then*

$$
\begin{aligned}
&{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2)=\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right) \\
&=\frac{1}{2}\left[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\right].
\end{aligned}
\tag{20}
$$

Proof. Since $\mathcal{W}$ is symmetric about $\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)$, we have $\mathcal{W}\left(2\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)-z\right)=\mathcal{W}(z)$, for all $z\in[\kappa_2,\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)]$. Putting $2\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)-\tau=z$, one obtains

$$
\begin{aligned}
{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2) &= \int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\left[\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)-\tau\right]^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}(\tau)d\tau \\
&= \int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}(z-\kappa_2)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}(2\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)-z)dz \\
&= \int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}(z-\kappa_2)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}(z)dz \\
&= \mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right).
\end{aligned}
$$

Hence, the proof is complete. □

Lemma 6. *Suppose that an open invex set $\Omega\subseteq\Re$, ϑ is a continuous bifunction, and $\vartheta:\Omega\times\Omega\to\Re$. Let a function $\mathcal{F}:\Omega\to\Re$ be differentiable and $\mathcal{F}'\in L[\kappa_2,\kappa_2+e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)]$ with $\kappa_1,\kappa_2\in\Omega$ and $e^{i\phi}\ \vartheta(\kappa_1,\kappa_2)>0$. If $\mathcal{W}:[\kappa_2,\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)]\to\Re$ is an integrable function, it is usually symmetric for $\kappa_2+\frac{1}{2}e^{i\phi}\vartheta(\kappa_1,\kappa_2)$, then*

$$
\begin{aligned}
&\frac{\mathcal{F}(\kappa_2)+\mathcal{F}\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)}{2}\left[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
&-\left[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
&=\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}\int_0^1\eta(\tau,v)\mathcal{F}'\left(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau,
\end{aligned}
\tag{21}
$$

where,

$$
\eta(\tau,v)=\int_0^{\tau}(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv+\int_1^{\tau}v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv,\quad \tau\in[0,1],
$$

with $0\le\phi\le\frac{\pi}{2}$ and $\mathcal{K}>0$.

Proof. Let

$$
\begin{aligned}
&\int_0^1 \eta(\tau,v)\mathcal{F}'\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau \\
&= \int_0^1 \Bigg[\int_0^\tau (1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv \\
&+\int_1^\tau v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\Bigg]\mathcal{F}'\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau \\
&= \int_0^1 \Bigg[\int_0^\tau (1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\Bigg]\mathcal{F}'\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau \\
&\quad +\int_0^1 \Bigg[\int_1^\tau v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\Bigg]\mathcal{F}'\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau \\
&:= \mathcal{P}_1+\mathcal{P}_2.
\end{aligned}
\tag{22}
$$

Now

$$
\begin{aligned}
\mathcal{P}_1 &= \int_0^1 \Bigg[\int_0^\tau (1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\Bigg]\mathcal{F}'\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau \\
&= \frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\Bigg[\bigg(\int_0^\tau (1-v)^{\frac{\rho}{\mathcal{K}}-1}\,\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\bigg)\mathcal{F}(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big|_0^1\Bigg] \\
&\quad -\frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int_0^1 (1-\tau)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\mathcal{F}\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau \\
&= \frac{\mathcal{F}(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int_0^1 (1-\tau)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau \\
&\quad -\frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int_0^1 (1-\tau)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\mathcal{F}\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau.
\end{aligned}
\tag{23}
$$

By using the change of variable technique, $z=\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)$ in the above equation, we have

$$
\begin{aligned}
\mathcal{P}_1 &= \frac{\mathcal{F}(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}{\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)-z\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}(z)dz \\
&\quad -\frac{1}{\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)-z\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{F}(z)\mathcal{W}(z)dz
\end{aligned}
$$

$$
= \frac{\mathcal{F}(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}{\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\,{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2)-\frac{1}{\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}}\,{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})(\kappa_2). \tag{24}
$$

For $\mathcal{P}_2$, we have

$$\begin{aligned}
\mathcal{P}_2 &= \int_0^1 \left[\int_1^\tau v^{\frac{\rho}{\mathcal{K}}-1} \mathcal{W}(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2))dv \right] \mathcal{F}'(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2))d\tau \\
&= \frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \left[\left(\int_1^\tau v^{\frac{\rho}{\mathcal{K}}-1} \mathcal{W}(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)dv) \right) \mathcal{F}(\kappa_2 + ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big|_0^1 \right] \\
&\quad - \frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \int_0^1 \tau^{\frac{\rho}{\mathcal{K}}-1} \mathcal{W}(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2))\mathcal{F}(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2))d\tau \\
&= \frac{\mathcal{F}(\kappa_2)}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \int_0^1 \tau^{\frac{\rho}{\mathcal{K}}-1} \mathcal{W}(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2))d\tau \\
&\quad - \frac{1}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \int_0^1 \tau^{\frac{\rho}{\mathcal{K}}-1} \mathcal{W}(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2))\mathcal{F}\Big(\kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)d\tau.
\end{aligned}$$

By using the change of variable technique, $z = \kappa_2 + \tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)$ in the above equation, we have

$$\begin{aligned}
\mathcal{P}_2 &= \frac{\mathcal{F}(\kappa_2)}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)} (z-\kappa_2)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}(z)dz \\
&\quad - \frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)} (z-\kappa_2)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{F}(z)\mathcal{W}(z)dz \\
&= \frac{\mathcal{F}(\kappa_2)}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}((\kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)) \\
&\quad - \frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})(\kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)).
\end{aligned} \tag{25}$$

By adding (24) and (25), utilizing (20), we obtain the desired results. □

5. Inequalities for Generalized $\mathcal{K}$-Fractional Conformable Integrals

In this section, we present some $\mathcal{K}$-analogues of Hermite-Hadamard-Fejér type inequalities for generalized conformable $\mathcal{K}$-fractional integrals.

Theorem 5. *Suppose that an open invex set $\Omega \subseteq \Re$, ϑ a continuous bifunction, and $\vartheta : \Omega \times \Omega \to \Re$. Let a function $\mathcal{F} : \Omega \to \Re$ be differentiable, and $\mathcal{F}' \in L[\kappa_2, \kappa_2 + e^{i\phi}\,\vartheta(\kappa_1,\kappa_2)]$ with $\kappa_1, \kappa_2 \in \Omega$ and $e^{i\phi}\ \vartheta(\kappa_1,\kappa_2) > 0$. If $\mathcal{W} : [\kappa_2, \kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)] \to \Re$ is an integrable function, which is usually symmetric for $\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)$. If $|\mathcal{F}'|$ is a ϕ-preinvex function on Ω, then we have:*

$$\begin{aligned}
&\Bigg| \frac{\mathcal{F}\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \left[\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
&- \frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \left[\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right] \Bigg| \\
&\le \frac{\|\mathcal{W}\|_\infty}{2^{\frac{\rho}{\mathcal{K}}+1}\left(\frac{\rho}{\mathcal{K}}+1\right)} \Big(|\mathcal{F}'(\kappa_2)| + |\mathcal{F}'(\kappa_1)|\Big),
\end{aligned} \tag{26}$$

where $0 \le \phi \le \frac{\pi}{2}$ *and* $\mathcal{K} > 0$.

Proof. Using Lemma 4, the modulus property, and ϕ-preinvexity of $|\mathcal{F}'|$ on Ω, we have

$$\begin{aligned}
&\Bigg|\frac{\mathcal{F}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}}\left[\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)+^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right]\\
&-\frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}}\left[\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)+^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right]\Bigg|\\
&=\Bigg|\int_0^{\frac{1}{2}}\left[\int_0^{\tau}v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv\right]\mathcal{F}'\left(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau\\
&\quad+\int_{\frac{1}{2}}^{1}\left[-\int_{\tau}^{1}(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv\right]\mathcal{F}'\left(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)d\tau\Bigg| \qquad (27)\\
&\le\int_0^{\frac{1}{2}}\left[\int_0^{\tau}v^{\frac{\rho}{\mathcal{K}}-1}\left|\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\right|dv\right]\left[(1-\tau)\left|\mathcal{F}'(\kappa_2)\right|+\tau\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau\\
&\quad+\int_{\frac{1}{2}}^{1}\left[\int_{\tau}^{1}(1-v)^{\frac{\rho}{\mathcal{K}}-1}\left|\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\right|dv\right]\left[(1-\tau)\left|\mathcal{F}'(\kappa_2)\right|+\tau\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau\\
&=\mathcal{Q}_1+\mathcal{Q}_2.
\end{aligned}$$

Change the order of integration in the first term of (27) to obtain the following result

$$\begin{aligned}
\mathcal{Q}_1&=\int_0^{\frac{1}{2}}\left[\int_0^{\tau}v^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)dv\right]\left[(1-\tau)\left|\mathcal{F}'(\kappa_2)\right|+\tau\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau\\
&=\int_0^{\frac{1}{2}}v^{\frac{\rho}{\mathcal{K}}-1}\left|\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\right|\int_v^{\frac{1}{2}}\left[(1-\tau)\left|\mathcal{F}'(\kappa_2)\right|+\tau\left|\mathcal{F}'(\kappa_1)\right|\right]d\tau dv \qquad (28)\\
&=\int_0^{\frac{1}{2}}v^{\frac{\rho}{\mathcal{K}}-1}\left|\mathcal{W}\left(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)\right|\left[\left(\frac{(1-v)^2}{2}-\frac{1}{8}\right)\left|\mathcal{F}'(\kappa_2)\right|+\left(\frac{1}{8}-\frac{v^2}{2}\right)\left|\mathcal{F}'(\kappa_1)\right|\right]dv.
\end{aligned}$$

By the change of variable technique $z=\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)$, for every $v\in[0,1]$.

$$\begin{aligned}
\mathcal{Q}_1=&\frac{|\mathcal{F}'(\kappa_2)|}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)}\left(\frac{1}{2}\left(1-\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^2-\frac{1}{8}\right)\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^{\frac{\rho}{\mathcal{K}}-1}|\mathcal{W}(z)|dz\\
&+\frac{|\mathcal{F}'(\kappa_1)|}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^2\right)\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^{\frac{\rho}{\mathcal{K}}-1}|\mathcal{W}(z)|dz. \qquad (29)
\end{aligned}$$

Consider $\|\mathcal{W}\|_\infty=\sup_{\tau\in[\kappa_2,\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)]}|\mathcal{W}(z)|$, we obtain

$$
\begin{aligned}
\mathcal{Q}_1 \leq\ & \frac{\|\mathcal{W}\|_\infty |\mathcal{F}'(\kappa_2)|}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)} \left(\frac{1}{2}\left(1-\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^2-\frac{1}{8}\right)\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^{\frac{\rho}{\mathcal{K}}-1} dz \\
& + \frac{\|\mathcal{W}\|_\infty |\mathcal{F}'(\kappa_1)|}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)} \left(\frac{1}{8}-\frac{1}{2}\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^2\right)\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^{\frac{\rho}{\mathcal{K}}-1} dz. \qquad (30)
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
\mathcal{Q}_2 \leq\ & \frac{\|\mathcal{W}\|_\infty |\mathcal{F}'(\kappa_2)|}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)} \left(\frac{1}{8}-\frac{1}{2}\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^2\right)\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^{\frac{\rho}{\mathcal{K}}-1} dz \\
& + \frac{\|\mathcal{W}\|_\infty |\mathcal{F}'(\kappa_1)|}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)} \int_{\kappa_2}^{\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)} \left(\frac{1}{2}\left(1-\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^2-\frac{1}{8}\right)\left(\frac{z-\kappa_2}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^{\frac{\rho}{\mathcal{K}}-1} dz. \qquad (31)
\end{aligned}
$$

By adding (30) and (31), then substituting in (27), we obtain the desired result. □

Corollary 1.
I. *Letting $\mathcal{K}=1$ in Theorem 5, then, we have a new result*

$$
\begin{aligned}
& \Bigg| \frac{\mathcal{F}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\rho+1}} \left[\mathcal{J}_{\rho}^{\kappa_2}\mathcal{W}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho}\mathcal{W}\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
& - \frac{1}{\left(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\right)^{\rho+1}} \left[\mathcal{J}_{\rho}^{\kappa_2}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)\right)\right]\Bigg| \qquad (32) \\
& \leq \frac{\|\mathcal{W}\|_\infty}{2^{\rho+1}(\rho+1)}\Big(|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|\Big).
\end{aligned}
$$

II. *Letting $\phi=0$ in Theorem 5, then, we have a new result*

$$
\begin{aligned}
& \Bigg| \frac{\mathcal{F}\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right)}{\left(\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \left[\mathcal{J}_{\rho,\mathcal{K}}^{\kappa_2}\mathcal{W}\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
& - \frac{1}{\left(\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \left[\mathcal{J}_{\rho,\mathcal{K}}^{\kappa_2}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right)\right]\Bigg| \qquad (33) \\
& \leq \frac{\|\mathcal{W}\|_\infty}{2^{\frac{\rho}{\mathcal{K}}+1}\left(\frac{\rho}{\mathcal{K}}+1\right)}\Big(|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|\Big).
\end{aligned}
$$

III. *Letting $\phi=0$ and $\mathcal{K}=1$ in Theorem 5, then, we have*

$$
\begin{aligned}
& \Bigg| \frac{\mathcal{F}\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right)}{\left(\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \left[\mathcal{J}_{\rho}^{\kappa_2}\mathcal{W}\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho}\mathcal{W}\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right)\right] \\
& - \frac{1}{\left(\vartheta(\kappa_1,\kappa_2)\right)^{\frac{\rho}{\mathcal{K}}+1}} \left[\mathcal{J}_{\rho}^{\kappa_2}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right) + {}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho}(\mathcal{F}\mathcal{W})\left(\kappa_2+\frac{1}{2}\vartheta(\kappa_1,\kappa_2)\right)\right]\Bigg| \qquad (34) \\
& \leq \frac{\|\mathcal{W}\|_\infty}{2^{\rho+1}(\rho+1)}\Big(|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|\Big).
\end{aligned}
$$

Theorem 6. *Suppose that an open invex set $\Omega \subseteq \Re$, ϑ is a continuous bifunction, and $\vartheta : \Omega\times\Omega \to \Re$. Let a function $\mathcal{F} : \Omega \to \Re$ be differentiable, and $\mathcal{F}' \in L[\kappa_2, \kappa_2 + e^{i\phi}\,\vartheta(\kappa_1,\kappa_2)]$ with*

$\kappa_1, \kappa_2 \in \Omega$ and $e^{i\phi}\ \vartheta(\kappa_1,\kappa_2) > 0$. If $\mathcal{W} : [\kappa_2, \kappa_2 + e^{i\phi}\vartheta(\kappa_1,\kappa_2)] \to \Re$ is an integrable function, which is usually symmetric for $\kappa_2 + \frac{e^{i\phi}}{2}\vartheta(\kappa_1,\kappa_2)$. If $|\mathcal{F}'|$ be a ϕ-preinvex function on Ω, then, we have:

$$\begin{aligned}&\Bigg|\frac{\mathcal{F}(\kappa_2)+\mathcal{F}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)}{2}\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\,\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\Big]\\&\quad-\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\,(\mathcal{F}\mathcal{W})(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\Big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\Big]\Bigg|\\&\leq\|\mathcal{W}\|_\infty\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\left(\frac{|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|}{2\rho}\right),\end{aligned}\tag{35}$$

where $0 \leq \phi \leq \frac{\pi}{2}$ and $\mathcal{K} > 0$.

Proof. Using Lemma 6 and the modulus property, we have

$$\begin{aligned}&\Bigg|\frac{\mathcal{F}(\kappa_2)+\mathcal{F}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)}{2}\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\,\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\Big]\\&\quad-\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\,(\mathcal{F}\mathcal{W})(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\Big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\Big]\Bigg|\\&=\Bigg|\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\int_0^1\Bigg(-\int_\tau^1(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\\&\quad+\int_0^\tau(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\Bigg)\mathcal{F}'\big(\kappa_2+\tau e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)d\tau\Bigg|.\end{aligned}\tag{36}$$

From ϕ-preinvexity of $|\mathcal{F}'|$ on Ω, we have

$$\begin{aligned}&\Bigg|\frac{\mathcal{F}(\kappa_2)+\mathcal{F}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)}{2}\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\,\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\Big]\\&\quad-\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\,(\mathcal{F}\mathcal{W})(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\Big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\Big]\Bigg|\\&\leq\Bigg|\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\int_0^1\Bigg(-\int_\tau^1(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\\&\quad+\int_0^\tau(1-v)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)dv\Bigg)\Bigg|\big((1-\tau)\big|\mathcal{F}'(\kappa_2)\big|+\tau\big|\mathcal{F}'(\kappa_1)\big|\big)d\tau.\end{aligned}\tag{37}$$

We have achieved this by adjusting the order of integration

$$
\begin{aligned}
&\Bigg|\frac{\mathcal{F}(\kappa_2)+\mathcal{F}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)}{2}\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\Big]\\
&\quad-\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\Big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\Big]\Bigg|\\
&\le\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\int_0^1(1-v)^{\frac{\rho}{\mathcal{K}}-1}\Big|\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\Big|\int_0^v\big[(1-\tau)\big|\mathcal{F}'(\kappa_2)\big|+\tau\big|\mathcal{F}'(\kappa_1)\big|\big]d\tau dv\\
&\quad+\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\int_0^1(1-v)^{\frac{\rho}{\mathcal{K}}-1}\big|\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\big|\int_v^1\big[(1-\tau)\big|\mathcal{F}'(\kappa_2)\big|+\tau\big|\mathcal{F}'(\kappa_1)\big|\big]d\tau dv\\
&=\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\int_0^1(1-v)^{\frac{\rho}{\mathcal{K}}-1}\big|\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\big|\int_0^1\big[(1-\tau)\big|\mathcal{F}'(\kappa_2)\big|+\tau\big|\mathcal{F}'(\kappa_1)\big|\big]d\tau dv\\
&=\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\left(\frac{|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|}{2}\right)\int_0^1(1-v)^{\frac{\rho}{\mathcal{K}}-1}\big|\mathcal{W}\big(\kappa_2+ve^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\big|dv
\end{aligned}\tag{38}
$$

By the change of variable technique, $z=\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)$, and using the fact that $\|\mathcal{W}\|_\infty=\sup_{\tau\in[\kappa_2,\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)]}|\mathcal{W}(z)|$, we have

$$
\begin{aligned}
&\Bigg|\frac{\mathcal{F}(\kappa_2)+\mathcal{F}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)}{2}\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\Big]\\
&\quad-\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})\Big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\Big]\Bigg|\\
&\le\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}}\|\mathcal{W}\|_\infty\left(\frac{|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|}{2}\right)\int_{\kappa_2}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\left(\frac{\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)-z}{e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\right)^{\frac{\rho}{\mathcal{K}}-1}dz\\
&=\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\frac{\rho}{\mathcal{K}}+1}\|\mathcal{W}\|_\infty\left(\frac{|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|}{2\rho}\right),
\end{aligned}\tag{39}
$$

which is the desired result. □

Corollary 2.
I. *Letting $\mathcal{K}=1$ in Theorem 6, then, we have a new result*

$$
\begin{aligned}
&\Bigg|\frac{\mathcal{F}(\kappa_2)+\mathcal{F}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)}{2}\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho}\mathcal{W}(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho}\mathcal{W}\big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)\Big]\\
&\quad-\Big[{}^{\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho}(\mathcal{F}\mathcal{W})(\kappa_2)+\mathcal{J}^{\kappa_2}_{\rho}(\mathcal{F}\mathcal{W})\Big(\kappa_2+e^{i\phi}\vartheta(\kappa_1,\kappa_2)\Big)\Big]\Bigg|\\
&\le\|\mathcal{W}\|_\infty\big(e^{i\phi}\vartheta(\kappa_1,\kappa_2)\big)^{\rho+1}\left(\frac{|\mathcal{F}'(\kappa_2)|+|\mathcal{F}'(\kappa_1)|}{2\rho}\right).
\end{aligned}\tag{40}
$$

II. *Letting $\phi = 0$ in Theorem 6, then, we have a new result*

$$\left|\frac{\mathcal{F}(\kappa_2) + \mathcal{F}(\kappa_2 + \vartheta(\kappa_1, \kappa_2))}{2}\left[{}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2) + \mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}\mathcal{W}(\kappa_2 + \vartheta(\kappa_1, \kappa_2))\right]\right.$$
$$\left. - \left[{}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{J}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})(\kappa_2) + \mathcal{J}^{\kappa_2}_{\rho,\mathcal{K}}(\mathcal{F}\mathcal{W})(\kappa_2 + \vartheta(\kappa_1, \kappa_2))\right]\right| \tag{41}$$
$$\leq \|\mathcal{W}\|_\infty(\vartheta(\kappa_1, \kappa_2))^{\frac{\rho}{\mathcal{K}}+1}\left(\frac{|\mathcal{F}'(\kappa_2)| + |\mathcal{F}'(\kappa_1)|}{2\rho}\right).$$

III. *Letting $\phi = 0$ and $\rho = 1$ along with $\mathcal{K} = 1$ in Theorem 6, then, we have a new result*

$$\left|\frac{\mathcal{F}(\kappa_2) + \mathcal{F}(\kappa_2 + \vartheta(\kappa_1, \kappa_2))}{2}\int_{\kappa_2}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{W}(z)dz - \int_{\kappa_2}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)}\mathcal{F}(z)\mathcal{W}(z)dz\right|$$
$$\leq \|\mathcal{W}\|_\infty(\vartheta(\kappa_1, \kappa_2))^2\left(\frac{|\mathcal{F}'(\kappa_2)| + |\mathcal{F}'(\kappa_1)|}{2}\right). \tag{42}$$

IV. *Letting $\vartheta(\kappa_1, \kappa_2) = \kappa_1 - \kappa_2$ in Corollary 2 part **III.**, then, we have a new result*

$$\left|\frac{\mathcal{F}(\kappa_2) + \mathcal{F}(\kappa_1)}{2}\int_{\kappa_2}^{\kappa_1}\mathcal{W}(z)dz - \int_{\kappa_2}^{\kappa_1}\mathcal{F}(z)\mathcal{W}(z)dz\right|$$
$$\leq \|\mathcal{W}\|_\infty(\kappa_1 - \kappa_2)^2\left(\frac{|\mathcal{F}'(\kappa_2)| + |\mathcal{F}'(\kappa_1)|}{2}\right). \tag{43}$$

6. Applications

6.1. Random Variable

Suppose that for $0 < \kappa_2 < \kappa_1$, $\mathcal{W} : [\kappa_2, \kappa_2 + \vartheta(\kappa_1, \kappa_2)] \to \Re^+$ is a continuous probability density of a continuous random variable X that is symmetric about $\kappa_2 + \frac{1}{2}\vartheta(\kappa_1, \kappa_2)$. Furthermore, for $r \in \Re$, suppose that the rth moment

$$E_r(X) = \int_{\kappa_2}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)} x^r\mathcal{W}(x)dx$$

is finite.

Letting $\mathcal{F}(x) = x^r$ on $[\kappa_2, \kappa_2 + \vartheta(\kappa_1, \kappa_2)]$ for $r \geq 2$, then, the function $|\mathcal{F}'(x)| = rx^{r-1}$ is a preinvex function. Therefore, using this function in Corollary 2 part **III.**, we have

$$\left|\frac{\kappa_2^r + (\kappa_2 + \vartheta(\kappa_1, \kappa_2))^r}{2} - E_r(x)\right| \leq \frac{r\|\mathcal{W}\|_\infty(\vartheta(\kappa_1, \kappa_2))^2}{2}\left(|\kappa_2|^{r-1} + |\kappa_1|^{r-1}\right), \tag{44}$$

since $\mathcal{W}$ is symmetric and $\int_{\kappa_2}^{\kappa_2+\vartheta(\kappa_1,\kappa_2)} \mathcal{W}(x)dx = 1$.

Corollary 3.
I. *Let $r = 1$ in (44), and $E(X)$ is the expectation of the random variable X, from the above inequality, we obtain the following known bound*

$$\left|\frac{2\kappa_2 + \vartheta(\kappa_1, \kappa_2)}{2} - E_1(x)\right| \leq \frac{\|\mathcal{W}\|_\infty(\vartheta(\kappa_1, \kappa_2))^2}{2}.$$

II. *Letting* $\vartheta(\kappa_1, \kappa_2) = \kappa_1 - \kappa_2$ *in Corollary 3 part* ***I.****, we obtain the following known bound*

$$\left|\frac{\kappa_1 + \kappa_2}{2} - E_1(x)\right| \leq \frac{\|\mathcal{W}\|_\infty (\kappa_1 - \kappa_2)^2}{2}.$$

6.2. Special Means

In the literature, the following means for real numbers $\kappa_1, \kappa_2 \in \Re$ are well known:

$$\mathcal{A}(\kappa_2, \kappa_1) = \frac{\kappa_2 + \kappa_1}{2} \quad arthimetic\ mean,$$

$$\mathcal{L}_r(\kappa_2, \kappa_1) = \left[\frac{\kappa_1^{r+1} - \kappa_2^{r+1}}{(\kappa_1 - \kappa_2)(r+1)}\right]^{\frac{1}{r}} \quad generalized\ \log - mean,\ r \in \mathcal{N},\ \kappa_2 < \kappa_1.$$

Consider $\mathcal{F}(z) = z^r$ for $z > 0,\ r \in \mathcal{N},\ \phi = 0,\ \mathcal{K} = \rho = 1$, and a differentiable symmetric to $\kappa_2 + \frac{1}{2}\vartheta(\kappa_1, \kappa_2)$ mapping $\mathcal{W} : [\kappa_2, \kappa_2 + \vartheta(\kappa_1, \kappa_2)] \to \Re^+$. Theorem 6 implies the following inequality

$$\begin{aligned}
&\left|\frac{\kappa_2^r + \left(\kappa_2 + \vartheta(\kappa_1, \kappa_2)\right)^r}{2} \int_{\kappa_2}^{\kappa_2 + \vartheta(\kappa_1, \kappa_2)} \mathcal{W}(z)dz - \int_{\kappa_2}^{\kappa_2 + \vartheta(\kappa_1, \kappa_2)} z^r \mathcal{W}(z)dz\right| \\
&\leq \frac{r\|\mathcal{W}\|_\infty \left(\vartheta(\kappa_1, \kappa_2)\right)^2}{2} \left(|\kappa_2|^{r-1} + |\kappa_1|^{r-1}\right).
\end{aligned}$$

So

$$\begin{aligned}
&\left|\mathcal{A}\left(\kappa_2^r, \left(\kappa_2 + \vartheta(\kappa_1, \kappa_2)\right)^r\right) \int_{\kappa_2}^{\kappa_2 + \vartheta(\kappa_1, \kappa_2)} \mathcal{W}(z)dz - \int_{\kappa_2}^{\kappa_2 + \vartheta(\kappa_1, \kappa_2)} z^r \mathcal{W}(z)dz\right| \\
&\leq r\|\mathcal{W}\|_\infty \left(\vartheta(\kappa_1, \kappa_2)\right)^2 \mathcal{A}\left(|\kappa_2|^{r-1}, |\kappa_1|^{r-1}\right).
\end{aligned} \tag{45}$$

Corollary 4.
I. *Letting* $\mathcal{W} = 1$ *in (45), then, we recapture the following result*

$$\left|\mathcal{A}\left(\kappa_2^r, \left(\kappa_2 + \vartheta(\kappa_1, \kappa_2)\right)^r\right) - \mathcal{L}_r^r\left(\kappa_2^r, \left(\kappa_2 + \vartheta(\kappa_1, \kappa_2)\right)^r\right)\right| \leq r\|\mathcal{W}\|_\infty \left(\vartheta(\kappa_1, \kappa_2)\right)^2 \mathcal{A}\left(|\kappa_2|^{r-1}, |\kappa_1|^{r-1}\right),\ \ for\, r \geq 2.$$

II. *Letting* $\vartheta(\kappa_1, \kappa_2) = \kappa_1 - \kappa_2$ *in (45), then, we recapture the following result*

$$\left|\mathcal{A}(\kappa_2^r, \kappa_1^r) \int_{\kappa_2}^{\kappa_1} \mathcal{W}(z)dz - \int_{\kappa_2}^{\kappa_1} z^r \mathcal{W}(z)dz\right| \leq r\|\mathcal{W}\|_\infty (\kappa_1 - \kappa_2)^2 \mathcal{A}\left(|\kappa_2|^{r-1}, |\kappa_1|^{r-1}\right),\ \ for\, r \geq 2.$$

III. *Letting* $\mathcal{W} = 1$ *in Corollary 4 part* ***II.****, then, we recapture the following result*

$$\left|\mathcal{A}(\kappa_2^r, \kappa_1^r) - \mathcal{L}_r^r(\kappa_2^r, \kappa_1^r)\right| \leq r(\kappa_1 - \kappa_2)^2 \mathcal{A}\left(|\kappa_2|^{r-1}, |\kappa_1|^{r-1}\right),\ \ for\, r \geq 2.$$

6.3. Examples

In this subsection, we use generalized conformable $\mathcal{K}$-fractional derivative to solve homogeneous and nonhomogeneous differential equations.

Example 1. *Consider the following homogeneous linear Cauchy-Euler* $\mathcal{K}$*-fractional differential equation*

$$\kappa_1 \frac{\tau^{\frac{2\rho}{\mathcal{K}}}}{\frac{\rho^2}{\mathcal{K}^2}} \mathcal{D}^0_{2\rho,\mathcal{K}} y(\tau) + \kappa_2 \frac{\tau^{\frac{\rho}{\mathcal{K}}}}{\frac{\rho}{\mathcal{K}}} \mathcal{D}^0_{\rho,\mathcal{K}} y(\tau) + \kappa_3 y(\tau) = 0, \quad 0 < \rho \leq 1, \quad \tau, \mathcal{K} > 0, \tag{46}$$

where κ_1, κ_2, and κ_3 are real constants.

We look for a solution of the form $y(\tau) = \tau^{\frac{s\rho}{\mathcal{K}}}$. Then, by definition of generalized conformable $\mathcal{K}$-fractional derivative, we have $\mathcal{D}^0_{\rho,\mathcal{K}}y(\tau) = s\frac{\rho}{\mathcal{K}}\tau^{\frac{(s-1)\rho}{\mathcal{K}}}$ and $\mathcal{D}^0_{2\rho,\mathcal{K}}y(\tau) = s(s-1)\frac{\rho^2}{\mathcal{K}^2}\tau^{\frac{(s-2)\rho}{\mathcal{K}}}$. Substitution of the formulas $\mathcal{D}^0_{\rho,\mathcal{K}}y(\tau)$, $\mathcal{D}^0_{2\rho,\mathcal{K}}y(\tau)$ and $y(\tau)$ into Equation (46) gives the auxiliary equation of Equation (46), which is $\kappa_1 s^2 + (\kappa_2 - \kappa_1)s + \kappa_3 = 0$. The auxiliary equation of Equation (46) has two roots with three possibilities for solution. The first one, if s_1 and s_2 are distinct real numbers, is the general solution of the form $y(\tau) = C_1\tau^{\frac{s_1\rho}{\mathcal{K}}} + C_2\tau^{\frac{s_2\rho}{\mathcal{K}}}$. If the roots are repeated $s_1 = s_2 = s$, then, the solution has the form $y(\tau) = C_1\tau^{\frac{s\rho}{\mathcal{K}}} + C_2\tau^{\frac{s\rho}{\mathcal{K}}}\ln\tau$. In the third probability, if the two roots are complex numbers, $s_{1,2} = \sigma \pm i\eta$ the general solution has the form $y(\tau) = C_1\tau^{\frac{\sigma\rho}{\mathcal{K}}}\cos\left(\frac{\rho}{\mathcal{K}}\eta\ln\tau\right) + C_2\tau^{\frac{\sigma\rho}{\mathcal{K}}}\sin\left(\frac{\rho}{\mathcal{K}}\eta\ln\tau\right)$.

Example 2. *Consider the following homogeneous linear $\mathcal{K}$-fractional differential equation*

$$\kappa_1\mathcal{D}^0_{2\rho,\mathcal{K}}y(\tau) + \kappa_2\mathcal{D}^0_{\rho,\mathcal{K}}y(\tau) + \kappa_3 y(\tau) = 0, \quad 0 < \rho \le 1, \quad \tau, \mathcal{K} > 0, \tag{47}$$

where κ_1, κ_2, and κ_3 are real constants.

We look for a solution of the form $y(\tau) = e^{\frac{s\mathcal{K}}{\rho}\tau^{\frac{\rho}{\mathcal{K}}}}$. Then, by definition of the generalized conformable $\mathcal{K}$-fractional derivative, we have $\mathcal{D}^0_{\rho,\mathcal{K}}y(\tau) = se^{\frac{s\mathcal{K}}{\rho}\tau^{\frac{\rho}{\mathcal{K}}}}$ and $\mathcal{D}^0_{2\rho,\mathcal{K}}y(\tau) = s^2e^{\frac{s\mathcal{K}}{\rho}\tau^{\frac{\rho}{\mathcal{K}}}}$. Substitution of the formulas $\mathcal{D}^0_{\rho,\mathcal{K}}y(\tau)$, $\mathcal{D}^0_{2\rho,\mathcal{K}}y(\tau)$ and $y(\tau)$ into Equation (47) gives the auxiliary equation of Equation (47), which is $\kappa_1 s^2 + \kappa_2 s + \kappa_3 = 0$. The auxiliary equation of Equation (47) has two roots with three possibilities for solution. The first one, if s_1 and s_2 are distinct real numbers, is the general solution of the form $y(\tau) = C_1e^{\frac{s_1\mathcal{K}}{\rho}\tau^{\frac{\rho}{\mathcal{K}}}} + C_2e^{\frac{s_2\mathcal{K}}{\rho}\tau^{\frac{\rho}{\mathcal{K}}}}$. If the roots are repeated $s_1 = s_2 = s$, then, the solution has the form $y(\tau) = (C_1 + \tau C_2)e^{\frac{s\mathcal{K}}{\rho}\tau^{\frac{\rho}{\mathcal{K}}}}$. The third probability, if the two roots are complex numbers, $s_{1,2} = \sigma \pm i\eta$ the general solution has the form $y(\tau) = e^{\frac{\sigma\mathcal{K}}{\rho}\tau^{\frac{\rho}{\mathcal{K}}}}\left[C_1\cos\left(\frac{\rho}{\mathcal{K}}\eta\tau\right) + C_2\sin\left(\frac{\rho}{\mathcal{K}}\eta\tau\right)\right]$.

7. Concluding Remarks

The authors of this article presented the left and right sides of generalized conformable $\mathcal{K}$-fractional derivatives and integrals on the left and right sides, respectively. We also addressed some new estimates for the lower and upper boundaries of the Hermite-Hadamard-Fej'er type inequality found for ϕ-preinvex functions using generalized conformable $\mathcal{K}$-fractional integrals, which offer fresh error bounds to the literature for the lower and higher boundaries of the Hermite-Hadamard-Fej'er type inequality for ϕ-preinvex functions in fractional domain. Using the results of this study, the reader can deduce a number of previously reported Hermite-Hadamard-Fejér type inequalities, as well as several new Hadamard and -Fejér–Hadamard type inequalities.

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