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Some New Midpoint and Trapezoidal-Type Inequalities for General Convex Functions in q-Calculus

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Abstract: The main objective of this study is to establish two important right q-integral equalities involving a right-quantum derivative with parameter $m \in [0,1]$. Then, utilizing these equalities, we derive some new variants for midpoint- and trapezoid-type inequalities for the right-quantum integral via differentiable (α, m)-convex functions. The fundamental benefit of these inequalities is that they may be transformed into q-midpoint- and q-trapezoid-type inequalities for convex functions, classical midpoint inequalities for convex functions and classical trapezoid-type inequalities for convex functions are transformed without having to prove each one independently. In addition, we present some applications of our results to special means of positive real numbers. It is expected that the ideas and techniques may stimulate further research in this field.

Keywords: midpoint inequalities; trapezoid inequalities; quantum calculus; (α, m)-convex functions

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1. Introduction

It is well known that modern investigation, directly or indirectly, involves the applications of convexity. The concept of convex sets (and, thus, of convex functions), has been widely generalized in numerous areas due to its widespread application and relevance. The concept of convexity, in its different forms, has played an important role in the advancement of various fields. Convex functions are powerful tools for proving a large class of inequalities. Today, the study of convex functions has evolved into a broader theory of functions including quasi-convex functions [1–3], log convex functions [4], coordinated convex functions [5,6], harmonically convex functions [7], GA-convex functions [8,9], and (α, m)-convex functions [10]. Convexity naturally gives rise to inequalities. The Hermite–Hadamard inequality is the first consequence of convex functions. A function $\mathfrak{F} : \mathfrak{I} \to \mathbb{R}$, where $\mathfrak{I}$ is a nonempty interval in $\mathbb{R}$ is called convex, if it satisfies the inequality

$$\mathfrak{F}(\mathfrak{w}\mathfrak{r} + (1-\mathfrak{w})\mathfrak{s}) \leq \mathfrak{w}\mathfrak{F}(\mathfrak{r}) + (1-\mathfrak{w})\mathfrak{F}(\mathfrak{s}),$$

where $\mathfrak{r}, \mathfrak{s} \in \mathfrak{I}$ and $\mathfrak{w} \in [0,1]$.

G. Toader first presented the concept of m-convexity in [11] and it is defined as follows:

Definition 1 ([11]). *A mapping $\mathfrak{F} : [0, \mathfrak{y}_2) \to \mathbb{R}$ is m-convex, if the inequality*

$$\mathfrak{F}(\mathfrak{w}\mathfrak{r} + m(1-\mathfrak{w})\mathfrak{s}) \leq \mathfrak{w}\mathfrak{F}(\mathfrak{r}) + m(1-\mathfrak{w})\mathfrak{F}(\mathfrak{s})$$

holds for all $\mathfrak{r}, \mathfrak{s} \in [0, \mathfrak{y}_2)$, $\mathfrak{w} \in [0,1]$ and $m \in [0,1]$.

Mihesan introduced the class of (α, m)-convex functions and stated it as:

Definition 2 ([10]). *A mapping $\mathfrak{F} : [0, \mathfrak{y}_2) \to \mathbb{R}$ is (α, m)-convex, if the inequality*

$$\mathfrak{F}(\mathfrak{w}\mathfrak{r} + m(1 - \mathfrak{w})\mathfrak{s}) \leq \mathfrak{w}^{\alpha}\mathfrak{F}(\mathfrak{r}) + m(1 - \mathfrak{w}^{\alpha})\mathfrak{F}(\mathfrak{s})$$

holds for all $\mathfrak{r}, \mathfrak{s} \in [0, \mathfrak{y}_2)$, $\mathfrak{w} \in [0, 1]$, $\alpha \in [0, 1]$ and $m \in [0, 1]$.

$\mathfrak{F}$ is also known to be convex if and only if it obeys the Hermite–Hadamard inequality, which is as follows:

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}\right) \leq \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1} \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r}) d\mathfrak{r} \leq \frac{\mathfrak{F}(\mathfrak{y}_1) + \mathfrak{F}(\mathfrak{y}_2)}{2},$$

where $\mathfrak{F} : \mathfrak{I} \to \mathbb{R}$ is a convex function and $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathfrak{I}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$. Convexity is mixed with other mathematical concepts, such as optimization [12], time scale [13,14], quantum, and post-quantum calculus [15].

On the other hand, in [15], Alp et al. proved the following version of the quantum Hermite–Hadamard type for convex functions using the left-quantum integrals:

$$\mathfrak{F}\left(\frac{q\mathfrak{y}_1 + \mathfrak{y}_2}{[2]_q}\right) \leq \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1} \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r} \leq \frac{q\mathfrak{F}(\mathfrak{y}_1) + \mathfrak{F}(\mathfrak{y}_2)}{[2]_q}. \tag{1}$$

Recently, Bermudo et al. [16] used the right-quantum integrals and proved the following variant of the Hermite–Hadamard type inequality for convex functions:

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1 + q\mathfrak{y}_2}{[2]_q}\right) \leq \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1} \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r} \leq \frac{\mathfrak{F}(\mathfrak{y}_1) + q\mathfrak{F}(\mathfrak{y}_2)}{[2]_q}. \tag{2}$$

For the left and right estimates of inequalities (1) and (2), one can consult [17–24]. In [25], Noor et al. established a generalized version of (1). In [26–29], the authors used convexity and coordinated convexity to prove Simpson's and Newton's type inequalities via q-calculus. For the study of Ostrowski's inequalities, one can consult [30–32]. For the theory and applications of calculus theory, one can consult [33,34].

Inspired by the ongoing studies, we derive some new inequalities of midpoint and trapezoid type inequalities for (α, m)-convex functions by utilizing quantum calculus. The fundamental benefit of these inequalities is that these can be turned into quantum midpoint- and trapezoid-type inequalities for convex functions [15,24], classical midpoint inequalities for convex functions [35], and the classical trapezoid type inequalities for convex functions [36] without having to prove each one separately.

The structure of this work is as follows: a quick introduction of the principles of q-calculus, as well as other related works in this subject, are presented in Section 2. We construct two pivotal identities in Section 3 that play a main role in the development of the primary results of the paper. In Sections 4 and 5, the midpoint- and trapezoid-type inequalities for q-differentiable functions are derived using q-integrals. Section 6 discusses the applications of special means. The relationship between the findings provided in this paper and similar discoveries in the literature is also considered. Section 7 finishes with some future research ideas.

2. Preliminaries and Definitions of q-Calculus

The definitions and properties of quantum integrals are presented first in this section. We also mention some well-known inequalities for quantum integrals. Throughout this paper, let $0 < q < 1$ be a constant.

The q-number or q-analogue of $n \in \mathbb{N}$ is given by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Definition 3 ([37]). *Let $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a continuous function. Then the left q-derivative of function $\mathfrak{F}$ at $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$ is defined by*

$$ {}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) = \begin{cases} \dfrac{\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{y}_1)}{(1-q)(\mathfrak{r} - \mathfrak{y}_1)}, & \text{if } \mathfrak{r} \neq \mathfrak{y}_1; \\ \lim\limits_{\mathfrak{r} \to \mathfrak{y}_1} {}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}), & \text{if } \mathfrak{r} = \mathfrak{y}_1. \end{cases} \tag{3}$$

The function $\mathfrak{F}$ is said to be q-differentiable function on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if ${}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})$ exists for all $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$.

Note that if $\mathfrak{y}_1 = 0$ and ${}_0D_q\mathfrak{F}(\mathfrak{r}) = D_q\mathfrak{F}(\mathfrak{r})$, then (3) reduces to

$$D_q\mathfrak{F}(\mathfrak{r}) = \begin{cases} \dfrac{\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(q\mathfrak{r})}{(1-q)\mathfrak{r}}, & \text{if } \mathfrak{r} \neq 0; \\ \lim\limits_{\mathfrak{r} \to 0} D_q\mathfrak{F}(\mathfrak{r}), & \text{if } \mathfrak{r} = 0, \end{cases}$$

which is the q-Jackson derivative—see [37–39] for more details.

Theorem 1 ([37]). *For the q-differentiable functions $\mathfrak{F}, \mathfrak{g} : J \to \mathbb{R}$, the following equalities hold:*

(i) *The product $\mathfrak{F}\mathfrak{g} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ is q-differentiable on $[\mathfrak{y}_1, \mathfrak{y}_2]$ with*

$$\begin{aligned} {}_{\mathfrak{y}_1}D_q(\mathfrak{F}\mathfrak{g})(\mathfrak{r}) &= \mathfrak{F}(\mathfrak{r}){}_{\mathfrak{y}_1}D_q\mathfrak{g}(\mathfrak{r}) + \mathfrak{g}(q\mathfrak{r} + (1-q)\mathfrak{r}){}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) \\ &= \mathfrak{g}(\mathfrak{r}){}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) + \mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{r}){}_{\mathfrak{y}_1}D_q\mathfrak{g}(\mathfrak{r}). \end{aligned}$$

(ii) *If $\mathfrak{g}(\mathfrak{r})\mathfrak{g}(q\mathfrak{r} + (1-q)\mathfrak{r}) \neq 0$, then $\mathfrak{F}/\mathfrak{g}$ is q-differentiable on $[\mathfrak{y}_1, \mathfrak{y}_2]$ with*

$$ {}_{\mathfrak{y}_1}D_q\left(\frac{\mathfrak{F}}{\mathfrak{g}}\right)(\mathfrak{r}) = \frac{\mathfrak{g}(\mathfrak{r}){}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(\mathfrak{r}){}_{\mathfrak{y}_1}D_q\mathfrak{g}(\mathfrak{r})}{\mathfrak{g}(\mathfrak{r})\mathfrak{g}(q\mathfrak{r} + (1-q)\mathfrak{r})}.$$

Definition 4 ([37]). *Let $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a continuous function. Then the left q-integral of function $\mathfrak{F}$ at $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$ is defined by*

$$\int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r}){}_{\mathfrak{y}_1}d_q\mathfrak{r} = (1-q)(z - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n\mathfrak{F}(q^n z + (1-q^n)\mathfrak{y}_1). \tag{4}$$

The function $\mathfrak{F}$ is said to be a q-integrable function on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if $\int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r}){}_{\mathfrak{y}_1}d_q\mathfrak{r}$ exists for all $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$.

Note that if $\mathfrak{y}_1 = 0$, then (4) reduces to

$$\int_0^z \mathfrak{F}(\mathfrak{r}){}_0d_q\mathfrak{r} = \int_0^z \mathfrak{F}(\mathfrak{r})d_q\mathfrak{r} = (1-q)z\sum_{n=0}^{\infty} q^n\mathfrak{F}(q^n z),$$

which is the q-Jackson integral (see [37–39] for more details).

Theorem 2 ([37]). *If $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ is a continuous function and $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$, then the following identities hold:*

(i) ${}_{\mathfrak{y}_1}D_q \int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r}){}_{\mathfrak{y}_1}d_q\mathfrak{r} = \mathfrak{F}(z)$;

(ii) $\displaystyle\int_c^z {}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1}d_q\mathfrak{r} = \mathfrak{F}(z) - \mathfrak{F}(c)$ *for* $c \in (\mathfrak{y}_1, z)$.

Bermudo et al. [16], on the other hand, defined a novel quantum derivative and quantum integral known as the right q-derivative and right q-integral:

Definition 5. *The right q-derivative of mapping* $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *is defined as:*

$$^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r}) = \frac{\mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{y}_2) - \mathfrak{F}(\mathfrak{r})}{(1-q)(\mathfrak{y}_2 - \mathfrak{r})}, \quad \mathfrak{r} \neq \mathfrak{y}_2.$$

If $\mathfrak{r} = \mathfrak{y}_2$, *we define* $^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2) = \lim_{\mathfrak{r}\to\mathfrak{y}_2} {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})$ *if it exists and it is finite.*

Definition 6. *The right q-definite integral of mapping* $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *on* $[\mathfrak{y}_1, \mathfrak{y}_2]$ *is defined as:*

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})^{\mathfrak{y}_2}d_q\mathfrak{r} = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{k=0}^{\infty} q^k\mathfrak{F}\Big(q^k\mathfrak{y}_1 + \Big(1-q^k\Big)\mathfrak{y}_2\Big). \tag{5}$$

Lemma 1 ([40])**.** *For continuous functions* $\mathfrak{F}, \mathfrak{g} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, *the following equality true:*

$$\begin{aligned}
&\int_0^c \mathfrak{g}(\mathfrak{w})^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + (1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} \\
= \quad & -\frac{\mathfrak{g}(\mathfrak{w})\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + (1-\mathfrak{w})\mathfrak{y}_2)}{\mathfrak{y}_2 - \mathfrak{y}_1}\Bigg|_0^c + \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_0^c D_q\mathfrak{g}(\mathfrak{w})\mathfrak{F}(q\mathfrak{w}\mathfrak{y}_1 + (1-q\mathfrak{w})\mathfrak{y}_2)\, d_q\mathfrak{w}.
\end{aligned}$$

3. Key Equalities

In this section, we prove two important quantum integral equalities utilizing the integration by parts method for quantum integrals, which are helpful to obtain our main results.

Lemma 2. *For* $m \in (0,1]$ *with* $0 < q < 1$, *assume that there is an arbitrary function* $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *such that* $^{\mathfrak{y}_2}D_q\mathfrak{F}$ *is continuous and integrable on* $[\mathfrak{y}_1, \mathfrak{y}_2]$. *Then, one has the identity:*

$$\begin{aligned}
q(m\mathfrak{y}_2 - \mathfrak{y}_1)\Bigg[&\int_0^{\frac{1}{[2]_q}} \mathfrak{w}\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} \\
&+ \int_{\frac{1}{[2]_q}}^1 \left(\mathfrak{w} - \frac{1}{q}\right) {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\Bigg] \\
&= \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r} - \mathfrak{F}\left(\frac{\mathfrak{y}_1 + qm\mathfrak{y}_2}{[2]_q}\right).
\end{aligned} \tag{6}$$

Proof. From fundamental properties of quantum integrals, we have

$$\begin{aligned}
&\Bigg[\int_0^{\frac{1}{[2]_q}} \mathfrak{w}\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} + \int_{\frac{1}{[2]_q}}^1 \left(\mathfrak{w} - \frac{1}{q}\right) {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\Bigg] \\
&= \Bigg[\int_0^{\frac{1}{[2]_q}} \mathfrak{w}\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} + \int_0^1 \left(\mathfrak{w} - \frac{1}{q}\right) {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} \\
&\quad - \int_0^{\frac{1}{[2]_q}} \left(\mathfrak{w} - \frac{1}{q}\right) {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\Bigg] \\
&= \mathfrak{I}_1 + \mathfrak{I}_2 - \mathfrak{I}_3.
\end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned}\mathfrak{I}_1 &= \int_0^{\frac{1}{[2]_q}} \mathfrak{w}\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} \\ &= -\mathfrak{w}\frac{\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)}{(m\mathfrak{y}_2 - \mathfrak{y}_1)}\Bigg|_0^{\frac{1}{[2]_q}} + \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)}\int_0^{\frac{1}{[2]_q}} \mathfrak{F}(\mathfrak{w}q\mathfrak{y}_1 + m(1-\mathfrak{w}q)\mathfrak{y}_2)d_q\mathfrak{w} \\ &= -\frac{1}{[2]_q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}\left(\frac{\mathfrak{y}_1 + qm\mathfrak{y}_2}{[2]_q}\right) + \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)}\int_0^{\frac{1}{[2]_q}} \mathfrak{F}(\mathfrak{w}q\mathfrak{y}_1 + m(1-\mathfrak{w}q)\mathfrak{y}_2)d_q\mathfrak{w}. \end{aligned} \tag{7}$$

Similarly, we have

$$\begin{aligned}\mathfrak{I}_2 &= \int_0^1 \left(\mathfrak{w} - \frac{1}{q}\right) {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} \\ &= -\frac{q-1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}(\mathfrak{y}_1) - \frac{1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}(m\mathfrak{y}_2) \\ &+ \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)}\int_0^1 \mathfrak{F}(\mathfrak{w}q\mathfrak{y}_1 + m(1-\mathfrak{w}q)\mathfrak{y}_2)d_q\mathfrak{w} \\ &= -\frac{q-1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}(\mathfrak{y}_1) - \frac{1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}(m\mathfrak{y}_2) \\ &+ \frac{1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)^2}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})^{m\mathfrak{y}_2}d_q\mathfrak{r} - \frac{1-q}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}(\mathfrak{y}_1) \end{aligned} \tag{8}$$

and

$$\begin{aligned}\mathfrak{I}_3 &= \int_0^{\frac{1}{[2]_q}} \left(\mathfrak{w} - \frac{1}{q}\right) {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w} \\ &= \frac{1}{q[2]_q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}\left(\frac{\mathfrak{y}_1 + qm\mathfrak{y}_2}{[2]_q}\right) - \frac{1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}(m\mathfrak{y}_2) \\ &+ \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)}\int_0^{\frac{1}{[2]_q}} \mathfrak{F}(\mathfrak{w}q\mathfrak{y}_1 + m(1-\mathfrak{w}q)\mathfrak{y}_2)d_q\mathfrak{w}. \end{aligned} \tag{9}$$

Thus from (7)–(9), we have

$$\mathfrak{I}_1 + \mathfrak{I}_2 - \mathfrak{I}_3 = -\frac{1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}\mathfrak{F}\left(\frac{\mathfrak{y}_1 + qm\mathfrak{y}_2}{[2]_q}\right) + \frac{1}{q(m\mathfrak{y}_2 - \mathfrak{y}_1)^2}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r} \tag{10}$$

and we obtained required equality (6) by multiplying $q(m\mathfrak{y}_2 - \mathfrak{y}_1)$ by both sides of (10). Thus, the proof is accomplished. □

Remark 1. *In Lemma 2, we have*

(i) If we set $m = 1$, then we find ([24] Lemma 2).
(i) If we set $m = 1$ and later taking $q \to 1^-$, then we find ([35] Lemma 2.1).

Lemma 3. *For $m \in (0,1]$ with $0 < q < 1$, assume that there is an arbitrary function $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ such that ${}^{\mathfrak{y}_2}D_q\mathfrak{F}$ is continuous and integrable on $[\mathfrak{y}_1, \mathfrak{y}_2]$. Then, one has the identity:*

$$\begin{aligned} &\frac{\mathfrak{F}(\mathfrak{y}_1) + q\mathfrak{F}(m\mathfrak{y}_2)}{[2]_q} - \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r} \\ = &\frac{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}{[2]_q}\int_0^1 (1 - [2]_q\mathfrak{w})\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1 + m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}. \end{aligned} \tag{11}$$

Proof. From fundamental properties of quantum integral, we have

$$\begin{aligned}
&\int_0^1 (1-[2]_q\mathfrak{w})\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\\
&= -\frac{(1-[2]_q\mathfrak{w})\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\Bigg|_0^1 - \frac{[2]_q}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_0^1 \mathfrak{F}(q\mathfrak{w}\mathfrak{y}_1+m(1-q\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\\
&= \frac{q\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(m\mathfrak{y}_2)}{(m\mathfrak{y}_2-\mathfrak{y}_1)} - \frac{[2]_q}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_0^1 \mathfrak{F}(q\mathfrak{w}\mathfrak{y}_1+m(1-q\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\\
&= \frac{q\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(m\mathfrak{y}_2)}{(m\mathfrak{y}_2-\mathfrak{y}_1)} - \frac{[2]_q(1-q)}{q(m\mathfrak{y}_2-\mathfrak{y}_1)}\sum_{n=0}^{\infty} q^{n+1}\mathfrak{F}(q^{n+1}\mathfrak{y}_1+m(1-q^{n+1})\mathfrak{y}_2)\\
&= \frac{q\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(m\mathfrak{y}_2)}{(m\mathfrak{y}_2-\mathfrak{y}_1)} - \frac{[2]_q(1-q)}{q(m\mathfrak{y}_2-\mathfrak{y}_1)}\left(\sum_{k=0}^{\infty} q^{k}\mathfrak{F}(q^{k}\mathfrak{y}_1+m(1-q^{k})\mathfrak{y}_2)-\mathfrak{F}(\mathfrak{y}_1)\right)\\
&= \frac{q\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(m\mathfrak{y}_2)}{(m\mathfrak{y}_2-\mathfrak{y}_1)} - \frac{[2]_q}{q(m\mathfrak{y}_2-\mathfrak{y}_1)^2}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})^{m\mathfrak{y}_2}d_q\mathfrak{r} + \frac{[2]_q(1-q)}{q(m\mathfrak{y}_2-\mathfrak{y}_1)}\mathfrak{F}(\mathfrak{y}_1)\\
&= \frac{\mathfrak{F}(\mathfrak{y}_1)+q\mathfrak{F}(m\mathfrak{y}_2)}{q(m\mathfrak{y}_2-\mathfrak{y}_1)} - \frac{[2]_q}{q(m\mathfrak{y}_2-\mathfrak{y}_1)^2}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}
\end{aligned} \tag{12}$$

and we obtain the required equality (11) by multiplying $\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}$ by both sides of (12). □

Remark 2. *In Lemma 3, we have*

(i) *If we set $m=1$, then we find ([24] Lemma 1).*
(i) *If we set $m=1$ and later taking limit as $q\to 1^-$, then we find ([36] Lemma 2.1).*

4. Midpoint-Type Inequalities for (α, m)-Convex Functions

In this section, we will derive midpoint-type inequalities for differentiable (α, m)-convex functions.

Theorem 3. *Under the conditions of Lemma 2, if $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}|$ is (α, m)-convex function over $[\mathfrak{y}_1, \mathfrak{y}_2]$, then we find the following midpoint type inequality:*

$$\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r} - \mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right|\\
&\le q(m\mathfrak{y}_2-\mathfrak{y}_1)[(A_1(q)+A_3(q))\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)| + m\ (A_2(q)+A_4(q))\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|],
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
A_1(q) &= \int_0^{\frac{1}{[2]_q}} \mathfrak{w}^{\alpha+1}d_q\mathfrak{w} = \frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}\\
A_2(q) &= \int_0^{\frac{1}{[2]_q}} \mathfrak{w}(1-\mathfrak{w}^{\alpha})\ d_q\mathfrak{w} = \frac{1}{[2]_q^{3}} - \frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}\\
A_3(q) &= \int_{\frac{1}{[2]_q}}^{1} \mathfrak{w}^{\alpha}\left(\frac{1}{q}-\mathfrak{w}\right)\ d_q\mathfrak{w} = \frac{1}{q[\alpha+1]_q} - \frac{1}{[\alpha+2]_q}\\
&\quad - \frac{1}{q[2]_q^{\alpha+1}[\alpha+1]_q} + \frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}\\
A_4(q) &= \int_{\frac{1}{[2]_q}}^{1} \left(\frac{1}{q}-\mathfrak{w}\right)(1-\mathfrak{w}^{\alpha})\ d_q\mathfrak{w} = \frac{1}{[2]_q^{3}} - \frac{1}{q[\alpha+1]_q} + \frac{1}{[\alpha+2]_q}\\
&\quad + \frac{1}{q[2]_q^{\alpha+1}[\alpha+1]_q} - \frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}.
\end{aligned}$$

Proof. By taking modulus in (6), and using (α, m)-convexity of $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}|$, we have

$$\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}-\mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right| \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\begin{array}{c}\int_0^{\frac{1}{[2]_q}}\ \mathfrak{w}\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)|\ d_q\mathfrak{w} \\ +\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)|\ d_q\mathfrak{w}\end{array}\right] \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\int_0^{\frac{1}{[2]_q}}\ \mathfrak{w}^{\alpha+1}\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|\ d_q\mathfrak{w}+\int_0^{\frac{1}{[2]_q}}\ m(\mathfrak{w}-\mathfrak{w}^{\alpha+1})\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\ d_q\mathfrak{w}\right] \\
&\quad+q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\begin{array}{c}\int_{\frac{1}{[2]_q}}^1\left(\frac{\mathfrak{w}^\alpha}{q}-\mathfrak{w}^{\alpha+1}\right)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|\ d_q\mathfrak{w} \\ +\int_{\frac{1}{[2]_q}}^1\ m\left(\frac{1}{q}-\mathfrak{w}\right)(1-\mathfrak{w}^\alpha)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\ d_q\mathfrak{w}\end{array}\right] \\
&=q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[(A_1(q)+A_3(q))\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|+m\ (A_2(q)+A_4(q))\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\right].
\end{aligned}$$

Thus, the proof is accomplished. □

Remark 3. *In Theorem 3, we have*

(i) *If we set $\alpha = m = 1$, then we find ([24] Theorem 1).*
(ii) *If we set $\alpha = m = 1$ and later taking the limit as $q \to 1^-$, then we find ([35] Theorem 2.2).*

Theorem 4. *Under the conditions of Lemma 2, if $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^r$, $r \geqslant 1$ is (α, m)-convex function over $[\mathfrak{y}_1, \mathfrak{y}_2]$, then we find the following midpoint type inequality:*

$$\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}-\mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right| \\
&\leq\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q^{\frac{3(r-1)}{r}}}\left[\begin{array}{c}(A_1(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+m\ A_2(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r)^{\frac{1}{r}} \\ +(A_3(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+m\ A_4(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r)^{\frac{1}{r}}\end{array}\right].
\end{aligned}\tag{14}$$

Proof. By taking modulus in (6), and using power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}-\mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right| \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\begin{array}{c}\int_0^{\frac{1}{[2]_q}}\ |\mathfrak{w}\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)|\ d_q\mathfrak{w} \\ +\int_{\frac{1}{[2]_q}}^1\left|\left(\mathfrak{w}-\frac{1}{q}\right)\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)\right|\ d_q\mathfrak{w}\end{array}\right] \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\ \mathfrak{w}d_q\mathfrak{w}\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{[2]_q}}\ \mathfrak{w}\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{[2]_q}}^1\ \left(\frac{1}{q}-\mathfrak{w}\right)\ d_q\mathfrak{w}\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{[2]_q}}^1\ \left(\frac{1}{q}-\mathfrak{w}\right)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

By applying (α, m)-convexity of $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^r$, we have

$$
\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}-\mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right| \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left(\frac{1}{[2]_q^3}\right)^{1-\frac{1}{r}} \\
&\times\left[\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^{\alpha+1}\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r\ d_q\mathfrak{w}+\int_0^{\frac{1}{[2]_q}}m\ (\mathfrak{w}-\mathfrak{w}^{\alpha+1})|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\int_{\frac{1}{[2]_q}}^1\left(\frac{\mathfrak{w}^{\alpha}}{q}-\mathfrak{w}^{\alpha+1}\right)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r\ d_q\mathfrak{w}+\int_{\frac{1}{[2]_q}}^1\ m\left(\frac{1}{q}-\mathfrak{w}\right)(1-\mathfrak{w}^{\alpha})\ \left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}}\right] \\
&=\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q^{\frac{3(r-1)}{r}}}\left[\begin{array}{c}(A_1(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+m\ A_2(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r)^{\frac{1}{r}} \\ +(A_3(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+m\ A_4(q)\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r)^{\frac{1}{r}}\end{array}\right].
\end{aligned}
$$

Thus, the proof is accomplished. □

Remark 4. *In Theorem 4, If we set $\alpha = m = 1$, then we find ([24] Theorem 2).*

Theorem 5. *Under the conditions of Lemma 2 and $r > 1$ is a real number, if $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^r$ is (α, m) convex function over $[\mathfrak{y}_1, \mathfrak{y}_2]$, then we find the following midpoint type inequality for $r^{-1}+s^{-1}=1$:*

$$
\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}-\mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right| \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\left(\frac{1}{[2]_q^{s+1}[s+1]_q}\right)^{\frac{1}{s}}\left(B_1(q)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+mC_1(q)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r\right)^{\frac{1}{r}}\right. \\
&\left.+(\eta(q))^{\frac{1}{s}}\left(B_2(q)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+mC_2(q)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r\right)^{\frac{1}{r}}\right],
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
B_1(q) &= \int_0^{\frac{1}{[2]_q}}\ \mathfrak{w}^{\alpha}d_q\mathfrak{w}=\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q} \\
B_2(q) &= \int_{\frac{1}{[2]_q}}^1\ \mathfrak{w}^{\alpha}d_q\mathfrak{w}=\frac{1}{[\alpha+1]_q}-\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q} \\
C_1(q) &= \int_0^{\frac{1}{[2]_q}}\ (1-\mathfrak{w}^{\alpha})\ d_q\mathfrak{w}=\frac{1}{[2]_q}-\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q} \\
C_2(q) &= \int_{\frac{1}{[2]_q}}^1\ (1-\mathfrak{w}^{\alpha})\ d_q\mathfrak{w}=\frac{q}{[2]_q}-\frac{1}{[\alpha+1]_q}+\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q} \\
\eta(q) &= \int_{\frac{1}{[2]_q}}^1\ \left(\frac{1}{q}-\mathfrak{w}\right)^s\ d_q\mathfrak{w}.
\end{aligned}
$$

Proof. Taking absolute value of (6) and using the Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}-\mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right| \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\int_0^{\frac{1}{[2]_q}}\ |\mathfrak{w}\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)|\ d_q\mathfrak{w}\right. \\
&\quad\left.+\int_{\frac{1}{[2]_q}}^1\left|\left(\mathfrak{w}-\frac{1}{q}\right)\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)\right|\ d_q\mathfrak{w}\right] \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\ \mathfrak{w}^s d_q\mathfrak{w}\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{[2]_q}}\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{[2]_q}}^1\ \left(\frac{1}{q}-\mathfrak{w}\right)^s\ d_q\mathfrak{w}\right)^{\frac{1}{s}}\left(\int_{\frac{1}{[2]_q}}^1\ \left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)\right|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

By applying (α,m)-convexity of $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^r$, we have

$$\begin{aligned}
&\left|\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{m\mathfrak{y}_2}d_q\mathfrak{r}-\mathfrak{F}\left(\frac{\mathfrak{y}_1+qm\mathfrak{y}_2}{[2]_q}\right)\right| \\
&\leq q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\ \mathfrak{w}^s d_q\mathfrak{w}\right)^{\frac{1}{s}}\right. \\
&\times\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^\alpha\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r\ d_q\mathfrak{w}+\int_0^{\frac{1}{[2]_q}}m\ (1-\mathfrak{w}^\alpha)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}} \\
&+\left(\int_{\frac{1}{[2]_q}}^1\ \left(\frac{1}{q}-\mathfrak{w}\right)^s\ d_q\mathfrak{w}\right)^{\frac{1}{s}} \\
&\quad\left.\times\times\left(\int_{\frac{1}{[2]_q}}^1\ \mathfrak{w}^\alpha\ |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r\ d_q\mathfrak{w}+\int_{\frac{1}{[2]_q}}^1 m\ (1-\mathfrak{w}^\alpha)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r\ d_q\mathfrak{w}\right)^{\frac{1}{r}}\right] \\
&=q(m\mathfrak{y}_2-\mathfrak{y}_1)\left[\left(\frac{1}{[2]_q^{s+1}[s+1]_q}\right)^{\frac{1}{s}}\left(B_1(q)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+mC_1(q)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^r\right)^{\frac{1}{r}}\right. \\
&\quad\left.+(\eta(q))^{\frac{1}{s}}\left(B_2(q)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)|^r+mC_2(q)|_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)|^r\right)^{\frac{1}{r}}\right]. \qquad (16)
\end{aligned}$$

Thus, the proof is accomplished. □

Remark 5. *In Theorem 5, if we set $\alpha=m=1$ and later taking the limit as $q\to1^-$, then we find ([35] Theorem 2.3).*

5. Trapezoid-Type Inequalities for (α,m)-Convex Functions

In this section, we will derive trapezoid-type inequalities for differentiable (α,m)-convex functions.

Theorem 6. *Under the conditions of Lemma 3, if $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}|$ is (α, m)-convex function over $[\mathfrak{y}_1, \mathfrak{y}_2]$, then we have the following trapezoid type inequality:*

$$\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+q\mathfrak{F}(m\mathfrak{y}_2)}{[2]_q}-\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}^{m\mathfrak{y}_2}d_q\mathfrak{r}\right|$$
$$\leq\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left[\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)\right|(K_1(q)-K_2(q))+m\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|(L_1(q)-L_2(q))\right], \tag{17}$$

where

$$K_1(q)=\int_0^{\frac{1}{[2]_q}}\left(\mathfrak{w}^\alpha-[2]_q\mathfrak{w}\right)\mathfrak{w}^{\alpha+1})d_q\mathfrak{w}=\frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q[\alpha+2]_q}$$
$$K_2(q)=\int_{\frac{1}{[2]_q}}^1\left(\mathfrak{w}^\alpha-[2]_q\mathfrak{w}\right)\mathfrak{w}^{\alpha+1})d_q\mathfrak{w}=\frac{1}{[\alpha+1]_q}-\frac{[2]_q}{[\alpha+2]_q}-\frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q[\alpha+2]_q}$$
$$L_1(q)=\int_0^{\frac{1}{[2]_q}}\left(\mathfrak{w}^\alpha-[2]_q\mathfrak{w}\right)(1-\mathfrak{w}^\alpha)d_q\mathfrak{w}=\frac{q}{[2]_q^2}-\frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q[\alpha+2]_q}$$
$$L_2(q)=\int_{\frac{1}{[2]_q}}^1\left(\mathfrak{w}^\alpha-[2]_q\mathfrak{w}\right)(1-\mathfrak{w}^\alpha)d_q\mathfrak{w}$$
$$=\frac{q[\alpha]_q}{[\alpha+1]_q[\alpha+2]_q}-\frac{q}{[2]_q^2}+\frac{1}{[\alpha+1]_q[2]_q^{\alpha+1}}-\frac{1}{[\alpha+2]_q[2]_q^{\alpha+1}}.$$

Proof. By taking modulus in (11), and using (α, m)-convexity of $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}|$, we have

$$\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+q\mathfrak{F}(m\mathfrak{y}_2)}{[2]_q}-\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}^{m\mathfrak{y}_2}d_q\mathfrak{r}\right|$$
$$=\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left|\int_0^1(1-[2]_q\mathfrak{w})\,{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\right|$$
$$\leq\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)\right|d_q\mathfrak{w}$$
$$\leq\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left(\begin{array}{c}\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\left|\mathfrak{w}^\alpha\right|\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)\right|d_q\mathfrak{w}\\+\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\left|m(1-\mathfrak{w}^\alpha)\right|\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|d_q\mathfrak{w}\end{array}\right)$$
$$=\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left[\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)\right|(K_1(q)-K_2(q))+m\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|(L_1(q)-L_2(q))\right].$$

Thus, the proof is accomplished. □

Remark 6. *In Theorem 6, we have*

(i) *If we set $\alpha = m = 1$, then we find ([24] Theorem 1).*
(ii) *If we set $\alpha = m = 1$ and later taking the limit as $q \to 1^-$, then we find ([36] Theorem 2.2).*

Theorem 7. *Under the conditions of of Lemma 3, if $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^r$, $r \geqslant 1$ is (α, m)-convex function over $[\mathfrak{y}_1, \mathfrak{y}_2]$, then we have the following trapezoid-type inequality:*

$$\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+q\mathfrak{F}(m\mathfrak{y}_2)}{[2]_q}-\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}^{m\mathfrak{y}_2}d_q\mathfrak{r}\right|$$
$$\leq\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left(\frac{2q}{[2]_q^2}\right)^{1-\frac{1}{r}}\left[\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r(K_1(q)-K_2(q))+m\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r(L_1(q)-L_2(q))\right]^{\frac{1}{r}}. \tag{18}$$

Proof. By taking modulus in (11) and using power mean inequality, we have

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+q\mathfrak{F}(m\mathfrak{y}_2)}{[2]_q}-\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}^{m\mathfrak{y}_2}d_q\mathfrak{r}\right| \\
&=\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left|\int_0^1(1-[2]_q\mathfrak{w})^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)d_q\mathfrak{w}\right| \\
&\leq\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left(\int_0^1\left|(1-[2]_q\mathfrak{w})\right|d_q\mathfrak{w}\right)^{1-\frac{1}{r}}\left(\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_1+m(1-\mathfrak{w})\mathfrak{y}_2)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}.
\end{aligned}$$

By applying (α,m)-convexity of $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^r$, we have

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+q\mathfrak{F}(m\mathfrak{y}_2)}{[2]_q}-\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}^{m\mathfrak{y}_2}d_q\mathfrak{r}\right| \\
&\leq\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left(\int_0^1\left|(1-[2]_q\mathfrak{w})\right|d_q\mathfrak{w}\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\mathfrak{w}^\alpha\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r d_q\mathfrak{w}+\int_0^1\left|(1-[2]_q\mathfrak{w})\right|m(1-\mathfrak{w}^\alpha)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}} \\
&=\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)}{[2]_q}\left(\frac{2q}{[2]_q^2}\right)^{1-\frac{1}{r}} \\
&\quad\times\left[\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r(K_1(q)-K_2(q))+m\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r(L_1(q)-L_2(q))\right]^{\frac{1}{r}}.
\end{aligned}$$

Thus, the proof is accomplished. □

Remark 7. *In Theorem 7, we have*

(i) If we set $\alpha=m=1$, then we find ([24] Theorem 2).
(ii) If we set $\alpha=m=1$ and later taking the limit as $q\to1^-$, then we find ([41] Theorem 6).

6. Application to Special Means

For any positive number $\mathfrak{y}_1,\mathfrak{y}_2\in\mathbb{R}$, we consider the following means:

(i) The arithmetic mean:

$$A(\mathfrak{y}_1,\mathfrak{y}_2)=\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}.$$

(ii) The harmonic mean:

$$H(\mathfrak{y}_1,\mathfrak{y}_2)=\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}.$$

(iii) The geometric mean:

$$G(\mathfrak{y}_1,\mathfrak{y}_2)=\sqrt{\mathfrak{y}_1\mathfrak{y}_2}.$$

Proposition 1. *For $\alpha,m\in[0,1]$ with $0<q<1$, let $\mathfrak{y}_1,\mathfrak{y}_2\in\mathbb{R}$ with $\mathfrak{y}_1<\mathfrak{y}_2$. Then we get*

$$\begin{aligned}
&\left|-H(q\mathfrak{y}_2m,\mathfrak{y}_1)+\frac{G^2(q\mathfrak{y}_2m,\mathfrak{y}_1)}{A(1,q)}\Upsilon_1\right| \\
\leq\ &-\frac{q(m\mathfrak{y}_2-\mathfrak{y}_1)G^2(q\mathfrak{y}_2m,\mathfrak{y}_1)}{A(1,q)}\left[\begin{array}{c}\frac{1}{\mathfrak{y}_1(q\mathfrak{y}_1+(1-q)m\mathfrak{y}_2)}(A_1(q)+A_3(q)) \\ +\frac{m}{\mathfrak{y}_2(q\mathfrak{y}_2+m(1-q)\mathfrak{y}_2)}(A_2(q)+A_4(q))\end{array}\right],
\end{aligned}\tag{19}$$

where

$$\Upsilon_1=\frac{1}{(m\mathfrak{y}_2-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{m\mathfrak{y}_2}\frac{1}{\mathfrak{r}}\,{}^{m\mathfrak{y}_2}d_q\mathfrak{r}=(1-q)\sum_{n=0}^{\infty}\frac{q^n}{q^n\mathfrak{y}_1+m(1-q^n)\mathfrak{y}_2}.$$

Proof. The inequality (13) for function $\mathfrak{F}(\mathfrak{r}) = \frac{2q\mathfrak{y}_2 m\mathfrak{y}_1}{([2]_q)\mathfrak{r}}$ leads to required result. □

If we take $\mathfrak{y}_1 = 1, \mathfrak{y}_2 = 2, q = 0.9, m = 0.9$ and $\alpha = 0.5$ in (19), we get

$$\left| -H(q\mathfrak{y}_2 m, \mathfrak{y}_1) + \frac{G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right| \approx 0.804$$

and

$$-\frac{q(m\mathfrak{y}_2 - \mathfrak{y}_1)G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \left[\begin{array}{c} \frac{1}{\mathfrak{y}_1(q\mathfrak{y}_1 + (1-q)m\mathfrak{y}_2)}(A_1(q) + A_3(q)) \\ + \frac{m}{\mathfrak{y}_2(q\mathfrak{y}_2 + m(1-q)\mathfrak{y}_2)}(A_2(q) + A_4(q)) \end{array} \right] = 1.283.$$

It is clear that $0.804 \leq 1.283$, which demonstrates the result described in Preposition 1.

Proposition 2. *For $\alpha, m \in [0,1]$ with $0 < q < 1$, let $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$. Then we get*

$$\begin{aligned} &\left| -G^2(\mathfrak{y}_2, qm\mathfrak{y}_1) + \frac{A(q\mathfrak{y}_2 m, \mathfrak{y}_1)G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right| \\ &\leq -\frac{q(m\mathfrak{y}_2 - \mathfrak{y}_1)A(q\mathfrak{y}_2 m, \mathfrak{y}_1)G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \\ &\quad \times \left[\begin{array}{c} \frac{1}{\mathfrak{y}_1(q\mathfrak{y}_1 + (1-q)m\mathfrak{y}_2)}(A_1(q) + A_3(q)) \\ + \frac{m}{\mathfrak{y}_2(q\mathfrak{y}_2 + (1-q)m\mathfrak{y}_2)}(A_2(q) + A_4(q)) \end{array} \right], \end{aligned} \tag{20}$$

where

$$\Upsilon_1 = \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)} \int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \frac{1}{\mathfrak{r}} \, {}^{m\mathfrak{y}_2}d_q\mathfrak{r} = (1-q) \sum_{n=0}^{\infty} \frac{q^n}{q^n\mathfrak{y}_1 + m(1-q^n)\mathfrak{y}_2}.$$

Proof. The inequality (13) for function $\mathfrak{F}(\mathfrak{r}) = \frac{q\mathfrak{y}_2 m\mathfrak{y}_1(qm\mathfrak{y}_1 + \mathfrak{y}_2)}{([2]_q)\mathfrak{r}}$ leads to required result. □

If we take $\mathfrak{y}_1 = 1, \mathfrak{y}_2 = 2, q = 0.9, m = 0.9$ and $\alpha = 0.5$ in (20), we get

$$\left| -G^2(\mathfrak{y}_2, qm\mathfrak{y}_1) + \frac{A(q\mathfrak{y}_2 m, \mathfrak{y}_1)G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right| \approx 0.487$$

and

$$-\frac{q(m\mathfrak{y}_2 - \mathfrak{y}_1)A(q\mathfrak{y}_2 m, \mathfrak{y}_1)G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \left[\begin{array}{c} \frac{1}{\mathfrak{y}_1(q\mathfrak{y}_1 + (1-q)m\mathfrak{y}_2)}(A_1(q) + A_3(q)) \\ + \frac{m}{\mathfrak{y}_2(q\mathfrak{y}_2 + (1-q)m\mathfrak{y}_2)}(A_2(q) + A_4(q)) \end{array} \right] = 1.989.$$

It is clear that $0.487 \leq 1.989$, which demonstrates the result described in Proposition 2.

Proposition 3. *For $(\alpha, m) \in [0,1]$ with $0 < q < 1$ and $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$. Then we get*

$$\begin{aligned} &\left| \frac{A(m\mathfrak{y}_2, q\mathfrak{y}_1)}{G^2(m\mathfrak{y}_2, \mathfrak{y}_1)A(1,q)} - \Upsilon_1 \right| \\ &\leq -\frac{q(m\mathfrak{y}_2 - \mathfrak{y}_1)}{A(1,q)} \left[\begin{array}{c} \frac{1}{\mathfrak{y}_1(q\mathfrak{y}_1 + (1-q)m\mathfrak{y}_2)}(K_1(q) - K_2(q)) \\ + \frac{m}{\mathfrak{y}_2(q\mathfrak{y}_2 + (1-q)m\mathfrak{y}_2)}(L_1(q) - L_2(q)) \end{array} \right], \end{aligned} \tag{21}$$

where

$$\Upsilon_1 = \frac{1}{(m\mathfrak{y}_2 - \mathfrak{y}_1)} \int_{\mathfrak{y}_1}^{m\mathfrak{y}_2} \frac{1}{\mathfrak{r}} \, {}^{m\mathfrak{y}_2}d_q\mathfrak{r} = (1-q) \sum_{n=0}^{\infty} \frac{q^n}{q^n\mathfrak{y}_1 + m(1-q^n)\mathfrak{y}_2}.$$

Proof. The inequality (17) for function $\mathfrak{F}(\mathfrak{r}) = \frac{1}{\mathfrak{r}}$ leads to required result. □

If we take $\mathfrak{y}_1 = 1, \mathfrak{y}_2 = 2, q = 0.9, m = 1$ and $\alpha = 0.1$ in (21), we get

$$\left| -G^2(\mathfrak{y}_2, qm\mathfrak{y}_1) + \frac{A(q\mathfrak{y}_2 m, \mathfrak{y}_1)G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right| \approx 0.232$$

and

$$-\frac{q(m\mathfrak{y}_2 - \mathfrak{y}_1)A(q\mathfrak{y}_2 m, \mathfrak{y}_1)G^2(q\mathfrak{y}_2 m, \mathfrak{y}_1)}{A(1,q)} \left[\begin{array}{l} \frac{1}{\mathfrak{y}_1(q\mathfrak{y}_1+(1-q)m\mathfrak{y}_2)}(A_1(q) + A_3(q)) \\ + \frac{m}{\mathfrak{y}_2(q\mathfrak{y}_2+(1-q)m\mathfrak{y}_2)}(A_2(q) + A_4(q)) \end{array} \right] = 0.271.$$

It is clear that $0.232 \leq 0.271$, which demonstrates the result described in Proposition 2.

7. Conclusions

In this paper, we proved some new quantum midpoint- and trapezoid-type inequalities for (α, m)-convex functions using the notions of q-calculus. We also proved that the newly established inequalities are the extension of comparable inequalities inside the literature. Finally, we gave some applications and examples for the presented results to show their worth. It is an interesting and new problem with which upcoming researchers can obtain similar inequalities for coordinated (α, m)-convex functions in their future work.

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