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Anticipated Backward Doubly Stochastic Differential Equations with Non-Lipschitz Coefficients

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Abstract: The work presented in this paper focuses on a type of differential equations called anticipated backward doubly stochastic differential equations (ABDSDEs) whose generators not only depend on the anticipated terms of the solution $(Y_\cdot, Z_\cdot)$ but also satisfy one kind of non-Lipschitz assumption. Firstly, we give the existence and uniqueness theorem. Further, two comparison theorems for the solutions of these equations are obtained after finding a new comparison theorem for backward doubly stochastic differential equations (BDSDEs) with non-Lipschitz coefficients.

Keywords: anticipated backward doubly stochastic differential equations; comparison theorems; non-Lipschitz coefficients

AMS Classification: 60F05; 60G15

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1. Introduction

In 1990, the pioneer research of Pardoux and Peng [1] proposed the theory of nonlinear backward stochastic differential Equations (BSDEs). Over the past 3 decades, BSDEs have attracted much attention from academia due to its wide application in lots of different fields of research, for example, financial mathematics (see El Karoui et al. [2]), stochastic optimal control, differential games and the theory of partial differential equations. Among others, a lot of effort has been made to relax the Lipschitz assumptions (see, e.g., [3–6]).

In order to obtain a probabilistic representation for a class of quasilinear stochastic partial differential equations, Pardoux and Peng [7] first presented a class of backward doubly stochastic differential Equations (BDSDEs in short) in the following

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s, t \in [0, T], \tag{1}$$

where the equations include a standard (forward) Itô integral dW_t, and a backward Itô integral $d\overleftarrow{B}_t$. They investigated the existence and uniqueness of solutions for BDSDE (1) under uniform Lipschitz generators. Then Shi et al. [8] gived a comparison theorem for BDSDEs with uniform Lipschitz condition on the generators. Refs. [9–11] have attempted to weaken the uniform Lipschitz assumption on the coefficients.

In 2009, Peng and Yang [12] introduced a new class of BSDEs called anticipated BSDEs (ABSDEs), whose generator involves not only the present values of the solutions but also the future situation. The authors proved ABSDEs have a unique solutions under uniform Lipschitz assumptions, obtained a comparison theorem for their solutions under some specific condition, and investigated the duality between anticipated BSDEs and delayed stochastic differential equations. Following the research of Peng and Yang [12], Zhang [13] studied the comparison theorems for one dimensional anticipated BSDEs under one kind of non-Lipschitz assumption. Zhou et al. [14] investigated the existence and uniqueness of this type equations under another non-Lipschitz conditions. Recently, Xu [15] and Zhang [16] introduced the following type of so-called anticipated BDSDES (ABDSDEs):

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) \mathrm{d}s \\ \qquad + \int_t^T g(s, Y_s, Z_s, Y_{s+\bar{\delta}(s)}, Z_{s+\bar{\zeta}(s)}) \mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s \mathrm{d}W_s, \ t \in [0, T], \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T+K], \end{cases} \tag{2}$$

where $\xi_\cdot, \eta_\cdot$ are given stochastic processes, and $\delta(\cdot), \bar{\delta}(\cdot), \zeta(\cdot)$ and $\bar{\zeta}(\cdot)$ are four given nonnegative deterministic continuous functions and for $\gamma(\cdot) = \delta(\cdot), \bar{\delta}(\cdot), \zeta(\cdot), \bar{\zeta}(\cdot)$ satisfying that:

(A1) there has a constant $K \geq 0$ such that, for each $t \in [0, T]$, $t + \gamma(t) \leqslant T + K$;

(A2) there has a constant $M \geqslant 0$ such that, for each $t \in [0, T]$ and for any nonnegative integrable $h(\cdot)$,

$$\int_t^T h(s + \gamma(s)) \mathrm{d}s \leqslant M \int_t^{T+K} h(s) \mathrm{d}s.$$

Xu [15] and Zhang [16] explored the existence and uniqueness of the solution for above equation, gave some comparison theorems, and investigated the duality between them and stochastic doubly differential equations with delay. Aidara [17,18] studied anticipated BDSDEs with one kind of non-Lipschitz coefficients, in which generator g does not depend on the anticipated term of y, z. They obtained the existence and uniqueness result and a comparison theorem in the one dimensional case. Recently, Wang and Yu [19] dealt with anticipated generalized backward doubly stochastic differential Equations (AGBDSDEs). Based on [15,16], we are concerned with anticipated BDSDEs under non-Lipschitz assumption. We will prove that under proper assumptions, the solution of the above ABDSDE with non-Lipschitz coefficients exists uniquely, and two comparison theorems are given for the one dimensional ABDSDEs with non-Lipschitz coefficients. These results are the cornerstones of ABDSDEs witn non-Lipschitz coefficients applied to some stochastic optimal control problems with delay effect.

This paper is divided into five sections as follows. In Section 2, some notations, assumptions and definition are given. We focus on the existence and uniqueness of the solutions of anticipated BDSDEs with non-Lipschitz coefficients in Section 3. In Section 4, we give two comparison theorems. Finally in Section 5, the conclusion and future work are presented.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. $T > 0, K \geq 0$ are two fixed constants. Let $\{W_t; 0 \leq t \leq T\}$ and $\{B_t; 0 \leq t \leq T\}$ be two mutually independent standard Brownian motions with values, respectively, in $\mathbb{R}^d$ and $\mathbb{R}^l$. For any $x, y \in R^n$ and $z \in R^{n \times m}$, we use $\langle x, y \rangle$ to denote the inner product of x and y, and $\mid y \mid$ to represent for the vector norm of y and $\mid z \mid := \sqrt{Tr(zz^*)}$ means the matrix norm of z, where z^* is the transpose of z. Set $\mathcal{N}$ to denote the class of P-null sets of $\mathcal{F}$. For all $t \in [0, T+K]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T+K}^B,$$

where for each process $\{\phi_t\}, \mathcal{F}_{s,t}^\eta = \sigma\{\phi_r - \phi_s; s \leq r \leq t\} \vee \mathcal{N}$. Notice that $\{\mathcal{F}_{0,t}^W, t \in [0, T+K]\}$ is increasing and $\{\mathcal{F}_{t,T}^B, t \in [0, T+K]\}$ is decreasing, therefor $\{\mathcal{F}_t, t \in [0, T+K]\}$ do not constitute a filtration. The following notations will be used throughout the paper: for each $t \geq 0$, $n \in N$ and $b > a \geq 0$,

(i) $L^2(\mathcal{F}_t; \mathbb{R}^n) \triangleq \{\eta : \eta \in \mathbb{R}^n \mid \eta$ is a $\mathcal{F}_t$-measurable random variable with $\mathbb{E}|\eta|^2 < \infty\}$;

(ii) $\mathcal{M}^2(a, b; \mathbb{R}^n) \triangleq \{\phi : \Omega \times [a, b] \to \mathbb{R}^n \mid \phi$ is a $\mathcal{F}_t$-progressively measurable processes such that $\| \phi \|_{\mathcal{M}^2}^2 = \mathbb{E}(\int_a^b |\phi_t|^2 dt) < \infty\}$;

(iii) $\mathcal{S}^2([a,b];\mathbb{R}^n) \triangleq \{\phi : \Omega \times [a,b] \to \mathbb{R}^n \mid \phi$ is a continuous and $\mathcal{F}_t$- progressively measurable processes such that $\| \phi \|_{\mathcal{S}^2}^2 = \mathbb{E}\left(\sup_{a \le t \le b} |\phi_t|^2 \right) < \infty$.

Let $f(t,\cdot,\cdot,\cdot,\cdot) : \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathcal{M}^2(t,T+K;\mathbb{R}^k) \times \mathcal{M}^2(t,T+K;\mathbb{R}^{k\times d}) \to L^2(\mathcal{F}_t; \mathbb{R}^k)$, $g(t,\cdot,\cdot,\cdot,\cdot) : \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathcal{M}^2(t,T+K;\mathbb{R}^k) \times \mathcal{M}^2(t,T+K;\mathbb{R}^{k\times d}) \to L^2(\mathcal{F}_t;\mathbb{R}^{k\times l})$. We make the following assumptions about (ξ, f, g):

(H1) $f(.,0,0,0,0) \in \mathcal{M}^2(0,T;\mathbb{R}^k), g(.,0,0,0,0) \in \mathcal{M}^2(0,T;\mathbb{R}^{k\times l})$.

(H2) For each $t \in [0,T]$, $y,\bar{y} \in \mathbb{R}^k$, $z,\bar{z} \in \mathbb{R}^{k\times d}$, $\theta(\cdot),\bar{\theta}(\cdot) \in \mathcal{M}^2(t,T+K;\mathbb{R}^k)$, $\vartheta(\cdot),\bar{\vartheta}(\cdot) \in \mathcal{M}^2(t,T+K;\mathbb{R}^{k\times d})$, $r,r' \in [t,T+K]$, we let

$$\begin{cases} |f(t,y,z,\theta(r),\vartheta(r')) - f(t,\bar{y},\bar{z},\bar{\theta}(r),\bar{\vartheta}(r'))|^2 \\ \leqslant C(|y-\bar{y}|^2 + \|z-\bar{z}\|^2 + \mathbb{E}^{\mathcal{F}_t}[|\theta(r)-\bar{\theta}(r)|^2 + \|\vartheta(r')-\bar{\vartheta}(r')\|^2]), \\ \|g(t,y,z,\theta(r),\vartheta(r')) - g(t,\bar{y},\bar{z},\bar{\theta}(r),\bar{\vartheta}(r'))\|^2 \\ \leqslant C(|y-\bar{y}|^2 + \mathbb{E}^{\mathcal{F}_t}[|\theta(r)-\bar{\theta}(r)|^2]) + \alpha_1\|z-\bar{z}\|^2 + \alpha_2\mathbb{E}^{\mathcal{F}_t}[\|\vartheta(r')-\bar{\vartheta}(r')\|^2], \end{cases}$$

where $C > 0, 0 < \alpha_1 < 1,\ 0 < \alpha_1 + \alpha_2 M < 1$ are three given constants.

(H3) For each $t \in [0,T]$, $y,\bar{y} \in \mathbb{R}^k$, $z,\bar{z} \in \mathbb{R}^{k\times d}$, $\theta(\cdot),\bar{\theta}(\cdot) \in \mathcal{M}^2(t,T+K;\mathbb{R}^k)$, $\vartheta(\cdot),\bar{\vartheta}(\cdot) \in \mathcal{M}^2(t,T+K;\mathbb{R}^{k\times d})$, $r,r' \in [t,T+K]$, we let

$$\begin{cases} |f(t,y,z,\theta(r),\vartheta(r')) - f(t,\bar{y},\bar{z},\bar{\theta}(r),\bar{\vartheta}(r'))|^2 \\ \leqslant \rho_1(t,|y-\bar{y}|^2) + \rho_2(r,\mathbb{E}^{\mathcal{F}_t}[|\theta(r)-\bar{\theta}(r)|^2]) + C(\|z-\bar{z}\|^2 + \mathbb{E}^{\mathcal{F}_t}[\|\vartheta(r')-\bar{\vartheta}(r')\|^2]), \\ \|g(t,y,z,\theta(r),\vartheta(r')) - g(t,\bar{y},\bar{z},\bar{\theta}(r),\bar{\vartheta}(r'))\|^2 \\ \leqslant \rho_1(t,|y-\bar{y}|^2) + \rho_2(r,\mathbb{E}^{\mathcal{F}_t}[|\theta(r)-\bar{\theta}(r)|^2]) + \alpha_1\|z-\bar{z}\|^2 + \alpha_2\mathbb{E}^{\mathcal{F}_t}[\|\vartheta(r')-\bar{\vartheta}(r')\|^2], \end{cases}$$

where $C > 0, 0 < \alpha_1 < 1,\ 0 < \alpha_1 + \alpha_2 M < 1$ are three given constants and for $i = 1,2$, $\rho_i : [0,T+K] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

- For fixed $t \in [0,T+K], \rho_i(t,.)$ is a concave and non-decreasing function such that $\rho_i(t,0) = 0$.
- For fixed u, $\int_0^{T+K} \rho_i(t,u)dt < +\infty$.
- For any $L > 0, S_2 > S_1 \ge 0$, the following ODE

$$\begin{cases} u' = -L(\rho_1(s,u) + M\rho_2(s,u)), \\ u(S_2) = 0 \end{cases}$$

has a unique solution $u(s) \equiv 0, s \in [S_1,S_2]$.

(H4) $(\xi_\cdot,\eta_\cdot) \in \mathcal{S}^2([T,T+K];\mathbb{R}^k) \times \mathcal{M}^2(T,T+K;\mathbb{R}^{k\times d})$.

Remark 1. 1. *It's easy to check that $\rho_1(t,u) = \rho_2(t,u) = Cu$ for $C > 0$ is an example of the function ρ_1 and ρ_2, and in this case the assumption (H3) degenerates to the assumption (H2).*

2. *If for $i = 1,2$, $\rho_i(t,u)$ has a linear growth that is $\rho_i(t,u) \le a_i(t) + b_i(t)u$ where $a_i(t) \ge 0, b_i(t) \ge 0$, with $\int_0^{T+K} a_i(t)dt < \infty$ and $\int_0^{T+K} b_i(t)dt < \infty$, it's easy to check that $\rho_i(t,u)$ satisfies assumption (H3). Similar assumptions were used in [3–6,9,11,14].*

3. *Similar non-Lipschitz assumption was also used in [14] when $g \not\equiv 0$ in the following form:*

$$\begin{aligned} &|f(t,y,z,\theta(r),\vartheta(r')) - f(t,\bar{y},\bar{z},\bar{\theta}(r),\bar{\vartheta}(r'))|^2 \\ &\leqslant \rho(t,|y-\bar{y}|^2) + \rho(r,\mathbb{E}^{\mathcal{F}_t}[|\theta(r)-\bar{\theta}(r)|^2]) + C(\|z-\bar{z}\|^2 + \mathbb{E}^{\mathcal{F}_t}[\|\vartheta(r')-\bar{\vartheta}(r')\|^2]). \end{aligned}$$

Definition 1. *A pair of processes $(Y_\cdot,Z_\cdot) : \Omega \times [0,T+K] \to \mathbb{R}^k \times \mathbb{R}^{k\times d}$ is a solution of ABDSDE (2) with non-Lipschitz coefficients, if $(Y_\cdot,Z_\cdot) \in \mathcal{S}^2([0,T+K],\mathbb{R}^k) \times \mathcal{M}^2(0,T+K,\mathbb{R}^{k\times d})$ and satisfies (2) and assumptions (H1), (H3) and (H4).*

3. Existence and Uniqueness Theorem

We can obtain directly the following existence and uniqueness result for ABDSDEs with uniform Lipschitz condition through combing the results given by Xu [15] and Zhang [16].

Lemma 1. *Let (H1), (H2) and (H4) hold. Then there exists a unique solution $(Y_\cdot, Z_\cdot) \in \mathcal{S}^2([0,T+K],\mathbb{R}^k) \times \mathcal{M}^2(0,T+K,\mathbb{R}^{k\times d})$ of ADDSDE (2).*

According to Lemma 1, we can construct the Picard-type iteration sequence of Equation (2) as follows:

$$\begin{cases} Y_t^0 = & 0, \\ Y_t^n = & \xi_T + \int_t^T f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)ds \\ & + \int_t^T g(s, Y_s^{n-1}, Z_s^n, Y_{s+\bar{\delta}(s)}^{n-1}, Z_{s+\bar{\zeta}(s)}^n)d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, t\in[0,T], \\ Y_t^n = & \xi_t, Z_t^n = \eta_t, t\in[T,T+K]. \end{cases} \tag{3}$$

In fact, for any $Y_\cdot^{n-1} \in \mathcal{S}^2([0,T+K],\mathbb{R}^k)$, according to Lemma 1, the ABDSDE (3) admits a unique solution $(Y_\cdot^n, Z_\cdot^n) \in \mathcal{S}^2([0,T+K],\mathbb{R}^k) \times \mathcal{M}^2(0,T+K,\mathbb{R}^{k\times d})$. We want to find the unique solution of ABDSDEs (2) through proving that the sequence $(Y_\cdot^n, Z_\cdot^n)_{n\geq 0}$ converges in $\mathcal{S}^2([0,T+K],\mathbb{R}^k) \times \mathcal{M}^2(0,T+K,\mathbb{R}^{k\times d})$. In order to achieve this goal, we need the following two lemmas.

Lemma 2. *Assume (A1), (A2), (H1), (H3) and (H4). Then, for any $0\leq t\leq T, n,m\geq 1$, we get*

$$\begin{aligned} \mathbb{E}|Y_t^{n+m} - Y_t^n|^2 \leq & e^{\frac{C(M+1)T}{1-\alpha_1-\alpha_2 M}}\left(\frac{1-\alpha_1-\alpha_2 M}{C(M+1)}+1\right)\Big(\int_t^T \rho_1(s,\mathbb{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2])ds \\ & + M\int_t^T \rho_2(s,\mathbb{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2])ds\Big). \end{aligned}$$

Proof. In view of Itô's formula, we have

$$\begin{aligned} & \mathbb{E}|Y_t^{n+m} - Y_t^n|^2 + \mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds = 2\mathbb{E}\int_t^T \langle Y_s^{n+m} - Y_s^n, \\ & f(s, Y_s^{n+m-1}, Z_s^{n+m}, Y_{s+\delta(s)}^{n+m-1}, Z_{s+\zeta(s)}^{n+m}) - f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)\rangle ds \\ & + \mathbb{E}\int_t^T \|g(s, Y_s^{n+m-1}, Z_s^{n+m}, Y_{s+\bar{\delta}(s)}^{n+m-1}, Z_{s+\bar{\zeta}(s)}^{n+m}) - g(s, Y_s^{n-1}, Z_s^n, Y_{s+\bar{\delta}(s)}^{n-1}, Z_{s+\bar{\zeta}(s)}^n)\|^2 ds. \end{aligned}$$

By the assumptions (H3), (A1), (A2), Young's inequality $2ab \leq \frac{1}{\theta}a^2 + \theta b^2$, and Jensen's inequality, for all $\theta > 0$, we get

$$\begin{aligned} & \mathbb{E}|Y_t^{n+m} - Y_t^n|^2 + \mathbb{E}\int_t^T \|Z_t^{n+m} - Z_t^n\|^2 ds \\ \leq & \frac{1}{\theta}\mathbb{E}\int_t^T |Y_t^{n+m} - Y_s^n|^2 ds + (\theta+1)\int_t^T \rho_1(s,\mathbb{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2])ds \\ & + (\theta+1)M\int_t^T \rho_2(s,\mathbb{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2])ds \\ & + (\theta C(M+1) + \alpha_1 + \alpha_2 M)\mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds. \end{aligned}$$

Choosing $\theta = \frac{1-\alpha_1-\alpha_2 M}{C(M+1)} > 0$, it follows from Gronwall's inequality that

$$\mathbb{E}|Y_t^{n+m} - Y_t^n|^2 \leq e^{\frac{C(M+1)T}{1-\alpha_1-\alpha_2 M}} \left(\frac{1-\alpha_1-\alpha_2 M}{C(M+1)} + 1 \right) (\int_t^T \rho_1(s, \mathbb{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2]) ds$$
$$+ M \int_t^T \rho_2(s, \mathbb{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2]) ds).$$

□

Lemma 3. *Assume that (A1), (A2), (H1), (H3) and (H4) hold. Then, there exists $T_1 \in [0, T]$ and a constant $L_1 \geq 0$ such that for any $t \in [T_1, T], n \geq 1, E|Y_t^n|^2 \leq L_1$.*

Proof. By applying Itô's formula, we get

$$\mathbb{E}|Y_t^n|^2 + \mathbb{E} \int_t^T \|Z_s^n\|^2 ds = \mathbb{E}|\xi|^2 + 2\mathbb{E} \int_t^T \langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) \rangle ds$$
$$+ \mathbb{E} \int_t^T \|g(s, Y_s^{n-1}, Z_s^n, Y_{s+\bar{\delta}(s)}^{n-1}, Z_{s+\bar{\zeta}(s)}^n)\|^2 ds.$$

From (H3), (A1), (A2) and Young's inequality $2ab \leq \frac{1}{\theta} a^2 + \theta b^2$, for each $\theta > 0$, we have

$$\begin{aligned}
&2\langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) \rangle \\
&\leq \frac{1}{\theta} |Y_s^n|^2 + \theta |f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)|^2 \\
&\leq \frac{1}{\theta} |Y_s^n|^2 + 2\theta (\rho_1(s, |Y_s^{n-1}|^2) + \rho_2(s+\delta(s), \mathbb{E}^{\mathcal{F}_s}[|Y_{s+\delta(s)}^{n-1}|^2])) \\
&+ 2\theta C(\|Z_s^n\|^2 + \mathbb{E}^{\mathcal{F}_s}[\|Z_{s+\zeta(s)}^n\|]) + 2\theta |f(s,0,0,0,0)|^2, \\
&\|g(s, Y_s^{n-1}, Z_s^n, Y_{s+\bar{\delta}(s)}^{n-1}, Z_{s+\bar{\zeta}(s)}^n)\|^2 \leq (1+\theta)(\rho_1(s, |Y_s^{n-1}|^2) + \rho_2(s+\bar{\delta}(s), \mathbb{E}^{\mathcal{F}_s}[|Y_{s+\bar{\delta}(s)}^{n-1}|^2])) \\
&+ (1+\theta)\alpha_1 \|Z_s^n\|^2 + (1+\theta)\alpha_2 \mathbb{E}^{\mathcal{F}_s}[\|Z_{s+\bar{\zeta}(s)}^n\|^2]) + \left(1 + \frac{1}{\theta}\right) \|g(s,0,0,0,0)\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}|Y_t^n|^2 + (1 - 2\theta C(1+M) - (1+\theta)(\alpha_1 + \alpha_2 M)) \mathbb{E} \int_t^T \|Z_s^n\|^2 ds \\
&\leq \mathbb{E}|\xi|^2 + \frac{1}{\theta} \mathbb{E} \int_t^T |Y_s^n|^2 ds + (3\theta+1) \int_t^T (\rho_1(s, \mathbb{E}|Y_s^{n-1}|^2) + M\rho_2(s, \mathbb{E}|Y_s^{n-1}|^2)) ds \\
&+ (3\theta+1) M \int_T^{T+K} \rho_2(s, \mathbb{E}|\xi_s|^2) ds + (2\theta CM + (1+\theta)\alpha_2) M \mathbb{E} \int_T^{T+K} \|\eta_s\|^2 ds \\
&+ \mathbb{E} \int_t^T (2\theta |f(s,0,0,0,0)|^2 + \left(1 + \frac{1}{\theta}\right) \|g(s,0,0,0,0)\|^2) ds.
\end{aligned}$$

We choose $\theta = \frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M} > 0$, then

$$\begin{aligned}
&\mathbb{E}|Y_t^n|^2 \leq \mathbb{E}|\xi|^2 + \frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}\mathbb{E}\int_t^T |Y_s^n|^2 ds \\
&+ \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M}\int_t^T (\rho_1(s,\mathbb{E}|Y_s^{n-1}|^2) + M\rho_2(s,\mathbb{E}|Y_s^{n-1}|^2))ds \\
&+ \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} M\int_T^{T+K} \rho_2(s,\mathbb{E}|\xi_s|^2)ds \\
&+ (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}CM + \frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_T^{T+K} \|\eta_s\|^2 ds \\
&+ \mathbb{E}\int_t^T (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2 + \frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds.
\end{aligned}$$

Now, in view of Gronwall's inequality, we derive

$$\mathbb{E}|Y_t^n|^2 \leq \delta_t + L\int_t^T (\rho_1(s,\mathbb{E}|Y_s^{n-1}|^2) + M\rho_2(s,\mathbb{E}|Y_s^{n-1}|^2))ds, \tag{4}$$

where

$$\begin{aligned}
\delta_t^1 =& e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T}(\mathbb{E}|\xi|^2 + 1 + \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} M\int_T^{T+K} \rho_2(s,\mathbb{E}|\xi_s|^2)ds \\
&+ (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}CM + \frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_T^{T+K} \|\eta_s\|^2 ds \\
&+ \mathbb{E}\int_t^T (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2 + \frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds),
\end{aligned}$$

and

$$L = \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T} > 0.$$

Let

$$\begin{aligned}
L_1 =& 2\delta_0^1 = e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T}(\mathbb{E}|\xi|^2 + 1 + \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} M\int_T^{T+K} \rho_2(s,\mathbb{E}|\xi_s|^2)ds \\
&+ (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}CM + \frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_T^{T+K} \|\eta_s\|^2 ds \\
&+ \mathbb{E}\int_0^T (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2 + \frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds),
\end{aligned}$$

By virtue of (H3),

$$\int_0^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds < \infty,$$

so we can find T_1 such that

$$\int_{T_1}^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds \leq \frac{\delta_0^1}{L}.$$

Indeed, if $\int_0^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds \leq \frac{\delta_0^1}{M}$, then we choose $T_1 = 0$. If

$$\int_0^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds > \frac{\delta_0^1}{L},$$

since $t \mapsto \int_t^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds$ is continuous, there exists $T_1 \in [0,T]$ such that

$$\int_{T_1}^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds = \frac{\delta_0^1}{L}.$$

Consequently, for any $t \in [T_1, T]$, from Equation (4) and $\rho_i(t,.)$ is increasing, we get

$$\begin{aligned}
\mathbb{E}|Y_t^1|^2 &\le \delta_t^1 \le 2\delta_0^1 = L_1, \\
\mathbb{E}|Y_t^2|^2 &\le \delta_t^1 + L\int_t^T (\rho_1(s,\mathbb{E}|Y_s^1|^2) + M\rho_2(s,\mathbb{E}|Y_s^1|^2))ds \\
&\le \delta_0^1 + L\int_t^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds \le 2\mu_0^1 = L_1, \\
\mathbb{E}|Y_t^3|^2 &\le \delta_t^1 + L\int_t^T (\rho_1(s,\mathbb{E}|Y_s^2|^2) + M\rho_2(s,\mathbb{E}|Y_s^2|^2))ds \\
&\le \delta_0^1 + L\int_t^T (\rho_1(s,L_1) + M\rho_2(s,L_1)ds \le 2\mu_0^1 = L_1.
\end{aligned}$$

By induction, for all $n \ge 1, t \in [T_1, T]$,

$$\mathbb{E}|Y_t^n|^2 \le L_1.$$

□

With the help of Lemmas 1 and 2, we can establish the existence and uniqueness theorem in the following.

Theorem 1. *Under (A1), (A2), (H1), (H3) and (H4). Then, ADDSDE (2) has a unique solution* $(Y_\cdot, Z_\cdot) \in \mathcal{S}^2([0, T+K], \mathbb{R}^k) \times \mathcal{M}^2(0, T+K, \mathbb{R}^{k\times d})$.

Proof. Existence. For any $n \ge 1$, and $t \in [0, T]$, we set

$$\varphi_0(t) = L\int_t^T (\rho_1(s,L_1) + M\rho_2(s,L_1))ds, \varphi_{n+1}(t) = L\int_t^T (\rho_1(s,\varphi_n(s)) + M\rho_2(s,\varphi_n(s)))ds.$$

Obviously, for all $t \in [T_1, T]$, we have

$$\begin{aligned}
\varphi_0(t) &= L\int_t^T (\rho_1(s,L_1) + M\rho_2(s,L_1))ds \le L_1, \\
\varphi_1(t) &= L\int_t^T (\rho_1(s,\varphi_0(s)) + M\rho_2(s,\varphi_0(s)))ds \\
&\le L\int_t^T (\rho_1(s,L_1) + M\rho_2(s,L_1))ds = \varphi_0(t) \le L_1, \\
\varphi_2(t) &= L\int_t^T (\rho_1(s,\varphi_1(s)) + M\rho_2(s,\varphi_1(s)))ds \\
&\le L\int_t^T (\rho_1(s,\varphi_0(s)) + M\rho_2(s,\varphi_0(s)))ds = \varphi_1(t) \le L_1.
\end{aligned}$$

Through induction, we get $\varphi_n(t)$ satisfies

$$0 \le \varphi_{n+1}(t) \le \varphi_n(t) \le \cdots \le \varphi_1(t) \le \varphi_0(t) \le L_1$$

for any $n \ge 1, t \in [T_1, T]$. Further, for any $n \ge 1, t, t' \in [T_1, T]$, we have

$$\begin{aligned}
|\varphi_n(t) - \varphi_n(t')| &= L\int_{t\wedge t'}^{t\vee t'} (\rho_1(s,\varphi_{n-1}(s)) + M\rho_2(s,\varphi_{n-1}(s)))ds \\
&\le L\int_{t\wedge t'}^{t\vee t'} (\rho_1(s,L_1) + M\rho_2(s,L_1))ds.
\end{aligned}$$

Since $t \mapsto \int_t^T (\rho_1(s,L_1) + M\rho_2(s,L_1))ds$ is a continuous mapping, we can obtain

$$\sup_n |\varphi_n(t) - \varphi_n(t')| \to 0 \quad as \quad |t - t'| \to 0.$$

Consequently, $\{\varphi_n(t)\}_{n\geq 1}$ is an equicontinuous family of function on $[T_1, T]$ and decreasing. Thus, by the Ascoli–Arzela theorem, $\{\varphi_n(t)\}_{n\geq 1}$ converges to a limit $\varphi(t)$, as $n \to 0$, satisfying

$$\varphi(t) = L\int_t^T (\rho_1(s, \varphi(s)) + M\rho_2(s, \varphi(s)))ds.$$

From (H3), it follows that $\varphi(t) = 0, t \in [T_1, T]$.

According to Lemmas 2 and 3, the definition of $\{\varphi_n(t)\}_{n\geq 1}$, and the fact that

$$e^{\frac{C(M+1)T}{1-\alpha_1-\alpha_2 M}}\left(\frac{1-\alpha_1-\alpha_2 M}{C(M+1)}+1\right) \leq L,$$

for any $t \in [T_1, T], n, m \geq 1$, we get

$$\begin{aligned}
\mathbb{E}|Y_t^{1,1+m} - Y_t^{1,1}|^2 &\leq e^{\frac{C(M+1)T}{1-\alpha_1-\alpha_2 M}}\left(\frac{1-\alpha_1-\alpha_2 M}{C(M+1)}+1\right)\int_t^T (\rho_1(s, \mathbb{E}[|Y_s^{1,m}|^2]) + M\rho_2(s, \mathbb{E}[|Y_s^{1,m}|^2]))ds \\
&\leq L\int_t^T (\rho_1(s, L_1) + M\rho_2(s, L_1))ds = \varphi_0(t) \leq L_1, \\
\mathbb{E}|Y_t^{1,2+m} - Y_t^{1,2}|^2 &\leq e^{\frac{C(M+1)T}{1-\alpha_1-\alpha_2 M}}\left(\frac{1-\alpha_1-\alpha_2 M}{C(M+1)}+1\right)\int_t^T (\rho_1(s, \mathbb{E}[|Y_s^{1,1+m} - Y_s^{1,1}|^2]) \\
&+ M\rho_2(s, \mathbb{E}[|Y_s^{1,1+m} - Y_s^{1,1}|^2]))ds \\
&\leq L\int_t^T (\rho_1(s, \varphi_0(s)) + M\rho_2(s, \varphi_0(s)))ds = \varphi_1(t) \leq L_1.
\end{aligned}$$

Through induction, we can give that

$$\mathbb{E}|Y_t^{1,n+m} - Y_t^{1,n}|^2 \leq \varphi_{n-1}(t) \leq L_1. \tag{5}$$

Notice that $\varphi_n(t) \to 0$ for any $t \in [T_1, T]$, as $n \to \infty$, thus we can know that $\{Y_\cdot^{1,n}\}$ is a Cauchy sequence in $\mathcal{M}^2(T_1, T+K; \mathbb{R}^k)$. Consequently, it is simple to check that $\{Z_\cdot^{1,n}\}$ is also a Cauchy sequence in $\mathcal{M}^2(T_1, T+K; \mathbb{R}^{k\times d})$. Let $Y_\cdot^1 = \lim_{n\to+\infty} Y_\cdot^{1,n}, Z_\cdot^1 = \lim_{n\to+\infty} Z_\cdot^{1,n}$, we obtain

$$\begin{cases}
Y_t^1 = \xi + \int_t^T f(s, Y_s^1, Z_s, Y_{s+\delta(s)}^1, Z_{s+\zeta(s)}^1)\mathrm{d}s \\
\qquad + \int_t^T g(s, Y_s^1, Z_s^1, Y_{s+\bar{\delta}(s)}^1, Z_{s+\bar{\zeta}(s)}^1)\mathrm{d}\overleftarrow{B}_s \\
\qquad - \int_t^T Z_s^1 \mathrm{d}W_s, \qquad t \in [T_1, T], \\
Y_t^1 = \xi_t, \quad Z_t^1 = \eta_t, \quad t \in [T, T+K].
\end{cases} \tag{6}$$

Applying Itô's formula, and using the assumptions (A1), (A2), (H3) and (H4), Young's inequality, Lemma 3 and Burkholder–Davis–Gundy's inequality, we can derive the limit $(Y_\cdot^1, Z_\cdot^1) \in \mathcal{S}^2([T_1, T+K]; \mathbb{R}^k) \times \mathcal{M}^2(T_1, T+K; \mathbb{R}^{k\times d})$.

Therefore, we have proved the existence of the solution to ABDSDE (2) on $[T_1, T]$. If $T_1 = 0$, then the existence is obtained.

If $T_1 \neq 0$, we will consider the following equation

$$\begin{cases}
Y_t^2 = Y_{T_1}^1 + \int_t^T f(s, Y_s^2, Z_s^2, Y_{s+\delta(s)}^2, Z_{s+\zeta(s)}^2)\mathrm{d}s \\
\qquad + \int_t^T g(s, Y_s^2, Z_s^2, Y_{s+\bar{\delta}(s)}^2, Z_{s+\bar{\zeta}(s)}^2)\mathrm{d}\overleftarrow{B}_s \\
\qquad - \int_t^T Z_s^2 \mathrm{d}W_s, \qquad t \in [0, T_1], \\
Y_t^2 = Y_t^1, \quad Z_t^2 = Z_t^1, \quad t \in [T_1, T+K].
\end{cases} \tag{7}$$

Let us introduce the approximate sequence which is similar to (3) for Equation (7). Through the similar process as in Lemmas 2 and 3, for each $t \in [0, T_1], n, m \geq 1$, we can prove that

$$\mathbb{E}|Y_t^{2,n+m} - Y_t^{2,n}|^2 \leq e^{\frac{C(M+1)T}{1-\alpha_1-\alpha_2 M}} \left(\frac{1-\alpha_1-\alpha_2 M}{C(M+1)} + 1 \right) \int_t^{T_1} (\rho_1(s, \mathbb{E}[|Y_s^{2,n+m-1} - Y_s^{2,n-1}|^2]) + M\rho_2(s, \mathbb{E}[|Y_s^{2,n+m-1} - Y_s^{2,n-1}|^2]))ds.$$

$$\mathbb{E}|Y_t^{2,n}|^2 \leq \delta_t^2 + L \int_t^T (\rho_1(s, \mathbb{E}|Y_s^{2,n-1}|^2) + M\rho_2(s, \mathbb{E}|Y_s^{2,n-1}|^2))ds,$$

where

$$\begin{aligned}\delta_t^2 =& e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T} (\mathbb{E}|Y_{T_1}^1|^2 + 1 + \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} M \int_{T_1}^{T+K} \rho_2(s, \mathbb{E}|Y_s^1|^2)ds \\ &+ (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M} CM + \frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_{T_1}^{T+K} |Z_s^1|^2 ds \\ &+ \mathbb{E}\int_t^T (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2 + \frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds),\end{aligned}$$

and

$$L = \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T} > 0.$$

Let

$$\begin{aligned}L_2 =& 2\delta_0^2 = 2e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T} (\mathbb{E}|Y_{T_1}^1|^2 + 1 + \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} M \int_{T_1}^{T+K} \rho_2(s, \mathbb{E}|Y_s^1|^2)ds \\ &+ (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M} CM + \frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_{T_1}^{T+K} |Z_s^1|^2 ds \\ &+ \mathbb{E}\int_0^T (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2 + \frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds).\end{aligned}$$

There exists $T_2 \in [0, T_1[$ with

$$\mathbb{E}|Y_t^{2,n}|^2 \leq L_2, n \geq 1, t \in [T_2, T_1],$$

where $T_2 = 0$ or $T_2 \in]0, T_1[$ with $\int_{T_2}^{T_1} (\rho_1(s, L_1) + M\rho_2(s, L_1)ds = \frac{\delta_0^2}{M}$. As proved above, ABDSDE (7) admits a solution on $[T_2, T + K]$. If $T_2 = 0$, the existence proof is complete. Otherwise, we can find a sequence $\{T_p, \delta_t^p, L_p, p \geq 1\}$ defined as

$$0 \leq T_p < T_{p-1} < \cdots < T_1 < T_0 = T,$$

$$\begin{aligned}\delta_t^p =& e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T} (\mathbb{E}|Y_{T_{p-1}}^{p-1}|^2 + 1 + \frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M} M \int_{T_{p-1}}^{T+K} \rho_2(s, \mathbb{E}|Y_s^{p-1}|^2)ds \\ &+ (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M} CM + \frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_{T_{p-1}}^{T+K} |Z_s^{p-1}|^2 ds \\ &+ \mathbb{E}\int_t^T (2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2 + \frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds),\end{aligned}$$

$$\begin{aligned}L_p =&2\delta_0^p = 2e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T}(\mathbb{E}|Y^{p-1}_{T_{p-1}}|^2+1+\frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M}M\int_{T_{p-1}}^{T+K}\rho_2(s,\mathbb{E}|Y_s^{p-1}|^2)ds\\&+(2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}CM+\frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_{T_{p-1}}^{T+K}|Z_s^{p-1}|^2ds\\&+\mathbb{E}\int_0^T(2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2+\frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds),\end{aligned}$$

and

$$\int_{T_p}^{T_{p-1}}(\rho_1(s,L_p)+M\rho_2(s,L_p)ds\geq\frac{\delta_0^p}{L}$$

through repeating the above procedure. Therefore, we can derive a solution to ABDSDE (2) on $[T_p, T+K]$ through iteration method. In the following, we will check there exists a finite number $p\geq 1$ with $T_p=0$. Denote

$$\begin{aligned}A = 2e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T}(1+\mathbb{E}\int_0^T(2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}|f(s,0,0,0,0)|^2\\+\frac{2C(1+M)+1}{1-\alpha_1-\alpha_2 M}\|g(s,0,0,0,0)\|^2)ds).\end{aligned}$$

Then, in view of (H1),

$$\begin{aligned}L_p =& 2e^{\frac{2C(1+M)+\alpha_1+\alpha_2 M}{1-\alpha_1-\alpha_2 M}T}(\mathbb{E}|Y^{p-1}_{T_{p-1}}|^2+\frac{3+2C(1+M)-2(\alpha_1+\alpha_2 M)}{2C(1+M)+\alpha_1+\alpha_2 M}M\int_{T_{p-1}}^{T+K}\rho_2(s,\mathbb{E}|Y_s^{p-1}|^2)ds\\&+(2\frac{1-\alpha_1-\alpha_2 M}{2C(1+M)+\alpha_1+\alpha_2 M}CM+\frac{2C(1+M)+1}{2C(1+M)+\alpha_1+\alpha_2 M}\alpha_2)\mathbb{E}\int_{T_{p-1}}^{T+K}|Z_s^{p-1}|^2ds)+A\\&\geq A>0, \quad \text{for all } p\geq 1.\end{aligned}$$

For any $p\geq 1$, since $\sup_{T_p\leq t\leq T+K}\mathbb{E}|Y_t^p|^2<\infty$, L_p is finite. So, we have

$$0<\frac{A}{L_p}\leq 1, \quad \text{for any } p\geq 1. \tag{8}$$

Since for $i=1,2$, $\rho_i(t,.)$ is a concave function, $\rho_i(t,0)=0$ while $t\in[0,T]$, we get

$$\rho_i(t,\lambda x)\geq\lambda\rho_i(t,x), \quad \text{for all any } \lambda\in[0,1] \quad \text{and} \quad \text{any} \quad x\in\mathbb{R}^+.$$

By the inequality (8), we have

$$\rho_i(t,A)=\rho_i\left(t,\frac{A}{L_p}L_p\right)\geq\frac{A}{L_p}\rho_i(t,L_p), \quad \text{for each } p\geq 1.$$

Therefore, for each $p\geq 1$, we can derive that

$$\begin{aligned}\int_{T_p}^T(\rho_1(s,A)+M\rho_2(s,A))ds &= \sum_{i=1}^p\int_{T_i}^{T_{i-1}}(\rho_1(s,A)+M\rho_2(s,A))ds\\&\geq\sum_{i=1}^p\frac{A}{L_i}\int_{T_i}^{T_{i-1}}(\rho_1(s,L_i)+M\rho_2(s,L_i))ds\geq\sum_{i=1}^p\frac{A}{L_i}\frac{\delta_0^i}{L}.\end{aligned}$$

By definition, $\frac{\delta_0^i}{L_i}=\frac{1}{2}$, for any $i\geq 1$. So, for any $p\geq 1$,

$$\int_{T_p}^T(\rho_1(s,A)+M\rho_2(s,A))ds\geq\frac{pA}{2L}. \tag{9}$$

Notice that $A > 0$, the right side of inequality (9) tends to $+\infty$ while $p \to +\infty$. Consequently, as $\int_0^T (\rho_1(s,A) + M\rho_2(s,A))ds$ is finite, so we can obtain a sufficient large p such that

$$\int_{T_p}^T (\rho_1(s,A) + M\rho_2(s,A))ds \geq \int_0^T (\rho_1(s,A) + M\rho_2(s,A))ds.$$

Therefore, there is a finite p with $T_p = 0$, and we assert the existence of solution on $[0, T+K]$.

Uniqueness. Set $(Y_\cdot, Z_\cdot), (Y'_\cdot, Z'_\cdot) \in \mathbb{S}^2([0,T+K];\mathbb{R}^k) \times \mathbb{M}^2(0,T+K;\mathbb{R}^{k\times d})$ be the two solutions of ABDSDE (2). Let $\beta > 0$. Applying the Itô's formula yields

$$\begin{aligned}
&\mathbb{E}|Y_t - Y'_t|^2 e^{\beta t} + \beta\mathbb{E}\int_t^T |Y_s - Y'_s|^2 e^{\beta s} ds + \mathbb{E}\int_t^T \|Z_s - Z'_s\|^2 e^{\beta s} ds = 2\mathbb{E}\int_t^T \langle Y_s - Y'_s, \\
&f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) - f(s, Y'_s, Z'_s, Y'_{s+\delta(s)}, Z'_{s+\zeta(s)})\rangle e^{\beta s} ds \\
&+ \mathbb{E}\int_t^T \|g(s, Y_s, Z_s, Y_{s+\bar{\delta}(s)}, Z_{s+\bar{\zeta}(s)}) - g(s, Y'_s, Z'_s, Y'_{s+\bar{\delta}(s)}, Z'_{s+\bar{\zeta}(s)})\|^2 e^{\beta s} ds.
\end{aligned}$$

By the assumptions (A1), (A2), (H3), Young's inequality $2ab \leq \frac{1}{\beta}a^2 + \beta b^2$, and Jensen's inequality, we have

$$\begin{aligned}
&\mathbb{E}|Y_t - Y'_t|^2 e^{\beta t} + (1 - \frac{1}{\beta}C(M+1) - \alpha_1 - \alpha_2 M)\mathbb{E}\int_t^T \|Z_s - Z'_s\|^2 e^{\beta s} ds \\
\leq &(\frac{1}{\beta} + 1)\int_t^T (\rho_1(s, \mathbb{E}[|Y_s - Y'_s|^2]) + M\rho_2(s, \mathbb{E}[|Y_s - Y'_s|^2])) e^{\beta s} ds.
\end{aligned}$$

Choosing $\beta > \frac{C(M+1)}{1-\alpha_1-\alpha_2 M} > 0$, we get

$$\begin{aligned}
&\mathbb{E}|Y_t - Y'_t|^2 + (1 - \frac{1}{\beta}C(M+1) - \alpha_1 - \alpha_2 M)\mathbb{E}\int_t^T \|Z_s - Z'_s\|^2 ds \\
&\leq (\frac{1}{\beta} + 1)e^{\beta T}\int_t^T (\rho_1(s, \mathbb{E}[|Y_s - Y'_s|^2]) + M\rho_2(s, \mathbb{E}[|Y_s - Y'_s|^2])) ds.
\end{aligned} \tag{10}$$

Therefore

$$\mathbb{E}|Y_t - Y'_t|^2 \leq \left(\frac{1}{\beta} + 1\right)e^{\beta T}\int_t^T (\rho_1(s, \mathbb{E}[|Y_s - Y'_s|^2]) + M\rho_2(s, \mathbb{E}[|Y_s - Y'_s|^2])) ds.$$

From the comparison Theorem for ODE, we can obtain

$$\mathbb{E}|Y_t - Y'_t|^2 \leq \gamma(t), \quad \text{for any t} \in [0, \text{T}],$$

where $\gamma(t)$ is the maximum left shift solution of the following equation:

$$\begin{cases} u' = -(\frac{1}{\beta} + 1)e^{\beta T}(\rho_1(t,u) + M\rho_2(t,u)); \\ u(T) = 0. \end{cases}$$

According to the assumption (H3), we have $\gamma(t) = 0, t \in [0,T]$. So $\mathbb{E}|Y_t - Y'_t|^2 = 0$, $t \in [0,T]$, this means $Y_t = Y'_t$, a.s. for all $t \in [0,T]$. It immediately derives from (10) that $Z_t = Z'_t$, a.s., for each $t \in [0,T]$. □

4. Comparison Theorems

In this part, we mainly focus on one dimensional ABDSDEs, that is, $k = 1$. Firstly, we propose one comparison theorem for BDSDEs under non-Lipschitz coefficients, which acts as a starting point for the following investigation. For $i = 1, 2$, assume that $\xi^i \in L^2(\mathcal{F}_T;\mathbb{R})$

and $f^i(t,y,z) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies (H1) and (H3). Then, the following BDSDE:

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i)\mathrm{d}s + \int_t^T g^i(s, Y_s^i, Z_s^i)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s^i \mathrm{d}W_s, \ \ t \in [0,T], \tag{11}$$

has a unique solution $(Y_\cdot^i, Z_\cdot^i) \in \mathcal{S}^2([0,T];\mathbb{R}) \times \mathcal{M}^2(0,T;\mathbb{R}^d)$ for $i = 1,2$, according to Theorem 3.4 in [10]. We can assert the following comparison theorem.

Lemma 4. *Let $(Y_\cdot^1, Z_\cdot^1)$ and $(Y_\cdot^2, Z_\cdot^2)$ be solutions of BDSDEs (11), respectively. Assume that* (1) $\xi_t^1 \geqslant \xi_t^2, a.s.$; (2) $f^1(t, Y_t^1, Z_t^1) \geqslant f^2(t, Y_t^1, Y_t^1)$ *or* $f^1(t, Y_t^2, Z_t^2) \geqslant f^2(t, Y_t^2, Y_t^2), a.s.,$ *for all* $t \in [0,T]$; (3) $g^1(t, Y_t^1, Z_t^1) = g^2(t, Y_t^1, Y_t^1)$ *or* $g^1(t, Y_t^2, Z_t^2) = g^2(t, Y_t^2, Y_t^2), a.s.,$ *for all* $t \in [0,T]$. *Then* $Y_t^1 \geqslant Y_t^2, a.s.,$ *for any* $t \in [0,T]$.

Proof. Let

$$\hat{Y}_t = Y_t^2 - Y_t^1, \ \ \hat{Z}_t = Z_t^2 - Z_t^1, \ \ \hat{\xi} = \xi^2 - \xi^1.$$
$$\hat{f}_t = f^2(t, Y_t^2, Z_t^2) - f^1(t, Y_t^1, Z_t^1), \ \ \hat{g}_t = g^2(t, Y_t^2, Z_t^2) - g^1(t, Y_t^1, Z_t^1).$$

In view of Itô-Meyer's formula and $\xi^1 \geqslant \xi^2$, a.s., we have

$$\mathbb{E}|\hat{Y}_t^+|^2 + \mathbb{E}\int_t^T 1_{\{\hat{Y}_s \geq 0\}}|\hat{Z}_s|^2 ds = 2\mathbb{E}\int_t^T \hat{Y}_s^+ \hat{f}_s ds + \mathbb{E}\int_t^T 1_{\{\hat{Y}_s \geq 0\}}|\hat{g}_s|^2 ds.$$

By the assumption (H3), Young's inequality $2ab \leq \frac{1}{\theta}a^2 + \theta b^2$, and Jensen's inequality, for each $\theta > 0$, we get

$$\begin{aligned}
2\mathbb{E}\int_t^T \hat{Y}_s^+ \hat{f}_s ds \leq & 2\mathbb{E}\int_t^T \hat{Y}_s^+ (f^2(s, Y_s^2, Z_s^2) - f^2(t, Y_s^1, Z_s^1)) ds \\
\leq & \frac{2C}{1-\alpha_1}\mathbb{E}\int_t^T |\hat{Y}_s^+|^2 ds + \frac{1-\alpha_1}{2C}\int_t^T 1_{\{\hat{Y}_s \geq 0\}}|f^2(s, Y_s^2, Z_s^2) - f^2(t, Y_s^1, Z_s^1)|^2 ds \\
\leq & \frac{2C}{1-\alpha_1}\mathbb{E}\int_t^T |\hat{Y}_s^+|^2 ds + \frac{1-\alpha_1}{2C}\int_t^T \rho_1(s, \mathbb{E}[|\hat{Y}_s^+|^2]) ds + \frac{1-\alpha_1}{2}\mathbb{E}\int_t^T 1_{\{\hat{Y}_s \geq 0\}}|\hat{Z}_s|^2 ds,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\int_t^T 1_{\{\hat{Y}_s \geq 0\}}|\hat{g}_s|^2 ds = & \mathbb{E}\int_t^T 1_{\{\hat{Y}_s \geq 0\}}|g^1(s, Y_s^2, Z_s^2) - g^1(t, Y_s^1, Z_s^1)|^2 ds \\
\leq & \int_t^T \rho_1(s, \mathbb{E}[|\hat{Y}_s^+|^2]) ds + \alpha_1 \mathbb{E}\int_t^T 1_{\{\hat{Y}_s \geq 0\}}|\hat{Z}_s|^2 ds.
\end{aligned}$$

Then, thanks to the above inequalities, we obtain

$$\mathbb{E}|\hat{Y}_t^+|^2 \leq \frac{2C}{1-\alpha_1}\mathbb{E}\int_t^T |\hat{Y}_s^+|^2 ds + (\frac{1-\alpha_1}{2C} + 1)\int_t^T \rho_1(s, \mathbb{E}[|\hat{Y}_s^+|^2]) ds.$$

From the Gronwall's inequality, we have

$$\mathbb{E}|\hat{Y}_t^+|^2 \leqslant e^{\frac{2C}{1-\alpha_1}}(\frac{1-\alpha_1}{2C} + 1)\int_t^T \rho_1(s, \mathbb{E}[|\hat{Y}_s^+|^2]) ds.$$

Using the same proof method about the uniqueness in Theorem 1, we can obtain

$$\mathbb{E}|\hat{Y}_t^+|^2 = 0, \quad for\ all\ t \in [0,T].$$

Hence

$$Y_t^1 \geqslant Y_t^2, \ a.s., \quad for\ any\ t \in [0,T].$$

□

For $i = 1, 2$, we first study a comparison theorem of anticipated BDSDEs of the following generalized version:

$$\begin{cases} Y_t^i = \xi_T^i + \int_t^T f^i(s, Y_s^i, Z_s^i, Y_{s+\delta^i(s)}^i, Z_{s+\zeta^i(s)}^i)\mathrm{d}s \\ \qquad + \int_t^T g^i(s, Y_s^i, Z_s^i, Y_{s+\overline{\delta}^i(s)}^i, Z_{s+\overline{\zeta}^i(s)}^i)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s^i \mathrm{d}W_s, \qquad t \in [0, T], \\ Y_t^i = \xi_t^i, \quad Z_t^i = \eta_t^i, \quad t \in [T, T+K]. \end{cases} \tag{12}$$

Let us assume that $\delta^i(\cdot), \overline{\delta}^i(\cdot), \zeta^i(\cdot), \overline{\zeta}^i(\cdot)$ satisfies (A1) and (A2), $\xi_\cdot^i \in \mathcal{S}^2([T, T+K]; \mathbb{R})$, $\eta_\cdot^i \in \mathcal{M}^2(T, T+K; \mathbb{R}^d)$, and (f^i, g^i) satisfies (H1) and (H3). Then, by Theorem 1, anticipated BDSDE (12) has a unique solution $(Y_\cdot^i, Z_\cdot^i) \in \mathcal{S}^2([0, T+K]; \mathbb{R}) \times \mathcal{M}^2(0, T+K; \mathbb{R}^d)$ for $i = 1, 2$.

Theorem 2. *Suppose $(Y_\cdot^1, Z_\cdot^1)$ and $(Y_\cdot^2, Z_\cdot^2)$ are solutions of ABDSDEs (12), respectively. Assume that (1) $\xi_t^1 \geqslant \xi_t^2, a.s.,$ for all $t \in [T, T+K]$; (2) $f^1(t, Y_t^1, Z_t^1, Y_{t+\delta^1(t)}^1, Z_{t+\zeta^1(t)}^1) \geqslant f^2(t, Y_t^1, Z_t^1, Y_{t+\delta^2(t)}^2, Z_{t+\zeta^2(t)}^2)$ or $f^1(t, Y_t^2, Z_t^2, Y_{t+\delta^1(t)}^1, Z_{t+\zeta^1(t)}^1) \geqslant f^2(t, Y_t^2, Y_t^2, Y_{t+\delta^2(t)}^2, Z_{t+\zeta^2(t)}^2), a.s.,$ for all $t \in [0, T]$; (3) $g^1(t, Y_t^1, Z_t^1, Y_{t+\overline{\delta}^1(t)}^1, Z_{t+\overline{\zeta}^1(t)}^1) = g^2(t, Y_t^1, Y_t^1, Y_{t+\overline{\delta}^2(t)}^2, Z_{t+\overline{\zeta}^2(t)}^2)$ or $g^1(t, Y_t^2, Z_t^2, Y_{t+\overline{\delta}^1(t)}^1, Z_{t+\overline{\zeta}^1(t)}^1) = g^2(t, Y_t^2, Y_t^2, Y_{t+\overline{\delta}^2(t)}^2, Z_{t+\overline{\zeta}^2(t)}^2), a.s.,$ for all $t \in [0, T]$.* Then $Y_t^1 \geqslant Y_t^2, a.s.,$ *for any* $t \in [0, T+K]$.

Proof. For $i = 1, 2$, set

$$F^i(t, y, z) = f^i(t, y, z, Y_{t+\delta^i(t)}^i, Z_{t+\zeta^i(t)}^i), G^i(t, y, z) = g^i(t, y, z, Y_{t+\overline{\delta}^i(t)}^i, Z_{t+\overline{\zeta}^i(t)}^i),$$

then $(Y_\cdot^i, Z_\cdot^i)$ is the unique solution of the following BDSDE,

$$Y_t^i = \xi_T^i + \int_t^T F^i(s, Y_s^i, Z_s^i)\mathrm{d}s + \int_t^T G^i(s, Y_s^i, Z_s^i)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s^i \mathrm{d}W_s, \ \ t \in [0, T].$$

According to Lemma 4, we can get

$$Y_t^1 \geqslant Y_t^2, a.s., \ for \ all \ t \in [0, T],$$

which implies

$$Y_t^1 \geqslant Y_t^2, a.s., \ for \ all \ t \in [0, T+K].$$

□

Example 1. *Let $f^1(t, \tilde{y}, \tilde{z}, \tilde{\theta}(r), \tilde{\vartheta}(\bar{r})) = |\tilde{y}| + |\tilde{z}| + \mathbb{E}^{\mathcal{F}_t}[|\tilde{\theta}(r)|] + \mathbb{E}^{\mathcal{F}_t}[\cos \tilde{\vartheta}(\bar{r})]$, $f^2(t, \tilde{y}, \tilde{z}, \tilde{\theta}(r), \tilde{\vartheta}(\bar{r})) = \tilde{y} + |\tilde{z}| + \mathbb{E}^{\mathcal{F}_t}[\sin \tilde{\theta}(r)] - 1$, $g^1(t, \tilde{y}, \tilde{z}, \tilde{\theta}(r), \tilde{\vartheta}(\bar{r})) = g^2(t, \tilde{y}, \tilde{z}, \tilde{\theta}(r), \tilde{\vartheta}(\bar{r})) = \tilde{y} + \frac{|\tilde{z}|}{2} + \mathbb{E}^{\mathcal{F}_t}[\tilde{\theta}(r)\tilde{\vartheta}(\bar{r})]$. Then by Theorem 3, we can derive $Y_t^1 \geqslant Y_t^2$, a.s.,* for any $t \in [0, T+K]$ *when the assumption (1) is satisfied.*

Secondly, for $i = 1, 2$, we will study a comparison theorem of anticipated BDSDEs of the following type:

$$\begin{cases} Y_t^i = \xi_T^i + \int_t^T f^i(s, Y_s^i, Z_s^i, Y_{s+\delta(s)}^i, Z_{s+\zeta^i(s)}^i)\mathrm{d}s \\ \qquad + \int_t^T g^i(s, Y_s^i, Z_s^i, Y_{s+\overline{\delta}^i(s)}^i, Z_{s+\overline{\zeta}^i(s)}^i)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s^i \mathrm{d}W_s, \qquad t \in [0, T], \\ Y_t^i = \xi_t^i, \quad Z_t^i = \eta_t^i, \quad t \in [T, T+K]. \end{cases} \tag{13}$$

Assume that $(\delta(\cdot), \zeta^i(\cdot), \overline{\delta}^i(\cdot), \overline{\zeta}^i(\cdot))$ satisfies (A1) and (A2), $\xi^i_\cdot \in \mathcal{S}^2([T, T+K]; \mathbb{R})$, $\eta^i_\cdot \in \mathcal{M}^2(T, T+K; \mathbb{R}^d)$ and (f^i, g^i) satisfies (H1) and (H3). Then, by Theorem 1, anticipated BDSDE (13) has a unique solution $(Y^i_\cdot, Z^i_\cdot) \in \mathcal{S}^2([0, T+K]; \mathbb{R}) \times \mathcal{M}^2(0, T+K; \mathbb{R}^d)$ for $i = 1, 2$. In order to obtain the following comparison theorem, we further assume $\rho_i(t, u)$ with linear growth, that is, for $i = 1, 2$, $\rho_i(t, u) \leq a_i(t) + b_i(t)u$ where $a_i(t) \geq 0, b_i(t) \geq 0$, with

$$\int_0^{T+K} a_i(t)dt < \infty \quad \text{and} \quad \int_0^{T+K} b_i(t)dt < \infty.$$

Theorem 3. *Suppose* $(Y^1_\cdot, Z^1_\cdot)$ *and* $(Y^2_\cdot, Z^2_\cdot)$ *are solutions of ABDSDEs* (13), *respectively. Assume that* (1) $\xi^1_t \geqslant \xi^2_t, a.s.,$ *for all* $t \in [T, T+K]$; (2) $f^1(t, Y^1_t, Z^1_t, Y^1_{t+\delta(t)}, Z^1_{t+\zeta^1(t)}) \geqslant f^2(t, Y^1_t, Y^1_t, Y^1_{t+\delta(t)}, Z^2_{t+\zeta^2(t)}), a.s.,$ *for all* $t \in [0, T]$; (3) for any $(t, y, z) \in [0, T] \times R \times R^d, \gamma \in L^2(\mathcal{F}_s; \mathbb{R}^d)$ and $s \in [t, T+K], f^2(t, y, z, \cdot, \gamma)$ is increasing, i.e., $f^2(t, y, z, y_2, \gamma) \geq f^2(t, y, z, y_1, \gamma)$, if $y_2 \geq y_1$ with $y_1, y_2 \in R$; (4) $g^1(t, Y^1_t, Z^1_t, Y^1_{t+\overline{\delta}^1(t)}, Z^1_{t+\overline{\zeta}^1(t)}) = g^2(t, Y^1_t, Z^1_t, Y^2_{t+\overline{\delta}^2(t)}, Z^2_{t+\overline{\zeta}^2(t)}), a.s.,$ *for all* $t \in [0, T]$. Then $Y^1_t \geqslant Y^2_t, a.s.,$ *for any* $t \in [0, T+K]$.

Proof. For $i = 1, 2$, set

$$F^i(t, y, z) = f^i(t, y, z, Y^i_{t+\delta(t)}, Z^i_{t+\zeta^i(t)}), G^i(t, y, z) = g^i(t, y, z, Y^i_{t+\overline{\delta}^i(t)}, Z^i_{t+\overline{\zeta}^i(t)}),$$

then $(Y^i_\cdot, Z^i_\cdot)$ be the unique solution of the following BDSDE,

$$\begin{cases} Y^i_t = \xi^i_T + \int_t^T F^i(s, Y^i_s, Z^i_s)\mathrm{d}s + \int_t^T G^i(s, Y^i_s, Z^i_s)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z^i_s \mathrm{d}W_s, \quad t \in [0, T], \\ Y^i_t = \xi^i_t, \quad Z^i_t = \eta^i_t, \quad t \in [T, T+K]. \end{cases}$$

Let $\overline{f}^2(t, y, z) = f^2(t, y, z, Y^1_{t+\delta(t)}, Z^2_{t+\zeta^i(t)})$, then the following BDSDE admits a unique solution (Y^3, Z^3),

$$\begin{cases} Y^3_t = \xi^2_T + \int_t^T \overline{f}^2(s, Y^3_s, Z^3_s)\mathrm{d}s + \int_t^T G^2(s, Y^3_s, Z^3_s)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z^3_s \mathrm{d}W_s, \quad t \in [0, T], \\ Y^3_t = \xi^2_t, \quad Z^3_t = \eta^2_t, \quad t \in [T, T+K]. \end{cases}$$

According to the assumptions (1), (2), (4) and Lemma 4, we can get $Y^1_t \geqslant Y^3_t, a.s.,$ for each $\mathrm{t} \in [0, \mathrm{T}]$, which implies $Y^1_t \geqslant Y^3_t, a.s.,$ for any $\mathrm{t} \in [0, \mathrm{T}+\mathrm{K}]$. The following BDSDE,

$$\begin{cases} Y^4_t = \xi^2_T + \int_t^T f^2(s, Y^4_s, Z^4_s, Y^3_{t+\delta(t)}, Z^2_{t+\zeta^2(t)})\mathrm{d}s + \int_t^T G^2(s, Y^4_s, Z^4_s)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z^4_s \mathrm{d}W_s, t \in [0, T], \\ Y^4_t = \xi^2_t, \quad Z^4_t = \eta^2_t, \quad t \in [T, T+K] \end{cases}$$

admits a unique solution $(Y^4_\cdot, Z^4_\cdot)$. From Lemma 4 and assumptions (3), we can get $Y^3_t \geqslant Y^4_t, a.s.,$ for all $\mathrm{t} \in [0, \mathrm{T}]$, which implies $Y^3_t \geqslant Y^4_t, a.s.,$ for any $\mathrm{t} \in [0, \mathrm{T}+\mathrm{K}]$. For $j \geq 5$, let

$$\begin{cases} Y^j_t = \xi^2_T + \int_t^T f^2(s, Y^j_s, Z^j_s, Y^{j-1}_{t+\delta(t)}, Z^2_{t+\zeta^2(t)})\mathrm{d}s + \int_t^T G^2(s, Y^j_s, Z^j_s)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z^j_s \mathrm{d}W_s, t \in [0, T], \\ Y^j_t = \xi^2_t, \quad Z^j_t = \eta^2_t, \quad t \in [T, T+K]. \end{cases}$$

According to Lemma 4 and by induction, we can get $Y^4_t \geqslant Y^5_t \geq \cdots \geq Y^{j-1}_t \geq Y^j_t, a.s.,$ for all $\mathrm{t} \in [0, \mathrm{T}+\mathrm{K}]$, hence, for all $j \geq 3$

$$Y^1_t \geqslant Y^j_t, a.s., \text{ for all } \mathrm{t} \in [0, \mathrm{T}+\mathrm{K}]. \tag{14}$$

For any $j \geq 4, \beta > 0$, apply Itô's formula to $e^{\beta t}|Y_t^j|^2$, in view of (H3), Young's inequality, Jensen's inequality and for $i = 1, 2$, $\rho_i(t, u)$ has a linear growth, we have

$$\begin{aligned}\mathbb{E}|Y_t^j|^2 e^{\beta t} + (1 - \alpha_1 - \frac{\alpha_1 + 2C}{\beta})\mathbb{E}\int_t^T |Z_s^j|^2 e^{\beta s} ds \leq & C_1 + (1 + \frac{3}{\beta})\mathbb{E}\int_t^T b_1(s) e^{\beta s}\mathbb{E}[|Y_s^j|^2]ds \\ & + \frac{2M}{\beta}\mathbb{E}\int_t^T b_2(s)e^{\beta s}\mathbb{E}[|Y_s^{j-1}|^2]ds,\end{aligned}$$

where

$$\begin{aligned}C_1 =& \mathbb{E}[e^{\beta T}|\xi_T^2|^2] + (1 + \frac{3}{\beta})\int_0^T a_1(s)e^{\beta s}\mathrm{d}s + (\alpha_2 + \frac{\alpha_2 + 2C}{\beta})M\mathbb{E}\int_0^{T+K}|Z_s^2|^2 e^{\beta s} ds \\ & + \frac{2M}{\beta}\mathbb{E}\int_T^{T+K} b_2(s)e^{\beta s}\mathbb{E}[|\xi_s^2|^2]ds + (1 + \frac{1}{\beta})M\int_0^{T+K}\rho_2(s, \mathbb{E}[|Y_s^2|^2])e^{\beta s}ds \\ & + \frac{2M}{\beta}\int_0^{T+K} a_2(s)e^{\beta s}ds + \frac{2}{\beta}\mathbb{E}\int_0^T |f^2(s, 0, 0, 0, 0)|^2 e^{\beta s}\mathrm{d}s + (1 + \beta)\int_0^T |g^2(s, 0, 0, 0, 0)|^2 e^{\beta s}ds.\end{aligned}$$

Now let us choose $\beta = \frac{\alpha_1 + 2C}{1 - \alpha_1}$. Then, there exists $C_2 > 0$, which is independent of j, such that for $j \geqslant 4$,

$$\mathbb{E}[|Y_t^j|^2] \leqslant C_2 + C_2\mathbb{E}\int_t^T b_1(s)|Y_s^j|^2\mathrm{d}s + C_2\mathbb{E}\int_t^T b_2(s)|Y_s^{j-1}|^2\mathrm{d}s,$$

which leads to

$$\sup_{j \geqslant 4}\mathbb{E}[|Y_t^j|^2] \leqslant C_2 + C_2\mathbb{E}\int_0^T b_2(s)|Y_t^3|^2\mathrm{d}t + C_2\int_t^T (b_1(s) + b_2(s))\sup_{j \geqslant 4}\mathbb{E}[|Y_s^j|^2]\mathrm{d}s,$$

Then Gronwall's inequality yields

$$\sup_{j \geqslant 4}\mathbb{E}[|Y_t^j|^2] \leqslant C_2 e^{C_2\int_t^T (b_1(s) + b_2(s))\mathrm{d}s}[1 + \mathbb{E}\int_0^T b_2(s)|Y_t^3|^2\mathrm{d}t],$$

which implies

$$\sup_{0 \leqslant t \leqslant T}\sup_{j \geqslant 4}\mathbb{E}[|Y_t^j|^2] \leqslant C_2 e^{C_2\int_0^T (b_1(s) + b_2(s))\mathrm{d}s}[1 + \int_0^T b_2(s)\mathbb{E}[\sup_{0 \leqslant t \leqslant T}|Y_t^3|^2]\mathrm{d}t] < \infty. \quad (15)$$

For $j \geqslant 4, p \geqslant 1$, Set

$$\begin{aligned}&\hat{Y}_t^j = Y_t^{j+p} - Y_t^j, \;\; \hat{Z}_t^j = Z_t^{j+p} - Z_t^j, \\ &\hat{F}_t^j = f^2(t, Y_t^{j+p}, Z_t^{j+p}, Y_{t+\delta(t)}^{j+p-1}, Z_{t+\zeta^2(t)}^2) - f^2(t, Y_t^j, Z_t^j, Y_{t+\delta(t)}^{j-1}, Z_{t+\zeta^2(t)}^2), \\ &\hat{G}_t^j = G^2(t, Y_t^{j+p}, Z_t^{j+p}) - G^2(t, Y_t^j, Z_t^j),\end{aligned}$$

Then for $j \geq 4$, $(\hat{Y}_\cdot^j, \hat{Z}_\cdot^j)$ satisfies

$$\begin{cases}\hat{Y}_t^j = \int_t^T \hat{F}_s^j\mathrm{d}s + \int_t^T \hat{G}_s^j\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s^j\mathrm{d}W_s, t \in [0, T], \\ \hat{Y}_t^j = 0, \quad \hat{Z}_t^j = 0, \quad t \in [T, T + K].\end{cases}$$

Applying Itô's formula to $e^{\beta t}|\hat{Y}_t^j|^2$ where $\beta > 0$, and from the assumption (H3), Young's inequality, Jensen's inequality, we can obtain

$$\begin{aligned}\mathbb{E}|\hat{Y}_t^j|^2 e^{\beta t} + \mathbb{E}\int_t^T |\hat{Z}_s^j|^2 e^{\beta s} ds \leq & \frac{1}{\beta}\int_t^T (\rho_1(s, \mathbb{E}[|\hat{Y}_s^j|^2]) + M\rho_2(s, \mathbb{E}[|\hat{Y}_s^{j-1}|^2]))e^{\beta s} ds \\ & + (\frac{C}{\beta} + \alpha_1)\mathbb{E}\int_t^T |\hat{Z}_s^j|^2 e^{\beta s} ds + \int_t^T \rho_1(s, \mathbb{E}[|\hat{Y}_s^j|^2])e^{\beta s} ds.\end{aligned}$$

Now let us choose $\beta = \frac{2C}{1-\alpha_1}$. Then there exists $C_3 > 0$, which is independent of j, p, such that

$$\mathbb{E}[|\hat{Y}_t^j|^2] + \mathbb{E}\int_t^T |\hat{Z}_s^j|^2 \mathrm{d}s \leqslant C_3 \int_t^T (\rho_1(s, \mathbb{E}[|\hat{Y}_s^j|^2]) + M\rho_2(s, \mathbb{E}[|\hat{Y}_s^{j-1}|^2]))ds. \tag{16}$$

Denote $h(t) = \limsup_{k\to\infty} \mathbb{E}[|\hat{Y}_t^j|^2]$. Then in view of (15) we have $0 \leqslant h(t) < \infty$. Then, apply Fatou's Lemma to the right-hand side of (16), we obtain

$$h(t) \leqslant C_3 \int_t^T (\rho_1(s, h(s)) + M\rho_2(s, h(s)))\mathrm{d}s.$$

Using the same proof method about the uniqueness in Theorem 1, we can obtain $h(t) = 0$, that is $\limsup_{j\to\infty} \mathbb{E}[|\hat{Y}_t^j|^2] = 0$, for all $t \in [0, T]$. Moreover, from (16) it follows that

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}[|\hat{Y}_t^j|^2] \leqslant C_3 \int_0^T (\rho(\mathbb{E}[|\hat{Y}_s^j|^2]) + \rho(\mathbb{E}[|\hat{Y}_s^{j-1}|^2]))\mathrm{d}s$$

Applying Fatou's Lemma again leads to

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}[|\hat{Y}_t^j|^2] \to 0 \ \text{ as } j \to \infty.$$

By taking $j \to \infty$ in (16), we use Fatou's Lemma again to obtain

$$\mathbb{E}\int_0^T |\hat{Z}_t^j|^2 \mathrm{d}t \to 0 \ \text{ as } j \to \infty.$$

Hence, we have shown that $(Y_\cdot^k, Z_\cdot^k)$ is a Cauchy sequence in $S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$, thus it is also a Cauchy sequence in $S^2(0, T + K; \mathbb{R}) \times M^2(0, T + K; \mathbb{R}^d)$. As a consequence, we can find $(Y_\cdot, Z_\cdot) \in S^2(0, T + K; \mathbb{R}) \times M^2(0, T + K; \mathbb{R}^d)$ such that $Y_t = \xi_t^2$, $Z_t = \eta_t^2$ for $T \leqslant t \leqslant T + K$ and

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}[|Y_t^j - Y_t|^2] + \mathbb{E}[\int_0^T |Z_t^j - Z_t|^2 \mathrm{d}t] \to 0 \ \text{ as } j \to \infty.$$

Therefore, it's easy to verify that, as $j \to \infty$,

$$\mathbb{E}\int_t^T |f^2(s, Y_s^j, Z_s^j, Y_{s+\delta(s)}^{j-1}, Z_{s+\zeta^2(s)}^2) - f^2(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta^2(s)}^2)|^2 \mathrm{d}s \to 0,$$

$$\mathbb{E}\int_t^T |g^2(s, Y_s^j, Z_s^j, Y_{s+\bar{\delta}^2(s)}^2, Z_{s+\bar{\zeta}^2(s)}^2) - g^2(s, Y_s, Z_s, Y_{s+\bar{\delta}^2(s)}^2, Z_{s+\bar{\zeta}^2(s)}^2)|^2 \mathrm{d}s \to 0,$$

$$\mathbb{E}[|\int_t^T Z_s^j \cdot \mathrm{d}W_s - \int_t^T Z_s \cdot \mathrm{d}W_s|^2] \to 0.$$

So, we conclude that $(Y_\cdot, Z_\cdot)$ solves the following ABDSDE:

$$\begin{cases} Y_t = \xi_T^2 + \int_t^T f^2(s, Y_s, Z_s, Y_{s+\delta(s)}, Z^i_{s+\zeta^2(s)})\mathrm{d}s \\ \qquad + \int_t^T g^2(s, Y_s, Z_s, Y^2_{s+\bar{\delta}^2(s)}, Z^2_{s+\bar{\zeta}^2(s)})\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s \mathrm{d}W_s, \qquad t \in [0,T], \\ Y_t = \xi_t^2, \quad Z_t = \eta_t^2, \quad t \in [T, T+K]. \end{cases}$$

Therefore, we can infer that $Y_t = Y_t^2, a.s.$, for all $t \in [0, T+K]$ from the uniqueness part of Theorem 1. Let $j \to \infty$ in (14) yields $Y_t^1 \geq Y_t^2, a.s.$, for all $t \in [0, T+K]$. $\square$

Example 2. *Let* $f^1(t,\tilde{y},\tilde{z},\tilde{\theta}(r),\tilde{\vartheta}(\bar{r})) = |\tilde{y}| + |\tilde{z}| + \mathbb{E}^{\mathcal{F}_t}[|\tilde{\theta}(r)|] + \mathbb{E}^{\mathcal{F}_t}[\arctan\tilde{\vartheta}(\bar{r})]$, $f^2(t,\tilde{y},\tilde{z},\tilde{\theta}(r),\tilde{\vartheta}(\bar{r})) = \tilde{y} - |\tilde{z}| + \mathbb{E}^{\mathcal{F}_t}[\tilde{\theta}(r)] - \frac{\pi}{2}$, $g^1(t,\tilde{y},\tilde{z},\tilde{\theta}(r),\tilde{\vartheta}(\bar{r})) = g^2(t,\tilde{y},\tilde{z},\tilde{\theta}(r),\tilde{\vartheta}(\bar{r})) = \mathrm{arccot}\tilde{y} + |\tilde{z}| + \mathbb{E}^{\mathcal{F}_t}[\tilde{\theta}(r) + \tilde{\vartheta}(\bar{r})]$. *Then by Theorem 3, we can derive* $Y_t^1 \geqslant Y_t^2$, *a.s.*, for any $t \in [0, T+K]$ *when the assumption (1) is satisfied.*

5. Conclusions

The purpose of this paper is to introduce and study a type of anticipated BDSDEs with non-Lipschitze coefficients. We first show that the adapted solution of this kind of ABDSDEs is existent and unique. Furthermore, we give two comparison theorems one dimensional situation. In our future publications, we will concentrate on investigating this interesting problem and pay much attention to the application of this kind of equation, especially in control such as [20–25].

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