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# Local Well-Posedness for the Magnetohydrodynamics in the Different Two Liquids Case

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**Abstract:** We consider the free boundary problem of MHD in the multi-dimensional case. This problem describes the motion of two incompressible fluids separated by a closed interface under the action of a magnetic field. This problem is overdetermined, and we find an equivalent system of equations which is uniquely solvable locally in time in the $L_p$- $L_q$ maximal regularity class, where $1 < p, q < \infty$ and $2/p + N/q < 1$. As a result, the original two-phase problem for the MHD is solvable locally in time.

**Keywords:** two-phase problem; magnetohydorodynamics; local well-posedness; $L_p$-$L_q$ maximal regularity

**MSC:** 35Q30; 35Q61

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## 1. Introduction

### 1.1. Formulation of the Problem

We consider a two-phase problem governing the motion of two incompressible electrically conducting capillary liquids separated by a sharp interface. Mathematical model of MHD is found in [1,2], the transmission conditions on the interface for the magnetic fields are found in [3–5], and the interface conditions for the incompressible viscous fluids are found in [6–10].

In this paper, the problem is formulated as follows: Let $\Omega_+$ and $\Omega_-$ be two domains in the $N$-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$). Assume that the boundary of each $\Omega_\pm$ consists of two connected components $\Gamma$ and $S_\pm$, where $\Gamma$ is the common boundary of $\Omega_\pm$. Throughout the paper, we assume that $\Gamma$ is a compact hypersurface of $C^2$ class, that $S_\pm$ are hypersurfaces of $C^2$ class, and that dist $(\Gamma, S_\pm) \geq d_\pm$ with some positive constants $d_\pm$, where the $\text{dist}(A, B)$ denotes the distance of any subsets $A$ and $B$ of $\mathbb{R}^N$ which is defined by setting $\text{dist}(A, B) = \inf\{|x - y| \mid x \in A, y \in B\}$. Let $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ and $\dot\Omega = \Omega_+ \cup \Omega_-$. The boundary of $\Omega$ is $S_+ \cup S_-$. We may consider the case that one of $S_\pm$ is an empty set or that both $S_\pm$ are empty sets. Let $\Gamma_t$ be an evolution of $\Gamma$ for time $t > 0$, which is assumed to be given by

$$\Gamma_t = \{x = y + h(y, t)\mathbf{n}(y) \mid y \in \Gamma\} \tag{1}$$

with an unknown function $h(y, t)$. We assume that $h|_{t=0} = h_0(y)$ is a given function. Let $\Omega_{t\pm}$ be two connected components of $\Omega \setminus \Gamma_t$ such that the boundary of $\Omega_{t\pm}$ consists of $\Gamma_t$ and $S_\pm$. Let $\mathbf{n}_t$ be the unit outer normal to $\Gamma_t$ oriented from $\Omega_{t+}$ into $\Omega_{t-}$, and let $\mathbf{n}_\pm$ be the unit outer normal to $S_\pm$, respectively. For any given functions $v_\pm$ defined in $\Omega_{t\pm}$, $v$ is defined by $v(x) = v_\pm(x)$ for $x \in \Omega_{t\pm}$, $t \geq 0$. Let

$$[[v]](x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_{t+}}} v_+(x) - \lim_{\substack{x\to x_0\\ x\in\Omega_{t-}}} v_-(x)$$

for every point $x_0 \in \Gamma_t$, which is the jump quantity of $v$ across $\Gamma_t$. Let $\dot\Omega_t = \Omega_{t+} \cup \Omega_{t-}$. The purpose of this paper is to prove the local in time unique solvability of the free boundary two-phase magnetohydrodynamical problem with interface conditions, which is formulated as follows:

$$\begin{aligned}
m(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mathrm{Div}\,(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H})) = 0, \quad \mathrm{div}\,\mathbf{v} = 0 \quad &\text{in } \dot\Omega_t,\\
[[(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H}))\mathbf{n}_t]] = \sigma\mathcal{H}(\Gamma_t)\mathbf{n}_t - \mathfrak{p}_0\mathbf{n}_t, \quad [[\mathbf{v}]] = 0, \quad V_{\Gamma_t} = \mathbf{v}_+\cdot\mathbf{n}_t \quad &\text{on } \Gamma_t,\\
\mu\partial_t\mathbf{H} + \mathrm{Div}\,\{\alpha^{-1}\mathrm{curl}\,\mathbf{H} - \mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v})\} = 0, \quad \mathrm{div}\,\mathbf{H} = 0 \quad &\text{in } \dot\Omega_t,\\
[[\{\alpha^{-1}\mathrm{curl}\,\mathbf{H} - \mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v})\}\mathbf{n}_t]] = 0 \quad &\text{on } \Gamma_t,\\
[[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0, \quad [[\mathbf{H}- <\mathbf{H},\mathbf{n}_t> \mathbf{n}_t]] = 0 \quad &\text{on } \Gamma_t,\\
\mathbf{v}_\pm = 0, \quad \mathbf{n}_\pm\cdot\mathbf{H}_\pm = 0, \quad (\mathrm{curl}\,\mathbf{H}_\pm)\mathbf{n}_\pm = 0 \quad &\text{on } S_\pm,\\
(\mathbf{v},\mathbf{H})|_{t=0} = (\mathbf{v}_0,\mathbf{H}_0) \quad \text{in } \dot\Omega, \quad h|_{t=0} = h_0 \quad &\text{on } \Gamma
\end{aligned} \tag{2}$$

for $t \in (0,T)$. Here, $\mathbf{v} = \mathbf{v}_\pm = (v_{\pm1}(x,t),\ldots,v_{\pm N}(x,t))^\top$ is the velocity vector field, where $M^\top$ stands for the transposed $M$, $\mathfrak{p} = \mathfrak{p}_\pm(x,t)$ the pressure field, and $\mathbf{H} = \mathbf{H}_\pm = (H_{\pm1}(x,t),\ldots,H_{\pm N}(x,t))^\top$ the magnetic field. The unknowns are $\mathbf{v}$, $\mathfrak{p}$, $\mathbf{H}$, and $\Gamma_t$, while $\mathbf{v}_0$ and $\mathbf{H}_0$ are prescribed $N$-component vectors. As for the remaining symbols, $\mathbf{T}(\mathbf{v},\mathfrak{p}) = \nu_\pm\mathbf{D}(\mathbf{v}_\pm) - \mathfrak{p}_\pm\mathbf{I}$ is the viscous stress tensor, $\mathbf{D}(\mathbf{v}_\pm) = \nabla\mathbf{v}_\pm + (\nabla\mathbf{v}_\pm)^\top$ is the doubled deformation tensor whose $(i,j)$th component is $\partial_j v_{\pm i} + \partial_i v_{\pm j}$ with $\partial_i = \partial/\partial x_i$, $\mathbf{I}$ the $N\times N$ unit matrix, $\mathbf{T}_M(\mathbf{H}) = \mathbf{T}_M(\mathbf{H}_\pm) = \mu_\pm(\mathbf{H}_\pm\otimes\mathbf{H}_\pm - \frac{1}{2}|\mathbf{H}_\pm|^2\mathbf{I})$ the magnetic stress tensor, $\mathrm{curl}\,\mathbf{v} = \mathrm{curl}\,\mathbf{v}_\pm = \nabla\mathbf{v}_\pm - (\nabla\mathbf{v}_\pm)^\top$ the doubled rotation tensor whose $(i,j)$th component is $\partial_j v_{\pm i} - \partial_i v_{\pm j}$, $V_{\Gamma_t}$ the velocity of the evolution of $\Gamma_t$ in the direction of $\mathbf{n}_t$, which is given by $V_{\Gamma_t} = (\partial_t h)\mathbf{n}\cdot\mathbf{n}_t$ in the case of (1), and $\mathcal{H}(\Gamma_t)$ is the mean curvature of $\Gamma_t$, which is given by $\mathcal{H}(\Gamma_t)\mathbf{n}_t = \Delta_{\Gamma_t}x$ for $x\in\Gamma_t$, where $\Delta_{\Gamma_t}$ is the Laplace Beltrami operator on $\Gamma_t$, $\mathfrak{p}_0$ the outside pressure. Moreover, $m = m_\pm$, $\mu = \mu_\pm$, $\nu = \nu_\pm$, and $\alpha = \alpha_\pm$, are positive constants describing respective the mass density, the magnetic permability, the kinematic viscosity, and conductivity. A positive constant $\sigma$ is the coefficient of the surface tension. Finally, for any matrix field $\mathbf{K}$ with $(i,j)$th component $K_{ij}$, the quantity $\mathrm{Div}\,K$ is an $N$-vector of functions with the $i$th component $\sum_{j=1}^N \partial_j K_{ij}$. For any $N$-vectors of functions $\mathbf{u} = (u_1,\ldots,u_N)^\top$ and $\mathbf{w} = (w_1,\ldots,w_N)^\top$, $\mathrm{div}\,\mathbf{u} = \sum_{j=1}^N \partial_j u_j$, $\mathbf{u}\cdot\nabla\mathbf{w}$ is an $N$-vector of functions with the $i$th component $\sum_{j=1}^N u_j\partial_j w_i$, and $\mathbf{u}\otimes\mathbf{w}$ an $N\times N$ matrix with the $(i,j)$th component $u_i w_j$. We notice that

$$\begin{aligned}
\Delta\mathbf{v} = -\mathrm{Div}\,\mathrm{curl}\,\mathbf{v} + \nabla\mathrm{div}\,\mathbf{v}, \quad \mathrm{rot}\,\mathrm{rot}\,\mathbf{H} = \mathrm{Div}\,\mathrm{curl}\,\mathbf{H},\\
\mathrm{Div}\,(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}) = \mathbf{v}\mathrm{div}\,\mathbf{H} - \mathbf{H}\mathrm{div}\,\mathbf{v} + \mathbf{H}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{H},\\
\mathrm{rot}\,\mathbf{v}\times\mathbf{H} = \mathrm{Div}\,(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}) \quad \text{(three-dimensional case)}.
\end{aligned} \tag{3}$$

In particular, in the three-dimensional case, the set of equations for the magnetic field in Equation (2) are written by

$$\begin{aligned}
\mu\partial_t\mathbf{H} + \mathrm{rot}\,(\alpha^{-1}\mathrm{rot}\,\mathbf{H} - \mu\mathbf{v}\times\mathbf{H}) = 0, \quad \mathrm{div}\,\mathbf{H} = 0 \quad &\text{in } \dot\Omega_t,\\
[[\mathbf{n}_t\times\{\alpha^{-1}\mathrm{rot}\,\mathbf{H} - \mu\mathbf{v}\times\mathbf{H})\}]] = 0, \quad [[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0, \quad [[\mathbf{H}- <\mathbf{H},\mathbf{n}_t> \mathbf{n}_t]] = 0 \quad &\text{on } \Gamma_t.
\end{aligned} \tag{4}$$

for $t \in (0,T)$. This is a standard description (cf. M. Padula and V. A. Solonnikov [4]), and so the set of equations for the magnetic field in Equation (2) is the $N$-dimensional mathematical description for the magnetic equations with transmission conditions.

In the equilibrium state, $\mathbf{v} = 0$, $\mathbf{H} = 0$, $\Gamma_t = \Gamma$, and $\mathfrak{p}$ is a constant state, and so we assume that

$$\mathfrak{p}_0 = \sigma\mathcal{H}(\Gamma). \tag{5}$$

Problem (2) is overdetermined, because there are too many equations for the magnetic fields $\mathbf{H}_\pm$. Instead of (2), we consider the following equivalent system:

$$\begin{aligned}
m(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mathrm{Div}\,(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H})) = 0, \quad \mathrm{div}\,\mathbf{v} = 0 \quad & \text{in } \dot\Omega,\\
[[(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H}))\mathbf{n}_t]] = \sigma\mathcal{H}(\Gamma_t)\mathbf{n}_t - \mathfrak{p}_0\mathbf{n}_t, \quad [[\mathbf{v}]] = 0, \quad V_{\Gamma_t} = \mathbf{v}_+\cdot\mathbf{n} \quad & \text{on } \Gamma_t,\\
\mu\partial_t\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} - \mathrm{Div}\,\mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}) = 0 \quad & \text{in } \dot\Omega_t,\\
[[\{\alpha^{-1}\mathrm{curl}\,\mathbf{H} - \mu(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v})\}\mathbf{n}_t]] = 0, \quad [[\mu\mathrm{div}\,\mathbf{H}]] = 0 \quad & \text{on } \Gamma_t,\\
[[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0, \quad [[\mathbf{H} - <\mathbf{H},\mathbf{n}_t>\mathbf{n}_t]] = 0 \quad & \text{on } \Gamma_t,\\
\mathbf{v}_\pm = 0, \quad \mathbf{n}_0\cdot\mathbf{H}_\pm = 0, \quad (\mathrm{curl}\,\mathbf{H}_\pm)\mathbf{n}_\pm = 0 \quad & \text{on } S_\pm,\\
(\mathbf{v},\mathbf{H})|_{t=0} = (\mathbf{v}_0,\mathbf{H}_0) \quad \text{in } \dot\Omega_0 &
\end{aligned} \tag{6}$$

for $t \in (0,T)$. Namely, two equations: $\mathrm{div}\,\mathbf{H}_\pm = 0$ in $\Omega_\pm$ is replaced with one boundary condition: $[[\mu\mathrm{div}\,\mathbf{H}]] = 0$ on $\Gamma$. Frolova and Shibata [11] proved that if for a solution of (6) $\mathrm{div}\,\mathbf{H} = 0$ initially, then $\mathrm{div}\,\mathbf{H} = 0$ in $\dot\Omega_t$ follows automatically for any $t > 0$ as long as the solution exists. Thus, the local well-posedness of Equation (2) follows from that of Equation (6) provided that the initial data $\mathbf{H}_0$ satisfy the divergence free condition: $\mathrm{div}\,\mathbf{H}_0 = 0$, which is a compatibility condition. This paper devotes itself to proving the local well-posedness of Equation (6) in the maximal $L_p$-$L_q$ regularity framework under the assumption that $h_0$ is small enough. It means that at the initial moment of time, the interface $\Gamma_t$ is very close to the reference interface $\Gamma$.

Since in problem (6) the domain $\dot\Omega_t$ and the interface $\Gamma_t$ are unknown, with the help of the Hanzawa coordinate transform (cf. Section 2.1), we reduce the free boundary problem to a problem in the given domain $\dot\Omega$. In Sections 2.2–2.4, we derive all the equations and boundary conditions to which Hanzawa transform maps (6). The main result is stated in Section 2.5 (Theorem 1). In Section 3, we formulate the maximal $L_p$-$L_q$ regularity theorems for corresponding linearized hydrodynamical (Theorem 2) and magnetic (Theorem 3) problems. The main result is proved in Section 5 by the fixed point theorem, on the base of the maximal $L_p$-$L_q$ regularity theorems for the corresponding linear problems and estimates of nonlinear terms (these estimates are given in Section 4).

### *1.2. Short History*

The equations of magnetodydrodynamics (MHD) can be found in [1,2,12]. The solvability of MHD equations was first obtained in [13]. The free boundary problem for MHD was first studied by Padula and Solonnikov [4] in the case when $\Omega_{-t}$ is a vacuum region in the three dimensional Euclidean space $\mathbb{R}^3$. They proved the local well-posedness in the $L_2$ framework and used Sobolve–Slobodetskii spaces of fractional order. Later on, the global well-posedness was proved by Solonnikov and Frolova [14]. Moreover, the $L_p$ approach to the same problem was calculated by Solonnikov [15,16]. In [4], by some technical reason, it was required that regularity class of the fluid be slightly higher than that of the magnetic field (cf. [4] (p. 331)). However, in this paper, we do not need this assumption; that is, we can solve the problem in the same regularity classes for the fluid and magnetic field. The different point of this paper compared to [4] appears in the iteration scheme (cf. (85) and (86)).

As a related topics, in [17,18] and references therein Kacprzyk proved the local and global well-posedness of the free boundary problem for the viscous nonhomogeneous incompressible MHD in the case where an incompressible fluid is occupied in a domain $\Omega_{-t}$ bounded by a free surface $\Gamma_t$ subjected to an electromagnetic field generated in a domain $\Omega_{+t}$ exterior to $\Omega_{-t}$ by some currents located on a fixed boundary $S_+$ of $\Omega_{+t}$. In [17,18], it is assumed that $S_- = \emptyset$. On the free surface, $\Gamma_t$, a free boundary condition without surface tension for the viscous fluid part and transmission conditions for electromagnetic fields part are imposed. Since the surface tension is not taken into account, the Lagrange transformation was applied, and so the viscous fluid part has one regularity higher than the electromagnetic fields part. An $L_2$ approach is applied and Sobolev–Slobodetskii spaces of fractional order are also used in [17,18]. Later, the local wellposedness of the same problem as in [17,18] was proved in the $L_p$ framework by Shibata and Zajaczkowski [5] and in the $L_p$ in time and $L_q$ in space framework by Oishi and Shibata [19].

### 1.3. Notation

Finally, we explain some symbols used throughout the paper. We denote the set of all natural numbers, real numbers, and complex numbers by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any multi-index $\kappa = (\kappa, \ldots, \kappa_N)$, $\kappa_j \in \mathbb{N}_0$, we set $\partial_x^\kappa = \partial_1^{\kappa_1} \ldots \partial_N^{\kappa_N}$, $|\kappa| = \kappa_1 + \ldots + \kappa_N$. For scalar $f$, and $N$-vector of functions, $\mathbf{g} = (g_1, \ldots, g_N)$, we set $\nabla^n f = (\partial_x^\kappa f \mid |\kappa| = n)$ and $\nabla^n \mathbf{g} = (\partial_x^\kappa g_j \mid |\kappa| = n, j = 1, \ldots, N)$. In particular, $\nabla^0 f = f$, $\nabla^0 \mathbf{g} = \mathbf{g}$, $\nabla^1 f = \nabla f$, and $\nabla^1 \mathbf{g} = \nabla \mathbf{g}$. For $1 \leq q \leq \infty$, $m \in \mathbb{N}$, $s \in \mathbb{R}$, and any domain $D \subset \mathbb{R}^N$, we denote by $L_q(D)$, $H_q^m(D)$, and $B_{q,p}^s(D)$ the standard Lebesgue, Sobolev, and Besov spaces, respectively, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{H_q^m(D)}$, and $\|\cdot\|_{B_{q,p}^s(D)}$ denote the norms of these spaces. We write $W_q^s(D) = B_{q,q}^s(D)$ and $H_q^0(D) = L_q(D)$. For $\mathcal{H} \in \{H_q^m, B_{q,p}^s\}$, the function spaces $\mathcal{H}(\dot\Omega)$ and their norms are defined by setting

$$\mathcal{H}(\dot\Omega) = \{f = f_\pm \mid f_\pm \in \mathcal{H}(\Omega_\pm)\}, \quad \|f\|_{\mathcal{H}(\dot D)} = \|f_+\|_{\mathcal{H}(D_+)} + \|f_-\|_{\mathcal{H}(D_-)}.$$

For any Banach space $X$ with the norm $\|\cdot\|_X$, $X^d$ denotes the $d$ product space defined by $\{x = (x_1, \ldots, x_d) \mid x_i \in X\}$, while the norm of $X^d$ is simply written by $\|\cdot\|_X$, that is $\|x\|_X = \sum_{j=1}^d \|x_j\|_X$. For any time interval $(a,b)$, $L_p((a,b), X)$ and $H_p^m((a,b), X)$ denote, respectively, the standard $X$-valued Lebesgue space and $X$-valued Sobolev space, while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{H_p^m((a,b),X)}$ denote their norms. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be, respectively, the Fourier transform and the Fourier inverse transform. Let $H_p^s(\mathbb{R}, X)$, $s > 0$, be the Bessel potential space of order $s$ defined by

$$H_p^s(\mathbb{R}, X) = \{f \in L_p(\mathbb{R}, X) \mid \|f\|_{H_p^s(\mathbb{R},X)} = \|\mathcal{F}^{-1}[(1 + |\tau|^2)^{s/2}\mathcal{F}[f](\tau)]\|_{L_p(\mathbb{R},X)} < \infty\}.$$

For any $N$-vector of functions, $\mathbf{u} = (u_1, \ldots, u_N)^\top$, $\nabla \mathbf{u}$ is regarded as an $N \times N$-matrix of functions whose $(i,j)$th component is $\partial_j u_i$. For any $m$-vector $V = (v_1, \ldots, v_m)$ and $n$-vector $W = (w_1, \ldots, w_n)$, $V \otimes W$ denotes an $m \times n$ matrix whose $(i,j)$th component is $V_i W_j$. For any $(mn \times N)$-matrix $A = (A_{ij,k} \mid i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, N)$, $AV \otimes W$ denotes an $N$-column vector whose $k^{\text{th}}$ component is the quantity: $\sum_{j=1}^m \sum_{j=1}^n A_{ij,k} v_i w_j$. Moreover, we define $AV \otimes W \otimes Z = (AV \otimes W) \otimes Z$. Inductively, we define $AV_1 \otimes \ldots \otimes V_n$ by setting $AV_1 \otimes \ldots \otimes V_n = (AV_1 \otimes \ldots \otimes V_{n-1}) \otimes V_n$ for $n \geq 4$.

Let $\mathbf{a} \cdot \mathbf{b} = \ < \mathbf{a}, \mathbf{b} > \ = \sum_{j=1}^N a_j b_j$ for any $N$-vectors $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots, b_N)$. For any $N$-vector $\mathbf{a}$, let $\mathbf{a}_\tau := \mathbf{a} - < \mathbf{a}, \mathbf{n} > \mathbf{n}$. For any two $N \times N$-matrices $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$, the quantity $\mathbf{A} : \mathbf{B}$ is defined by $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^N A_{ij} B_{ij}$. For any domain $G$ with boundary $\partial G$, we set

$$(\mathbf{u}, \mathbf{v})_G = \int_G \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)}\, dx, \quad (\mathbf{u}, \mathbf{v})_{\partial G} = \int_{\partial G} \mathbf{u} \cdot \overline{\mathbf{v}(x)}\, d\sigma,$$

where $\overline{\mathbf{v}(x)}$ is the complex conjugate of $\mathbf{v}(x)$, and $d\sigma$ denotes the surface element of $\partial G$. Given $1 < q < \infty$, let $q' = q/(q-1)$. Throughout the paper, the letter $C$ denotes generic constants and $C_{a,b,\ldots}$ the constant which depends on $a$, $b$, …. The values of constants $C$, $C_{a,b,\ldots}$ may be changed from line to line.

## 2. Hanzawa Transform and Statement of Main Result

### 2.1. Hanzawa Transform

Let $\mathbf{n}$ be the unit normal to $\Gamma$ oriented from $\Omega_+$ into $\Omega_-$. Since $\Gamma_t$ is unknown, we assume that the $\Gamma_t$ is represented by (1). Our task is to find not only $\mathbf{v}$, $\mathfrak{p}$, and $\mathbf{H}$ but also $h$. We know the existence of an $N$-vector, $\tilde{\mathbf{n}}$, of $C^2$ functions defined on $\mathbb{R}^N$ such that

$$\mathbf{n} = \tilde{\mathbf{n}} \quad \text{on } \Gamma, \quad \operatorname{supp} \tilde{\mathbf{n}} \subset U_\Gamma, \quad \|\tilde{\mathbf{n}}\|_{H^2_\infty(\mathbb{R}^N)} \leq C \tag{7}$$

with some constant $C$. Here, we set $U_\Gamma = \bigcup_{x_0 \in \Gamma}\{x \in \mathbb{R}^N \mid |x - x_0| < \alpha\}$ with some small constant $\alpha > 0$. We will construct $\tilde{\mathbf{n}}$ in Section 2.3 below. We may assume that

$$\operatorname{dist}(\operatorname{supp} \tilde{\mathbf{n}}, S_\pm) \geq 3d_\pm/4.$$

Let $H_h$ be an extension function of $h$ such that $h = H_h$ on $\Gamma$. In fact, we take $H_h$ as a solution of the harmonic equation:

$$(-\Delta + \lambda_0)H_h = 0 \quad \text{in } \dot{\Omega}, \quad H_h|_\Gamma = h \tag{8}$$

with some large positive number $\lambda_0$ which guarantees the unique solvability of (8). In this case, if $h$ satisfies the regularity condition:

$$h \in H^1_p((0,T), W^{2-1/q}_q(\Gamma)) \cap L_p((0,T), W^{3-1/q}_q(\Gamma)), \tag{9}$$

then $H_h$ satisfies the regularity condition:

$$H_h \in H^1_p((0,T), H^2_q(\Omega)) \cap L_p((0,T), H^3_q(\Omega)), \tag{10}$$

and possesses the estimate:

$$\begin{aligned}
&\|\partial_t^i H_h\|_{L_p((0,T),H^{3-i}_q(\Omega))} \leq C\|\partial_t^i h\|_{L_p((0,T),W^{3-i-1/q}_q(\Gamma))} \quad (i = 0,1), \\
&\|\partial_t^i h\|_{L_p((0,T),W^{3-1/q-i}_q(\Gamma))} \leq C\|\partial_t^i H_h\|_{L_p((0,T),H^{3-i}_q(\Omega))} \quad (i = 0,1).
\end{aligned} \tag{11}$$

To transform Equation (6) to the problem in a domain with fixed boundary and interface, we use the Hanzawa transformation defined by

$$x = y + H_h(t,y)\tilde{\mathbf{n}}(y) := \Xi_h(t,y). \tag{12}$$

Let $\delta > 0$ be a small number such that

$$|\Xi_h(y_1,t) - \Xi_h(y_2,t)| \leq (1/2)|y_1 - y_2| \tag{13}$$

provided that

$$\sup_{0<t<T} \|\bar{\nabla} H_h(\cdot,t)\|_{L_\infty(\Omega)} \leq \delta. \tag{14}$$

Henceforth, we use the symbol: $\bar{\nabla} H_h = (\partial_x^\alpha H_h \mid |\alpha| \leq 1) = (H_h, \nabla H_h)$. From (13) and (14), the map $x = \Xi(y,t)$ is injective. Under suitable regularity condition on $H_h$, for example, $H_h \in C^{1+\alpha}$ for each $t \in (0,T)$ with some small $\alpha > 0$, the map $x = \Xi_h(y,t)$ becomes an open and closed map, so that $\{x = \Xi_h(y,t) \mid y \in \Omega\} = \Omega$, because $x = \Xi_h(y,t)$ is an identity map on $\Omega \setminus U_\Gamma$. We assume that the initial surface $\Gamma_0$ is given by

$$\Gamma_0 = \{x = y + h_0(y)\mathbf{n} \mid y \in \Gamma\}$$

with a given sufficiently small function $h_0$. Let $H_{h_0}$ be an extension of $h_0$ which is given by a unique solution of Equation (8), where $H_h$ and $h$ are replaced with $H_{h_0}$ and $h_0$, respectively.

Let $\Xi_{h_0} = y + H_{h_0}\tilde{\mathbf{n}}$, and set

$$\begin{aligned}\mathbf{u}(y,t) = \mathbf{v}(\Xi_h^{-1}(y,t),t), \quad \mathfrak{q}(y,t) = \mathfrak{p}(\Xi_h^{-1}(y,t),t), \quad \mathbf{G}(y,t) = \mathbf{H}(\Xi_h^{-1}(y,t),t),\\ \Gamma_t = \{x = \Xi_h(y,t) \mid y \in \Gamma\}, \quad \Omega_{t\pm} = \{x = \Xi_h(y,t) \mid y \in \Omega_\pm\},\\ \mathbf{u}_0(y) = \mathbf{v}_0(\Xi_{h_0}^{-1}(y)), \quad \mathbf{G}_0(y) = \mathbf{H}_0(\Xi_{h_0}^{-1}(y)).\end{aligned} \tag{15}$$

Noting that $x = y$ near $S_\pm$, we have

$$\begin{aligned}\mathbf{u}_\pm = 0, \quad \mathbf{n}_\pm \cdot \mathbf{G}_\pm = 0, \quad (\operatorname{curl}\mathbf{G}_\pm)\mathbf{n}_\pm = 0 \quad \text{on } S_\pm \times (0,T),\\ (\mathbf{u},\mathbf{G},h)|_{t=0} = (\mathbf{u}_0,\mathbf{G}_0,h_0) \quad \text{in } \dot\Omega \times \Gamma, \quad H_h|_{t=0} = h_0 \quad \text{on } \Gamma.\end{aligned}$$

In what follows, we derive equations and interface conditions for $\mathbf{u}$, $\mathfrak{q}$, and $\mathbf{G}$.

### 2.2. Derivation of Equations

In this subsection, we derive equations obtained by the Hanzawa transformation: $x = y + H_h(y,t)\tilde{\mathbf{n}}(y)$ from the first, second, and third equations in Equation (6). We assume that $H_h$ satisifies (14) with a small positive number $\delta > 0$. We have

$$\frac{\partial x}{\partial y} = \mathbf{I} + \frac{\partial(H_h\tilde{\mathbf{n}})}{\partial y}$$

and then, choosing $\delta > 0$ in (14) small enough, we see that there exists an $N \times N$ matrix, $V_0(\mathbf{K})$, of bounded real analytic functions defined on $U_\delta = \{\mathbf{K} \in \mathbb{R}^{N+1} \mid |\mathbf{K}| \leq \delta\}$ with $V_0(0) = 0$ such that

$$\frac{\partial y}{\partial x} = \left(\frac{\partial x}{\partial y}\right)^{-1} = \mathbf{I} + \mathbf{V}_0(\bar\nabla H_h). \tag{16}$$

Here, we use the symbol $\mathbf{K} = (\kappa_0, \kappa_1, \ldots, \kappa_N)$, where $\kappa_0, \kappa_1, \ldots, \kappa_N$ are independent variables corresponding to $H_h, \partial H_h/\partial y_1, \ldots, \partial H_h/\partial y_N$, respectively. Let $V_{0ij}(\mathbf{K})$ be the $(i,j)$ th component of $V_0(\mathbf{K})$. Then, by the chain rule, we have

$$\frac{\partial}{\partial x_j} = \sum_{k=1}^N (\delta_{jk} + V_{0jk}(\mathbf{K}))\frac{\partial}{\partial y_k}, \quad \nabla_x = (\mathbf{I} + \mathbf{V}_0(\mathbf{K}))\nabla_y. \tag{17}$$

Since $V_{0jk}(0) = 0$, we write

$$V_{0jk}(\mathbf{K}) = \int_0^1 \frac{d}{d\theta}(V_{0jk}(\theta\mathbf{K}))\,d\theta = \tilde V_{0jk}(\mathbf{K})\mathbf{K} \quad \text{with} \quad \tilde V_{0jk}(\mathbf{K}) = \int_0^1 V_{0jk}'(\theta\mathbf{K})\,d\theta,$$

where $V_{0jk}'$ denotes the dirivative of $V_{0jk}$ with respect to $\mathbf{K}$. In particular,

$$\begin{aligned}\operatorname{curl}_{ij}(\mathbf{v}) = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = \operatorname{curl}_{ij}(\mathbf{u}) + V_{Cij}(\mathbf{K})\nabla\mathbf{u},\\ D_{ij}(\mathbf{v}) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = D_{ij}(\mathbf{u}) + V_{Dij}(\mathbf{K})\nabla\mathbf{u},\end{aligned} \tag{18}$$

with

$$\begin{aligned}V_{Cij}(\mathbf{K})\nabla\mathbf{u} = \sum_{k=1}^N (V_{0jk}(\mathbf{K})\frac{\partial u_i}{\partial y_j} - V_{0ik}(\mathbf{K})\frac{\partial u_j}{\partial y_k}),\\ V_{Dij}(\mathbf{K})\nabla\mathbf{u} = \sum_{k=1}^N (V_{0jk}(\mathbf{K})\frac{\partial u_i}{\partial y_j} + V_{0ik}(\mathbf{K})\frac{\partial u_j}{\partial y_k}).\end{aligned}$$

Here and in the following, for an $N \times N$ matrix $A$, $A_{ij}$ denotes its $(i,j)$ th component and $(A_{ij})$ denotes an $N \times N$ matrix whose $(i,j)$ th component is $A_{ij}$. To obtain the first equation in (72) in Section 2.5 below, we make the pressure term linear. From $\nabla\mathfrak{p} = (\mathbf{I}+\mathbf{V}_0(\mathbf{K}))\nabla\mathfrak{q}$, it follows that

$$\frac{\partial\mathfrak{q}}{\partial y_j} = \sum_{k=1}^N(\delta_{jk}+\frac{\partial x_k}{\partial y_j})\frac{\partial\mathfrak{p}}{\partial x_k}.$$

Let $\tilde{\mathbf{n}} = (\tilde{n}_1,\ldots\tilde{n}_N)^\top$, then

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t}u_i(y+H_h\tilde{\mathbf{n}},t) = \frac{\partial u_i}{\partial t}+\sum_{j=1}^N\frac{\partial u_i}{\partial y_j}\frac{\partial H_h}{\partial t}\tilde{n}_j. \tag{19}$$

Thus, the first equation in (6) is transformed to

$$\begin{aligned}\frac{\partial\mathfrak{q}}{\partial y_m} &= -m\sum_{i=1}^N(\delta_{mi}+\frac{\partial x_m}{\partial y_i})\{\frac{\partial u_i}{\partial t}+\sum_{j=1}^N\frac{\partial u_i}{\partial y_j}\frac{\partial H_h}{\partial t}\tilde{n}_j+\sum_{j,k=1}^N u_j(\delta_{jk}+V_{0jk}(\bar{\nabla}H_h))\frac{\partial u_i}{\partial y_k}\}\\ &\quad+\sum_{i,j,k=1}^N(\delta_{mi}+\frac{\partial x_m}{\partial y_i})(\delta_{jk}+V_{0jk}(\bar{\nabla}H_h))\frac{\partial}{\partial y_k}\{\nu(D_{ij}(\mathbf{u})+V_{Dij}(\bar{\nabla}H_h)\nabla\mathbf{u})+T_{Mij}(\mathbf{G})\}\\ &= -m\partial_t u_m+\sum_{k=1}^N\frac{\partial}{\partial y_k}(\nu D_{mk}(\mathbf{u}))+f_{1m}(\mathbf{u},\mathbf{G},H_h)\end{aligned}$$

with

$$\begin{aligned}f_{1m}(\mathbf{u},\mathbf{G},H_h) &= -m\Big(\sum_{j=1}^N\frac{\partial u_m}{\partial y_j}\frac{\partial H_h}{\partial t}\tilde{n}_j+\sum_{j,k=1}^N u_j(\delta_{jk}+V_{0jk}(\bar{\nabla}H_h))\frac{\partial u_m}{\partial y_k}\Big)\\ &\quad-m\sum_{i=1}^N\frac{\partial(H_h\tilde{n}_m)}{\partial y_i}\Big(\frac{\partial u_i}{\partial t}+\sum_{j=1}^N\frac{\partial u_i}{\partial y_j}\frac{\partial H_h}{\partial t}\tilde{n}_j+\sum_{j,k=1}^N u_j(\delta_{jk}+V_{0jk}(\bar{\nabla}H_h))\frac{\partial u_i}{\partial y_k}\Big)\\ &\quad+\sum_{j,k=1}^N(\delta_{jk}+V_{0jk}(\bar{\nabla}H_h))\frac{\partial}{\partial y_k}(\nu(D_{mj}(\mathbf{u})+V_{D_{mj}}(\bar{\nabla}H_h)\nabla\mathbf{u})+T_{Mmj}(\mathbf{G}))\\ &\quad+\sum_{i,j,k=1}^N\frac{\partial(H_h\tilde{n}_m)}{\partial y_i}(\delta_{jk}+V_{0jk}(\bar{\nabla}H_h))\frac{\partial}{\partial y_k}(\nu(D_{ij}(\mathbf{u})+V_{D_{ij}}(\bar{\nabla}H_h)\nabla\mathbf{u})+T_{Mij}(\mathbf{G})).\end{aligned} \tag{20}$$

Thus, setting $\mathbf{f}_1(\mathbf{u},\mathbf{G},H_h) = (f_{11}(\mathbf{u},\mathbf{G},H_h),\ldots,f_{1N}(\mathbf{u},\mathbf{G},H_h))^\top$, we have

$$m\partial_t\mathbf{u}-\mathrm{Div}\,\mathbf{T}(\mathbf{u},\mathfrak{q}) = \mathbf{f}_1(\mathbf{u},\mathbf{G},H_h)\quad\text{in }\dot{\Omega}\times(0,T). \tag{21}$$

Since $V_{0jk}(0)=0$ and $V_{\mathbf{D}ij}(0)=0$, we may write

$$\begin{aligned}\mathbf{f}_1(\mathbf{u},\mathbf{G},H_h) &= f_0^1\bar{\nabla}H_h\otimes\partial_t\mathbf{u}+\mathcal{F}_0^1(\bar{\nabla}H_h)\partial_t H_h\otimes\nabla\mathbf{u}+\mathcal{F}_1^1(\bar{\nabla}H_h)\mathbf{u}\otimes\nabla\mathbf{u}\\ &\quad+\mathcal{F}_2^1(\bar{\nabla}H_h)\bar{\nabla}H_h\otimes\nabla^2\mathbf{u}+\mathcal{F}_3^1(\bar{\nabla}H_h)\bar{\nabla}^2H_h\otimes\nabla\mathbf{u}+\mathcal{F}_4^1(\bar{\nabla}H_h)\mathbf{G}\otimes\nabla\mathbf{G},\end{aligned} \tag{22}$$

where $f_0^1$ is a bounded function and $\mathcal{F}_j^1(\mathbf{K})$ are some matrices of bounded analytic functions defined on $U_\delta$. Here and in the following, we write $\bar{\nabla}^k H_h = (\partial_y^\alpha H_h \mid |\alpha|\leq k)$ for $k\geq 2$ and $\bar{\nabla}H_h = (\partial_y^\alpha H_h \mid |\alpha|\leq 1)$.

We next consider the divergence free condition: $\mathrm{div}\,\mathbf{v}=0$. By (17),

$$\mathrm{div}\,\mathbf{v} = \sum_{j=1}^N\frac{\partial v_j}{\partial x_j} = \sum_{j,k=1}^N(\delta_{jk}+V_{0jk}(\bar{\nabla}H_h))\frac{\partial u_j}{\partial y_k}. \tag{23}$$

Let $J = \det(\partial x/\partial y)$, choosing $\delta > 0$ small enough in (14), we have

$$J = 1 + J_0(\bar{\nabla} H_h), \tag{24}$$

where $J_0(\mathbf{K})$ is a real analytic function defined on $U_\delta$ such that $J_0(0) = 0$. Using this symbol, we have

$$\begin{aligned}(\mathrm{div}\,_x\mathbf{v}_\pm, \varphi)_{\Omega_{t\pm}} = -(\mathbf{v}_\pm, \nabla_x\varphi)_{\Omega_{t\pm}} &= -\sum_{j=1}^{N}(Ju_{\pm,j}, \sum_{k=1}^{N}(\delta_{jk} + V_{0jk}(\bar{\nabla} H_h))\frac{\partial\varphi}{\partial y_k})_{\Omega_\pm} \\ &= (\sum_{j,k=1}^{N}\frac{\partial}{\partial y_k}\{J(\delta_{jk} + V_{0jk}(\bar{\nabla} H_h))u_{\pm j}\}, \varphi)_{\Omega_\pm},\end{aligned}$$

so that

$$\mathrm{div}\,\mathbf{v}_\pm = J^{-1}\sum_{j,k=1}^{N}\frac{\partial}{\partial y_k}\{J(\delta_{jk} + V_{0jk}(\bar{\nabla} H_h))u_{\pm j}\}. \tag{25}$$

Combining (23)–(25) yields

$$\mathrm{div}\,\mathbf{u} = g(\mathbf{u}, H_h) = \mathrm{div}\,\mathbf{g}(\mathbf{u}, H_h) \quad \text{in } \dot{\Omega} \times (0, T), \tag{26}$$

where

$$\begin{aligned} g(\mathbf{u}, H_h) &= \sum_{j,k=1}^{N} V_{0jk}(\bar{\nabla} H_h)\frac{\partial u_{\pm j}}{\partial y_k} + J_0(\bar{\nabla} H_h)\{\mathrm{div}\,\mathbf{u} + \sum_{j,k=1}^{N} V_{0jk}(\bar{\nabla} H_h)\frac{\partial u_j}{\partial y_k}\}, \\ \mathbf{g}(\mathbf{u}, H_h)|_k &= \sum_{j=1}^{N} V_{0jk}(\bar{\nabla} H_h)u_j + J_0(\bar{\nabla} H_h)\sum_{j=1}^{N}(\delta_{jk} + V_{0jk}(\bar{\nabla} H_h))u_j. \end{aligned} \tag{27}$$

Since $V_{0jk}(0) = J_0(0) = 0$, we may write

$$g(\mathbf{u}, H_h) = \mathcal{G}_1(\bar{\nabla} H_h)\bar{\nabla} H_h \otimes \nabla\mathbf{u}, \quad \mathbf{g}(\mathbf{u}, H_h) = \mathcal{G}_2(\bar{\nabla} H_h)\bar{\nabla} H_h \otimes \mathbf{u}, \tag{28}$$

where $\mathcal{G}_i(\mathbf{K})$ are some matrices of bounded analytic functions defined on $U_\delta$.

We next consider the third equation in Equation (6). By (19),

$$\mu\partial_t\mathbf{H} = \mu\partial_t\mathbf{G} + \mu\sum_{j,k=1}^{N}\tilde{n}_j\frac{\partial\mathbf{G}}{\partial y_j}\frac{\partial H_h}{\partial t}.$$

Moreover,

$$\begin{aligned}\Delta &= \sum_{j=1}^{N}(\sum_{k=1}^{N}(\delta_{jk} + V_{0jk}(\bar{\nabla} H_h))\frac{\partial}{\partial y_k})(\sum_{\ell=1}^{N}(\delta_{j\ell} + V_{0j\ell}(\bar{\nabla} H_h))\frac{\partial}{\partial y_\ell}) \\ &= \sum_{j=1}^{N}\{\frac{\partial^2}{\partial y_j^2} + \sum_{\ell=1}^{N}\frac{\partial}{\partial y_j}(V_{0j\ell}(\bar{\nabla} H_h)\frac{\partial}{\partial y_\ell}) + \sum_{\ell,k=1}^{N} V_{0jk}(\bar{\nabla} H_h)\frac{\partial}{\partial y_k}((\delta_{j\ell} + V_{0j\ell}(\bar{\nabla} H_h))\frac{\partial}{\partial y_\ell}) \\ &= \Delta + V_{\Delta 2}(\bar{\nabla} H_h)\nabla^2 + V_{\Delta 1}(\bar{\nabla} H_h)\nabla\end{aligned}$$

with

$$\begin{aligned} V_{\Delta 2}(\bar{\nabla} H_h)\nabla^2 &= 2\sum_{j,k=1}^{N} V_{0jk}(\bar{\nabla} H_h)\frac{\partial^2}{\partial y_j\partial y_k} + \sum_{j,k,\ell=1}^{N} V_{0jk}(\bar{\nabla} H_h)V_{0j\ell}(\bar{\nabla} H_h)\frac{\partial^2}{\partial y_k\partial y_\ell}, \\ V_{\Delta 1}(\bar{\nabla} H_h)\nabla &= \sum_{j,k=1}^{N}\frac{\partial V_{0j\ell}(\bar{\nabla} H_h)}{\partial y_j}\frac{\partial}{\partial y_k} + \sum_{j,k,\ell=1}^{N} V_{0jk}(\bar{\nabla} H_h)\frac{\partial V_{0j\ell}(\bar{\nabla} H_h)}{\partial y_k}\frac{\partial}{\partial y_\ell}. \end{aligned}$$

Thus, setting

$$\begin{aligned}\mathbf{f}_2(\mathbf{u},\mathbf{G},H_h) = &-\mu\sum_{j,k=1}^{N}\tilde{n}_j\frac{\partial\mathbf{G}}{\partial y_j}\frac{\partial H_h}{\partial t} + \alpha^{-1}V_{\Delta 2}(\bar{\nabla}H_h)\nabla^2\mathbf{G}_m + \alpha^{-1}V_{\Delta 1}(\bar{\nabla}H_h)\nabla\mathbf{G} \\ &+\mu\sum_{j,k=1}^{N}(\delta_{jk}+V_{0jk}(\mathbf{K}))\frac{\partial}{\partial y_k}(\mathbf{u}\otimes\mathbf{G}-\mathbf{G}\otimes\mathbf{u}),\end{aligned} \tag{29}$$

we have

$$\mu\partial_t\mathbf{G} - \alpha^{-1}\Delta\mathbf{G} = \mathbf{f}_2(\mathbf{u},\mathbf{G},H_h) \quad \text{in } \dot{\Omega}\times(0,T). \tag{30}$$

Since $V_{0jk}(0) = 0$, we may write

$$\begin{aligned}\mathbf{f}_2(\mathbf{u},\mathbf{G},H_h) = &f_2\nabla\mathbf{G}\otimes\partial_t H_h + \mathcal{F}_1^2(\bar{\nabla}H_h)\bar{\nabla}H_h\otimes\nabla^2\mathbf{G} + \mathcal{F}_2^2(\bar{\nabla}H_h)\bar{\nabla}^2H_h\otimes\nabla\mathbf{G} \\ &+\mathcal{F}_3^2(\bar{\nabla}H_h)\nabla\mathbf{u}\otimes\mathbf{G} + \mathcal{F}_4^2(\bar{\nabla}H_h)\mathbf{u}\otimes\nabla\mathbf{G}.\end{aligned} \tag{31}$$

where $f_2$ is a bounded function, $\mathcal{F}_j^2(\mathbf{K})$ are some matrices of bounded analytic functions defined on $U_\delta$.

### 2.3. The Unit Outer Normal and the Laplace Beltrami Operator on $\Gamma_t$

Since $\Gamma$ is a compact hypersurface of $C^3$ class, we have the following lemma.

**Lemma 1.** *For any constant $M_1 \in (0,1)$, there exist a finite number $n \in \mathbb{N}$, constants $M_2 > 0$, $d, d' \in (0,1)$, $n$ $N$-vectors of functions $\Phi^\ell \in C^3(\mathbb{R}^N)^N$, $n$ points $x^\ell \in \Gamma$ and two domains $\mathcal{O}_\pm$ such that the following assertions hold:*

(i) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi^\ell(x) \in \mathbb{R}^N$ are bijective for $j \in \mathbb{N}$.*

(ii) *$\Omega = (\bigcup_{\ell=1}^n \Phi^\ell(B_d)) \cup \mathcal{O}_+ \cup \mathcal{O}_-$, $B_{d'}(x^\ell) \subset \Phi^\ell(B_d) \subset \Omega$, $B_{d'}(x^\ell) \cap \Omega_\pm \subset \Phi^\ell(B_d \cap \mathbb{R}^N_\pm) \subset \Omega_\pm$ and $\Gamma \cap B_{d'}(x^\ell) \subset \Phi^\ell(B_d \cap \mathbb{R}^N_0)$, where $B_d = \{x \in \mathbb{R}^N \mid |x| < d\}$, $B_{d'}(x^\ell) = \{x \in \mathbb{R}^N \mid |x - x^\ell| < d'\}$, $\mathbb{R}^N_\pm = \{x = (x_1,\ldots,x_N) \mid \pm x_N > 0\}$, and $\mathbb{R}^N_0 = \{x = (x_1,\ldots,x_N) \in \mathbb{R}^N \mid x_N = 0\}$.*

(iii) *There exist $n$ $C^\infty$ functions $\zeta^\ell$ such that* $\operatorname{supp}\zeta^\ell \subset B_{d'}(x^\ell)$ *and $\sum_{\ell=1}^n \zeta^\ell = 1$ on $\Gamma$.*

(iv) *$\nabla\Phi^\ell = \mathcal{A}^\ell + B^\ell$, $\nabla(\Phi^\ell)^{-1} = \mathcal{A}^{\ell,-1} + B^{\ell,-1}$ where $\mathcal{A}^\ell$ are $N\times N$ constant orthogonal matrices and $B^\ell$ are $N\times N$ matrices of $C^3(\mathbb{R}^N)$ functions satisfying the conditions: $\|B_\ell\|_{L_\infty(\mathbb{R}^N)} \le M_1$ and $\|\nabla B_\ell\|_{H^1_\infty(\mathbb{R}^N)} \le M_2$ for $\ell = 1,\ldots,n$.*

In what follows, we write $B_{d'}(x^\ell)$ simply by $B^\ell$ and set $V_0 = B_d \cap \mathbb{R}^N_0$. The index $\ell$ runs from 1 through $n$. Recall that $\Gamma \cap B^\ell \subset \Phi^\ell(V_0)$, $\sum_{\ell=1}^n \zeta^\ell = 1$ on $\Gamma$, and $\operatorname{supp}\zeta^\ell \subset B^\ell \subset \Phi^\ell(B_d) \subset \Omega$. Let

$$\tau_j(u) = \frac{\partial\Phi^\ell(u)}{\partial u_j} = A_j^\ell + \mathcal{B}_j^\ell(u)$$

for $j = 1,\ldots,N$ and $u = (u_1,\ldots,u_N) \in \mathbb{R}^N$. By Lemma 1, $A_j^\ell$ are $N$-constant vectors, and $\mathcal{B}_j^\ell(u)$ are $N$ vector of functions such that

$$A_j^\ell \cdot A_k^\ell = \delta_{jk}, \quad \|\mathcal{B}_j^\ell\|_{L_\infty(\mathbb{R}^N)} \le M_1, \quad \|\nabla\mathcal{B}_j^\ell\|_{H^1_\infty(\mathbb{R}^N)} \le M_2, \tag{32}$$

where $\delta_{jk}$ are the Kronecker delta symbols defined by $\delta_{jj} = 1$ and $\delta_{jk} = 0$ for $j \ne k$. Notice that $\{\tau_j(u',0)\}_{j=1}^{N-1}$, $u' = (u_1,\ldots,u_{N-1},0) \in V_0$, forms a basis of the tangent space of $\Gamma \cap B^\ell$. Let $g_{ij}^\ell(u) = \tau_i^\ell(u)\cdot\tau_j^\ell(u)$, $G^\ell(u)$ an $N\times N$ matrix whose $(i,j)$ th component is $g_{ij}^\ell(u)$, $g^\ell(u) = \sqrt{\det G^\ell(u))}$, and $g_\ell^{ij}(u)$ the $(i,j)$ th component of $(G^\ell)^{-1}$, respectively. $G^\ell(u',0)$

is a first fundamental matrix of the tangent space of $\Gamma \cap B^\ell$. By (32) there exist functions $\tilde{g}^\ell(u)$, $\tilde{g}^\ell_{ij}(u)$ and $\tilde{g}^{ij}_\ell(u)$ such that

$$g^\ell_{ij}(u) = \delta_{ij} + \tilde{g}^\ell_{ij}(u), \quad g^\ell(u) = 1 + \tilde{g}^\ell(u), \quad g^{ij}_\ell(u) = \delta_{ij} + \tilde{g}^{ij}_\ell(u),$$
$$\|(\tilde{g}^\ell_{ij}, \tilde{g}^\ell, \tilde{g}^{ij}_\ell)\|_{L_\infty(\mathbb{R}^N)} \leq CM_1, \quad \|\nabla(\tilde{g}^\ell_{ij}, \tilde{g}^\ell, \tilde{g}^{ij}_\ell)\|_{H^1_\infty(\mathbb{R}^N)} \leq C_{M_2}. \tag{33}$$

Here, the constant $C_{M_2}$ is a generic constant depending on $M_2$. We may assume that $0 < M_1 < 1 \leq M_2$.

We now define an extension of $\mathbf{n}$ to $\mathbb{R}^N$ satisfying (7). Let $\varphi_{i,j}(u) = \partial\Phi^\ell_i(u)/\partial u_j$ with $\Phi^\ell = (\Phi^\ell_1, \ldots, \Phi^\ell_N)^\top$, and let $\mathcal{N}^\ell_i(u)$ be $N \times (N-1)$ functions defined by setting

$$\mathcal{N}^\ell_i(u) = (-1)^{i+N} \det \begin{pmatrix} \varphi_{1,1} & \cdots & \varphi_{1,N-1} \\ \vdots & \ddots & \vdots \\ \varphi_{i-1,1} & \cdots & \varphi_{i-1,N-1} \\ \varphi_{i+1,1} & \cdots & \varphi_{i+1,N-1} \\ \vdots & \ddots & \vdots \\ \varphi_{N,1} & \cdots & \varphi_{N,N-1} \end{pmatrix}$$

for $i = 1 = 1, \ldots, N-1$. We set $\mathcal{N}^\ell = (\mathcal{N}^\ell_1, \ldots, \mathcal{N}^\ell_N)^\top$, then

$$< \mathcal{N}^\ell, \frac{\partial \Phi^\ell}{\partial u_k} > \; = 0 \quad \text{for } k = 1, \ldots, N-1,$$

because

$$0 = \det \begin{pmatrix} \varphi_{1,1} & \cdots & \varphi_{1,N-1} & \varphi_{1,k} \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_{N1} & \cdots & \varphi_{N,N-1} & \varphi_{N,k} \end{pmatrix} = \sum_{j=1}^N \mathcal{N}^\ell_j \varphi_{j,k} =< \mathcal{N}^\ell, \frac{\partial \Phi^\ell}{\partial u_k} >$$

for $k = 1, \ldots, N-1$. Let $\tilde{\mathbf{n}}^\ell = \mathcal{N}^\ell / |\mathcal{N}^\ell|$, and then

$$< \tilde{\mathbf{n}}^\ell, \tau^\ell_j(u) > \; = 0 \quad \text{for } j = 1 \ldots N-1 \text{ and } u \in \mathbb{R}^N.$$
$$\tilde{\mathbf{n}}^\ell \circ (\Phi^\ell)^{-1} = \mathbf{n} \quad \text{on } \Gamma \cap B^\ell. \tag{34}$$

Moreover, by (32) $\|\nabla \tilde{\mathbf{n}}^\ell\|_{H^1_\infty(\mathbb{R}^N)} \leq C_{M_2}$ for some constant $C_{M_2}$ depending on $M_2$. Let

$$\tilde{\mathbf{n}} = \sum_{\ell=1}^n \zeta^\ell \tilde{\mathbf{n}}^\ell \circ (\Phi^\ell)^{-1},$$

then $\tilde{\mathbf{n}}$ satisfies the properties given in (7).

Next, we give a representation formula for $\mathbf{n}_t$. Since $\Gamma_t \cap B^\ell$ is represented by $x = \Phi^\ell(u', 0) + H_h(\Phi^\ell(u', 0), t)\mathbf{n}(\Phi^\ell(u', 0))$ for $(u', 0) \in V_0$, setting $\tilde{H}^\ell_h = H_h(\Phi^\ell(u), t)$, we define $\tau^\ell_t = (\tau^\ell_{t1}(u), \ldots, \tau^\ell_{tN-1})^\top$ by

$$\tau^\ell_{tj}(u) = \frac{\partial}{\partial u_j}(\Phi^\ell(u) + \tilde{H}^\ell_h(u, t)\tilde{\mathbf{n}}^\ell(u)). \tag{35}$$

Notice that $\{\tau_{tj}^{\ell}(u',0)\}_{j=1}^{N-1}$ forms a basis of the tangent space $\Gamma_t$ locally. To obtain a formula of $\tilde{\mathbf{n}}_t$, we set $\tilde{\mathbf{n}}_t^{\ell} = a(\tilde{\mathbf{n}}^{\ell} + \sum_{j=1}^{N-1}\tau_j^{\ell}b_j)$, and we choose $a$ and $b_j$ in such a way that $|\tilde{\mathbf{n}}_t^{\ell}| = 1$ and $< \tilde{\mathbf{n}}_t^{\ell}, \tau_t^{\ell} > \ = 0$. From $|\tilde{\mathbf{n}}_t^{\ell}|^2 = 1$, it follows that

$$1 = a^2(\tilde{\mathbf{n}}^{\ell} + \sum_{j=1}^{N-1} b_j\tau_j^{\ell}) \cdot (\tilde{\mathbf{n}}^{\ell} + \sum_{k=1}^{N-1} b_k\tau_k^{\ell}) = a^2(1 + \sum_{j,k=1}^{N-1} g_{jk}^{\ell}(u)b_jb_k),$$

so that

$$a = (1 + \sum_{j,k=1}^{N-1} g_{jk}^{\ell}(u)b_jb_k)^{-1/2}. \tag{36}$$

From $< \tilde{\mathbf{n}}_t^{\ell}, \tau_t^{\ell} > \ = 0$ and (35), it follows that

$$0 = (\tilde{\mathbf{n}}^{\ell} + \sum_{k=1}^{N-1} b_k\tau_k^{\ell}) \cdot (\tau_j^{\ell} + \tilde{H}_h^{\ell}\frac{\partial \tilde{\mathbf{n}}^{\ell}}{\partial u_j} + \frac{\partial \tilde{H}_h^{\ell}}{\partial u_j}\tilde{\mathbf{n}}^{\ell}) = \sum_{j,k=1}^{N-1} g_{jk}^{\ell}b_k + \frac{\partial \tilde{H}_h^{\ell}}{\partial u_j} + \sum_{k=1}^{N-1} b_k < \tau_k^{\ell}, \frac{\partial \tilde{\mathbf{n}}^{\ell}}{\partial u_j} > \tilde{H}_h^{\ell},$$

where we have used the first formula in (34), and $< \tilde{\mathbf{n}}^{\ell}, \partial\tilde{\mathbf{n}}^{\ell}/\partial u_j > \ = 0$ which follows from $|\tilde{\mathbf{n}}^{\ell}|^2 = 1$. Setting $L^{\ell} = \ < \partial\tilde{\mathbf{n}}^{\ell}/\partial u_j, \tau_k^{\ell} >$, we have

$$\nabla'\tilde{H}_h^{\ell} = -(G^{\ell} + L^{\ell}\tilde{H}_h^{\ell})\mathbf{b}$$

where we have set $\mathbf{b} = (b_1, \ldots, b_{N-1})^{\top}$ and $\nabla'\tilde{H}_h^{\ell} = (\partial\tilde{H}_h^{\ell}/\partial u_1, \ldots, \partial\tilde{H}_h^{\ell}/\partial u_{N-1})^{\top}$. We now introduce a symbol $O_{\ell}^2$ which denotes a generic term of the form:

$$O_{\ell}^2 = a^{\ell}(u, \bar{\nabla}'\tilde{H}_h^{\ell})\bar{\nabla}'\tilde{H}_h^{\ell} \otimes \bar{\nabla}'\tilde{H}_h^{\ell}$$

with some matrix $a^{\ell}(u, \mathbf{K}')$ which is defined on $\mathbb{R}^N \times U'_{\delta}$ and satisfies the conditions:

$$\begin{aligned}
\|a^{\ell}\|_{L_\infty(\mathbb{R}^N \times U'_{\delta})} &\le C_{M_2}, \\
|\nabla_u a^{\ell}(u, \bar{\nabla}'\tilde{H}_h^{\ell})| &\le C_{M_2}|\bar{\nabla}_u^2 H_h^{\ell}| \\
|\nabla_u^2 a^{\ell}(u, \bar{\nabla}'\tilde{H}_h^{\ell})| &\le C_{M_2}(|\bar{\nabla}_u^3\tilde{H}_h^{\ell}| + |\bar{\nabla}_u^2\tilde{H}_h^{\ell}|^2), \\
|\partial_t a^{\ell}(u, \bar{\nabla}'\tilde{H}_h^{\ell})| &\le C_{M_2}|\bar{\nabla}_u^1\partial_t\tilde{H}_h^{\ell}|, \\
|\nabla_u\partial_t a^{\ell}(u, \bar{\nabla}'\tilde{H}_h^{\ell})| &\le C_{M_2}(|\bar{\nabla}_u^2\partial_t\tilde{H}_h^{\ell}| + |\bar{\nabla}_u^2\tilde{H}_h^{\ell}||\bar{\nabla}_u^1\partial_t\tilde{H}_h^{\ell}|)
\end{aligned}$$

provided that (14) holds with some small number $\delta > 0$, where we have set $\nabla_u = (\partial/\partial u_1, \ldots, \partial/\partial u_N)$, $\bar{\nabla}_u^k a = (\partial^{\alpha}a/\partial u^{\alpha} \mid |\alpha| \le k)$, and $\mathbf{K}' = (k_0, k_1, \ldots, k_{N-1}) \in U'_{\delta} = \{\mathbf{K}' \in \mathbb{R}^N \mid |\mathbf{K}'| < \delta\}$. Choosing $\delta > 0$ small enough in (14) and using (33) with small $M_1$, we see that $(G^{\ell} + L^{\ell}\tilde{H}_h^{\ell})^{-1} = (\mathbf{I} + (G^{\ell})^{-1}L^{\ell}\tilde{H}_h^{\ell})^{-1}(G^{\ell})^{-1}$ exists, and then

$$\mathbf{b} = -(\mathbf{I} + (G^{\ell})^{-1}L^{\ell}\tilde{H}_h^{\ell})^{-1}(G^{\ell})^{-1}\nabla'\tilde{H}_h^{\ell}.$$

Therefore, we have

$$\begin{aligned}
\tilde{\mathbf{n}}_t^{\ell} &= (1 + < G^{\ell}(\mathbf{I} + (G^{\ell})^{-1}L^{\ell}\tilde{H}_h^{\ell})^{-1}(G^{\ell})^{-1}\nabla'\tilde{H}_h^{\ell}, (\mathbf{I} + (G^{\ell})^{-1}L^{\ell}\tilde{H}_h^{\ell})^{-1}(G^{\ell})^{-1}\nabla'\tilde{H}_h^{\ell} >)^{-1/2} \\
&\quad \times (\tilde{\mathbf{n}}^{\ell} - < (\mathbf{I} + (G^{\ell})^{-1}L^{\ell}\tilde{H}_h^{\ell})^{-1}(G^{\ell})^{-1}\nabla'\tilde{H}_h^{\ell}, \tilde{\tau}^{\ell} >) \\
&= \tilde{\mathbf{n}}^{\ell} - < (G^{\ell})^{-1}\nabla' H_h^{\ell}, \tilde{\tau}^{\ell} > + O_{\ell}^2.
\end{aligned} \tag{37}$$

Since

$$\frac{\partial\tilde{H}_h^{\ell}}{\partial u_j} = \sum_{k=1}^{N} \frac{\partial\Phi_k^{\ell}}{\partial u_j}\frac{\partial H_h}{\partial y_k} \circ \Phi^{\ell},$$

setting

$$< G^{-1}\nabla_\Gamma H_h, \tau > \; = \sum_{\ell=1}^{n} \zeta^\ell < (G^\ell)^{-1}\nabla' \tilde{H}_h^\ell, \tilde{\tau}^\ell > \circ (\Phi^\ell)^{-1},$$

by (37) we see that there exists a matrix of functions, $\mathbf{V_n}(y, \mathbf{K})$, defined on $\mathbb{R}^N \times U_\delta$ such that

$$\mathbf{n}_t = \mathbf{n} - < G^{-1}\nabla_\Gamma H_h, \tau > + \mathbf{V_n}(\cdot, \bar{\nabla} H_h)\bar{\nabla} H_h \otimes \bar{\nabla} H_h \quad \text{on } \Gamma \tag{38}$$

and $\mathbf{V_n}(y, \mathbf{K})$ satisfies the following conditions: supp $\mathbf{V_n}(y, \mathbf{K}) \subset U_\Gamma$ for any $\mathbf{K} \in U_\delta$, and

$$\begin{aligned}
\|\mathbf{V_n}\|_{L_\infty(\mathbb{R}^N \times U_\delta)} &\leq C_{M_2}, \\
|\nabla \mathbf{V_n}(y, \bar{\nabla} H_h)| &\leq C|\bar{\nabla}^2 H_h| \\
|\nabla^2 \mathbf{V_n}(y, \bar{\nabla} H_h)| &\leq C(|\bar{\nabla}^3 H_h| + |\bar{\nabla}^2 H_h|^2), \\
|\partial_t \mathbf{V_n}(y, \bar{\nabla} H_h)| &\leq C|\bar{\nabla}\partial_t H_h|, \\
|\nabla \partial_t \mathbf{V_n}(y, \bar{\nabla} H_h)| &\leq C(|\bar{\nabla}^2 \partial_t H_h| + |\bar{\nabla}^2 H_h||\bar{\nabla}\partial_t H_h|)
\end{aligned}$$

provided that (14) holds with some small $\delta > 0$.

We next represent $\Delta_{\Gamma_t}$. Let $G_t = (g_{ijt})$ be the first fundamental form, and set $g_t = \sqrt{\det G_t}$ and $G_t^{-1} = (g_t^{ij})$. Then, $\Delta_{\Gamma_t}$ is given by setting

$$\Delta_{\Gamma_t} f = \frac{1}{g_t} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial u_i}(g_t g_t^{ij} \frac{\partial f}{\partial u_j}) \quad \text{on } V_0. \tag{39}$$

Since $< \tilde{\mathbf{n}}^\ell, \partial \tilde{\mathbf{n}}^\ell / \partial u_j > \; = 0$ and $< \partial \Phi^\ell / \partial u_j, \tilde{\mathbf{n}} > \; = \; < \tau_j^\ell, \tilde{\mathbf{n}} > \; = 0$, in view of (35), setting

$$\alpha_{ij}^\ell = \; < \tau_i^\ell, \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_j} > + < \tau_j^\ell, \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_i} >, \quad \beta_{ij}^\ell = \; < \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_i}, \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_j} >,$$

we have

$$g_{tij}^\ell = \; < \tau_{ti}^\ell, \tau_{tj}^\ell > \; = g_{ij}^\ell + \alpha_{ij}^\ell \tilde{H}_h^\ell + \beta_{ij}^\ell (\tilde{H}_h^\ell)^2 + < \frac{\partial \tilde{H}_h^\ell}{\partial u_i}, \frac{\partial \tilde{H}_h^\ell}{\partial u_j} > .$$

Notice that $\alpha_{ij}^\ell$ and $\beta_{ij}^\ell$ are all bounded $C^2$ functions. The function $f$ is bounded $C^2$. It means that $f$ is a $C^2$ function and $f$ and its derivatives up to order 2 are all bounded. Let $g_t^\ell = \sqrt{\det(g_{tij}^\ell)}$ and $(G_t^\ell)^{-1} = (g_t^{ij\ell})$, and then by (14) with small $\delta > 0$ and (33), we have the representation formulas:

$$g_t^\ell = g^\ell + \gamma_0^\ell(u)\tilde{H}_h^\ell + O_\ell^2, \quad \frac{1}{g_t^\ell} = \frac{1}{g^\ell} + \gamma_1^\ell(u)\tilde{H}_h^\ell + O_\ell^2, \quad g_t^{ij\ell} = g_\ell^{ij} + \gamma_{ij}^\ell(u)\tilde{H}_h^\ell + O_\ell^2,$$

where $\gamma_0^\ell(u)$, $\gamma_1^\ell(u)$, and $\gamma_{ij}^\ell(u)$ are some bounded $C^2$ functions defined on $\mathbb{R}^N$. In view of (39), setting

$$\begin{aligned}
V_{\Delta ij}^{1\ell} &= \gamma_{ij}^\ell(u)\tilde{H}_h^\ell + O_\ell^2, \\
V_{\Delta j}^{2\ell} &= \sum_{i=1}^{N-1} \Big( \frac{\partial}{\partial u_i}(\gamma_{ij}^\ell(u)\tilde{H}_h^\ell) + \frac{\partial}{\partial u_i} O_\ell^2 + \frac{1}{g_\ell}(\partial_i(\gamma_0^\ell(u)\tilde{H}_h^\ell) + \partial_i O_\ell^2) \\
&\quad + (\gamma_1^\ell(u)\tilde{H}_h^\ell + O_\ell^2)(\partial_i g^\ell + \partial_i(\gamma_0^\ell(u)\tilde{H}_h^\ell) + \partial_i O_\ell^2) \Big)
\end{aligned} \tag{40}$$

we have

$$\Delta_{\Gamma_t} = \Delta_\Gamma + \dot{\Delta}_{\Gamma_t} \quad \text{on } \Gamma \cap B^\ell, \tag{41}$$

where $\dot{\Delta}_{\Gamma_t}$ is an operator defined by setting

$$\dot{\Delta}_{\Gamma_t} f = \sum_{\ell=1}^{n} \zeta^\ell \Big( \sum_{i,j=1}^{N-1} V^{1\ell}_{\Delta ij} \frac{\partial^2 (f(\Phi^\ell(u',0))}{\partial u_i \partial u_j} + \sum_{j=1}^{N-1} V^{2\ell}_{\Delta j} \frac{\partial (f(\Phi^\ell(u',0))}{\partial u_j} \Big) \circ (\Phi^\ell)^{-1}. \tag{42}$$

We finally derive a formula for the curvature. Recall that $\mathcal{H}(\Gamma_t)\mathbf{n}_t = \Delta_{\Gamma_t} x$ for $x \in \Gamma_t$. For $x \in \Gamma_t$, $x$ is represented by $x = \Phi^\ell(u',0) + \tilde{H}^\ell_h(u',0,t)\tilde{\mathbf{n}}^\ell(u',0)$ locally. By (41) and (42), we have

$$\begin{aligned} < \mathcal{H}(\Gamma_t)\mathbf{n}_t, \mathbf{n} > \;&=\; < \Delta_\Gamma (y + H_h \mathbf{n}), \mathbf{n} > \\ + \sum_{\ell=1}^{n} \zeta^\ell \Big( \sum_{i,j=1}^{N-1} < V^{1\ell}_{\Delta ij} \frac{\partial^2}{\partial u_i \partial u_j} (\Phi^\ell + \tilde{H}^\ell_h \mathbf{n}^\ell), \mathbf{n}^\ell > &+ \sum_{j=1}^{N-1} < V^{2\ell}_{\Delta j} \frac{\partial}{\partial u_j} (\Phi^\ell + \tilde{H}^\ell_h \mathbf{n}^\ell), \mathbf{n}^\ell > \Big) \end{aligned}$$

on $\Gamma$. Since $\Delta_\Gamma y = H_h(\Gamma)\mathbf{n}$ for $y \in \Gamma$ and

$$\begin{aligned} < \Delta_\Gamma \mathbf{n}, \mathbf{n} > &= \sum_{\ell=1}^{n} \zeta^\ell \sum_{i,j=1}^{N-1} g^{ij}_\ell < \frac{\partial^2 \mathbf{n}^\ell}{\partial u_i \partial u_j}, \mathbf{n}^\ell > \\ &= - \sum_{\ell=1}^{\infty} \zeta^\ell \sum_{i,j=1}^{N-1} g^{ij}_\ell < \frac{\partial \mathbf{n}^\ell}{\partial u_i}, \frac{\partial \mathbf{n}^\ell}{\partial u_j} > = - < G^{-1} \nabla_\Gamma \mathbf{n}, \nabla_\Gamma \mathbf{n} >, \end{aligned}$$

as follows from $< \partial \tilde{\mathbf{n}}^\ell / \partial u_i, \tilde{\mathbf{n}}^\ell > \; = 0$, we have

$$< \Delta_\Gamma (y + H_h \mathbf{n}), \mathbf{n} > = H_h(\Gamma) + \Delta_\Gamma H_h - < G^{-1} \nabla_\Gamma \mathbf{n}, \nabla_\Gamma \mathbf{n} > H_h.$$

Moreover, by (42) and (40), we have

$$\begin{aligned} < V^{1\ell}_{\Delta ij} \frac{\partial^2}{\partial u_i \partial u_j} \Phi^\ell, \tilde{\mathbf{n}}^\ell > \;&=\; < \gamma^\ell_{ij} \frac{\partial^2}{\partial u_i \partial u_j} \Phi^\ell, \tilde{\mathbf{n}}^\ell > \tilde{H}^\ell_h + O^2_\ell, \\ < V^{1\ell}_{\Delta ij} \frac{\partial^2}{\partial u_i \partial u_j} (\tilde{H}^\ell_h \tilde{\mathbf{n}}^\ell), \tilde{\mathbf{n}}^\ell > \;&=\; V^{1\ell}_{\Delta ij} \frac{\partial^2 \tilde{H}^\ell_h}{\partial u_i \partial u_j} + < V^{1\ell}_{\Delta ij} \frac{\partial^2 \tilde{\mathbf{n}}^\ell}{\partial u_i \partial u_j}, \tilde{\mathbf{n}}^\ell > \tilde{H}^\ell_h \\ = V^{1\ell}_{\Delta ij} \frac{\partial^2 \tilde{H}^\ell_h}{\partial u_i \partial u_j} - < (\frac{\partial}{\partial u_i} V^{1\ell}_{\Delta ij}) \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_j}, \tilde{\mathbf{n}}^\ell > \tilde{H}^\ell_h &- < V^{1\ell}_{\Delta ij} \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_j}, \frac{\partial \tilde{\mathbf{n}}^\ell}{\partial u_i} > \tilde{H}^\ell_h, \\ < V^{2\ell}_{\Delta j} \frac{\partial}{\partial u_j} (\tilde{H}^\ell_h \tilde{\mathbf{n}}^\ell), \tilde{\mathbf{n}}^\ell > = V^{2\ell}_{\Delta j} \frac{\partial \tilde{H}^\ell_h}{\partial u_j}. \end{aligned}$$

Combination of these formulas gives

$$< H_h(\Gamma_t)\mathbf{n}_t, \mathbf{n} > = H_h(\Gamma) + \Delta_\Gamma H_h + a(y) H_h + \mathbf{V}_s(y, \bar{\nabla} H_h) \bar{\nabla} H_h \otimes \bar{\nabla}^2 H_h, \tag{43}$$

where $a(y)$ is a bounded $C^1$ function, and $\mathbf{V}_s = \mathbf{V}_s(y, \mathbf{K})$ are some matrices of functions defined on $\mathbb{R}^N \times U_\delta$ such that

$$\begin{aligned} &\text{supp}\, \mathbf{V}_s(y, \mathbf{K}) \subset U_\Gamma \quad \text{for any } \mathbf{K} \in U_\delta, \quad \sup_{t \in (0,T)} \|\mathbf{V}_s(\cdot, \bar{\nabla} H_h)\|_{L_\infty(\Omega)} \le C_{M_2}, \\ &|\nabla \mathbf{V}_s(y, \bar{\nabla} H_h)| \le C_{M_2} |\bar{\nabla}^2 H_h(y,t)|, \quad |\partial_t \mathbf{V}_s(y, \bar{\nabla} H_h)| \le C_{M_2} |\bar{\nabla} \partial_t H_h(y,t)|. \end{aligned} \tag{44}$$

Under the assumption that (14) holds with some small constant $\delta > 0$.

### 2.4. Derivation of Transmission Conditions and Kinematic Condition

At first, we consider the kinematic condition: $V_{\Gamma_t} = \mathbf{v}_+ \cdot \mathbf{n}_t$. Note that $\mathbf{v}_+ = \mathbf{v}_-$ on $\Gamma_t$. Since

$$V_{\Gamma_t} = \frac{\partial x}{\partial t} \cdot \mathbf{n}_t = \frac{\partial H_h}{\partial t} \mathbf{n} \cdot \mathbf{n}_t,$$

it follows from (38) that

$$\partial_t h + < \nabla_\Gamma h \perp \mathbf{u}_+ > - \mathbf{u}_+ \cdot \mathbf{n} = < \mathbf{u}_+ - \frac{\partial H_h}{\partial t}\mathbf{n}, \mathbf{V}_n(\cdot, \bar{\nabla} H_h)\bar{\nabla} H_h \otimes \bar{\nabla} H_h > . \tag{45}$$

Here, we introduce the symbol

$$< \nabla_\Gamma h \perp \mathbf{u}_+ > = \sum_{\ell=1}^n \zeta^\ell \Big( \sum_{i,j=1}^{N-1} g_\ell^{ij} \frac{(\partial h \circ \Phi^\ell)}{\partial u_j} < \tilde{\tau}_i^\ell, \mathbf{u}_+ \circ \Phi^\ell > \Big).$$

If we move $< \nabla_\Gamma h \perp \mathbf{u}_+ >$ to the right-hand side in proving the local well-posedness by using a standard fixed point argument, we have to assume the smallness of the initial velocity field $\mathbf{u}_0$ as well as the smallness of the initial height $h_0$. However, this is not satisfactory. We have to treat at least the large initial velocity case for the local well-posedness. To avoid the smallness assumption of the initial velocity field, we use an idea due to Padula and Solonnikov [4]. Let $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\dot\Omega)$ be an initial velocity field and $\mathbf{u}_0^+ = \mathbf{u}_0|_{\Omega_+}$. We know that $[[\mathbf{u}_0]] = 0$ on $\Gamma$, which follows from the compatibility conditions. Let $\tilde{\mathbf{u}}_0^+$ be an extension of $\mathbf{u}_0^+$ to $\mathbb{R}^N$ such that $\tilde{\mathbf{u}}_0^+ = \mathbf{u}_0^+$ in $\Omega_+$ and

$$\|\tilde{\mathbf{u}}_0^+\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} \leq C \|\mathbf{u}_0^+\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)}. \tag{46}$$

Let

$$\mathbf{u}_\kappa = \frac{1}{\kappa} \int_0^\kappa T_0(s) \tilde{\mathbf{u}}_0^+ \, ds,$$

where $\{T_0(s)\}_{s \geq 0}$ is a $C^0$ analytic semigroup generated by $-\Delta + \lambda_0$ with large $\lambda_0$ in $\mathbb{R}^N$, that is

$$T_0(s) f = \mathcal{F}^{-1}\big[e^{-s(|\xi|^2 + \lambda_0)} \hat{f}(\xi)\big](x).$$

Here, $\hat{f}$ denotes the Fourier transform of $f$ and $\mathcal{F}^{-1}$ the inverse Fourier transform. We know that

$$\begin{aligned} \|T_0(\cdot)\tilde{\mathbf{u}}_0^+\|_{L_p((0,\infty), H^2_q(\mathbb{R}^N))} + \|\partial_t T_0(\cdot)\tilde{\mathbf{u}}_0^+\|_{L_p((0,\infty), L_q(\mathbb{R}^N))} + \|T_0(\cdot)\tilde{\mathbf{u}}_0^+\|_{L_\infty((0,\infty), B^{2(1-1/p)}_{q,p}(\mathbb{R}^N))} \\ \leq C \|\mathbf{u}_0^+\|_{B^{2(1-1/p}_{q,p}(\mathbb{R}^N)}, \end{aligned} \tag{47}$$

which yields that

$$\begin{aligned} \|\mathbf{u}_\kappa\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} &\leq C \|\mathbf{u}_0^+\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)}, \\ \|\mathbf{u}_\kappa\|_{H^2_q(\mathbb{R}^N)} &\leq C \kappa^{-1/p} \|\mathbf{u}_0^+\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)}. \end{aligned} \tag{48}$$

As a kinematic condition, we use the following equation:

$$\partial_t h + < \nabla_\Gamma h \perp \mathbf{u}_\kappa > - \mathbf{u} \cdot \mathbf{n} = d(\mathbf{u}, H_h), \tag{49}$$

where

$$d(\mathbf{u}, H_h) = \; < \nabla_\Gamma H_h \perp \mathbf{u} - \mathbf{u}_\kappa > + < \mathbf{u} - \frac{\partial H_h}{\partial t}\mathbf{n}, \mathbf{V}_\mathbf{n}(\cdot, \bar{\nabla} H_h)\bar{\nabla} H_h \otimes \bar{\nabla} H_h > . \tag{50}$$

Let $\mathcal{E}_\mp$ be an the extension map, which is acting on $\mathbf{u}_\pm \in H^2_q(\Omega_\pm)$ and satisfying the properties: $\mathcal{E}_\mp(\mathbf{u}_\pm) \in H^2_q(\Omega)$, $\mathcal{E}_\mp(\mathbf{u}_\pm) = \mathbf{u}_\pm$ in $\Omega_\pm$,

$$(\partial_x^\alpha \mathcal{E}_\mp(\mathbf{u}_\pm))(x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega_\pm}} \partial_x^\alpha \mathbf{u}_\pm(x) \tag{51}$$

for $x_0 \in \Gamma$ and $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq 1$, and

$$\|\mathcal{E}_{\mp}(\mathbf{u}_{\pm})\|_{H_q^\ell(\Omega)} \leq C_{\ell,q}\|\mathbf{u}_{\pm}\|_{H_q^\ell(\Omega_{\pm})} \tag{52}$$

for $\ell = 0, 1, 2$. Note that

$$[[\partial_x^\alpha \mathbf{u}]] = \partial_x^\alpha \mathcal{E}_-(\mathbf{u}_+)|_\Gamma - \partial_x^\alpha \mathcal{E}_+(\mathbf{u}_-)|_\Gamma \tag{53}$$

for $|\alpha| \leq 1$ on $\Gamma$. For the notational simplicity, we write

$$tr[\mathbf{u}] = \mathcal{E}_-(\mathbf{u}_+) - \mathcal{E}_+(\mathbf{u}_-). \tag{54}$$

Then, we have

$$\begin{aligned} \partial_x^\alpha\, tr[\mathbf{u}]|_\Gamma &= [[\partial_x^\alpha \mathbf{u}]] \\ \|tr[\mathbf{u}]\|_{H_q^i(\Omega)} &\leq C(\|\mathbf{u}_+\|_{H_q^i(\Omega)} + \|\mathbf{u}_-\|_{H_q^i(\Omega)}) = C\|\mathbf{u}\|_{H_q^i(\dot\Omega)} \end{aligned} \tag{55}$$

for $i = 0, 1, 2$.

Next, we consider the interface conditions:

$$[[(\mathbf{T}(\mathbf{v}, \mathfrak{p}) + \mathbf{T}_M(\mathbf{H}_h))\mathbf{n}_t]] = \sigma H_h(\Gamma_t)\mathbf{n}_t \quad \text{on } \Gamma_t. \tag{56}$$

Let

$$\mathbf{\Pi}_t \mathbf{d} = \mathbf{d} - < \mathbf{d}, \mathbf{n}_t > \mathbf{n}_t, \quad \mathbf{\Pi}_0 \mathbf{d} = \mathbf{d} - < \mathbf{d}, \mathbf{n} > \mathbf{n} \tag{57}$$

The following lemma was given in Solonnikov [20].

**Lemma 2.** *If $\mathbf{n}_t \cdot \mathbf{n} \neq 0$, then for arbitrary vector $\mathbf{d}$, $\mathbf{d} = 0$ is equivalent to*

$$\mathbf{\Pi}_0 \mathbf{\Pi}_t \mathbf{d} = 0 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{d} = 0. \tag{58}$$

In view of Lemma 2, the interface condition (56) is equivalent to the following two conditions:

$$\mathbf{\Pi}_0 \mathbf{\Pi}_t [[\nu(\mathbf{D}(\mathbf{u}) + \mathbf{V}_D(\mathbf{K})\nabla \mathbf{u})) + \mathbf{T}_M(\mathbf{G})]]\mathbf{n}_t = 0, \tag{59}$$

$$\mathbf{n} \cdot ([[\nu(\mathbf{D}(\mathbf{u}) + \mathbf{V}_D(\mathbf{K})\nabla \mathbf{u}) - \mathfrak{q}\mathbf{I} + \mathbf{T}_M(\mathbf{G})]]\mathbf{n}_t - \sigma H_h(\Gamma_t)\mathbf{n}_t) = 0. \tag{60}$$

Here and hereafter, $\mathbf{V}_D(\mathbf{K})\nabla \mathbf{u}$ is the $N \times N$ matrix with $(i, j)$ components $V_{Dij}(\mathbf{K})\nabla \mathbf{u}$ (cf. (18)). Noting that $\mathbf{\Pi}_0\mathbf{\Pi}_0 = \mathbf{\Pi}_0$, we see that the condition (59) takes the form

$$\mathbf{\Pi}_0[[\nu \mathbf{D}(\mathbf{u})]]\mathbf{n} = \mathbf{h}_1'(\mathbf{u}, \mathbf{G}, H_h) \tag{61}$$

with

$$\begin{aligned} \mathbf{h}_1'(\mathbf{u}, \mathbf{G}, H_h) = {} & \mathbf{\Pi}_0(\mathbf{\Pi}_0 - \mathbf{\Pi}_t)[[\nu \mathbf{D}(\mathbf{u})]]\mathbf{n}_t + \mathbf{\Pi}_0[[\nu \mathbf{D}(\mathbf{u})]](\mathbf{n} - \mathbf{n}_t) \\ & - \mathbf{\Pi}_0 \mathbf{\Pi}_t [[\nu \mathbf{V}_D(\mathbf{K})\nabla \mathbf{u} + \mathbf{T}_M(\mathbf{G})]]\mathbf{n}_t. \end{aligned} \tag{62}$$

On the other hand, by (43) we see that Equation (60) can be written in the form

$$\mathbf{n} \cdot [[\nu \mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}]]\mathbf{n} - \sigma(\Delta_\Gamma h + ah) = h_{1N}(\mathbf{u}, \mathbf{G}, H_h) + \sigma \mathbf{V}_s(\cdot, \bar\nabla H_h)\bar\nabla H_h \otimes \bar\nabla^2 H_h. \tag{63}$$

with

$$h_{1N}(\mathbf{u}, \mathbf{G}, H_h) = (\mathbf{n} \cdot \mathbf{n}_t)^{-1}\{\mathbf{n} \cdot [[\nu \mathbf{D}(\mathbf{u})]](\mathbf{n} - \mathbf{n}_t) - \mathbf{n} \cdot [[\nu \mathbf{V}_D(\mathbf{K})\nabla \mathbf{u} + \mathbf{T}_M(\mathbf{G})]]\mathbf{n}_t\}. \tag{64}$$

In particular, by setting

$$\mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_h) = (\mathbf{h}_1'(\mathbf{u}, \mathbf{G}, H_h), h_{1N}(\mathbf{u}, \mathbf{G}, H_h) + \sigma \mathbf{V}_s(\cdot, \bar\nabla H_h)\bar\nabla H_h \otimes \bar\nabla^2 H_h),$$

in view of (18), (38), and (55), we obtain

$$\mathbf{h}_1(\mathbf{u},\mathbf{G},H_h) = \mathbf{V}^1_{\mathbf{h}}(\cdot,\bar\nabla H_h)\bar\nabla H_h\otimes\nabla tr[\mathbf{u}] + a(y)tr[\mathbf{G}]\otimes tr[\mathbf{G}] \\ + \mathbf{V}^2_{\mathbf{h}}(\cdot,\bar\nabla H_h)\bar\nabla H_h\otimes tr[\mathbf{G}]\otimes tr[\mathbf{G}] + \mathbf{V}_s(\cdot,\bar\nabla H_h)\bar\nabla H_h\otimes\bar\nabla^2 H_h. \tag{65}$$

Here, $a(y)$ is an $N$-vector of bounded $C^2$ functions, $\mathbf{V}^i_{\mathbf{h}}(\cdot,\mathbf{K})$ $(i = 1,2)$ are some matrices of functions defined on $\mathbb{R}^N\times U_\delta$ and satisfying the conditions: $\|\mathbf{V}^i_{\mathbf{h}}\|_{L_\infty(\mathbb{R}^N\times U_\delta)}\leq C$, $\operatorname{supp}\mathbf{V}^i_{\mathbf{h}}(y,\mathbf{K})\subset U_\Gamma$,

$$|\nabla\mathbf{V}^i_{\mathbf{h}}(y,\bar\nabla H_h)|\leq C|\bar\nabla^2 H_h(y,t)|,\quad |\partial_t\mathbf{V}^i_{\mathbf{h}}(y,\bar\nabla H_h)|\leq C|\bar\nabla\partial_t H_h(y,t)|. \tag{66}$$

provided that (14) holds with some small $\delta>0$.

From (18), we see that the interface condition: $[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}_h + \mu(\mathbf{v}\otimes\mathbf{H}_h - \mathbf{H}_h\otimes\mathbf{v})]]\mathbf{n}_t = 0$ takes the form

$$[[\alpha^{-1}\mathrm{curl}\,\mathbf{G}]]\mathbf{n} = \mathbf{h}_2(\mathbf{u},\mathbf{G},H_h), \tag{67}$$

where

$$\mathbf{h}_2(\mathbf{u},\mathbf{G},H_h) = [[\alpha^{-1}\mathrm{curl}\,\mathbf{G}]](\mathbf{n}-\mathbf{n}_t) - [[\alpha^{-1}\mathbf{V}_C(\mathbf{K})\nabla\mathbf{u}]]\mathbf{n}_t - [[\mu(\mathbf{u}\otimes\mathbf{G}-\mathbf{G}\otimes\mathbf{u})]]\mathbf{n}_t.$$

Here, $\mathbf{V}_C(\mathbf{K})\nabla\mathbf{u}$ is the $N\times N$ matrix with $(i,j)$ components $V_{Cij}(\mathbf{K})\nabla\mathbf{u}$, which are given in (18). In particular, in view of (18), (38), and (55), we may write

$$\mathbf{h}_2(\mathbf{u},\mathbf{G},H_h) = \mathbf{V}^3_{\mathbf{h}}(\cdot,\bar\nabla H_h)\bar\nabla H_h\otimes\nabla tr[\mathbf{u}](\mathcal{E}_-(\mathbf{u}_+) + b(y)tr[\mathbf{u}]\otimes tr[\mathbf{G}] \\ + \mathbf{V}^4_{\mathbf{h}}(\cdot,\bar\nabla H_h)\bar\nabla H_h\otimes tr[\mathbf{u}]\otimes tr[\mathbf{G}]. \tag{68}$$

Here, $b(y)$ is an $N$-vector of $C^2$ functions, and $\mathbf{V}^i_{\mathbf{h}}(\cdot,\mathbf{K})$ $(i=3,4)$ are some matrices of functions defined on $\mathbb{R}^N\times U_\delta$ satisfying the same conditions as those stated in (66) provided that (14) holds with some small $\delta>0$.

From (23), we see that the interface condition: $[[\mu\mathrm{div}\,\mathbf{H}_h]] = 0$ can be written in the form

$$[[\mu\mathrm{div}\,\mathbf{G}]] = h_3(\mathbf{u},\mathbf{G},H_h), \tag{69}$$

where

$$h_3(\mathbf{u},\mathbf{G},H_h) = -\mu\sum_{j,k=1}^N\tilde V_{0jk}(\mathbf{K})\mathbf{K}\frac{\partial}{\partial y_k}tr[\mathbf{u}]_j,$$

$V_{0jk}(\mathbf{K}) = \tilde V_{0jk}(\mathbf{K})\mathbf{K}$ are the symbols given in (17), and $tr[\mathbf{u}] = (tr[\mathbf{u}]_1,\ldots,tr[\mathbf{u}]_N)$.

Finally, the interface conditions: $[[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0$ and $[[\mathbf{H}- <\mathbf{H},\mathbf{n}_t>\mathbf{n}_t]] = 0$ can be written in the form

$$[[\mu\mathbf{G}\cdot\mathbf{n}]] = k_1(\mathbf{G},H_h),\quad [[\mathbf{G}- <\mathbf{G},\mathbf{n}>\mathbf{n}]] = \mathbf{k}_2(\mathbf{G},H_h), \tag{70}$$

where

$$k_1(\mathbf{G},H_h) = [[\mu\mathbf{G}\cdot(\mathbf{n}-\mathbf{n}_t)]],\quad \mathbf{k}_2(\mathbf{G},H_h) = [[<\mathbf{G},\mathbf{n}_t-\mathbf{n}>\mathbf{n}_t]] + [[<\mathbf{G},\mathbf{n}>(\mathbf{n}_t-\mathbf{n})]].$$

In particular, in view of (38) and (55), we obtain

$$(k_1(\mathbf{G},H_h),\mathbf{k}_2(\mathbf{G},H_h)) = \mathbf{V}^5_{\mathbf{k}}(\cdot,\bar\nabla H_h)\bar\nabla H_h\otimes tr[\mathbf{G}]. \tag{71}$$

Here $\mathbf{V}^5_{\mathbf{k}}(\cdot,\mathbf{K})$ are matrices of functions defined on $\mathbb{R}^N\times U_\delta$ and satisfying the same conditions as stated in (66), provided that (14) holds with sufficiently small $\delta>0$.

### 2.5. Statement of the Local Well-Posedness Theorem

Summing up the results obtained in Sections 2.2–2.4, we see that Equation (6) is transformed to the following equations:

$$\begin{cases} m\partial_t\mathbf{u} - \mathrm{Div}\,\mathbf{T}(\mathbf{u},\mathfrak{q}) = \mathbf{f}_1(\mathbf{u},\mathbf{G},H_h), & \text{in } \dot\Omega\times(0,T), \\ \mathrm{div}\,\mathbf{u} = g(\mathbf{u},H_h) = \mathrm{div}\,\mathbf{g}(\mathbf{u},H_h) & \text{in } \dot\Omega\times(0,T), \\ \partial_t h + < \nabla_\Gamma h \perp \mathbf{u}_\kappa > -\mathbf{n}\cdot\mathbf{u} = d(\mathbf{u},H_h) & \text{on } \Gamma\times(0,T), \\ [[\mathbf{u}]] = 0, \quad [[\mathbf{T}(\mathbf{u},\mathfrak{q})\mathbf{n}]] - \sigma(\Delta_\Gamma h + ah)\mathbf{n} = \mathbf{h}_1(\mathbf{u},\mathbf{G},H_h), & \text{on } \Gamma\times(0,T), \\ \mu\partial_t\mathbf{G} - \alpha^{-1}\Delta\mathbf{G} = \mathbf{f}_2(\mathbf{u},\mathbf{G},H_h) & \text{in } \dot\Omega\times(0,T), \\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{G}]]\mathbf{n} = \mathbf{h}_2(\mathbf{u},\mathbf{G},H_h), \quad [[\mu\mathrm{div}\,\mathbf{G}]] = h_3(\mathbf{u},\mathbf{G},H_h) & \text{on } \Gamma\times(0,T), \\ [[\mu\mathbf{G}\cdot\mathbf{n}]] = k_1(\mathbf{G},H_h), \quad [[\mathbf{G}- < \mathbf{G},\mathbf{n} > \mathbf{n}]] = \mathbf{k}_2(\mathbf{G},H_h) & \text{on } \Gamma\times(0,T), \\ \mathbf{u}_\pm = 0, \quad \mathbf{n}_\pm\cdot\mathbf{G}_\pm = 0, \quad (\mathrm{curl}\,\mathbf{G}_\pm)\mathbf{n}_\pm = 0 & \text{on } S_\pm\times(0,T), \\ (\mathbf{u},\mathbf{G},h)|_{t=0} = (\mathbf{u}_0,\mathbf{G}_0,h_0) & \text{in } \dot\Omega\times\dot\Omega\times\Gamma, \end{cases} \tag{72}$$

where $H_h$ is a solution of Equation (8).

The purpose of this paper is to prove the following local in time unique existence theorem.

**Theorem 1.** *Let $2 < p < \infty$, $N < q < \infty$, $2/p + N/q < 1$ and $B > 0$. Assume that condition (5) holds. There exist a small number $\epsilon$ and a small time $T > 0$ depending on $B$ such that if initial data $h_0 \in B^{3-1/p-1/q}_{q,p}(\Gamma)$ satisfy the smallness condition $\|h_0\|_{B^{3-1/p-1/q}_{q,p}} \leq \epsilon$, and $(\mathbf{u}_0,\mathbf{G}_0) \in B^{2(1-1/p)}_{q,p}(\dot\Omega)^{2N}$ satisfies $\|(\mathbf{u}_0,\mathbf{G}_0)\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \leq B$ and the compatibility conditions:*

$$\begin{aligned} \mathrm{div}\,\mathbf{u}_0 = 0 \quad & \text{in } \dot\Omega, \\ [[(\nu\mathbf{D}(\mathbf{u}_0) + \mathbf{T}_M(\mathbf{G}_0)\mathbf{n}]]_\tau = 0, \quad [[\mathbf{u}_0]] = 0 \quad & \text{on } \Gamma, \\ [[\{\alpha^{-1}\mathrm{curl}\,\mathbf{G}_0 + \mu(\mathbf{u}_0\otimes\mathbf{G}_0 - \mathbf{G}_0\otimes\mathbf{u}_0)\}\mathbf{n}]] = 0, \quad [[\mu\mathrm{div}\,\mathbf{G}_0]] = 0 \quad & \text{on } \Gamma, \\ [[\mu\mathbf{G}_0\cdot\mathbf{n}]] = 0, \quad [[\mathbf{G}_0- < \mathbf{G}_0,\mathbf{n} > \mathbf{n}]] = 0 \quad & \text{on } \Gamma, \\ \mathbf{u}_{0\pm} = 0, \quad \mathbf{n}_0\cdot\mathbf{G}_{0\pm} = 0, \quad (\mathrm{curl}\,\mathbf{G}_{0\pm})\mathbf{n}_\pm = 0 \quad & \text{on } S_\pm, \end{aligned} \tag{73}$$

*then Equation (72) admits unique solution $\mathbf{u}$, $\mathfrak{q}$, $\mathbf{G}$, and $h$ with the following properties:*

$$\begin{aligned} &\mathbf{u} \in H^1_p((0,T),L_q(\dot\Omega)^N) \cap L_p((0,T),H^2_q(\dot\Omega)^N), \\ &\mathfrak{q} \in L_p((0,T),H^1_q(\dot\Omega) + \hat H^1_q(\Omega), \\ &\mathbf{G} \in H^1_p((0,T),L_q(\dot\Omega)^N) \cap L_p((0,T),H^2_q(\dot\Omega)^N), \\ &h \in H^1_p((0,T),W^{2-1/q}_q(\Gamma)^N) \cap L_p((0,T),W^{3-1/q}_q(\Gamma)), \\ &\|H_h\|_{L_\infty((0,T),H^1_\infty(\Omega))} \leq \delta. \end{aligned}$$

*This solution satisfies the estimate:*

$$\begin{aligned} &\|(\mathbf{u},\mathbf{G})\|_{L_p((0,T),H^2_q(\dot\Omega))} + \|\partial_t(\mathbf{u},\mathbf{G})\|_{L_p((0,T),L_q(\dot\Omega))} \\ &\quad + \|h\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))} + \|\partial_t h\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))} + \|\partial_t h\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))} \leq f(B). \end{aligned}$$

*Here, $\delta$ is a constant appearing in (14), and $f(B)$ is a polynomial of $B$.*

## 3. Linear Theory

Since the coupling of the velocity field and the magnetic field in (6) is semilinear, the linearized equations are decoupled. Namely, we consider the two linearized equations: one is the Stokes equations with transmission conditions on $\Gamma$ and nonslip conditions on $S_\pm$, and another is the system of the heat equations with transmission conditions on $\Gamma$ and the perfect wall conditions on $S_\pm$. We assume that $\Gamma$ is a compact hypersurface of $C^3$ class and that $S_\pm$ are hypersurfaces of $C^2$ class.

### 3.1. Two-Phase Problem for the Stokes Equations

This subsection is devoted to presenting the $L_p$-$L_q$ maximal regularity for the two-phase problem of the Stokes equations with transmission conditions given as follows:

$$\begin{cases} m\partial_t \mathbf{u} - \mathrm{Div}\,\mathbf{T}(\mathbf{u}, \mathfrak{q}) = \mathbf{f}_1 & \text{in } \dot\Omega \times (0,T), \\ \mathrm{div}\,\mathbf{u} = g = \mathrm{div}\,\mathbf{g} & \text{in } \dot\Omega \times (0,T), \\ \partial_t h + < \nabla_\Gamma h \perp \mathbf{w}_\kappa > -\mathbf{n}\cdot\mathbf{u} = d & \text{on } \Gamma \times (0,T), \\ [[\mathbf{u}]] = 0, \quad [[\mathbf{T}(\mathbf{u}, \mathfrak{q})\mathbf{n}]] - \sigma(ah + \Delta_\Gamma h)\mathbf{n} = \mathbf{h} & \text{on } \Gamma \times (0,T), \\ \mathbf{u}_\pm = 0, & \text{on } S_\pm \times (0,T), \\ (\mathbf{u}, h)|_{t=0} = (\mathbf{u}_0, h_0) & \text{in } \dot\Omega \times \Gamma. \end{cases} \tag{74}$$

Assumptions for Equation (74) are the following:

(a.1) $a$ is a bounded $C^1$ functions defined in $\Omega$.
(a.2) $\mathbf{w}_\kappa$ is a family of $N$-vector of functions defined on $\Gamma$ for $\kappa \in (0,1)$ and such that

$$|\mathbf{w}_\kappa(x)| \le m_1, \quad |\mathbf{w}_\kappa(x) - \mathbf{w}_\kappa(y)| \le m_1|x-y|^b \text{ for any } x, y \in \Gamma,$$
$$\|\mathbf{w}_\kappa\|_{W^{2-1/r}_r(\Gamma)} \le m_2\kappa^{-c}.$$

Here, $m_1$, $m_2$, $b$, and $c$ are positive constants and $r \in (N, \infty)$.

**Theorem 2.** *Let $1 < p < \infty$, $1 < q \le r$, $2/p + 1/q \neq 1, 2$, and $T > 0$. Assume that the assumptions (a.1) and (a.2) are satisfied. Then, there exists a constant $\gamma_0 > 0$ such that the following assertion holds: Let $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\dot\Omega)$ and $h_0 \in B^{3-1/p-1/q}_{q,p}(\Gamma)$. Let $\mathbf{f}$, $g$, $\mathbf{g}$, $\mathbf{h} = (\mathbf{h}', h_N)$, and $d$ appearing in the right-hand side of Equation (74) be given functions satisfying the following conditions:*

$$\mathbf{f} \in L_p((0,T), L_q(\dot\Omega)^N), \quad e^{-\gamma t} g \in L_p(\mathbb{R}, H^1_q(\dot\Omega)) \cap H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega)),$$
$$e^{-\gamma t}\mathbf{g} \in H^1_p(\mathbb{R}, L_q(\dot\Omega)^N), \quad e^{-\gamma t}\mathbf{h} \in L_p(\mathbb{R}, H^1_q(\Omega)^N) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)^N),$$
$$d \in L_p((0,T), W^{2-1/q}_q(\Gamma))$$

*for any $\gamma \ge \gamma_0$. Assume that $\mathbf{u}_0$, $g$, and $\mathbf{h}$ satisfy the following compatibility conditions:*

$$\mathrm{div}\,\mathbf{u}_0 = g|_{t=0} \quad \text{on } \dot\Omega, \tag{75}$$
$$[[\nu\mathbf{D}(\mathbf{u}_0)\mathbf{n}]]_\tau = \mathbf{h}_\tau|_{t=0} \quad \text{on } \Gamma \quad \text{provided } 2/p + 1/q < 1, \tag{76}$$
$$[[\mathbf{u}_0]] = 0 \quad \text{on } \Gamma, \quad \mathbf{u}_{0,\pm} = 0 \quad \text{on } S_\pm \quad \text{provided } 2/p + 1/q < 2, \tag{77}$$

*where $\mathbf{d}_\tau = \mathbf{d} - < \mathbf{d}, \mathbf{n} > \mathbf{n}$. Then, Equation (74) admits unique solutions $\mathbf{u}$, $\mathfrak{q}$, and $h$ with*

$$\mathbf{u} \in L_p((0,T), H^2_q(\dot\Omega)^N) \cap H^1_p((0,T), L_q(\dot\Omega)^N), \quad \mathfrak{q} \in L_p((0,T), H^1_q(\dot\Omega) + \hat H^1_q(\Omega)),$$
$$h \in L_p((0,T, W^{3-1/q}_q(\Gamma)) \cap H^1_p((0,T), W^{2-1/q}_q(\Gamma))$$

*possessing the estimates:*

$$\begin{aligned}
&\|\partial_t \mathbf{u}\|_{L_p((0,T),L_q(\dot\Omega))} + \|\mathbf{u}\|_{L_p((0,T),H^2_q(\dot\Omega))} + \|\partial_t h\|_{L_p((0,\infty),W^{2-1/q}_q(\Gamma))} + \|h\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))}\\
&\quad\le Ce^{\gamma\kappa^{-c}T}\{\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \kappa^{-c}\|h_0\|_{B^{3-1/p-/q}_{q,p}(\Gamma)} + \|\mathbf{f}\|_{L_p((0,T),L_q(\dot\Omega))}\\
&\quad+\|e^{-\gamma t}g\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{g}\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{h}\|_{L_p(\mathbb{R},H^1_q(\Omega))}\\
&\quad+(1+\gamma^{1/2})(\|e^{-\gamma t}g\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathbf{h}\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}) + \|d\|_{L_p((0,T),H^2_q(\Omega))}\}
\end{aligned}$$

*for any $\gamma \ge \gamma_0$ with some constant $C > 0$ independent of $\gamma$.*

**Remark 1.** (1) *Theorem 2 has been proved in Shibata and Saito [21]. The reason why we assume that $\Gamma$ is a compact in this paper is that the weak Neumann problem is uniquely solvable. Namely, if we consider the weak Neumann problem:*

$$(m^{-1}\nabla u, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for any } \varphi \in \hat H^1_{q'}(\Omega) \tag{78}$$

*where*

$$\hat H^1_{q'}(\Omega) = \{\varphi \in L_{q',\mathrm{loc}}(\Omega) \mid \nabla\varphi \in L_{q'}(\Omega)\}, \quad q' = q/(q-1),$$

*then for any $\mathbf{f} \in L_q(\Omega)^N$, problem (78) admits a unique solution $u \in \hat H^1_q(\Omega)$ satisfying the estimate: $\|\nabla u\|_{L_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)}$ with some constant $C > 0$. If $\Gamma$ is unbounded, then in general we have to assume that the weak Neumann problem is uniquely solvable except for a few cases where $\Gamma$ is flat, that is $\Gamma = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}$, or $\Gamma$ is asymptotically flat.*

### *3.2. Two-Phase Problem for the Linear Electromagnetic Field Equations*

This subsection is devoted to presenting the $L_p$-$L_q$ maximal regularity for the linear electromagnetic field equations. The problem is formulated by the following equations:

$$\begin{cases}
\mu\partial_t\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = \mathbf{f} & \text{in } \dot\Omega \times (0,T),\\
[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} = \mathbf{h}', \quad [[\mu \mathrm{div}\,\mathbf{H}]] = h_N & \text{on } \Gamma \times (0,T),\\
[[\mathbf{H} - <\mathbf{H},\mathbf{n}>\mathbf{n}]] = \mathbf{k}', \quad [[\mu\mathbf{H}\cdot\mathbf{n}]] = k_N & \text{on } \Gamma \times (0,T),\\
\mathbf{n}_\pm \cdot \mathbf{H}_\pm = 0, \quad (\mathrm{curl}\,\mathbf{H}_\pm)\mathbf{n}_\pm = 0 & \text{on } S_\pm \times (0,T),\\
\mathbf{H}|_{t=0} = \mathbf{H}_0 & \text{in } \dot\Omega.
\end{cases} \tag{79}$$

**Theorem 3.** *Let $1 < p, q < \infty$, $2/p + 1/q \ne 1, 2$, and $T > 0$. There exists a constant $\gamma_0$ such that the following assertion holds: Let $\mathbf{H}_0 \in B^{2(1-1/p)}_{q,p}(\dot\Omega)$ and let $\mathbf{f}$, $\mathbf{h} = (\mathbf{h}', h_N)$, and $\mathbf{k} = (\mathbf{k}', k_N)$ be given functions appearing in the right-hand side of Equation (79) and satisfying the following conditions:*

$$\begin{aligned}
&\mathbf{f} \in L_p((0,T), L_q(\dot\Omega)^N), \quad e^{-\gamma t}\mathbf{h} \in L_p(\mathbb{R}, H^1_q(\Omega)^N) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)^N),\\
&e^{-\gamma t}\mathbf{k} \in L_p(\mathbb{R}, H^2_q(\Omega)^N) \cap H^1_p(\mathbb{R}, L_q(\Omega)^N)
\end{aligned}$$

*for any $\gamma \ge \gamma_0$. Assume that $\mathbf{f}$, $\mathbf{h}$, and $\mathbf{k}$ satisfy the following compatibility conditions:*

$$[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}_0]]\mathbf{n} = \mathbf{h}'|_{t=0}, \quad [[\mu \mathrm{div}\,\mathbf{H}_0]] = h_N|_{t=0} \quad \text{on } \Gamma, \quad [[\mathrm{curl}\,\mathbf{H}_{0\pm})\mathbf{n}_\pm = 0 \quad \text{on } S_\pm \tag{80}$$

*provided that $2/p + 1/q < 1$;*

$$[[\mathbf{H}_0 - <\mathbf{H}_0,\mathbf{n}>\mathbf{n}]] = \mathbf{k}'|_{t=0}, \quad [[\mu\mathbf{H}_0\cdot\mathbf{n}]] = k_N|_{t=0} \quad \text{on } \Gamma, \quad \mathbf{n}_\pm \cdot \mathbf{H}_{0\pm} = 0 \quad \text{on } S_\pm \tag{81}$$

*provided that* $2/p + 1/q < 2$. *Then, problem (79) admits a unique solution* $\mathbf{H}$ *with*

$$\mathbf{H} \in L_p((0,T), H^2_q(\dot\Omega)^N) \cap H^1_p((0,T), L_q(\dot\Omega)^N)$$

*possessing the estimate:*

$$\|\partial_t \mathbf{H}\|_{L_p((0,T),L_q(\dot\Omega))} + \|\mathbf{H}\|_{L_p((0,T),H^2_q(\dot\Omega))} \leq Ce^{\gamma T}\{\|\mathbf{H}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|\mathbf{f}\|_{L_p(\mathbb{R},L_q(\dot\Omega))}$$
$$+ \|e^{-\gamma t}\mathbf{h}\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|e^{-\gamma t}\mathbf{h}\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\mathbf{k}\|_{L_p(\mathbb{R},H^2_q(\Omega))} + \|e^{-\gamma t}\partial_t \mathbf{k}\|_{L_p(\mathbb{R},L_q(\Omega))}\}$$

*for any* $\gamma \geq \gamma_0$ *with some constant* $C > 0$ *independent of* $\gamma$.

**Remark 2.** *Theorem 3 was proved by Froloba and Shibata [11] under the assumption that* $\Omega$ *is a uniformly* $C^3$ *domain. Of course, if* $\Gamma$ *is a compact hypersurface of* $C^3$ *class, then* $\Omega$ *is a uniform* $C^3$ *domain.*

## 4. Estimates of Nonlinear Terms

First of all, we give an iteration scheme to prove Theorem 1 by the Banach fixed point theorem. For a given $h$ satisfying (9), let $H_h$ be a unique solution of Equation (8) satisfying (10) and (11). Let $\mathbf{U}_T$ be a space defined by

$$\begin{aligned}\mathbf{U}_T = \{(\mathbf{u}, \mathbf{G}, h) \mid &(\mathbf{u}, \mathbf{G}) \in H^1_p((0,T), L_q(\dot\Omega)^{2N}) \cap L_p((0,T), H^2_q(\dot\Omega)^{2N}),\\ &h \in L_p((0,T), W^{3-1/q}_q(\Gamma)) \cap H^1_p((0,T), W^{2-1/q}_q(\Gamma)),\\ &(\mathbf{u}, \mathbf{G}, h)|_{t=0} = (\mathbf{u}_0, \mathbf{G}_0, h_0) \quad \text{in } \dot\Omega \times \dot\Omega \times \Gamma,\\ &E_T(\mathbf{u}, \mathbf{G}, h) \leq L, \quad \|H_h\|_{L_\infty((0,T),H^1_\infty(\Omega))} \leq \delta\},\end{aligned} \tag{82}$$

where we have set

$$\begin{aligned}E_T(\mathbf{u}, \mathbf{G}, h) &= E^1_T(\mathbf{u}) + E^1_T(\mathbf{G}) + E^2_T(h) + \|\partial_t h\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))},\\ E^1_T(\mathbf{w}) &= \|\mathbf{w}\|_{L_p((0,T),H^2_q(\dot\Omega))} + \|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\dot\Omega))} \quad \mathbf{w} \in \{\mathbf{u}, \mathbf{G}\},\\ E^2_T(h) &= \|h\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))} + \|\partial_t h\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}.\end{aligned} \tag{83}$$

For initial data $\mathbf{u}_0$, $\mathbf{G}_0$, and $h_0$, we assume that

$$\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \leq B, \quad \|\mathbf{G}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \leq B, \quad \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)} \leq \epsilon. \tag{84}$$

Here, $B$ is a given positive number. Since we mainly consider the case where $\mathbf{u}_0$ and $\mathbf{G}_0$ are large, we may assume that $B > 1$ in the following, and we shall choose $L > 0$ large enough and $\epsilon > 0$ small enough eventually. So we may assume that $0 < \epsilon < 1 < L$. For any given $(\mathbf{u}, \mathbf{G}, h) \in \mathbf{U}_T$, let $(\mathbf{v}, \mathfrak{q}, \rho)$ be a solution of the problem:

$$\begin{cases} m\partial_t \mathbf{v} - \mathrm{Div}\, \mathbf{T}(\mathbf{v}, \mathfrak{q}) = \mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h), & \text{in } \dot\Omega \times (0,T),\\ \mathrm{div}\, \mathbf{v} = g(\mathbf{u}, H_h) = \mathrm{div}\, \mathbf{g}(\mathbf{u}, H_h) & \text{in } \dot\Omega \times (0,T),\\ \partial_t \rho + < \nabla_\Gamma \rho \perp \mathbf{u}_\kappa > - \mathbf{n} \cdot \mathbf{v}_+ = d(\mathbf{u}, H_h) & \text{on } \Gamma \times (0,T),\\ [[\mathbf{v}]] = 0, \quad [[\mathbf{T}(\mathbf{v}, \mathfrak{q})\mathbf{n}]] - \sigma(\Delta_\Gamma \rho + a\rho)\mathbf{n} = \mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_h), & \text{on } \Gamma \times (0,T),\\ \mathbf{v}_\pm = 0 & \text{on } S_\pm \times (0,T),\\ (\mathbf{v}, \rho)|_{t=0} = (\mathbf{u}_0, h_0) & \text{in } \dot\Omega \times \Gamma. \end{cases} \tag{85}$$

Let $\mathbf{H}$ be a solution of the problem:

$$\begin{cases} \mu\partial_t\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} = \mathbf{f}_2(\mathbf{u}, \mathbf{G}, H_h) & \text{in } \dot\Omega \times (0,T), \\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} = \mathbf{h}_2(\mathbf{u}, \mathbf{G}, H_h), \quad [[\mu\mathrm{div}\,\mathbf{H}]] = h_3(\mathbf{u}, \mathbf{G}, H_h) & \text{on } \Gamma \times (0,T), \\ [[\mu\mathbf{H}\cdot\mathbf{n}]] = k_1(\mathbf{G}, H_\rho), \quad [[\mathbf{H} - <\mathbf{H}, \mathbf{n}> \mathbf{n}]] = \mathbf{k}_2(\mathbf{G}, H_\rho) & \text{on } \Gamma \times (0,T), \\ \mathbf{n}_\pm \cdot \mathbf{H}_\pm = 0, \quad (\mathrm{curl}\,\mathbf{H}_\pm)\mathbf{n}_\pm = 0 & \text{on } S_\pm \times (0,T), \\ \mathbf{H}|_{t=0} = \mathbf{G}_0 & \text{in } \dot\Omega. \end{cases} \tag{86}$$

Notice that to define $\mathbf{H}$ we use not only $H_h$ but also $H_\rho$ unlike Padula and Solonnikov [4] to avoid their technical assumption that the velocity field is slightly more regular than the magnetic field.

In this section, we shall demonstrate the estimates of the nonlinear terms appearing in the right sides of Equations (85) and (86). Since $(\mathbf{u}, \mathbf{G}, h) \in \mathbf{U}_T$, we have

$$E_T(\mathbf{u}, \mathbf{G}, h) \leq L, \tag{87}$$

$$\|H_h\|_{L_\infty((0,T), H^1_\infty(\Omega))} \leq \delta. \tag{88}$$

Below, we assume that $2 < p < \infty$, $N < q < \infty$ and $2/p + N/q < 1$. We use the following inequalities which follow from Sobolev's inequalities.

$$\begin{aligned} \|f\|_{L_\infty(\dot\Omega)} &\leq C\|f\|_{H^1_q(\dot\Omega)}, \\ \|fg\|_{H^1_q(\dot\Omega)} &\leq C\|f\|_{H^1_q(\dot\Omega)}\|g\|_{H^1_q(\dot\Omega)}, \\ \|fg\|_{H^2_q(\dot\Omega)} &\leq C(\|f\|_{H^2_q(\dot\Omega)}\|g\|_{H^1_q(\dot\Omega)} + \|f\|_{H^1_q(\dot\Omega)}\|g\|_{H^2_q(\dot\Omega)}), \\ \|fg\|_{W^{1-1/q}_q(\Gamma)} &\leq C\|f\|_{W^{1-1/q}_q(\Gamma)}\|g\|_{W^{1-1/q}_q(\Gamma)}, \\ \|fg\|_{W^{2-1/q}_q(\Gamma)} &\leq C(\|f\|_{W^{2-1/q}_q(\Gamma)}\|g\|_{W^{1-1/q}_q(\Gamma)} + \|f\|_{W^{1-1/q}_q(\Gamma)}\|g\|_{W^{2-1/q}_q(\Gamma)}). \end{aligned} \tag{89}$$

For any $C^k$ function, $f(u)$, defined for $|u| \leq \sigma$, we consider a composite function $f(u(x))$, and then for $N < q < \infty$, we have

$$\begin{aligned} \|\nabla(f(u)vw)\|_{L_q(\dot\Omega)} &\leq C\|(f, f')\|_{L_\infty}(1 + \|\nabla u\|_{L_q(\dot\Omega)})\|v\|_{H^1_q(\dot\Omega)}\|w\|_{H^1_q(\dot\Omega)}; \\ \|\nabla^2(f(u)vw)\|_{L_q(\dot\Omega)} &\leq C\|(f, f', f'')\|_{L_\infty}(\|\nabla v\|_{H^1_q(\dot\Omega)}\|w\|_{H^1_q(\dot\Omega)} + \|v\|_{H^1_q(\dot\Omega)}\|\nabla w\|_{H^1_q(\dot\Omega)} \\ &\quad + \|\nabla u\|_{H^1_q(\dot\Omega)}(1 + \|\nabla u\|_{L_q(\dot\Omega)}))\|v\|_{H^1_q(\dot\Omega)}\|w\|_{H^1_q(\dot\Omega)}), \end{aligned} \tag{90}$$

provided that $\|u\|_{L_\infty(\dot\Omega)} \leq \sigma$. We use the following estimate of the time trace proved by a real interpolation theorem:

$$\begin{aligned} &\|\mathbf{w}\|_{L_\infty((0,T), B^{2(1-1/p)}_{q,p}(\dot\Omega))} \\ &\quad \leq C\{\|\mathbf{w}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|\mathbf{w}\|_{L_p((0,T), H^2_q(\dot\Omega))} + \|\partial_t\mathbf{w}\|_{L_p((0,T), L_q(\dot\Omega))}\} \leq C(B + L) \end{aligned} \tag{91}$$

for $\mathbf{w} \in \{\mathbf{u}, \mathbf{G}\}$,

$$\begin{aligned} &\|h\|_{L_\infty((0,T), B^{3-1/p-1/q}_{q,p}(\Gamma))} \\ &\quad \leq C\{\|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)} + \|h\|_{L_p((0,T), W^{3-1/q}_q(\Gamma))} + \|\partial_t h\|_{L_p((0,T), W^{2-1/q}_q(\Gamma))}\} \leq CL. \end{aligned} \tag{92}$$

Then, we have

$$\|h\|_{L_\infty((0,T), W^{2-1/q}_q(\Gamma))} \leq \|h_0\|_{W^{2-1/q}_q(\Gamma)} + T^{1/p'}\|\partial_t h\|_{L_p((0,T), W^{2-1/q}_q(\Gamma))} \leq \epsilon + T^{1/p'}L.. \tag{93}$$

In what follows, we assume that $0 < \epsilon = T = \kappa < 1$ and $1 \le B, L$. In particular, $\epsilon + LT^{1/p'} \le T + LT^{1/p'} \le 2LT^{p1/p'}$. We assume that $LT^{1/p'} \le 1$, and so by (93),

$$\|h\|_{L_\infty((0,T),W_q^{2-1/q}(\Gamma))} \le LT^{1/p'}, \quad \|h\|_{L_\infty((0,T),W_q^{2-1/q}(\Gamma))} \le 1. \tag{94}$$

We first estimate $\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h)$. In view of (22), we may write

$$\begin{aligned}\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h) = \mathbf{V}_{\mathbf{f}_1}(\cdot, \bar\nabla H_h)(\bar\nabla H_h \otimes (\partial_t \mathbf{u}, \nabla^2 \mathbf{u}) \\ + \partial_t H_h \otimes \nabla \mathbf{u} + \mathbf{u} \otimes \nabla \mathbf{u} + \bar\nabla^2 H_h \otimes \nabla \mathbf{u} + \mathbf{G} \otimes \nabla \mathbf{G}),\end{aligned} \tag{95}$$

where $\mathbf{V}_{\mathbf{f}_1}(y, \mathbf{K})$ is a matrix of bounded functions defined on $\Omega \times \{\mathbf{K} \in \mathbb{R}^{N+1} \mid |\mathbf{K}| \le \delta\}$. Applying (11), (88), and (89), we have

$$\begin{aligned}&\|\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h)\|_{L_q(\dot\Omega)} \le C\{\|h\|_{W_q^{2-1/q}(\Gamma)}\|(\partial_t \mathbf{u}, \nabla^2 \mathbf{u})\|_{L_q(\dot\Omega)} + \|\partial_t h\|_{W_q^{1-1/q}(\Gamma)}\|\nabla \mathbf{u}\|_{L_q(\dot\Omega)} \\ &+ \|\mathbf{u}\|^2_{H_q^1(\dot\Omega)} + \|h\|_{W_q^{2-1/q}(\dot\Omega)}\|\nabla \mathbf{u}\|_{H_q^1(\Omega)} + \|\mathbf{G}\|^2_{H_q^1(\dot\Omega)}\}.\end{aligned}$$

For a maximal regularity term $f$ and a lower order term $g$, we have

$$\|fg\|_{L_p((0,T))} \le \|f\|_{L_p((0,T))}\|g\|_{L_\infty((0,T))}.$$

Only for a lower order term $g$, we use the estimate

$$\|g\|_{L_p((0,T))} \le T^{1/p}\|g\|_{L_\infty((0,T))}.$$

Thus, using (11), we have

$$\begin{aligned}&\|\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h)\|_{L_p((0,T),L_q(\dot\Omega))} = \|\|\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h)\|_{L_q(\dot\Omega)}\|_{L_p((0,T))} \\ &\le C\{\|h\|_{L_\infty((0,T),W_q^{2-1/q}(\Gamma))}(\|\partial_t \mathbf{u}\|_{L_p((0,T),L_q(\dot\Omega))} + \|\mathbf{u}\|_{L_p((0,T),H_q^2(\dot\Omega))}) \\ &+ T^{1/p}(\|\partial_t h\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))}\|\mathbf{u}\|_{L_\infty((0,T),H_q^1(\dot\Omega))} \\ &+ \|\mathbf{u}\|^2_{L_\infty((0,T),H_q^1(\dot\Omega))} + \|\mathbf{G}\|^2_{L_\infty((0,T),H_q^1(\dot\Omega))})\}.\end{aligned} \tag{96}$$

By (11), (87), (84), (91), and (92), we arrive at

$$\begin{aligned}&\|\mathbf{w}\|_{L_\infty((0,T),H_q^1(\dot\Omega))} \le C(B+L) \quad \text{for } \mathbf{w} \in \{\mathbf{u}, \mathbf{G}\}, \\ &\|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\dot\Omega))} + \|\mathbf{w}\|_{L_p((0,T),H_q^2(\dot\Omega))} \le L \quad \text{for } \mathbf{w} \in \{\mathbf{u}, \mathbf{G}\}, \\ &\|\partial_t h\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))} \le CE_T(\mathbf{u}, \mathbf{G}, h) \le CL, \\ &\|\partial_t h\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} + \|h\|_{L_p((0,T),W_q^{3-1/q}(\Gamma))} \le L.\end{aligned} \tag{97}$$

Thus, by (94) and (97)

$$\|\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h)\|_{L_p((0,T),L_q(\dot\Omega))} \le C\{T^{1/p'}L^2 + T^{1/p}(L+B)^2\}.$$

Since $1/p' > 1/p$ as follows from $1 < p' = p/(p-1) < 2 < p < \infty$, we have

$$\|\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h)\|_{L_p((0,T),L_q(\dot\Omega))} \le CT^{1/p}(B+L)^2. \tag{98}$$

We next estimate $d(\mathbf{u}, H_h)$ given in (50). We shall prove that

$$\begin{aligned}
\|d(\mathbf{u}, H_h)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))} &\leq CL(B+L)T^{1/p'}, \\
\|d(\mathbf{u}, H_h)\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} &\leq C_s L^2(B+L)T^{\frac{s}{p'(1+s)}},
\end{aligned} \tag{99}$$

where $s \in (0, 1-2/p)$. Here and in the following, $C_s$ is a generic constant depending on $s$, whose value may change from line to line.

In fact, by (11), (88), (89), and (90),

$$\begin{aligned}
&\|d(\mathbf{u}, H_h)\|_{W_q^{1-1/q}(\Gamma)} \leq C\{\|h\|_{W_q^{2-1/q}(\Gamma)}\|\mathbf{u}_+ - \mathbf{u}_\kappa\|_{H_q^1(\Omega_+)} \\
&\quad + (1 + \|h\|_{W_q^{2-1/q}(\Gamma)})\|h\|^2_{W_q^{2-1/q}(\Gamma)}(\|\mathbf{u}\|_{H_q^1(\Omega_+)} + \|\partial_t h\|_{W_q^{1-1/q}(\Gamma)})\}; \\
&\|d(\mathbf{u}, H_h)\|_{W_q^{2-1/q}(\Gamma)} \leq C\{\|h\|_{W_q^{3-1/q}(\Gamma)}\|\mathbf{u} - \mathbf{u}_\kappa\|_{H_q^1(\Omega_+)} + \|h\|_{W_q^{2-1/q}(\Gamma)}\|\mathbf{u} - \mathbf{u}_\kappa\|_{H_q^2(\Omega_+)} \\
&\quad + \|h\|_{W_q^{3-1/q}(\Gamma)}(1 + \|h\|_{W_q^{2-1/q}(\Gamma)})\|h\|^2_{W_q^{2-1/q}(\Gamma)}(\|\mathbf{u}\|_{H_q^1(\Omega_+)} + \|\partial_t h\|_{W_q^{1-1/q}(\Gamma)}) \\
&\quad + (\|\mathbf{u}\|_{H_q^2(\Omega_+)} + \|\partial_t h\|_{W_q^{2-1/q}(\Gamma)})\|h\|^2_{W_q^{2-1/q}(\Gamma)} \\
&\quad + (\|\mathbf{u}\|_{H_q^1(\Omega_+)} + \|\partial_t h\|_{W_q^{1-1/q}(\Gamma)})\|h\|_{W_q^{3-1/q}(\Gamma)}\|h\|_{W_q^{2-1/q}(\Gamma)}\}.
\end{aligned} \tag{100}$$

By (48), we have

$$\begin{aligned}
\|\mathbf{u}_+ - \mathbf{u}_\kappa\|_{L_\infty((0,T),H_q^1(\Omega_+))} &\leq C(\|\mathbf{u}\|_{L_\infty((0,T),H_q^1(\dot\Omega))} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}) \\
&\leq C(L+B),
\end{aligned} \tag{101}$$

and so by (94) and (97)

$$\begin{aligned}
\|d(\mathbf{u}, H_h)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))} &\leq C\{T^{1/p'}L(B+L) + (1 + T^{1/p'}L)(T^{1/p'}L)^2(B+L)\} \\
&\leq CT^{1/p'}L(B+L),
\end{aligned}$$

which shows the first inequality in (99).

To prove the second inequality in (99), we use the estimates:

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_\kappa\|_{L_p((0,T),H_q^2(\Omega_+))} &\leq C(L+B), \\
\|\mathbf{u} - \mathbf{u}_\kappa\|_{L_\infty((0,T),H_q^1(\Omega_+))} &\leq C_s T^{\frac{s}{p'(1+s)}}(L+B).
\end{aligned} \tag{102}$$

Here, $s$ is a fixed constant $0 < s < 1 - 2/p$. In fact, by (48) and (84)

$$\|\mathbf{u} - \mathbf{u}_\kappa\|_{H_q^2(\Omega_+)} \leq C(\|\mathbf{u}(\cdot, t)\|_{H_q^2(\dot\Omega)} + \kappa^{-1/p}B).$$

Consequently, by (87) and (97), we have

$$\|\mathbf{u} - \mathbf{u}_\kappa\|_{L_p((0,T),H_q^2(\Omega_+))} \leq C(L + T^{1/p}\kappa^{-1/p}B) \leq C(L+B),$$

because we have taken $\kappa = T$. This shows the first inequality in (102). For $t \in (0, T)$ by (47), (84), and (87)

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\kappa\|_{L_q(\Omega_+)} &\leq \|\mathbf{u} - \mathbf{u}_0\|_{L_q(\Omega+)} + \|\mathbf{u}_0 - \mathbf{u}_\kappa\|_{L_q(\Omega_+)} \\ &\leq \int_0^T \|\mathbf{u}_t(\cdot, t)\|_{L_q(\Omega_+)}\, dt + \frac{1}{\kappa}\int_0^\kappa \|T_0(s)\tilde{\mathbf{u}}_0^+ - \mathbf{u}_0\|_{L_q(\Omega_+)}\, ds \\ &\leq T^{1/p'} L + \frac{1}{\kappa}\int_0^\kappa \Big(\int_0^s \|\partial_r T_0(r)\tilde{\mathbf{u}}_0^+\|_{L_q(\Omega_+)}\, dr\Big)\, ds \\ &\leq T^{1/p'} L + C\kappa^{1/p'} B \leq C(L+B)T^{1/p'}. \end{aligned}$$

By real interpolation,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\kappa\|_{H^1_q(\Omega_+)} &\leq C_s \|\mathbf{u} - \mathbf{u}_\kappa\|^{s/(1+s)}_{L_q(\Omega_+)} \|\mathbf{u} - \mathbf{u}_\kappa\|^{1/(1+s)}_{W^{1+s}_q(\Omega_+)} \\ &\leq C_s \|\mathbf{u} - \mathbf{u}_\kappa\|^{s/(1+s)}_{L_q(\Omega_+)} \|\mathbf{u} - \mathbf{u}_\kappa\|^{1/(1+s)}_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \end{aligned}$$

for any $s \in (0, 1 - 2/p))$, which, combined with (48) and (91), yields the second inequality in (102).

Applying (94), (97), and (102) to the second inequality in (100) yields

$$\begin{aligned} &\|d(\mathbf{u}, H_h)\|_{L_p((0,T), W^{2-1/q}_q(\Gamma))} \\ &\quad\leq C\{L(B+L)T^{1/p'} + C_s T^{\frac{s}{p'(1+s)}} L(L+B) \\ &\qquad + LLT^{1/p'}(L+B) + LLT^{1/p'} + (L+B)LLt^{1/p'}\} \\ &\quad\leq CT^{\frac{s}{p'(1+s)}} L^2(L+B), \end{aligned}$$

which proves the second inequality in (99).

We now estimate $g(\mathbf{u}, H_h)$, $\mathbf{g}(\mathbf{u}, H_h)$ and $\mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_h)$ given in (27), and (65), respectively. We have to extend them to the whole time line $\mathbb{R}$. Let $\mathcal{E}_\mp$ be the extension maps given in Section 2 (cf. (51) and (52)). For $\mathbf{w} \in \{\mathbf{u}, \mathbf{G}\}$, let $\tilde{\mathbf{w}}_{0\pm} \in B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)$ be extensions of $\mathbf{w}_{0\pm}$ to $\mathbb{R}^N$ such that

$$\tilde{\mathbf{w}}_{0\pm} = \mathcal{E}_\mp(\mathbf{w}_{0\pm}) \quad \text{in } \Omega, \quad \|\tilde{\mathbf{w}}_{0\pm}\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} \leq C\|\mathbf{w}_{0\pm}\|_{B^{2(1-1/p)}_{q,p}(\Omega_\pm)} \leq CB.$$

Let $\gamma_0$ be a large positive number appearing in Theorems 2 and 3, and we fix $\gamma_1$ in such a way that $\gamma_1 > \gamma_0$. Let $T_v(t)\mathbf{w}_{0\pm}$ be defined by setting

$$T_v(t)\mathbf{w}_{0\pm} = e^{-(2\gamma_1 - \Delta)t}\tilde{\mathbf{w}}_{0\pm} = \mathcal{F}^{-1}[e^{-(|\xi|^2 + 2\gamma_1)t}\mathcal{F}[\tilde{\mathbf{w}}_{0\pm}](\xi)].$$

In particular, $T_v(0)\mathbf{w}_{0\pm} = \mathbf{w}_{0\pm}$ in $\Omega_\pm$, $T_v(0)\mathbf{w}_{0\pm} = \mathcal{E}_\mp(\mathbf{w}_{0\pm})$ in $\Omega$, and

$$\|e^{\gamma_1 t}T_v(t)\mathbf{w}_{0\pm}\|_{H^1_p((0,\infty), L_q(\Omega))} + \|e^{\gamma_1 t}T_v(t)\mathbf{w}_{0\pm}\|_{L_p((0,\infty), H^2_q(\Omega))} \leq CB. \tag{103}$$

We also construct a similar extension for $H_h$. Let $\mathbf{W}$, $P$, and $\Xi$ be solutions of the equations:

$$\begin{aligned} \partial_t \mathbf{W} + \lambda_0 \mathbf{W} - \mathrm{Div}\, \mathbf{T}(\mathbf{W}, P) = 0, \quad \mathrm{div}\, \mathbf{W} = 0 \quad &\text{in } \dot\Omega \times (0, \infty), \\ \partial_t \Xi + \lambda_0 \Xi - \mathbf{W} \cdot \mathbf{n} = 0 \quad &\text{on } \Gamma \times (0, \infty), \\ [[\mathbf{T}(\mathbf{W}, P)\mathbf{n}]] - \sigma(\Delta_\Gamma \Xi)\mathbf{n} = 0, \quad [[\mathbf{W}]] = 0 \quad &\text{on } \Gamma \times (0, \infty), \\ \mathbf{W}_\pm = 0 \quad &\text{on } S_\pm \times (0, \infty), \\ (\mathbf{W}, \Xi)|_{t=0} = (0, h_0) \quad &\text{in } \dot\Omega \times \Gamma. \end{aligned}$$

For large $\lambda_0 > 0$, we know the unique existence of $\mathbf{W}$, $P$, and $\Xi$ such that

$$\mathbf{W} \in H^1_p((0,\infty), L_q(\dot\Omega)^N) \cap L_p((0,\infty), H^2_q(\dot\Omega)^N),$$
$$\Xi \in H^1_p((0,\infty), W^{2-1/q}_q(\Gamma)) \cap L_p((0,\infty), W^{3-1/q}_q(\Gamma)),$$

and the following estimate:

$$\begin{aligned}
&\|e^{\gamma_1 t}\mathbf{W}\|_{H^1_p((0,\infty),L_q(\dot\Omega))} + \|e^{\gamma_1 t}\mathbf{W}\|_{L_p((0,\infty),H^2_q(\dot\Omega))} + \|e^{\gamma_1 t}\mathbf{W}\|_{L_\infty((0,\infty),W^{2(1-1/p)}_{q,p}(\dot\Omega))} \\
&\quad + \|e^{\gamma_1 t}\Xi\|_{H^1_p((0,\infty),W^{2-1/q}_q(\Gamma))} + \|e^{\gamma_1 t}\Xi\|_{L_p((0,\infty),W^{3-1/q}_q(\Gamma))} + \|e^{\gamma_1 t}\Xi\|_{H^1_\infty(0,\infty),W^{1-1/q}_q(\Gamma))} \\
&\quad \le C\|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)} \le C\epsilon
\end{aligned}$$

holds. Let $T_h(t)h_0 = H_\Xi(x,t)$, where $H_\Xi$ is a unique solution of (8) with $h = \Xi$,. then by (11) we have

$$\begin{aligned}
&\|e^{\gamma_1 t}T_h(\cdot)h_0\|_{H^1_p((0,\infty),H^2_q(\Omega))} + \|e^{\gamma_1 t}T(\cdot)h_0\|_{L_p((0,\infty),H^3_q(\Omega))} \\
&\qquad + \|e^{\gamma_1 t}T_h(\cdot)h_0\|_{L_\infty((0,\infty),B^{3-1/p}_{q,p}(\Omega))} + \|e^{\gamma_1 t}T_h(\cdot)h_0\|_{H^1_\infty((0,\infty),H^1_q(\Omega))} \le C\epsilon.
\end{aligned} \tag{104}$$

In what follows, a generic constant $C$ depends on $\gamma_1$ when we use (103) and (104), but $\gamma_1$ is eventually fixed in such a way that the estimates given in Theorems 2 and 3 hold, and so we do not mention the dependence on $\gamma_1$.

For a function $f(t)$ defined on $(0,T)$, we define an extension $e_T[f]$ of $f$ by setting

$$e_T[f] = \begin{cases} 0 & \text{for } t < 0, \\ f(t) & \text{for } 0 < t < T, \\ f(2T-t) & \text{for } T < t < 2T, \\ 0 & \text{for } t > 2T. \end{cases}$$

Obviously, $e_T[f] = f$ for $t \in (0,T)$ and $e_T[f]$ vanishes for $t \notin (0,2T)$. Moreover, if $f|_{t=0} = 0$, then

$$\partial_t e_T[f] = \begin{cases} 0 & \text{for } t < 0, \\ \partial_t f(t) & \text{for } 0 < t < T, \\ -(\partial_t f)(2T-t) & \text{for } T < t < 2T, \\ 0 & \text{for } t > 2T. \end{cases} \tag{105}$$

If $f \in L_p((0,T), X)$ with some Banach space $X$ and $f|_{t=0} = 0$, then

$$\|e_T[f]\|_{L_p(\mathbb{R},X)} \le 2\|f\|_{L_p((0,T),X)} \quad (1 \le p \le \infty),$$
$$\|e_T[f]\|_{L_p(\mathbb{R},X)} \le 2T^{1/p}\|f\|_{L_\infty((0,T),X)} \quad (1 \le p < \infty).$$

Moreover, if $f|_{t=0} = 0$, then $e_T[f](t) = \int_0^t \partial_t e_T[f]\,ds$, and so

$$\|e_T[f]\|_{L_\infty(\mathbb{R},X)} \le 2(2T)^{1/p'}\|f\|_{L_p((0,T),X)} \quad (1 < p < \infty,\ \ p' = p/(p-1)),$$

because $e_T[f]$ vanishes for $t \notin (0,2T)$.

Let $\psi \in C^\infty(\mathbb{R})$ equal one for $t > -1$ and zero for $t < -2$. Under these preparations, for $\mathbf{w} \in \{\mathbf{u}, \mathbf{G}\}$ and $H_h$, we define the extensions $\mathcal{E}_1[\mathbf{w}_\pm]$, $\mathcal{E}_1[tr[\mathbf{w}]]$, and $\mathcal{E}_2[H_h]$ by setting

$$\begin{aligned}
\mathcal{E}_1[\mathbf{w}_\pm] &= e_T[\mathbf{u}_\pm - T_v(t)\mathbf{w}_{0\pm}] + \psi(t)T_v(|t|)\mathbf{w}_{0\pm}, \\
\mathcal{E}_1[tr[\mathbf{w}]] &= e_T[tr[\mathbf{w}] - T_v(t)tr[\mathbf{w}_0]] + \psi(t)T_v(|t|)tr[\mathbf{w}_0], \\
\mathcal{E}_2[H_h] &= e_T[H_h - T_h(t)h_0] + \psi(t)T_h(|t|)h_0.
\end{aligned} \tag{106}$$

Here, we have set $tr[\mathbf{w}_0] = \tilde{\mathbf{w}}_{0+} - \tilde{\mathbf{w}}_{0-}$. Notice that $\mathbf{w}_\pm - T_v(t)\mathbf{w}_{0\pm} = 0$ for $t = 0$, $tr[\mathbf{w}] - T_v(t)tr[\mathbf{w}_0]$ for $t = 0$, and $H_h - T_h(t)h_0 = 0$ for $t = 0$. Obviously,

$$\mathcal{E}_1[\mathbf{u}_\pm] = \mathbf{u}_\pm, \quad \mathcal{E}_1[tr[\mathbf{w}]] = tr[\mathbf{w}], \quad \mathcal{E}_2[H_h] = H_h \quad \text{for } 0 < t < T. \tag{107}$$

By (97), (103), and (104), we have

$$\begin{aligned}
&\|e^{-\gamma t}\mathcal{E}_1[\mathbf{w}]\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))} + \|e^{-\gamma t}\partial_t\mathcal{E}_1[\mathbf{w}]\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \leq C(e^{2(\gamma-\gamma_1)}B + L),\\
&\|e^{-\gamma t}\mathcal{E}_1[\mathbf{w}]\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))} \leq Ce^{2(\gamma-\gamma_1)}B + C_s(B+L)T^{\frac{s}{p'(1+s)}},\\
&\|\mathcal{E}_2[H_h]\|_{L_p(\mathbb{R},H^3_q(\dot\Omega))} + \|\mathcal{E}_2[H_h]\|_{H^1_p(\mathbb{R},H^2_q(\dot\Omega))} + \|\partial_t\mathcal{E}_2[H_h]\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))} \leq C(\epsilon + L) \leq 2CL,\\
&\|\mathcal{E}_2[H_h]\|_{L_\infty(\mathbb{R},H^2_q(\dot\Omega))} \leq C(\epsilon + LT^{1/p'}) \leq 2CLT^{1/p'},
\end{aligned} \tag{108}$$

where $\mathbf{w} \in \{\mathbf{u}, \mathbf{G}, tr[\mathbf{u}], tr[\mathbf{G}]\}$. In fact, the first and third inequalites in (108) follow from (103), (104), and (87). To prove the second inequality in (108), we observe that

$$\begin{aligned}
\|e_T[\mathbf{w} - T_v(t)\mathbf{w}_0]\|_{L_q(\dot\Omega)} &\leq \int_0^t \|\partial_s e_T[\mathbf{w} - T_v(t)\mathbf{w}_0]\|_{L_q(\dot\Omega)}\,ds\\
&\leq T^{1/p'}(\|\partial_t\mathbf{w}\|_{L_p((0,T),L_q(\dot\Omega))} + \|\partial_t T_v(t)\mathbf{w}_0\|_{L_p(0,2T),L_q(\dot\Omega))}\\
&\leq T^{1/p'}(L+B);\\
\|e_T[\mathbf{w} - T_v(t)\mathbf{w}_0]\|_{W^{1+s}_q(\dot\Omega)} &\leq C_s\|\|e_T[\mathbf{w} - T_v(t)\mathbf{w}_0]\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \leq C_s(B+L),
\end{aligned}$$

for any $s \in (0, 1-2/p)$. Thus, using the inequality:

$$\|v\|_{H^1_q(\dot\Omega)} \leq C_s\|v\|^{s/(1+s)}_{L_q(\dot\Omega)}\|v\|^{1/(1+s)}_{W^{1+s}_q(\dot\Omega)},$$

we have the second inequality in (108). By (104) and (94),

$$\begin{aligned}
\|\psi(t)T_h(|t|)h_0\|_{L_\infty(\mathbb{R},H^1_\infty(\Omega))} &\leq C\epsilon,\\
\|e_T[H_h - T_h(t)h_0]\|_{L_\infty(\mathbb{R},H^1_\infty(\Omega))} &\leq C(\|H_h\|_{L_\infty((0,T),H^2_q(\Omega))} + \|T(\cdot)h_0\|_{L_\infty((0,\infty),H^2_q(\Omega))})\\
&\leq C(\epsilon + T^{1/p'}L),
\end{aligned}$$

and so we have the last inequality in (108).

From (106), we see that $\|\mathcal{E}_2[H_h]\|_{H^1_\infty(\Omega)} \leq C(\|H_h\|_{H^1_\infty(\Omega)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)})$. Choosing $\delta$ in (14) and $\epsilon$ in Theorem 1 smaller, we may assume that

$$\sup_{t\in\mathbb{R}}\|\mathcal{E}_2[H_h]\|_{H^1_\infty(\Omega)} \leq \delta. \tag{109}$$

In addition,

$$\|\mathcal{E}_2[H_h]\|_{L_\infty(\mathbb{R},H^2_q(\dot\Omega))} \leq CLT^{1/p'}, \quad \|\mathcal{E}_2[H_h]\|_{L_\infty(\mathbb{R},H^2_q(\dot\Omega))} \leq 1. \tag{110}$$

To estimate $H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega))$ norm, we use the following lemma.

**Lemma 3.** *Let $1 < p < \infty$ and $N < q < \infty$. Let*

$$f \in L_\infty(\mathbb{R}, H^1_q(\dot\Omega)) \cap H^1_\infty(\mathbb{R}, L_q(\dot\Omega)), \quad g \in H^{1/2}_p(\mathbb{R}, H^1_q(\dot\Omega)) \cap L_p(\mathbb{R}, H^1_q(\dot\Omega)).$$

*Then, we have*

$$\|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|fg\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}$$
$$\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))})^{1/2}\|f\|^{1/2}_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}(\|g\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|g\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}).$$

**Proof.** To prove Lemma 3, we use the fact that

$$H^{1/2}_p(\mathbb{R},L_q(\dot\Omega)) \cap L_p(\mathbb{R},H^{1/2}_q(\dot\Omega)) = (L_p(\mathbb{R},L_q(\dot\Omega)), H^1_p(\mathbb{R},L_q(\dot\Omega)) \cap L_p(\mathbb{R},H^1_q(\dot\Omega)))_{[1/2]}$$

where $(\cdot,\cdot)_{[1/2]}$ denotes a complex interpolation functor of order $1/2$. We have

$$\begin{aligned}&\|fg\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))} + \|fg\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\\ &\quad\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))}\|g\|_{L_p((\mathbb{R},H^1_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\|g\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))})\\ &\quad\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))})(\|g\|_{L_p((\mathbb{R},H^1_q(\dot\Omega))} + \|g\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))}).\end{aligned}$$

Moreover,

$$\|fg\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \leq C\|f\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\|g\|_{L_p(\mathbb{R},L_q(\dot\Omega))}.$$

Thus, by complex interpolation, we have

$$\begin{aligned}&\|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|fg\|_{L_p(\mathbb{R},H^{1/2}_q(\dot\Omega))}\\ &\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))})^{1/2}\\ &\qquad\times \|f\|^{1/2}_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}(\|g\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|g\|_{L_p(\mathbb{R},H^{1/2}_q(\dot\Omega))}).\end{aligned}$$

Moreover, we have

$$\|fg\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \leq C\|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\|g\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}.$$

Thus, combining these two inequalities gives the required estimate, which completes the proof of Lemma 3. □

**Lemma 4.** *Let $1 < p, q < \infty$. Then,*

$$H^1_p(\mathbb{R},L_q(\dot\Omega)) \cap L_p(\mathbb{R},H^2_q(\dot\Omega)) \subset H^{1/2}_p(\mathbb{R},H^1_q(\dot\Omega))$$

*and*

$$\|u\|_{H^{1/2}_p(\mathbb{R},H^1_q(\dot\Omega))} \leq C(\|u\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))} + \|\partial_t u\|_{L_p(\mathbb{R},L_q(\dot\Omega))}).$$

**Proof.** For a proof, see Shibata [10] (Proposition 1). □

We now estimate $\mathbf{h}_1(\mathbf{u},\mathbf{G},H_h)$. In view of (65), we define an extension of $\mathbf{h}_1(\mathbf{u},\mathbf{G},H_h)$ to the whole time interval $\mathbb{R}$ by setting $\tilde{\mathbf{h}}_1(\mathbf{u},\mathbf{G},H_h) = A^1 + A^2 + A^3$ with

$$\begin{aligned}A^1 &= \mathbf{V}^1_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_h])\bar\nabla\mathcal{E}_2[H_h]\otimes\nabla\mathcal{E}_1[tr[\mathbf{u}]],\\ A^2 &= a(\cdot)\mathcal{E}_1[tr[\mathbf{G}]]\otimes\mathcal{E}_1[tr[\mathbf{G}]] + \mathbf{V}^2_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_h])\bar\nabla\mathcal{E}_2[H_h])\otimes\mathcal{E}_1[tr[\mathbf{G}]]\otimes\mathcal{E}_1[tr[\mathbf{G}]],\\ A^3 &= \mathbf{V}_s(\cdot,\bar\nabla\mathcal{E}_2[H_h])\bar\nabla\mathcal{E}_2[H_h]\otimes\bar\nabla^2\mathcal{E}_2[H_h].\end{aligned} \tag{111}$$

Obviously, $\tilde{\mathbf{h}}_1(\mathbf{u}, \mathbf{G}, H_h) = \mathbf{h}_1(\mathbf{u}, \mathbf{G}, H_h)$ for $t \in (0, T)$. To estimate $A^1$, for notational simplicity we set $\mathcal{V}^1 = \mathbf{V}^1_{\mathbf{h}}(\cdot, \bar\nabla\mathcal{E}_2[H_h])\bar\nabla\mathcal{E}_2[H_h]$. By (108) and (109),

$$\|\partial_t\mathcal{V}^1\|_{L_\infty(\mathbb{R}, L_q(\dot\Omega))} \leq C\|\partial_t\mathcal{E}_2[H_h]\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))} \leq CL,$$
$$\|\mathcal{V}^1\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))} \leq C\|\mathcal{E}_2[H_h]\|_{L_\infty(\mathbb{R}, H^2_q(\dot\Omega))} \leq CLT^{1/p'},$$

and so, we have

$$(\|\partial_t\mathcal{V}^1\|_{L_\infty(\mathbb{R}, L_q(\dot\Omega))} + \|\mathcal{V}^1\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))})^{1/2}\|\mathcal{V}^1\|^{1/2}_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))} \leq CLT^{1/(2p')}. \tag{112}$$

Thus, by (108) and (112) and Lemmas 3 and 4, we have

$$\begin{aligned}
&\|e^{-\gamma t}A^1\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega))} + \|e^{-\gamma t}A^1\|_{L_p(\mathbb{R}, H^1_q(\dot\Omega))} \\
&\quad\leq CLT^{1/(2p')}(\|e^{-\gamma t}\nabla\mathcal{E}_1[tr[\mathbf{u}]]\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega))} + \|e^{-\gamma t}\nabla\mathcal{E}_1[tr[\mathbf{u}]]\|_{L_p(\mathbb{R}, H^1_q(\dot\Omega))} \\
&\quad\leq CT^{1/(2p')}L(e^{2(\gamma-\gamma_1)}B + L).
\end{aligned} \tag{113}$$

Since

$$\|e^{-\gamma t}\bar\nabla^2\mathcal{E}_2[H_h]\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega))} \leq \|e^{-\gamma t}\mathcal{E}_2[H_h]\|_{H^1_p(\mathbb{R}, H^2_q(\dot\Omega))} \leq C(e^{2(\gamma-\gamma_1)}\epsilon + L);$$
$$\|e^{-\gamma t}\bar\nabla^2\mathcal{E}_2[H_h]\|_{L_p(\mathbb{R}, H^1_q(\dot\Omega))} \leq C\|e^{-\gamma t}\mathcal{E}_2[H_h]\|_{L_p(\mathbb{R}, H^3_q(\dot\Omega))} \leq C(e^{2(\gamma-\gamma_1)}\epsilon + L)$$

as follows from (87), the third formula of (104) and (106), employing the same argument as in proving (113), we have

$$\|e^{-\gamma t}A^3\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega))} + \|e^{-\gamma t}A^3\|_{L_p(\mathbb{R}, H^1_q(\dot\Omega))} \leq CT^{1/(2p')}L(e^{2(\gamma-\gamma_1)}\epsilon + L). \tag{114}$$

We now estimate $A^2$. For this purpose, we use the following esitmate which follows from complex interpolation theory:

$$\|f\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega))} \leq C\|f\|^{1/2}_{H^1_p(\mathbb{R}, L_q(\dot\Omega))}\|f\|^{1/2}_{L_p(\mathbb{R}, L_q(\dot\Omega))}. \tag{115}$$

Let

$$A^2_1 = \mathcal{E}_1[tr[\mathbf{G}]] \otimes \mathcal{E}_1[tr[\mathbf{G}]], \quad A^2_2 = \mathbf{V}^2_{\mathbf{h}}(\cdot, \bar\nabla\mathcal{E}_2[H_h])\bar\nabla\mathcal{E}_2[H_h] \otimes A^2_1.$$

We further divide $A^2_1 = A^2_{11} + A^2_{12} + A^2_{21} + A^2_{22}$, where

$$\begin{aligned}
&A^2_{11} = \mathcal{A}_1 \otimes \mathcal{A}_1, \quad A^2_{12} = \mathcal{A}_1 \otimes \mathcal{A}_2, \quad A^2_{21} = \mathcal{A}_2 \otimes \mathcal{A}_1, \quad A^2_{22} = \mathcal{A}_2 \otimes \mathcal{A}_2, \\
&\mathcal{A}_1 = \psi(t)T_v(|t|)tr[\mathbf{G}_0], \quad \mathcal{A}_2 = e_T[tr[\mathbf{G}] - T_v(t)tr[\mathbf{G}_0]].
\end{aligned}$$

Using (89), we obtain

$$\begin{aligned}
&\|e^{-\gamma t}A^2_{11}\|_{H^i_p(\mathbb{R}, L_q(\dot\Omega))} \leq C\|e^{-\gamma t}\mathcal{A}_1\|_{H^i_p(\mathbb{R}, L_q(\dot\Omega))}\|\mathcal{A}_1\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))} \quad (i = 0, 1), \\
&\|e^{-\gamma t}A^2_{12}\|_{H^1_p(\mathbb{R}, L_q(\dot\Omega))} \leq C(\|\mathcal{A}_1\|_{H^1_p(\mathbb{R}, L_q(\dot\Omega))}\|\mathcal{A}_2\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))} \\
&\qquad\qquad + \|\mathcal{A}_1\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))}\|\mathcal{A}_2\|_{H^1_p(\mathbb{R}, L_q(\dot\Omega))}), \\
&\|e^{-\gamma t}A^2_{12}\|_{L_p(\mathbb{R}, L_q(\dot\Omega))} \leq C\|\mathcal{A}_1\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))}\|\mathcal{A}_2\|_{L_p(\mathbb{R}, L_q(\dot\Omega))}, \\
&\|e^{-\gamma t}A^2_{22}\|_{H^i_p(\mathbb{R}, L_q(\dot\Omega))} \leq C\|\mathcal{A}_2\|_{H^i_p(\mathbb{R}, L_q(\dot\Omega))}\|\mathcal{A}_2\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))} \quad (i = 0, 1)
\end{aligned} \tag{116}$$

where we have set $H^0_p = L_p$ and used the fact that $|e^{-\gamma t}\mathcal{A}_2| \leq |\mathcal{A}_2|$, which follows from $\mathcal{A}_2 = 0$ for $t \notin (0, 2T)$.

By (55), (87) , (97), (103), and (105), we have

$$
\begin{aligned}
\|e^{-\gamma t}\mathcal{A}_1\|_{H^i_p(\mathbb{R},L_q(\dot\Omega))} &\le Ce^{2(\gamma-\gamma_1)}B; \quad \|\mathcal{A}_1\|_{H^i_p(\mathbb{R},L_q(\dot\Omega))} \le CB; \quad \|\mathcal{A}_1\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))} \le CB;\\
\|\mathcal{A}_2\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))} &\le C(L+B); \quad \|\mathcal{A}_2\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))} \le C(L+B);\\
\|\mathcal{A}_2\|_{L_p(\mathbb{R},L_q(\dot\Omega))} &\le CT^{1/p}(\|tr[\mathbf{G}]\|_{L_\infty((0,T),L_q(\dot\Omega))} + \|T_v(\cdot)tr[\mathbf{G}_0]\|_{L_\infty((0,T),L_q(\dot\Omega))})\\
&\le C(L+B)T^{1/p}.
\end{aligned} \tag{117}
$$

Notice that $A^2_{12}$ and $A^2_{21}$ have the same estimate. In view of (115), combining estimates in (116) and (117), we have

$$
\begin{aligned}
&\|e^{-\gamma t}A^2_1\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))}\\
&\quad\le C(e^{2(\gamma-\gamma_1)}B^2 + (B(L+B))^{1/2}(B(L+B)T^{1/p})^{1/2} + (B+L)^2T^{1/(2p)})\\
&\quad\le C(e^{2(\gamma-\gamma_1)}B^2 + (L+B)^2T^{1/(2p)}).
\end{aligned} \tag{118}
$$

In addition, by (89)

$$
\begin{aligned}
\|e^{-\gamma t}A^2_1\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \le C\{&\|e^{-\gamma t}\mathcal{A}_1\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\|\mathcal{A}_1\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\\
&+2\|\mathcal{A}_1\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\|\mathcal{A}_2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \|\mathcal{A}_2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\|\mathcal{A}_2\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\}.
\end{aligned}
$$

By (87), (97), (103), and (105), we have

$$
\begin{aligned}
\|e^{-\gamma t}\mathcal{A}_1\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} &\le Ce^{2(\gamma-\gamma_1)}B;\\
\|\mathcal{A}_2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} &\le T^{1/p}\|\mathcal{A}_2\|_{L_\infty((0,2T),H^1_q(\dot\Omega))}\\
&\le T^{1/p}(\|tr\mathbf{G}\|_{L_\infty((0,T),H^1_q(\dot\Omega))} + \|T_v(\cdot)tr[\mathbf{G}_0]\|_{L_\infty((0,T),H^1_q(\dot\Omega))}\\
&\le CT^{1/p}(L+B).
\end{aligned} \tag{119}
$$

Using (117) and (119), we obtain

$$
\|e^{-\gamma t}A^2_1\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \le C(e^{2(\gamma-\gamma_1)}B^2 + (L+B)^2T^{1/p}). \tag{120}
$$

Moreover, by Lemma 3 and (112), we arrive at

$$
\begin{aligned}
&\|e^{-\gamma t}A^2_2\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|A^2_2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\\
&\quad\le CLT^{1/(2p')}(\|A^2_1\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|A^2_1\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}),
\end{aligned}
$$

which, combined with (118) and (120), yields

$$
\begin{aligned}
&\|e^{-\gamma t}A^2\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}A^2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\\
&\quad\le C(e^{2(\gamma-\gamma_1)}B^2 + (L+B)^2T^{1/(2p)} + LT^{1/(2p')}(e^{2(\gamma-\gamma_1)}B^2 + (L+B)^2T^{1/(2p)}))\\
&\quad\le C(e^{2(\gamma-\gamma_1)}B^2 + L(B^2+L^2)T^{1/(2p)}),
\end{aligned} \tag{121}
$$

where we use the assumption $LT^{1/2p'} \le 1$.

Combining (113), (114), and (121) yields

$$
\begin{aligned}
&\|\tilde{\mathbf{h}}_1(\mathbf{u},\mathbf{G},H_h)\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|\tilde{\mathbf{h}}_1(\mathbf{u},\mathbf{G},H_h)\|_{Lp(\mathbb{R},H^1_q(\dot\Omega))}\\
&\quad\le C(e^{2(\gamma-\gamma_1)}B^2 + L(L^2+B^2+e^{2(\gamma-\gamma_1)}B)T^{1/(2p)}),
\end{aligned} \tag{122}
$$

where we have used the facts: $1/(2p) < 1/(2p')$, $\epsilon e^{2(\gamma-\gamma_1)} < Be^{2(\gamma-\gamma_1)}$, and $L \le L^2$.

We finally consider $g(\mathbf{u}, H_h)$ and $\mathbf{g}(\mathbf{u}, H_h)$. In view of (28), we set

$$\begin{aligned} \tilde{g}(\mathbf{u}, H_h) &= \mathcal{G}_1(\bar{\nabla}\mathcal{E}_2[H_h])\bar{\nabla}\mathcal{E}_2[H_h] \otimes \nabla\mathcal{E}_1[\mathbf{u}], \\ \tilde{\mathbf{g}}(\mathbf{u}, H_h) &= \mathcal{G}_2(\bar{\nabla}\mathcal{E}_2[H_h])\bar{\nabla}\mathcal{E}_2[H_h] \otimes \mathcal{E}_1[\mathbf{u}]. \end{aligned} \tag{123}$$

Obviously,

$$\tilde{g}(\mathbf{u}, H_h) = g(\mathbf{u}, H_h), \quad \tilde{\mathbf{g}}(\mathbf{u}, H_h) = \mathbf{g}(\mathbf{u}, H_h) \quad \text{for } t \in (0, T)$$

and $\operatorname{div} \tilde{\mathbf{g}}(\mathbf{u}, H_h) = \tilde{g}(\mathbf{u}, H_h)$ as follows from (25)–(27). Employing the same argument as that in proving (113), we have

$$\|e^{-\gamma t}\tilde{g}(\mathbf{u}, H_h)\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot{\Omega}))} + \|e^{-\gamma t}\tilde{g}(\mathbf{u}, H_h)\|_{L_p(\mathbb{R}, H^1_q(\dot{\Omega}))} \leq CT^{1/(2p')}L(L + e^{2(\gamma-\gamma_1)}B). \tag{124}$$

By (109), we have

$$\|\partial_t\tilde{\mathbf{g}}(\mathbf{u}, H_h)\|_{L_q(\dot{\Omega})} \leq C\{\|\bar{\nabla}\mathcal{E}_2[H_h]\|_{H^1_q(\Omega)}\|\partial_t\mathcal{E}_1[\mathbf{u}]\|_{L_q(\dot{\Omega})} + \|\partial_t\bar{\nabla}\mathcal{E}_2[H_h]\|_{L_q(\Omega)}\|\mathcal{E}_1[\mathbf{v}]\|_{H^1_q(\dot{\Omega})}\}.$$

Thus, using (89), (91), (92), (94), (103)–(105), and (109), we have

$$\begin{aligned} &\|e^{-\gamma t}\partial_t\tilde{\mathbf{g}}(\mathbf{u}, H_h)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad \leq C(\|H_h\|_{L_\infty((0,T), H^2_q(\Omega))} + \|T_h(\cdot)h_0\|_{L_\infty((0,\infty), H^2_q(\Omega))}) \\ &\qquad \times (\|\partial_t\mathbf{u}\|_{L_p((0,T), L_q(\Omega))} + \|e^{-\gamma t}T_v(|\cdot|)\mathbf{u}_0\|_{H^1_p((-2,\infty), L_q(\dot{\Omega}))}) \\ &\quad + (T^{1/p}\|\partial_t H_h\|_{L_\infty((0,T), H^1_q(\Omega))} + \|\partial_t T_h(\cdot)h_0\|_{L_p((0,\infty), H^1_q(\Omega))}) \\ &\qquad \times (\|\mathbf{u}\|_{L_\infty((0,T), H^1_q(\dot{\Omega}))} + \|e^{-\gamma t}T_v(|\cdot|)\mathbf{u}_0\|_{L_\infty((-2,\infty), H^1_q(\dot{\Omega}))}) \\ &\quad \leq C\{\epsilon + LT^{1/p'})(L + e^{2(\gamma-\gamma_1)}B) + (T^{1/p}L + \epsilon)(L + e^{2(\gamma-\gamma_1)}B)\} \\ &\quad \leq CL(L + e^{2(\gamma-\gamma_1)}B)T^{1/p}. \end{aligned} \tag{125}$$

We now apply Theorem 2 to Equation (85) and use the estimate in Theorem 2 with $\gamma = \gamma_1$. Then, assuming that $1 \leq B \leq L$, noting that $s/(p'(1+s)) < 1/(2p) < 1/(2p')$ and using (98), (99), and (122), we arrive at

$$\begin{aligned} &E^1_T(\mathbf{v}) + \|\nabla\mathfrak{q}\|_{L_p((0,T), L_q(\dot{\Omega}))} + E^2_T(\rho) \\ &\quad \leq C(1 + \gamma_1^{1/2})e^{\gamma_1 T^{1-1/p}}\{B^2 + T^{-1/p}\epsilon + L^3 T^{\frac{s}{p'(1+s)}}\}. \end{aligned} \tag{126}$$

Here and in the following $s \in (0, 1 - 2/p)$ and $\gamma_1$ are fixed, and so we do not take care of the dependence of constants on $s$ and $\gamma_1$.

By the third equation of (48), (85), and (99), we have

$$\begin{aligned} \|\partial_t\rho\|_{L_p((0,T), W^{1-1/q}_q(\Gamma))} &\leq C\{\|\rho\|_{L_\infty((0,T), W^{2-1/q}_q(\Gamma))}B + \|\mathbf{v}\|_{L_\infty((0,T), H^1_q(\dot{\Omega}))} + L(L+B)T^{1/p'}\} \\ &\leq C\{(\epsilon + T^{1/p'}E^2_T(\rho))B + B + E^1_T(\mathbf{v}) + L(L+B)T^{1/p'})\} \\ &\leq C\{B + E^1_T(\mathbf{v}) + E^2_T(\rho) + L(L+B)T^{1/p'}\}, \end{aligned}$$

where we used the facts that $\epsilon B \leq B$ and $T^{1/p'}B \leq T^{1/p'}L \leq 1$. In combination with (126), it gives us

$$\begin{aligned} &E^1_T(\mathbf{v}) + E^2_T(\rho) + \|\partial_t\rho\|_{L_\infty((0,T), W^{1-1/q}_q(\Gamma))} \\ &\quad \leq C[(1 + \gamma_1^{1/2})e^{\gamma_1 T^{1-1/p}}\{B^2 + T^{-1/p}\epsilon + L^3 T^{\frac{s}{p'(1+s)}}\} + B + L(L+B)T^{1/p'}]. \end{aligned}$$

Noting that $0 < T = \epsilon < 1$ and $T^{-1/p}\epsilon = T^{1-1/p} < 1 < B^2$, we have

$$E^1_T(\mathbf{v}) + E^2_T(\rho) + \|\partial_t\rho\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))} \leq M_1(B^2 + L^3T^{\frac{s}{p'(1+s)}}) \tag{127}$$

for some positive constant $M_1$ depending on $s$ and $\gamma_1$ provided that $0 < T < 1$, $\epsilon = \kappa = T$, $T^{1/p'}L \leq 1$, and $L > B \geq 1$.

We now estimate $\mathbf{H}$ by using Theorem 3 with the constant $\gamma_1$ given above. Let $\mathbf{f}_2(\mathbf{u}, \mathbf{G}, H_h)$ be a nonlinear term given in (29). Recalling the formula in (31) and employing the same argument as that in proving (98), we have

$$\|\mathbf{f}_2(\mathbf{u}, \mathbf{G}, H_h)\|_{L_p((0,T),L_q(\dot\Omega))} \leq CT^{1/p}(L + B)^2. \tag{128}$$

We next consider $\mathbf{h}_2(\mathbf{u}, \mathbf{G}, H_h)$ and $h_3(\mathbf{u}, \mathbf{G}, H_h)$ given in (67) and in (69), respectively. Let $\tilde{\mathbf{h}}_2(\mathbf{u}, \mathbf{G}, H_h)$ and $\tilde h_3(\mathbf{u}, \mathbf{G}, H_h)$ be their extension to $\mathbb{R}$ with respect to $t$ defined by setting

$$\begin{aligned}\tilde{\mathbf{h}}_2(\mathbf{u}, \mathbf{G}, H_h) &= \mathbf{V}^3_{\mathbf{h}}(\cdot\bar\nabla\mathcal{E}_2[H_h])\bar\nabla\mathcal{E}_2[H_h] \otimes \bar\nabla\mathcal{E}_1[tr[\mathbf{u}]] + b(y)\mathcal{E}_1[tr[\mathbf{u}]] \otimes \mathcal{E}_1[tr[\mathbf{G}]] \\ &\quad + \mathbf{V}^4_{\mathbf{h}}(\cdot, \bar\nabla\mathcal{E}_2[H_h])\bar\nabla\mathcal{E}_2[H_h] \otimes \mathcal{E}_1[tr[\mathbf{u}]] \otimes \mathcal{E}_1[tr[\mathbf{G}]]; \\ \tilde h_3(\mathbf{u}, \mathbf{G}, H_h) &= \mu \sum_{j,k=1}^N \tilde V_{0jk}(\nabla\mathcal{E}_2[H_h])\nabla\mathcal{E}_2[H_h]\frac{\partial}{\partial y_k}\mathcal{E}_1[tr[\mathbf{u}]]_j.\end{aligned} \tag{129}$$

Employing the same argument as in proving (113), we have

$$\begin{aligned}&\|e^{-\gamma t}(\tilde{\mathbf{h}}_2(\mathbf{u}, \mathbf{G}, H_h), \tilde h_3(\mathbf{u}, \mathbf{G}, H_h))\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} \\ &\quad + \|e^{-\gamma t}(\tilde{\mathbf{h}}_2(\mathbf{u}, \mathbf{G}, H_h), \tilde h_3(\mathbf{u}, \mathbf{G}, H_h))\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \\ &\quad \leq CT^{1/(2p')}L(e^{2(\gamma-\gamma_1)}B + L).\end{aligned} \tag{130}$$

We finally consider $k_1(\mathbf{G}, H_\rho)$ and $\mathbf{k}_2(\mathbf{G}, H_\rho)$ given in (70). In view of (127), choosing $L$ so large that $M_1B^2 < L/2$ and $T$ so small that $M_1L^3T^{s/(p'(1+s))} \leq L/2$, we have

$$E^2_T(\rho) + \|\partial_t\rho\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))} \leq L. \tag{131}$$

In particular, we have

$$\|\mathcal{E}_2[H_\rho]\|_{L_\infty(\mathbb{R},H^2_q(\Omega)} \leq C(\epsilon + LT^{1/p'}). \tag{132}$$

Thus, choosing $\epsilon = T$ sufficiently small, we may also assume that

$$\sup_{t\in\mathbb{R}} \|\mathcal{E}_2[H_\rho](\cdot, t)\|_{H^1_\infty(\Omega)} \leq \delta_2, \tag{133}$$

and that

$$\|\mathcal{E}_2[H_\rho]\|_{L_\infty(\mathbb{R},H^2_q(\Omega))} \leq LT^{1/p'}, \quad \|\mathcal{E}_2[H_\rho]\|_{L_\infty(\mathbb{R},H^2_q(\Omega))} \leq 1. \tag{134}$$

In view of (71), we define the extensions of $k_1(\mathbf{G}, H_\rho)$ and $\mathbf{k}_2(\mathbf{G}, H_\rho)$ by setting

$$(\tilde k_1(\mathbf{G}, H_\rho), \tilde{\mathbf{k}}_2(\mathbf{G}, H_\rho)) = \mathbf{V}^5_{\mathbf{k}}(\cdot, \bar\nabla\mathcal{E}_2[H_\rho])\bar\nabla\mathcal{E}_2[H_\rho] \otimes \mathcal{E}_1[tr[\mathbf{G}]]. \tag{135}$$

Obviously, $(\tilde k_1(\mathbf{G}, H_\rho), \tilde{\mathbf{k}}_2(\mathbf{G}, H_\rho)) = (k_1(\mathbf{G}, H_\rho), \mathbf{k}_2(\mathbf{G}, H_\rho))$ for $t \in (0, T)$. By (90) and (133), we have

$$\begin{aligned}&\|(\tilde k_1(\mathbf{G}, H_\rho), \tilde{\mathbf{k}}_2(\mathbf{G}, H_\rho)\|_{H^2_q(\dot\Omega)} \\ &\quad \leq C\{\|\bar\nabla\mathcal{E}_2[H_\rho]\|_{H^2_q(\dot\Omega)}\|\mathcal{E}_1[tr[\mathbf{G}]]\|_{H^1_q(\dot\Omega)} + \|\bar\nabla\mathcal{E}_2[H_\rho]\|_{H^1_q(\dot\Omega)}\|\nabla\mathcal{E}_1[tr[\mathbf{G}]]\|_{H^1_q(\dot\Omega)} \\ &\quad + \|\bar\nabla\mathcal{E}_2[H_\rho]\|_{H^2_q(\dot\Omega)}(1 + \|\bar\nabla\mathcal{E}_2[H_\rho]\|_{H^1_q(\dot\Omega)})\|\bar\nabla\mathcal{E}_2[H_\rho]\|_{H^1_q(\dot\Omega)}\|\mathcal{E}_1[tr[\mathbf{G}]]\|_{H^1_q(\dot\Omega)}\}.\end{aligned}$$

By (104), (108), (131), and (134), we have

$$\begin{aligned}
&\|e^{-\gamma t}(\tilde{k}_1(\mathbf{G}, H_\rho), \tilde{\mathbf{k}}_2(\mathbf{G}, H_\rho))\|_{L_p(\mathbb{R}, H^2_q(\dot\Omega)}\\
&\quad \leq C\{(\|\rho\|_{L_p((0,T), W^{3-1/q}_q(\Gamma))} + \epsilon)(e^{2(\gamma-\gamma_1)}B + (L+B)T^{\frac{s}{p'(1+s)}})\\
&\qquad\qquad + L^2T^{1/p'}(L + e^{2(\gamma-\gamma_1)}B)\}\\
&\quad \leq C\{e^{2(\gamma-\gamma_1)}B\|\rho\|_{L_p((0,T), W^{3-1/q}_q(\Gamma))} + L^2(L + e^{2(\gamma-\gamma_1)}B)T^{\frac{s}{p'(1+s)}}\}.
\end{aligned} \tag{136}$$

By (103), (104), (108), (131), and (132), we have

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t(\tilde{k}_1(\mathbf{G}, H_\rho), \tilde{\mathbf{k}}_2(\mathbf{G}, H_\rho))\|_{L_p(\mathbb{R}, L_q(\dot\Omega)}\\
&\quad \leq C\{\|\partial_t \mathcal{E}_2[H_\rho]\|_{L_p(\mathbb{R}, H^1_q(\Omega))}\|e^{-\gamma t}\mathcal{E}_1[tr[\mathbf{G}]]\|_{L_\infty(\mathbb{R}, H^1_q(\dot\Omega))}\\
&\qquad + \|\mathcal{E}_2[H_\rho]\|_{L_\infty(\mathbb{R}, H^2_q(\Omega))}\|e^{-\gamma t}\partial_t\mathcal{E}_1[tr[\mathbf{G}]]\|_{L_p(\mathbb{R}, L_q(\dot\Omega))}\\
&\quad \leq C\{(T^{1/p}L + \epsilon)(e^{2(\gamma-\gamma_1)}B + (L+B)T^{\frac{s}{p'(1+s)}}) + (\epsilon + T^{1/p'}L)(L + e^{2(\gamma-\gamma_1)}B)\}\\
&\quad \leq C(T^{1/p} + T^{1/p'})L(L + e^{2(\gamma-\gamma_1)}B) \leq CL(L + e^{2(\gamma-\gamma_1)}B)T^{1/p}.
\end{aligned} \tag{137}$$

Applying the estimate in Theorem 3 with $\gamma = \gamma_1$ to Equation (86) and using (128), (130), (136), and (137), we have

$$E^1_T(\mathbf{H}) \leq Ce^{\gamma_1 T}\{B(1 + \|\rho\|_{L_p((0,T), W^{3-1/q}_q(\Gamma))}) + L^2(L+B)T^{\frac{s}{p'(1+s)}}\},$$

which, combined with (126), yields

$$\begin{aligned}
&E^1_T(\mathbf{v}) + E^2_T(\rho) + \|\partial_t\rho\|_{L_p((0,T), W^{1-1/q}_q(\Gamma))} + E^1_T(\mathbf{H})\\
&\quad \leq M_1(B^2 + L^3T^{\frac{s}{p'(1+s)}}) + Ce^{\gamma_1 T}(B(1 + M_1B^2 + M_1L^3T^{\frac{s}{p'(1+s)}}) + L^2(L+B)T^{\frac{s}{p'(1+s)}})\\
&\quad \leq M_1B^2 + Ce^{\gamma_1}B(1 + M_1B^2) + \{M_1L^3 + Ce^{\gamma_1}M_1L^3 + L^2(L+B)\}T^{\frac{s}{p'(1+s)}},
\end{aligned} \tag{138}$$

provided that $0 < \epsilon = T = \kappa < 1$, $L > 1$, and $B > 1$. Choosing $L > 0$ so large that $L/2 \geq M_1B^2 + Ce^{\gamma_1}B(1 + M_1B^2)$ and $T > 0$ so small that $L/2 \geq \{M_1L^3 + Ce^{\gamma_1}M_1L^3 + L^2(L+B)\}T^{\frac{s}{p'(1+s)}}$, and setting $L = f(B) = 2(M_1B^2 + Ce^{\gamma_1}B(1 + M_1B^2))$, we see that $E_T(\mathbf{v}, \mathbf{H}, \rho) \leq L$. If we define a map $\Phi$ by $\Phi(\mathbf{u}, \mathbf{G}, h) = (\mathbf{v}, \mathbf{H}, \rho)$, then, $\Phi$ maps $U_T$ into itself.

## 5. Estimates of the Difference of Nonlinear Terms and Completion of the Proof of Theorem 1

Let $(\mathbf{u}_i, \mathbf{G}_i, h_i) \in \mathbf{U}_T$ $(i = 1, 2)$. In this section, we shall estimate $E_T(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{H}_1 - \mathbf{H}_2, \rho_1 - \rho_2)$ with $(\mathbf{v}_i, \mathbf{H}_i, \rho_i) = \Phi(\mathbf{u}_i, \mathbf{G}_i, h_i)$ $(i = 1, 2)$, and then we shall prove that $\Phi$ is a contraction map on $U_T$ with a suitable choice of $\epsilon > 0$. For notational simplicity, we set

$$\begin{aligned}
&\tilde{\mathbf{v}} = \mathbf{v}_1 - \mathbf{v}_2, \quad \tilde{\mathbf{H}} = \mathbf{H}_1 - \mathbf{H}_2, \quad \tilde\rho = \rho_1 - \rho_2,\\
&\mathcal{F}_1 = \mathbf{f}_1(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{f}_1(\mathbf{u}_2, \mathbf{G}_2, H_{h_2}), \quad \mathfrak{g} = g(\mathbf{u}_1, H_{h_1}) - g(\mathbf{u}_2, H_{h_2}),\\
&\mathcal{G} = \mathbf{g}(\mathbf{u}_1, H_{h_1}) - \mathbf{g}(\mathbf{u}_2, H_{h_2}), \mathcal{D} = d(\mathbf{u}_1, H_{H_1}) - d(\mathbf{u}_2, H_{h_2}),\\
&\mathcal{H}_1 = \mathbf{h}_1(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{h}_1(\mathbf{u}_2, \mathbf{G}_2, H_{h_2}), \quad \mathcal{F}_2 = \mathbf{f}_2(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{f}_2(\mathbf{u}_2, \mathbf{G}_2, H_{h_2}),\\
&\mathcal{H}_2 = \mathbf{h}_2(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{h}_2(\mathbf{u}_2, \mathbf{G}_2, H_{h_2}), \quad \mathcal{H}_3 = \mathbf{h}_3(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{h}_3(\mathbf{u}_2, \mathbf{G}_2, H_{h_2}),\\
&\mathcal{K}_1 = k_1(\mathbf{G}_1, H_{\rho_1}) - k_1(\mathbf{G}_2, H_{\rho_2}), \quad \mathcal{K}_2 = \mathbf{k}_2(\mathbf{G}_1, H_{\rho_1}) - \mathbf{k}_2(\mathbf{G}_2, H_{\rho_2}).
\end{aligned}$$

Then, $\tilde{\mathbf{v}}$ and $\tilde{\rho}$ satisfy the following equations with some pressure term $Q$ :

$$\begin{cases} m\partial_t\tilde{\mathbf{v}} - \mathrm{Div}\,\mathbf{T}(\tilde{\mathbf{v}}, Q) = \mathcal{F}_1, & \text{in } \dot{\Omega}\times(0,T), \\ \mathrm{div}\,\tilde{\mathbf{v}} = \mathfrak{g} = \mathrm{div}\,\mathcal{G} & \text{in } \dot{\Omega}\times(0,T), \\ \partial_t\tilde{\rho} + < \nabla_\Gamma\tilde{\rho} \perp \mathbf{u}_\kappa > - \mathbf{n}\cdot\tilde{\mathbf{v}}_+ = \mathcal{D} & \text{on } \Gamma\times(0,T), \\ [[\tilde{\mathbf{v}}]] = 0, \quad [[\mathbf{T}(\tilde{\mathbf{v}}, Q)\mathbf{n}]] - \sigma(\Delta_\Gamma\tilde{\rho} + a\tilde{\rho})\mathbf{n} = \mathcal{H}_1, & \text{on } \Gamma\times(0,T), \\ \tilde{\mathbf{v}}_\pm = 0 & \text{on } S_\pm\times(0,T), \\ (\tilde{\mathbf{v}}, \tilde{\rho})|_{t=0} = (0,0) & \text{in } \dot{\Omega}\times\Gamma. \end{cases} \tag{139}$$

In addition, $\tilde{\mathbf{H}}$ satisfies the following equations:

$$\begin{cases} \mu\partial_t\tilde{\mathbf{H}} - \alpha^{-1}\Delta\tilde{\mathbf{H}} = \mathcal{F}_2 & \text{in } \dot{\Omega}\times(0,T), \\ [[\alpha^{-1}\mathrm{curl}\,\tilde{\mathbf{H}}]]\mathbf{n} = \mathcal{H}_2 \quad [[\mu\mathrm{div}\,\tilde{\mathbf{H}}]] = \mathcal{H}_3 & \text{on } \Gamma\times(0,T), \\ [[\mu\tilde{\mathbf{H}}\cdot\mathbf{n}]] = \mathcal{K}_1, \quad [[\tilde{\mathbf{H}} - < \tilde{\mathbf{H}}, \mathbf{n} > \mathbf{n}]] = \mathcal{K}_2 & \text{on } \Gamma\times(0,T), \\ \mathbf{n}_\pm\cdot\tilde{\mathbf{H}}_\pm = 0, \quad (\mathrm{curl}\,\tilde{\mathbf{H}}_\pm)\mathbf{n}_\pm = 0 & \text{on } S_\pm\times(0,T), \\ \tilde{\mathbf{H}}|_{t=0} = 0 & \text{in } \dot{\Omega}. \end{cases} \tag{140}$$

We have to estimate the nonlinear terms appearing in the right side of equations (139) and (140). We start with estimating $\mathcal{F}_1$. As was written in (95), we write

$$\mathbf{f}_1(\mathbf{u}, \mathbf{G}, H_h) = \mathbf{V}_{\mathbf{f}_1}(\bar{\nabla}H_h)\mathbf{f}_3(\mathbf{u}, \mathbf{G}, H_h),$$

where

$$\mathbf{f}_3(\mathbf{u}, \mathbf{G}, H_h) = \bar{\nabla}H_h\otimes(\partial_t\mathbf{u}, \nabla^2\mathbf{u}) + \partial_tH_h\otimes\nabla\mathbf{u} + \mathbf{u}\otimes\nabla\mathbf{u} + \bar{\nabla}^2H_h\otimes\nabla\mathbf{u} + \mathbf{G}\otimes\nabla\mathbf{G}.$$

Consequently, we can write $\mathcal{F}_1$ as follows:

$$\begin{aligned} \mathcal{F}_1 &= (\mathbf{V}_{\mathbf{f}_1}(\bar{\nabla}H_{h_1}) - \mathbf{V}_{\mathbf{f}_1}(\bar{\nabla}H_{h_2}))\mathbf{f}_3(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) \\ &\qquad + \mathbf{V}_{\mathbf{f}_1}(\bar{\nabla}H_{h_2})(\mathbf{f}_3(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{f}_3(\mathbf{u}_2, \mathbf{G}_2, H_{h_2})); \\ &\mathbf{f}_3(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{f}_3(\mathbf{u}_2, \mathbf{G}_2, H_{h_2}) \\ &\quad = \bar{\nabla}(H_{h_1} - H_{h_2})\otimes(\partial_t\mathbf{u}_1, \nabla^2\mathbf{u}_1) + \bar{\nabla}H_{h_2}\otimes(\partial_t(\mathbf{u}_1 - \mathbf{u}_2), \nabla^2(\mathbf{u}_1 - \mathbf{u}_2)) \\ &\quad + \partial_t(H_{h_1} - H_{h_2})\otimes\nabla\mathbf{u}_1 + \partial_tH_{h_2}\otimes\nabla(\mathbf{u}_1 - \mathbf{u}_2) \\ &\quad + (\mathbf{u}_1 - \mathbf{u}_2)\otimes\nabla\mathbf{u}_1 + \mathbf{u}_2\otimes\nabla(\mathbf{u}_1 - \mathbf{u}_2) + \bar{\nabla}^2(H_{h_1} - H_{h_2})\otimes\nabla\mathbf{u}_1 + \bar{\nabla}^2H_{h_2}\otimes\nabla(\mathbf{u}_1 - \mathbf{u}_2) \\ &\quad + (\mathbf{G}_1 - \mathbf{G}_2)\otimes\nabla\mathbf{G}_1 + \mathbf{G}_2\otimes\nabla(\mathbf{G}_1 - \mathbf{G}_2). \end{aligned}$$

Since we may write

$$\mathbf{V}_{\mathbf{f}_1}(\bar{\nabla}H_{h_1}) - \mathbf{V}_{\mathbf{f}_1}(\bar{\nabla}H_{h_2}) = \int_0^1 (d_{\mathbf{K}}\mathbf{V}_{\mathbf{f}_1})(\bar{\nabla}H_{h_2} + \theta\bar{\nabla}(H_{h_1} - H_{h_2}))\,d\theta\bar{\nabla}(H_{h_1} - H_{h_2}), \tag{141}$$

where $d_{\mathbf{K}}\mathbf{V}_{\mathbf{f}_1}f$ is the derivative of $\mathbf{V}_{\mathbf{f}_1}(\mathbf{K})$ with respect to $\mathbf{K}$, noting that $H_{h_1} - H_{h_2} = 0$ for $t = 0$ and using (94) and (88), we have

$$\|\mathbf{V}^1_{\mathbf{f}_1}(\bar{\nabla}H_{h_1}) - \mathbf{V}^1_{\mathbf{f}_1}(\bar{\nabla}H_{h_2})\|_{L_\infty((0,T),H^1_q(\Omega))} \leq T^{1/p'}\|\partial_th_1 - \partial_th_2\|_{L_p((0,T),H^2_q(\Omega))}. \tag{142}$$

Since $\mathbf{f}_3(\mathbf{u}, \mathbf{G}, H_h)$ satisfies the estimate (98), replacing $h$, $\mathbf{u}$, and $\mathbf{G}$ with $h_1$, $\mathbf{u}_1$, and $\mathbf{G}_1$, we have

$$\|\mathbf{f}_3(\mathbf{u}_1, \mathbf{G}_1, H_{h_1})\|_{L_p((0,T),L_q(\dot{\Omega}))} \leq C\{T^{1/p}(L+B)^2 + (\epsilon + T^{1/p'}L)L\} \leq CT^{1/p}(L+B)^2. \tag{143}$$

By (11) and (89), we have

$$
\begin{aligned}
&\|\mathbf{f}_3(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{f}_3(\mathbf{u}_2, \mathbf{G}_2, H_{h_2})\|_{L_q(\dot\Omega)} \\
&\quad \le C\{\|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)} \|(\partial_t \mathbf{u}_1, \nabla^2 \mathbf{u}_1)\|_{L_q(\dot\Omega)} \\
&\quad + \|h_2\|_{W_q^{2-1/q}(\Gamma)} (\|\partial_t(\mathbf{u}_1 - \mathbf{u}_2), \nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_q(\dot\Omega)}) \\
&\quad + \|\partial_t(h_1 - h_2)\|_{W_q^{1-1/q}(\Gamma)} \|\nabla \mathbf{u}_1\|_{L_q(\dot\Omega)} + \|\partial_t h_2\|_{W_q^{1-1/q}(\Gamma)} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_q(\dot\Omega)} \\
&\quad + \|(\mathbf{u}_1, \mathbf{u}_2)\|_{H_q^1(\dot\Omega)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^1(\dot\Omega)} + \|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)} \|\mathbf{u}_1\|_{H_Q^2(\dot\Omega)} \\
&\quad + \|h_2\|_{W_q^{2-1/q}(\Gamma)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^2(\dot\Omega)} + \|(\mathbf{G}_1, \mathbf{G}_2)\|_{H_q^1(\dot\Omega)} \|\mathbf{G}_1 - \mathbf{G}_2)\|_{H_q^1(\dot\Omega)}\}.
\end{aligned}
$$

Since

$$
\|h_1 - h_2\|_{L_\infty((0,T), W_q^{2-1/q}(\Gamma)} \le T^{1/p'} \|\partial_t(h_1 - h_2)\|_{L_p((0,T), W_q^{2-1/q}(\Gamma))'} \tag{144}
$$

noting that $\mathbf{u}_1 - \mathbf{u}_2 = 0$ and $\mathbf{G}_1 - \mathbf{G}_2 = 0$ at $t = 0$, by (91), (94), and (97), we have

$$
\begin{aligned}
&\|\mathbf{f}_3(\mathbf{u}_1, \mathbf{G}_1, H_{h_1}) - \mathbf{f}_3(\mathbf{u}_2, \mathbf{G}_2, H_{h_2}))\|_{L_p((0,T), L_q(\dot\Omega))} \\
&\quad \le C\{T^{1/p'} \|\partial_t(h_1 - h_2)\|_{L_p((0,T), W_q^{2-1/q}(\Gamma))} L + L T^{1/p'} E_T^1(\mathbf{u}_1 - \mathbf{u}_2) \\
&\quad + T^{1/p}(\|\partial_t(h_1 - h_2)\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))}(B + L) + L E_T^1(\mathbf{u}_1 - \mathbf{u}_2) + (B + L) E_T^1(\mathbf{u}_1 - \mathbf{u}_2)) \\
&\quad + T^{1/p'}(\|\partial_t(h_1 - h_2)\|_{L_p((0,T), W_q^{2-1/q}(\Gamma))} L + L\|\mathbf{u}_1 - \mathbf{u}_2\|_{L_p((0,T), H_q^2(\dot\Omega))}) \\
&\quad + T^{1/p}(B + L) E_T^2(\mathbf{G}_1 - \mathbf{G}_2),
\end{aligned}
$$

which, combined with (142) and (143), leads to

$$
\|\mathcal{F}_1\|_{L_p((0,T), L_q(\dot\Omega))} \le C T^{1/p}(L + B) E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2). \tag{145}
$$

Here, we have used the estimate:

$$
T^{1/p'} T^{1/p}(L + B)^2 \le 2(T^{1/p'} L) T^{1/p}(L + B) \le 2 T^{1/p}(L + B),
$$

which follows from $B \le L$ and $T^{1/p'} L \le 1$.

We next consider the difference $\mathcal{D}$. In view of (50), we write

$$
\begin{aligned}
\mathcal{D} = \; & < \nabla_\Gamma(H_{h_1} - H_{h_2}) \perp \mathbf{u}_1 - \mathbf{u}_\kappa > + < \nabla_\Gamma H_{h_2} \perp \mathbf{u}_1 - \mathbf{u}_2 > \\
& + < \mathbf{u}_1 - \mathbf{u}_2 - \frac{\partial}{\partial t}(H_{h_1} - H_{h_2})\mathbf{n}, \mathbf{V}_\mathbf{n}(\cdot, \bar\nabla H_1) \bar\nabla H_1 \otimes \bar\nabla H_1 > \\
& + < \mathbf{u}_2 - \frac{\partial}{\partial t} H_{h_2} \mathbf{n}, (\tilde{\mathbf{V}}_\mathbf{n}(\cdot, \bar\nabla H_{h_1}) - \tilde{\mathbf{V}}_\mathbf{n}(\cdot, \bar\nabla H_{h_2})) \otimes \bar\nabla H_{h_1} > \\
& + < \mathbf{u}_2 - \frac{\partial}{\partial t} H_{h_2} \mathbf{n}, \mathbf{V}_\mathbf{n}(\cdot, \bar\nabla H_{h_2}) \bar\nabla H_{h_2} \otimes \bar\nabla(H_{h_1} - H_{h_2}) >,
\end{aligned}
$$

where we have set $\mathbf{V}_\mathbf{n}(\cdot, \mathbf{K})\mathbf{K} = \tilde{\mathbf{V}}_\mathbf{n}(\cdot, \mathbf{K})$. We have

$$
\|\mathcal{D}\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))} \le C(L + B) T^{1/p'} E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2); \tag{146}
$$

$$
\|\mathcal{D}\|_{L_p((0,T), W_q^{2-1/q}(\Gamma))} \le C L(L + B) T^{\frac{s}{p'(1+s))}} E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2). \tag{147}
$$

In fact, noting that the difference: $\tilde{\mathbf{V}}_{\mathbf{n}}(\cdot, \bar{\nabla} H_{h_1}) - \tilde{\mathbf{V}}_{\mathbf{n}}(\cdot, \bar{\nabla} H_{h_2})$ has the similar formula to that in (141), by (11), (88), (89), and (90), we have

$$
\begin{aligned}
&\|\mathcal{D}\|_{W_q^{1-1/q}(\Gamma)} \\
&\leq C\{\|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma))}\|\mathbf{u}_1 - \mathbf{u}_\kappa\|_{H_q^1(\Omega_+)} + \|h_2\|_{W_q^{2-1/q}(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^1(\Omega_+)} \\
&+ (\|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^1(\Omega_+)} + \|\partial_t(h_1 - h_2)\|_{W_q^{1-1/q}(\Gamma)})(1 + \|h_1\|_{W_q^{2-1/q}(\Gamma)})\|h_1\|^2_{W_q^{2-1/q}(\Gamma)} \\
&+ (\|\mathbf{u}_2\|_{H_q^1(\Omega_+)} + \|\partial_t h_2\|_{W_q^{1-1/q}(\Gamma)})(1 + \|h_1\|_{W_q^{2-1/q}(\Gamma)} + \|h_2\|_{W_q^{2-1/q}(\Gamma)}) \\
&\qquad\qquad \times \|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1\|_{W_q^{2-1/q}(\Gamma)} \\
&+ (\|\mathbf{u}_2\|_{H_q^1(\Omega_+)} + \|\partial_t h_2\|_{W_q^{1-1/q}(\Gamma)})(1 + \|h_2\|_{W_q^{2-1/q}(\Gamma)})\|h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)}\}.
\end{aligned}
$$

Thus, by (91), (94), (102), and (144), we have

$$
\begin{aligned}
&\|\mathcal{D}\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))} \\
&\quad \leq C\{T^{1/p'} E_T^2(h_1 - h_2) T^{\frac{s}{p'(1+s)}}(L + B) + L T^{1/p'} E_T^1(\mathbf{u}_1 - \mathbf{u}_2) \\
&\quad + L T^{1/p'}(E_T^1(\mathbf{u}_1 - \mathbf{u}_2) + \|\partial_t(h_1 - h_2)\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))}) \\
&\quad + (L + B) T^{1/p'}(\epsilon + L T^{1/p'}) E_T^2(h_1 - h_2)\},
\end{aligned}
$$

which leads to the inequality in (146), because $T^{\frac{s}{p'(1+s)}} < 1$ as follows from $0 < T < 1$.

By (11), (88), (89), and (90), we obtain

$$
\begin{aligned}
\|\mathcal{D}\|_{W_q^{2-1/q}(\Gamma)} &\leq C[\|h_1 - h_2\|_{W_q^{3-1/q}(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_\kappa\|_{H_q^1(\Omega_+)} + \|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_\kappa\|_{H_q^2(\Omega_+)} \\
&+ \|h_2\|_{W_q^{3-1/q}(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^1(\Omega_+)} + \|h_2\|_{W_q^{2-1/q}(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^2(\Omega_+)} \\
&+ (\|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^2(\Omega_+)} + \|\partial_t(h_1 - h_2)\|_{W_q^{2-1/q}(\Gamma)})(1 + \|h_1\|_{W_q^{2-1/q}(\Gamma)})\|h_1\|^2_{W_q^{2-1/q}(\Gamma)} \\
&+ (\|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^1(\Omega_+)} + \|\partial_t(h_1 - h_2)\|_{W_q^{1-1/q}(\Gamma)}) \\
&\quad \times (\|h_1\|_{W_q^{3-1/q}(\Gamma)}\|h_1\|_{W_q^{2-1/q}(\Gamma)} + \|h_1\|_{W_q^{3-1/q}(\Gamma)}(1 + \|h_1\|_{W_q^{2-1/q}(\Gamma)})\|h_1\|^2_{W_q^{2-1/q}(\Gamma)}) \\
&+ (\|\mathbf{u}_2\|_{H_q^2(\Omega_+)} + \|\partial_t h_2\|_{W_q^{2-1/q}(\Gamma)})(1 + \|h_1\|_{W_q^{2-1/q}(\Gamma)} + \|h_2\|_{W_q^{2-1/q}(\Gamma)}) \\
&\qquad\qquad \times \|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1\|_{W_q^{2-1/q}(\Gamma)} \\
&+ (\|\mathbf{u}_2\|_{H_q^1(\Omega_+)} + \|\partial_t h_2\|_{W_q^{1-1/q}(\Gamma)})\{\|h_1 - h_2\|_{W_q^{3-1/q}(\Gamma)}\|h_1\|_{W_q^{2-1/q}(\Gamma)} \\
&+ \|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1\|_{W_q^{3-1/q}(\Gamma)} + (\|h_1\|_{W_q^{3-1/q}(\Gamma)} + \|h_2\|_{W_q^{3-1/q}(\Gamma)}) \\
&\quad \times (1 + \|h_1\|_{W_q^{2-1/q}(\Gamma)} + \|h_2\|_{W_q^{2-1/q}(\Gamma)})\|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1\|_{W_q^{2-1/q}(\Gamma)}\} \\
&+ (\|\mathbf{u}_2\|_{H_q^2(\Omega_+)} + \|\partial_t h_2\|_{W_q^{2-1/q}(\Gamma)})(1 + \|h_2\|_{W_q^{2-1/q}(\Gamma)})\|h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)} \\
&+ (\|\mathbf{u}_2\|_{H_q^1(\Omega_+)} + \|\partial_t h_2\|_{W_q^{1-1/q}(\Gamma)}) \\
&\quad \times \{\|h_2\|_{W_q^{3-1/q}(\Gamma)}\|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)} + \|h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1 - h_2\|_{W_q^{3-1/q}(\Gamma)} \\
&\quad + \|h_2\|_{W_q^{3-1/q}(\Gamma)}(1 + \|h_2\|_{W_q^{2-1/q}(\Gamma)})\|h_2\|_{W_q^{2-1/q}(\Gamma)}\|h_1 - h_2\|_{W_q^{2-1/q}(\Gamma)}\}].
\end{aligned}
$$

Since

$$\begin{aligned}
&\|\mathbf{u}_1-\mathbf{u}_2\|_{H^1_q(\Omega_+)} \\
&\quad\le C_s T^{\frac{s}{p'(1+s)}}\|\partial_t(\mathbf{u}_1-\mathbf{u}_2)\|^{s/(1+s)}_{L_p((0,T),L_q(\Omega_+))}\|\mathbf{u}_1-\mathbf{u}_2\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)} \\
&\quad\le C_s T^{\frac{s}{p'(1+s)}} E^1_T(\mathbf{u}_1-\mathbf{u}_2),
\end{aligned} \tag{148}$$

by (94), (97), and (102), we obtain

$$\begin{aligned}
&\|\mathcal{D}\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))} \\
&\quad\le C[T^{\frac{s}{p'(1+s)}}(L+B)\|h_1-h_2\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))} \\
&\quad+T^{1/p'}\|\partial_t(h_1-h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}(B+L) \\
&\quad+LT^{\frac{s}{p'(1+s)}}E^1_T(\mathbf{u}_1-\mathbf{u}_2)+LT^{1/p'}\|\mathbf{u}_1-\mathbf{u}_2\|_{L_p((0,T),H^2_q(\dot\Omega_+))} \\
&\quad+(\|\mathbf{u}_1-\mathbf{u}_2\|_{L_p((0,T),H^2_q(\dot\Omega_+))}+\|\partial_t(h_1-h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))})LT^{1/p'} \\
&\quad+(E^1_T(\mathbf{u}_1-\mathbf{u}_2)+\|\partial_t(h_1-h_2)\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))})L^2T^{1/p'} \\
&\quad+LT^{1/p'}\|\partial_t(h_1-h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}+(L+B)\{\|h_1-h_2\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))}LT^{1/p'} \\
&\quad+T^{1/p'}\|\partial_t(h_1-h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}L\}+LT^{1/p'}\|\partial_t(h_1-h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))} \\
&\quad+(L+B)(LT^{1/p'}\|\partial_t(h_1-h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}+LT^{1/p'}\|h_1-h_2\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))} \\
&+\quad LT^{1/p'}\|\partial_t(h_1-h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))})],
\end{aligned}$$

which yields (147).

We next conider $\mathcal{H}_1$. In view of (111), we set

$$\tilde{\mathcal{H}}_1=\mathcal{H}^1_1+a(y)\mathcal{H}^2_1+\mathcal{H}^3_1+\mathcal{H}^4_1$$

with

$$\begin{aligned}
\mathcal{H}^1_1&=(\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_1}])-\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}]))\otimes\nabla\mathcal{E}_1[tr[\mathbf{u}_1]] \\
&\quad+\tilde{\mathbf{V}}_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}])\bar\nabla\mathcal{E}_2[H_{h_2}]\otimes\nabla(\mathcal{E}_1[tr[\mathbf{u}_1]]-\mathcal{E}_1[tr[\mathbf{u}_2]]), \\
\mathcal{H}^2_1&=(\mathcal{E}_1[tr[\mathbf{G}_1]]-\mathcal{E}_1[tr[\mathbf{G}_2]])\otimes\mathcal{E}_1[tr[\mathbf{G}_1]] \\
&\quad+\mathcal{E}_1[tr[\mathbf{G}_2]]\otimes(\mathcal{E}_1[tr[\mathbf{G}_1]]-\mathcal{E}_1[tr[\mathbf{G}_2]]), \\
\mathcal{H}^3_1&=(\tilde{\mathbf{V}}^2_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_1}])-\tilde{\mathbf{V}}^2_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}]))\otimes\mathcal{E}_1[tr[\mathbf{G}_1]]\otimes\mathcal{E}_1[tr[\mathbf{G}_1]] \\
&\quad+\mathbf{V}^2_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}])\bar\nabla\mathcal{E}_2[H_{h_2}]\otimes\mathcal{H}^2_1, \\
\mathcal{H}^4_1&=(\tilde{\mathbf{V}}_s(\cdot,\bar\nabla\mathcal{E}_2[H_{h_1}])-\tilde{\mathbf{V}}_s(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}]))\otimes\bar\nabla^2\mathcal{E}_2[H_{h_1}] \\
&\quad+\mathbf{V}_s(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}])\bar\nabla\mathcal{E}_2[H_{h_2}]\otimes\bar\nabla^2(\mathcal{E}_2[H_{h_1}]-\mathcal{E}_2[H_{h_2}]),
\end{aligned}$$

where we have set

$$\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot,\mathbf{K})=\mathbf{V}^1_{\mathbf{h}}(\cdot,\mathbf{K})\mathbf{K},\quad \tilde{\mathbf{V}}^2_{\mathbf{h}}(\cdot,\mathbf{K})=\mathbf{V}^2_{\mathbf{h}}(\cdot,\mathbf{K})\mathbf{K},\quad \tilde{\mathbf{V}}_s(\cdot,\mathbf{K})=\mathbf{V}_s(\cdot,\mathbf{K})\mathbf{K}.$$

We see that $\tilde{\mathcal{H}}_1$ is defined for $t\in\mathbb{R}$ and $\tilde{\mathcal{H}}_1=\mathcal{H}_1$ for $t\in(0,T)$. Writing

$$\begin{aligned}
&\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_1}])-\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}]) \\
&=\int_0^1(d_K\tilde{\mathbf{V}}^1_{\mathbf{h}})(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}]+\theta\bar\nabla(\mathcal{E}_2[H_{h_1}]-\mathcal{E}_2[H_{h_2}]))\,d\theta\,\bar\nabla(\mathcal{E}_2[H_{h_1}]-\mathcal{E}_2[H_{h_2}]).
\end{aligned}$$

Since $\mathcal{E}_2[H_{h_1}] - \mathcal{E}_2[H_{h_2}] = e_T[H_{h_1} - H_{h_2}]$, by (11), (89), (108), (110), and (144), we have

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t(\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot, \bar{\nabla}\mathcal{E}_2[H_{h_1}]) - \tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot, \bar{\nabla}\mathcal{E}_2[H_{h_2}]))\|_{L_\infty(\mathbb{R}, L_q(\dot{\Omega}))} \\
&\quad \le C\{(\|\partial_t\bar{\nabla}\mathcal{E}_2[H_{h_1}]\|_{L_\infty(\mathbb{R}, L_q(\Omega))} + \|\partial_t\bar{\nabla}\mathcal{E}_2[H_{h_2}]\|_{L_\infty(\mathbb{R}, L_q(\Omega))})\|h_1 - h_2\|_{L_\infty((0,T), W^{2-1/q}_q(\Gamma))} \\
&\quad + (\|\bar{\nabla}\mathcal{E}_2[H_{h_1}]\|_{L_\infty(\mathbb{R}, H^1_q(\Omega))} + \|\bar{\nabla}\mathcal{E}_2[H_{h_2}]\|_{L_\infty(\mathbb{R}, H^1_q(\Omega))})\|\partial_t(h_1 - h_2)\|_{L_\infty((0,T), W^{1-1/q}_q(\Gamma))} \\
&\quad \le CLT^{1/p'}(\|\partial_t(h_1 - h_2)\|_{L_p((0,T), W^{2-1/q}_q(\Gamma))} + \|\partial_t(h_1 - h_2)\|_{L_\infty((0,T), W^{1-1/q}_q(\Gamma))}\}.
\end{aligned} \tag{149}$$

By (110) and (144), we deduce

$$\begin{aligned}
&\|e^{-\gamma t}(\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot, \bar{\nabla}\mathcal{E}_2[H_{h_1}]) - \tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot, \bar{\nabla}\mathcal{E}_2[H_{h_2}]))\|_{L_\infty(\mathbb{R}, H^1_q(\Omega))} \\
&\quad \le C(1 + \|\bar{\nabla}\mathcal{E}_2[H_{h_1}]\|_{L_\infty(\mathbb{R}, H^1_q(\Omega))} + \|\bar{\nabla}\mathcal{E}_2[H_{h_2}]\|_{L_\infty(\mathbb{R}, H^1_q(\Omega))}]\|\bar{\nabla}e[H_{h_1} - H_{h_2}]\|_{L_\infty(\mathbb{R}, H^1_q(\Omega))} \\
&\quad \le CT^{1/p'}\|\partial_t(h_1 - h_2)\|_{L_p((0,T), W^{2-1/q}_q(\Gamma))}.
\end{aligned} \tag{150}$$

By Lemma 4 and (108), we have

$$\begin{aligned}
&\|\nabla\mathcal{E}_1[tr[\mathbf{u}_1]]\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot{\Omega}))} + \|\nabla\mathcal{E}_1[tr[\mathbf{u}_1]]\|_{L_p(\mathbb{R}, H^1_q(\dot{\Omega}))} \\
&\quad \le C(\|\mathcal{E}_1[tr[\mathbf{u}_1]]\|_{L_p(\mathbb{R}, H^2_q(\dot{\Omega}))} + \|\partial_t\mathcal{E}_1[tr[\mathbf{u}_1]]\|_{L_p(\mathbb{R}, L_q(\dot{\Omega}))} \\
&\quad \le C(B + L).
\end{aligned}$$

Thus, setting

$$\mathcal{H}^{11}_1 = (\tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot, \bar{\nabla}\mathcal{E}_2[H_{h_1}]) - \tilde{\mathbf{V}}^1_{\mathbf{h}}(\cdot, \bar{\nabla}\mathcal{E}_2[H_{h_2}])) \otimes \nabla\mathcal{E}_1[tr[\mathbf{u}_1]],$$

by Lemma 3 we have

$$\begin{aligned}
&\|e^{-\gamma t}\mathcal{H}^{11}_1\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot{\Omega}))} + \|e^{-\gamma t}\mathcal{H}^{11}_1\|_{L_p((\mathbb{R}, H^1_q(\dot{\Omega}))} \\
&\quad \le CL(B + L)T^{1/p'}(\|\partial_t(h_1 - h_2)\|_{L_p((0,T), W^{2-1/q}_q(\Gamma))} + \|\partial_t(h_1 - h_2)\|_{L_\infty((0,T), W^{1-1/q}_q(\Gamma))}).
\end{aligned} \tag{151}$$

Noticing that $\mathcal{E}_1[tr[\mathbf{u}_1]] - \mathcal{E}_1[tr[\mathbf{u}_2]] = e_T[tr[\mathbf{u}_1] - tr[\mathbf{u}_2]]$, by (55) and Lemma 4, we have

$$\begin{aligned}
&\|e^{-\gamma t}\nabla(\mathcal{E}_1[tr[\mathbf{u}_1]] - \mathcal{E}_2[tr[\mathbf{u}_2]])\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot{\Omega}))} + \|e^{-\gamma t}\nabla(\mathcal{E}_1[tr[\mathbf{u}_1]] - \mathcal{E}_2[tr[\mathbf{u}_2]])\|_{L_p(\mathbb{R}, H^1_q(\dot{\Omega}))} \\
&\quad \le C(\|\mathbf{u}_1 - \mathbf{u}_2\|_{L_p((0,T), H^2_q(\dot{\Omega}))} + \|\partial_t(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_p((0,T), L_q(\dot{\Omega}))}).
\end{aligned}$$

Thus, setting $\mathcal{H}^{12}_1 = \tilde{\mathbf{V}}_{\mathbf{h}}(\cdot, \bar{\nabla}\mathcal{E}_2[H_{h_2}])\bar{\nabla}\mathcal{E}_2[H_{h_2}] \otimes \nabla(\mathcal{E}_1[tr[\mathbf{u}_1]] - \mathcal{E}_1[tr[\mathbf{u}_2]])$, by (112) and Lemma 3, we have

$$\begin{aligned}
&\|e^{-\gamma t}\mathcal{H}^{12}_1\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot{\Omega}))} + \|e^{-\gamma t}\mathcal{H}^{12}_1\|_{L_p(\mathbb{R}, H^1_q(\dot{\Omega}))} \\
&\quad \le CLT^{1/(2p')}(\|\mathbf{u}_1 - \mathbf{u}_2\|_{L_p((0,T), H^2_q(\dot{\Omega}))} + \|\partial_t(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_p((0,T), L_q(\dot{\Omega}))}),
\end{aligned}$$

which, combined with (151), yields that

$$\begin{aligned}
&\|e^{-\gamma t}\mathcal{H}^1_1\|_{H^{1/2}_p(\mathbb{R}, L_q(\dot{\Omega}))} + \|e^{-\gamma t}\mathcal{H}^1_1\|_{L_p(\mathbb{R}, H^1_q(\dot{\Omega}))} \\
&\quad \le CL(B + L)T^{1/(2p')}E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2).
\end{aligned} \tag{152}$$

We next consider $\mathcal{H}_1^2$. Since $\mathcal{E}_1[tr[\mathbf{G}_1]] - \mathcal{E}_2[tr[\mathbf{G}_2]] = e_T[tr[\mathbf{G}_1] - tr[\mathbf{G}_2]]$, we have

$$\begin{aligned}
&\|e^{-\gamma t}(\mathcal{E}_1[tr[\mathbf{G}_1]] - \mathcal{E}_1[tr[\mathbf{G}_2]])\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))} \leq C\|\mathbf{G}_1 - \mathbf{G}_2\|_{H^1_p((0,T),L_q(\dot\Omega))};\\
&\|e^{-\gamma t}(\mathcal{E}_1[tr[\mathbf{G}_1]] - \mathcal{E}_1[tr[\mathbf{G}_2]])\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\
&\quad \leq CT^{1/p}(\|\mathbf{G}_1 - \mathbf{G}_2\|_{L_p((0,T),H^2_q(\dot\Omega))} + \|\partial_t(\mathbf{G}_1 - \mathbf{G}_2)\|_{L_p((0,T),L_q(\dot\Omega))});\\
&\|e^{-\gamma t}(\mathcal{E}_1[tr[\mathbf{G}_1]] - \mathcal{E}_1[tr[\mathbf{G}_2]])\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\\
&\quad \leq \|\mathbf{G}_1 - \mathbf{G}_2\|^{s/(1+s)}_{L_\infty((0,T),L_q(\dot\Omega))}\|\mathbf{G}_1 - \mathbf{G}_2\|^{1/(1+s)}_{L_\infty((0,T),W^s_q(\dot\Omega))}\\
&\quad \leq CT^{\frac{s}{p'(1+s)}}(\|\mathbf{G}_1 - \mathbf{G}_2\|_{L_p((0,T),H^2_q(\dot\Omega))} + \|\partial_t(\mathbf{G}_1 - \mathbf{G}_2)\|_{L_p((0,T),L_q(\dot\Omega))}).
\end{aligned} \tag{153}$$

On the other hand, we have

$$\begin{aligned}
\|\mathcal{E}_1[tr[\mathbf{G}_i]]\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))} &\leq C(L+B);\\
\|\mathcal{E}_1[tr[\mathbf{G}_i]]\|_{L_p(\mathbb{R},L_q(\dot\Omega))} &\leq C(L+B);\\
\|\mathcal{E}_1[tr[\mathbf{G}_i]]\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))} &\leq C(L+B)
\end{aligned} \tag{154}$$

for $i = 1, 2$, and therefore by (115), we have

$$\|e^{-\gamma t}\mathcal{H}_1^2\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} \leq C(L+B)T^{1/(2p)}E^1_T(\mathbf{G}_1 - \mathbf{G}_2). \tag{155}$$

By (89), (91), and (154), we deduce

$$\begin{aligned}
&\|e^{-\gamma t}\mathcal{H}_1^2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \leq C(L+B)\|\mathbf{G}_1 - \mathbf{G}_2\|_{L_p((0,T),H^1_q(\dot\Omega))}\\
&\quad \leq C(L+B)T^{1/p}(\|\partial_t(\mathbf{G}_1 - \mathbf{G}_2)\|_{L_p((0,T),L_q(\dot\Omega))} + \|\mathbf{G}_1 - \mathbf{G}_2\|_{L_p((0,T),H^2_q(\dot\Omega))}),
\end{aligned}$$

which, combined with (155), yields

$$\|e^{-\gamma t}\mathcal{H}_1^2\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathcal{H}_1^2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \leq C(L+B)T^{1/(2p)}E^1_T(\mathbf{G}_1 - \mathbf{G}_2). \tag{156}$$

Since

$$\|\mathcal{E}_1[tr[\mathbf{G}_1]] \otimes \mathcal{E}_1[tr[\mathbf{G}_1]]\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|\mathcal{E}_1[tr[\mathbf{G}_1]] \otimes \mathcal{E}_1[tr[\mathbf{G}_1]]\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \leq C(L+B)^2,$$

by setting

$$\mathcal{H}_1^{31} = (\tilde{\mathbf{V}}^2_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_1}]) - \tilde{\mathbf{V}}^2_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}])\mathcal{E}_1[tr[\mathbf{G}_1]] \otimes \mathcal{E}_1[tr[\mathbf{G}_1]],$$

and using Lemma 3, (149), and (150), we have

$$\begin{aligned}
&\|e^{-\gamma t}\mathcal{H}_1^{31}\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))}\\
&\quad \leq CL(L+B)^2T^{1/p'}(\|\partial_t(h_1 - h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))} + \|\partial_t(h_1 - h_2)\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))}).
\end{aligned} \tag{157}$$

By (11), (110), (108), and (144), we arrive at

$$\begin{aligned}
\|e^{-\gamma t}\mathcal{H}_1^{31}\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} &\leq C(1 + \|\bar\nabla\mathcal{E}_2[H_{h_1}]\|_{L_\infty(\mathbb{R},H^1_q(\Omega))} + \|\bar\nabla\mathcal{E}_2[H_{h_1}]\|_{L_\infty(\mathbb{R},H^1_q(\Omega))})\\
&\quad \times \|\bar\nabla e_T[H_{h_1} - H_{h_2}]\|_{L_p((0,T),H^1_q(\dot\Omega))}\|\mathcal{E}[tr[\mathbf{G}_1]]\|^2_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\\
&\leq C(L+B)^2T^{1/p'}\|\partial_t(h_1 - h_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))},
\end{aligned}$$

which, combined with (157), yields that

$$\|e^{-\gamma t}\mathcal{H}_1^{31}\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathcal{H}_2^{31}\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \\ \leq CL(L+B)^2T^{1/p'}(\|\partial_t(h_1-h_2)\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} + \|\partial_t(h_1-h_2)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))}). \tag{158}$$

Setting $\mathcal{H}_1^{32} = \mathbf{V}_\mathbf{h}^2(\cdot,\bar\nabla\mathcal{E}_2[H_{h_2}])\bar\nabla\mathcal{E}_2[H_{h_2}]\otimes\mathcal{H}_1^2$, by Lemma 3, (112), and (156), we have

$$\|e^{-\gamma t}\mathcal{H}_1^{32}\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathcal{H}_1^{32}\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \leq CL(L+B)T^{1/2}E_T^1(\mathbf{G}_1-\mathbf{G}_2),$$

where we have used $1/p+1/p'=1$. Combined with (158), it yields

$$\|e^{-\gamma t}\mathcal{H}_1^3\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathcal{H}_1^3\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \leq CL(L+B)^2T^{1/p'}E_T^1(\mathbf{G}_1-\mathbf{G}_2). \tag{159}$$

Since

$$\|\bar\nabla^2\mathcal{E}_1[H_{h_1}]\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|\bar\nabla^2\mathcal{E}_1[H_{h_1}]\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \\ \leq C(\|h_1\|_{H_p^1((0,T),W_q^{2-1/q}(\Gamma))} + \|h_1\|_{L_p((0,T),W_q^{3-1/q}(\Gamma))} \\ + \|T_h(\cdot)h_0\|_{H_p^1((0,\infty),H_q^2(\dot\Omega))} + \|T_h(\cdot)h_0\|_{L_p((0,\infty),H_q^3(\dot\Omega))} \\ \leq C(L+\epsilon) \leq 2CL;$$
$$\|e^{-\gamma t}\bar\nabla^2(\mathcal{E}_1[H_{h_1}]-\mathcal{E}_2[H_{h_2}])\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\bar\nabla^2(\mathcal{E}_1[H_{h_1}]-\mathcal{E}_2[H_{h_2}])\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \\ \leq C(\|h_1-h_2\|_{H_p^1((0,T),W_q^{2-1/q}(\Gamma))} + \|h_1-h_2\|_{L_p((0,T),W_q^{3-1/q}(\Gamma))}),$$

by Lemma 3, (112), (149), and (150), we have

$$\|e^{-\gamma t}\mathcal{H}_1^4\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\mathcal{H}_1^4\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \\ \leq CL^2T^{1/(2p')}(E_T^2(h_1-h_2) + \|\partial_t(h_1-h_2)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))}),$$

which, combined with (152), (156), and (159), yields

$$\|e^{-\gamma t}\tilde{\mathcal{H}}_1\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{H}}_1\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \\ \leq CL(B+L)^2T^{1/(2p)}E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2), \tag{160}$$

where we have used the fact that $1/p<1/p'$.

We now consider $\mathfrak{g}$ and $\mathcal{G}$. In view of (123), we set

$$\tilde{\mathfrak{g}} = \mathcal{G}_1(\bar\nabla\mathcal{E}_2[H_{h_1}])\bar\nabla\mathcal{E}_2[H_{h_1}]\otimes\nabla\mathcal{E}_1[\mathbf{u}_1] - \mathcal{G}_1(\bar\nabla\mathcal{E}_2[H_{h_2}])\bar\nabla\mathcal{E}_2[H_{h_2}]\otimes\nabla\mathcal{E}_1[\mathbf{u}_2],$$
$$\tilde{\mathcal{G}} = \mathcal{G}_2(\bar\nabla\mathcal{E}_2[H_{h_1}])\bar\nabla\mathcal{E}_2[H_{h_1}]\otimes\mathcal{E}_1[\mathbf{u}_1] - \mathcal{G}_2(\bar\nabla\mathcal{E}_2[H_{h_2}])\bar\nabla\mathcal{E}_2[H_{h_2}]\otimes\mathcal{E}_1[\mathbf{u}_2].$$

Then, $\tilde{\mathfrak{g}}$ and $\tilde{\mathcal{G}}$ are defined for $t\in\mathbb{R}$ and $\mathfrak{g}=\tilde{\mathfrak{g}}$, $\mathcal{G}=\tilde{\mathcal{G}}$ for $t\in(0,T)$. Employing the same argument as in proving (152), we have

$$\|e^{-\gamma t}\tilde{\mathfrak{g}}\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathfrak{g}}\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \\ \leq CL(B+L)T^{1/(2p')}E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2). \tag{161}$$

To estimate $\tilde{\mathcal{G}}$, we write $\tilde{\mathcal{G}} = G_1+G_2$ with

$$G_1 = (\tilde{\mathcal{G}}_2(\bar\nabla\mathcal{E}_2[H_{h_1}[) - \tilde{\mathcal{G}}_2[\bar\nabla\mathcal{E}_2[H_{h_2}]))\otimes\mathcal{E}_1[\mathbf{u}_1],$$
$$G_2 = \mathcal{G}_2(\bar\nabla\mathcal{E}_2[H_{h_2}])\bar\nabla\mathcal{E}_2[H_{h_2}]\otimes(\mathcal{E}_1[\mathbf{u}_1]-\mathcal{E}_2[\mathbf{u}_2]),$$

where we have set $\tilde{\mathcal{G}}_2(\mathbf{K}) = \tilde{\mathcal{G}}_2(\mathbf{K})\mathbf{K}$. To estimate $\partial_t G_1$, we write

$$\partial_t G_1 = \int_0^1 (d_{\mathbf{K}}\tilde{\mathcal{G}}_2)(\bar{\nabla}\mathcal{E}_2[H_{h_2}] + \theta\bar{\nabla}(\mathcal{E}_2[H_{h_1}] - \mathcal{E}_2[H_{h_2}])]\, d\theta\bar{\nabla}(\mathcal{E}_2[H_{h_1}] - \mathcal{E}_2[H_{h_2}]) \otimes \partial_t\mathcal{E}_1[\mathbf{u}_1]$$
$$+ \Big(\int_0^1 (d_{\mathbf{K}}\tilde{\mathcal{G}}_2)(\bar{\nabla}\mathcal{E}_2[H_{h_2}] + \theta\bar{\nabla}(\mathcal{E}_2[H_{h_1}] - \mathcal{E}_2[H_{h_2}])]\, d\theta\bar{\nabla}\partial_t(\mathcal{E}_2[H_{h_1}] - \mathcal{E}_2[H_{h_2}])\Big) \otimes \mathcal{E}_1[\mathbf{u}_1]$$
$$+ \Big(\int_0^1 (d^2_{\mathbf{K}}\tilde{\mathcal{G}}_2)(\bar{\nabla}\mathcal{E}_2[H_{h_2}] + \theta\bar{\nabla}(\mathcal{E}_2[H_{h_1}] - \mathcal{E}_2[H_{h_2}])]\partial_t((1-\theta)\bar{\nabla}\mathcal{E}_2[H_{h_2}] + \theta\bar{\nabla}\mathcal{E}_2[H_{h_1}])\, d\theta\Big)$$
$$\otimes \bar{\nabla}(\mathcal{E}_2[H_{h_1}] - \mathcal{E}_2[H_{h_2}]) \otimes \mathcal{E}_1[\mathbf{u}_1].$$

By (11), (89), (108), (109), and (144), we have

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t G_1\|_{L_p(\mathbb{R},L_q(\dot{\Omega}))} \\
&\quad\le C\{\|h_1 - h_2\|_{L_\infty((0,T),W_q^{2-1/q}(\Gamma))}\|\partial_t\mathcal{E}_1[\mathbf{u}_1]\|_{L_p(\mathbb{R},L_q(\dot{\Omega}))} \\
&\qquad + T^{1/p}\|\partial_t(h_1 - h_2)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))}\|\mathcal{E}_1[\mathbf{u}_1]\|_{L_\infty((0,T),H_q^1(\dot{\Omega}))} \\
&\qquad + T^{1/p}(\|\partial_t\mathcal{E}_2[H_{h_1}]\|_{L_\infty(\mathbb{R},W_q^{1-1/q}(\Gamma))} + \|\partial_t\mathcal{E}_2[H_{h_2}]\|_{L_\infty(\mathbb{R},W_q^{1-1/q}(\Gamma))}) \\
&\qquad\quad \times \|h_1 - h_2\|_{L_\infty((0,T),W_q^{2-1/q}(\Gamma))}\|\mathcal{E}_1[\mathbf{u}_1]\|_{L_\infty((0,T),H_q^1(\dot{\Omega}))} \\
&\quad\le C\{T^{1/p'}\|\partial_t(h_1 - h_2)\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}(L+B) \\
&\qquad + TL\|\partial_t(h_1 - h_2)\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}(L+B) \\
&\qquad + T^{1/p}\|\partial_t(h_1 - h_2)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))}(L+B)\} \\
&\quad\le CT^{1/p}(L+B)(\|\partial_t(h_1 - h_2)\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} + \|\partial_t(h_1 - h_2)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))}),
\end{aligned} \tag{162}$$

where we have used $T^{1/p'}L \le 1$. Since $\mathcal{E}_1[\mathbf{u}_1] - \mathcal{E}_2[\mathbf{u}_2] = e_T[\mathbf{u}_1 - \mathbf{u}_2]$, writing

$$\partial_t G_2 = \mathcal{G}_2(\bar{\nabla}\mathcal{E}_2[H_{h_2}])\bar{\nabla}\mathcal{E}_2[H_{h_2}] \otimes \partial_t e_T[\mathbf{u}_1 - \mathbf{u}_2] + \mathcal{G}_2(\bar{\nabla}\mathcal{E}_2[H_{h_2}])\bar{\nabla}\partial_t\mathcal{E}_2[H_{h_2}]) \otimes e_T[\mathbf{u}_1 - \mathbf{u}_2]$$
$$+ (d_{\mathbf{K}}\mathcal{G}_2)(\bar{\nabla}\mathcal{E}_2[H_{h_2}])\partial_t\bar{\nabla}\mathcal{E}_2[H_{h_2}] \otimes \bar{\nabla}\mathcal{E}_2[H_{h_2}] \otimes e_T[\mathbf{u}_1 - \mathbf{u}_2],$$

by (108)–(110) and (148)

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t G_2\|_{L_p(\mathbb{R},L_q(\dot{\Omega}))} \\
&\quad\le CLT^{1/p'}\|\partial_t(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_p((0,T),L_q(\dot{\Omega}))} + \|\partial_t\mathcal{E}_2[H_{h_2}]\|_{L_p(\mathbb{R},W_q^{2-1/q}(\Gamma))}\|\mathbf{u}_1 - \mathbf{u}_2\|_{L_\infty((0,T),L_q(\dot{\Omega}))} \\
&\quad\le C\{LT^{1/p'}\|\partial_t(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_p((0,T),L_q(\dot{\Omega}))} + T^{\frac{s}{p'(1+s)}}LE_T^1(\mathbf{u}_1 - \mathbf{u}_2)\},
\end{aligned}$$

which, combined with (162), yields that

$$\|e^{-\gamma t}\partial_t\tilde{\mathcal{G}}\|_{L_p(\mathbb{R},L_q(\dot{\Omega}))} \le C(L+B)T^{\frac{s}{p'(1+s)}}E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2). \tag{163}$$

Applying Theorem 2 to Equation (139) and using (145), (147), (160), (161), and (163), we have

$$E_T(\mathbf{v}_1 - \mathbf{v}_2) + E_T^2(\rho_1 - \rho_2) \le C(1 + \gamma_1^{1/2})e^{\gamma_1}L^3T^{\frac{s}{p'(1+s)}}E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2), \tag{164}$$

provided that $LT^{1/p'} \le 1$, $0 < T = \kappa = \epsilon < 1$, and $L > B \ge 1$.

Moreover, by the third equation of (139), and (146), we have

$$\|\partial_t(\rho_1 - \rho_2)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))}$$

$$\begin{aligned}
&\le C(B\|\rho_1-\rho_2\|_{L_\infty((0,T),W^{2-1/q}_q(\Gamma))}+\|\mathbf{v}_1-\mathbf{v}_2\|_{L_\infty((0,T),H^1_q(\dot\Omega))}\\
&+T^{1/p'}(L+B)E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2))\\
&\le C(BT^{1/p'}\|\partial_t(\rho_1-\rho_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}+\|\mathbf{v}_1-\mathbf{v}_2\|_{L_\infty((0,T),H^1_q(\dot\Omega))}\\
&+T^{1/p'}(L+B)E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2)),
\end{aligned}$$

which, combined with (164) and $BT^{1/p'}\le 1$, yields that

$$\begin{aligned}
&E^1_T(\mathbf{v}_1-\mathbf{v}_2)+E^2_T(\rho_1-\rho_2)+\|\partial_t(\rho_1-\rho_2)\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))}\\
&\quad\le C((1+\gamma_1^{1/2})e^{\gamma_1}L^3T^{\frac{s}{p'(1+s)}}+(L+B)T^{1/p'})E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2)\\
&\quad\le M_2L^3T^{\frac{s}{p'(1+s)}}E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2),
\end{aligned}\tag{165}$$

with some constant $M_2$ depending on $s\in(0,1-2/p)$ and $\gamma_1>0$, provided that $LT^{1/p'}\le 1$, $1\le B\le L$, and $0<T=\kappa=\epsilon<1$.

Now, we consider $\tilde{\mathbf{H}}=\mathbf{H}_1-\mathbf{H}_2$. We first consider $\mathcal{F}_2$. In view of (31), we may write

$$\mathbf{f}_2(\mathbf{u},\mathbf{G},H_\rho)=\mathbf{V}^2_{\mathbf{f}}(\bar\nabla H_\rho)\mathbf{f}_4(\mathbf{u},\mathbf{G},H_\rho),$$

where

$$\mathbf{f}_4(\mathbf{u},\mathbf{G},H_\rho)=\nabla\mathbf{G}\otimes\partial_tH_\rho+\bar\nabla H_\rho\otimes\nabla^2\mathbf{G}+\bar\nabla^2H_\rho\otimes\nabla\mathbf{G}+\nabla\mathbf{u}\otimes\mathbf{G}+\mathbf{u}\otimes\nabla\mathbf{G},$$

where $\mathbf{V}^2_{\mathbf{f}}(\mathbf{K})$ is some matrix of smooth functions of $\mathbf{K}$ for $|\mathbf{K}|<\delta$. Then, employing the same argument as in proving (145), we have

$$\|\mathcal{F}_2\|_{L_p((0,T),L_q(\dot\Omega))}\le CT^{1/p}(L+B)E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2),\tag{166}$$

provided that $T^{1/p'}L\le 1$, $1<B\le L$, and $0<T=\epsilon=\kappa<1$.

Concerning $\mathcal{H}_2$ and $\mathcal{H}_3$, in view of (129), we define $\tilde{\mathcal{H}}_2$ and $\tilde{\mathcal{H}}_3$ by setting $\tilde{\mathcal{H}}_2=\mathcal{B}_1+b(y)\mathcal{B}_2+\mathcal{B}_3$ with

$$\begin{aligned}
\mathcal{B}_1&=(\tilde{\mathbf{V}}^3_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_1}])-\tilde{\mathbf{V}}^3_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_2}]))\otimes\nabla\mathcal{E}_1[tr[\mathbf{u}_1]]\\
&\quad+\mathbf{V}^3_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_2}])\bar\nabla\mathcal{E}_{\rho_2}[H_{\rho_2}]\otimes\nabla(\mathcal{E}_1[tr[\mathbf{u}_1]]-\mathcal{E}_1[tr[\mathbf{u}_2]]);\\
\mathcal{B}_2&=(\mathcal{E}_1[tr[\mathbf{u}_1]]-\mathcal{E}_1[tr[\mathbf{u}_2]])\otimes\mathcal{E}_1[tr[\mathbf{G}_1]]+\mathcal{E}_1[tr[\mathbf{u}_2]]\otimes(\mathcal{E}_1[tr[\mathbf{G}_1]]-\mathcal{E}_1[tr[\mathbf{G}_2]]);\\
\mathcal{B}_3&=(\tilde{\mathbf{V}}^4_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_1}])-\tilde{\mathbf{V}}^4_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_2}]))\mathcal{E}_1[tr[\mathbf{u}_1]]\otimes\mathcal{E}_1[tr[\mathbf{G}_1]]\\
&\quad+\mathbf{V}^4_{\mathbf{h}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_2}])\bar\nabla\mathcal{E}_2[H_{\rho_2}]\otimes\mathcal{B}_2,
\end{aligned}$$

where we have set

$$\tilde{\mathbf{V}}^3_{\mathbf{h}}(\cdot,\mathbf{K})=\mathbf{V}^3_{\mathbf{h}}(\cdot,\mathbf{K})\mathbf{K},\quad \tilde{\mathbf{V}}^4_{\mathbf{h}}(\cdot,\mathbf{K})=\mathbf{V}^4_{\mathbf{h}}(\cdot,\mathbf{K})\mathbf{K};$$

and

$$\begin{aligned}
\tilde{\mathcal{H}}_3&=-\mu\sum_{j,k=1}^N(\tilde V_{0jk}(\nabla\mathcal{E}_2[H_{\rho_1}])-\tilde V_{0jk}(\nabla\mathcal{E}_2[h_{\rho_2}]))\frac{\partial}{\partial y_k}\mathcal{E}_1[tr[\mathbf{u}_1]]_j\\
&\quad-\sum_{j,k=1}^N V_{0jk}(\nabla\mathcal{E}_2[H_{\rho_2}])\nabla\mathcal{E}_2[H_{\rho_2}]\frac{\partial}{\partial y_k}(\mathcal{E}_1[tr[\mathbf{u}_1]]_j-\mathcal{E}_1[tr[\mathbf{u}_2]]_j).
\end{aligned}$$

Obviously, $\tilde{\mathcal{H}}_i$ is defined for $t \in \mathbb{R}$, and $\tilde{\mathcal{H}}_i = \mathcal{H}_i$ for $t \in (0,T)$ for $i = 3,4$. Employing the same argument as in proving (152) and (160), we have

$$\begin{aligned} &\|e^{-\gamma t}\tilde{\mathcal{H}}_3\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{H}}_3\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \\ &\quad \leq CL(B+L)T^{1/(2p')}E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2); \\ &\|e^{-\gamma t}\tilde{\mathcal{H}}_2\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{H}}_2\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \\ &\quad \leq CL(B+L)^2T^{1/(2p)}E_T(\mathbf{u}_1-\mathbf{u}_2,\mathbf{G}_1-\mathbf{G}_2,h_1-h_2), \end{aligned} \tag{167}$$

provided that $T^{1/p'}L \leq 1$, $0 < \epsilon = T < 1$, $1 \leq L$, and $1 \leq B$.

We finally consider $\mathcal{K}_1$ and $\mathcal{K}_2$. As was mentioned in (109), we may assume that

$$\sup_{t\in\mathbb{R}} \|\mathcal{E}_2[H_{\rho_i}]\|_{H^1_\infty(\Omega)} \leq \delta \quad (i=1,2).$$

In view of (135), we set $\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_1 + \tilde{\mathcal{K}}_2$ with

$$\begin{aligned} \tilde{\mathcal{K}}_1 &= (\tilde{\mathbf{V}}^5_{\mathbf{k}}(\cdot\bar\nabla\mathcal{E}_2[H_{\rho_1}]) - \tilde{\mathbf{V}}^5_{\mathbf{k}}(\cdot\bar\nabla\mathcal{E}_2[H_{\rho_2}]))\otimes\mathcal{E}_1[tr[\mathbf{G}_1]] \\ \tilde{\mathcal{K}}_2 &= \mathbf{V}^5_{\mathbf{k}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_2}])\bar\nabla\mathcal{E}_2[H_{\rho_2}]\otimes(\mathcal{E}_1[tr[\mathbf{G}_1]]-\mathcal{E}_2[tr[\mathbf{G}_2]]), \end{aligned}$$

where we have set $\tilde{\mathbf{V}}^5_{\mathbf{k}}(\cdot,\mathbf{K}) = \mathbf{V}^5_{\mathbf{k}}(\cdot,\mathbf{K})\mathbf{K}$. Obviously, $\tilde{\mathcal{K}}$ is defined for $t \in \mathbb{R}$ and $\tilde{\mathcal{K}} = (\mathcal{K}_1,\mathcal{K}_2)$ for $t \in (0,T)$. To estimate $\tilde{\mathcal{K}}_1$, we write

$$\begin{aligned} &\tilde{\mathbf{V}}^5_{\mathbf{k}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_1}]) - \tilde{\mathbf{V}}^5_{\mathbf{k}}(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_2}]) \\ &\quad = \int_0^1 (d_{\mathbf{K}}\tilde{\mathbf{V}}^5_{\mathbf{k}})(\cdot,\bar\nabla\mathcal{E}_2[H_{\rho_2}]+\theta\bar\nabla(\mathcal{E}_2[H_{\rho_1}]-\mathcal{E}_2[H_{\rho_2}]))\,d\theta\bar\nabla e_T[H_{\rho_1}-H_{\rho_2}], \end{aligned}$$

and then, by (90), we have

$$\begin{aligned} &\|\tilde{\mathcal{K}}_1\|_{H^2_q(\dot\Omega)} \\ &\quad\leq C\{\|\mathcal{E}_1[tr[\mathbf{G}_1]]\|_{H^2_q(\dot\Omega)}\|\bar\nabla e_T[H_{\rho_1}-H_{\rho_2}]\|_{H^1_q(\dot\Omega)} \\ &\quad+ \|\mathcal{E}_1[tr[\mathbf{G}_1]]\|_{H^1_q(\dot\Omega)}\|\bar\nabla e_T[H_{\rho_1}-H_{\rho_2}]\|_{H^2_q(\Omega)} \\ &\quad+ (\|\bar\nabla\mathcal{E}_2[H_{\rho_1}]\|_{H^2_q(\Omega)} + \|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^2_q(\Omega)})(1+\|\bar\nabla\mathcal{E}_2[H_{\rho_1}]\|_{H^1_q(\Omega)} + \|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^1_q(\Omega)}) \\ &\qquad\qquad \times \|\bar\nabla e_T[H_{\rho_1}-H_{\rho_2}]\|_{|H^1_q(\Omega)}\|\mathcal{E}_1[tr[\mathbf{G}_1]]\|_{H^1_q(\dot\Omega)}\}. \end{aligned}$$

Noting that $e_T[H_{\rho_1}-H_{\rho_2})$ vanishes for $t \notin (0,2T)$, we have

$$\|\bar\nabla(H_{\rho_1}-H_{\rho_2})\|_{L_\infty(\mathbb{R},H^1_q(\Omega)} \leq CT^{1/p'}\|\partial_t(\rho_1-\rho_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}.$$

Thus, by (108) and (134),

$$\begin{aligned} \|e^{-\gamma t}\tilde{\mathcal{K}}_1\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))} &\leq C\{T^{1/p'}(L+e^{2(\gamma-\gamma_1)}B)\|\partial_t(\rho_1-\rho_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))} \\ &\quad+ (Be^{2(\gamma-\gamma_1)}+(L+B)T^{\frac{s}{p'(1+s)}})\|\rho_1-\rho_2\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))} \\ &\quad+ L(L+B)T^{1/p'}\|\partial_t(\rho_1-\rho_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}\} \\ &\leq C(e^{2(\gamma-\gamma_1)}B+L(L+B)T^{\frac{s}{p'(1+s)}})E^2_T(\rho_1-\rho_2). \end{aligned}$$

Using (89), we have

$$\|\tilde{\mathcal{K}}_2\|_{H^2_q(\dot\Omega)}$$

$$\leq C\{\|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^2_q(\Omega)}\|e_T[tr[\mathbf{G}_1]-tr[\mathbf{G}_2]]\|_{H^1_q(\dot\Omega)}$$
$$+\|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^1_q(\Omega)}\|e_T[tr[\mathbf{G}_1]-tr[\mathbf{G}_2]]\|_{H^2_q(\dot\Omega)}$$
$$+\|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^2_q(\Omega)}(1+\|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^1_q(\Omega)})\|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^1_q(\Omega)}\|e_T[tr[\mathbf{G}_1]-tr[\mathbf{G}_2]]\|_{H^1_q(\dot\Omega)}.$$

Employing the same argument as in (153), we have

$$\|tr[\mathbf{G}_1]-tr[\mathbf{G}_2]\|_{L_\infty((0,T),H^1_q(\dot\Omega))}\leq CT^{\frac{s}{p'(1+s)}}E^1_T(\mathbf{G}_1-\mathbf{G}_2),$$

for some $s\in(0,1-2/p)$, and so we have

$$\begin{aligned}\|e^{-\gamma t}\tilde{\mathcal{K}}_2\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))}&\leq C\{LT^{\frac{s}{p'(1+s)}}E^1_T(\mathbf{G}_1-\mathbf{G}_2)+LT^{1/p'}\|\mathbf{G}_1-\mathbf{G}_2\|_{L_p((0,T),H^2_q(\dot\Omega))}\}\\&\leq CLT^{\frac{s}{p'(1+s)}}E^1_T(\mathbf{G}_1-\mathbf{G}_2).\end{aligned}$$

By (89) and (90), we have

$$\begin{aligned}&\|\partial_t\tilde{\mathcal{K}}_1\|_{L_q(\dot\Omega)}\\&\leq C\{(\|\partial_t\bar\nabla\mathcal{E}_2[H_{\rho_1}]\|_{L_q(\dot\Omega)}+\|\partial_t\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{L_q(\dot\Omega)})\|\bar\nabla e_T[H_{\rho_1}-H_{\rho_2}]\|_{H^1_q(\dot\Omega)}\|\mathcal{E}_1[tr[\mathbf{G}_1]]\|_{H^1_q(\dot\Omega)}\\&\quad+\|\partial_t\bar\nabla e_T[H_{\rho_1}-H_{\rho_2}]\|_{L_q(\dot\Omega)}\|\mathcal{E}_1[tr[\mathbf{G}_1]]\|_{H^1_q(\dot\Omega)}\\&\quad+\|\bar\nabla e_T[H_{\rho_1}-H_{\rho_2}]\|_{H^1_q(\dot\Omega)}\|\partial_t\mathcal{E}_1[tr[\mathbf{G}_1]]\|_{L_q(\dot\Omega)};\\&\|\partial_t\tilde{\mathcal{K}}_2\|_{L_q(\dot\Omega)}\leq C\{\|\partial_t\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{L_q(\dot\Omega)}\|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^1_q(\dot\Omega)}\|e_T[tr[\mathbf{G}_1]-tr[\mathbf{G}_2]]\|_{H^1_q(\dot\Omega)}\\&\quad+\|\partial_t\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{L_q(\dot\Omega)}\|e_T[tr[\mathbf{G}_1]-tr[\mathbf{G}_2]]\|_{H^1_q(\dot\Omega)}\\&\quad+\|\bar\nabla\mathcal{E}_2[H_{\rho_2}]\|_{H^1_q(\dot\Omega)}\|\partial_t e_T[tr[\mathbf{G}_1]-tr[\mathbf{G}_2]]\|_{L_q(\dot\Omega)}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\|e^{-\gamma t}\partial_t\tilde{\mathcal{K}}_1\|_{L_p(\mathbb{R},L_q(\dot\Omega))}+\|e^{-\gamma t}\partial_t\tilde{\mathcal{K}}_2\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\&\leq C\{TL\|\partial_t(\rho_1-\rho_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}(B+L)\\&\quad+T^{1/p}\|\partial_t(\rho_1-\rho_2)\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))}(B+L)\\&\quad+T^{1/p'}\|\partial_t(\rho_1-\rho_2)\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}(B+L)+T^{1/p}LT^{\frac{s}{p'(1+s)}}E^1_T(\mathbf{G}_1-\mathbf{G}_2)\\&\quad+T^{1/p}LT^{\frac{s}{p'(1+s)}}E^1_T(\mathbf{G}_1-\mathbf{G}_2)+LT^{1/p'}\|\partial_t(\mathbf{G}_1-\mathbf{G}_2)\|_{L_p((0,T),L_q(\dot\Omega))}\}\\&\leq CT^{1/p}(L+B)(\tilde E^2_T(\rho_1-\rho_2)+E^2_T(\mathbf{G}_1-\mathbf{G}_2)),\end{aligned}$$

where we have set $\tilde E^2_T(\rho_1-\rho_2)=E^2_T(\rho_1-\rho_2)+\|\partial_t(\rho_1-\rho_2)\|_{L_\infty((0,T),W^{1-1/q}_q(\Gamma))}$. Putting these inequalities together, we arrive at

$$\begin{aligned}&\|e^{-\gamma t}\tilde{\mathcal{K}}\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))}+\|e^{-\gamma t}\partial_t\tilde{\mathcal{K}}\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\&\leq C(e^{2(\gamma-\gamma_1)}B+L(L+B)T^{\frac{s}{p'(1+s)}})\tilde E^2_T(\rho_1-\rho_2)\\&\quad+C(L+B)T^{T^{\min(\frac{1}{p'},\frac{1}{p}+\frac{s}{p'(1+s)})}}E^1_T(\mathbf{G}_1-\mathbf{G}_2).\end{aligned}\tag{168}$$

Applying Theorem 3 to Equations (140) and using (166), (167), and (168), we obtain

$$\begin{aligned}
&\|e^{-\gamma t}(\mathbf{H}_1 - \mathbf{H}_2)\|_{L_p((0,T),H^2_q(\dot\Omega))} + \|e^{-\gamma t}\partial_t(\mathbf{H}_1 - \mathbf{H}_2)\|_{L_p((0,T),L_q(\dot\Omega))} \\
&\quad \leq Ce^{\gamma_1}\{(B + L(L+B)T^{\frac{s}{p'(1+s)}})\tilde{E}^2_T(\rho_1 - \rho_2) \\
&\qquad + L(B+L)^2 T^{1/(2p)} E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2)\}.
\end{aligned} \tag{169}$$

Combining (165) and (169) yields that

$$E_T(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{H}_1 - \mathbf{H}_2, \rho_1 - \rho_2) \leq \mathcal{N}_T(L,B) E_T(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{G}_1 - \mathbf{G}_2, h_1 - h_2)$$

with

$$\mathcal{N}_T(L,B) = (Ce^{\gamma_1}(B + L(L+B)T^{\frac{s}{p'(1+s)}}) + 1)M_2 L^3 T^{\frac{s}{p'(1+s)}} + Ce^{\gamma_1} L(B+L)^2 T^{1/(2p)}).$$

Thus, choosing $T$ so small that $\mathcal{N}_T(L,B) \leq 1/2$, we see that $\Phi$ is a contraction map from $U_T$ into itself, and so there is a unique fixed point $(\mathbf{u}, \mathbf{G}, h) \in U_T$ of the map $\Phi$. This $(\mathbf{u}, \mathbf{G}, h)$ solves Equation (72) uniquely and possessing the properties mentioned in Theorem 1. This completes the proof of Theorem 1.

## 6. Concluding Remark

(1) A future work will be to show a global well-posedness for the system (2).
(2) The maximal regularity of some other models of MHD (cf. [22,23]) can be considered.

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