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Solution of Integral Equations Using Some Multiple Fixed Point Results in Special Kinds of Distance Spaces

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Abstract: In this paper, we explore some extensions of multiple fixed point results for various distance spaces such as s-distance space, (s,q)-distance space, and balanced distance space. Some examples are also discussed for the elaboration of these generalized structures. An application of our result that demonstrates the existence of a unique solution of a system of integral equations is also provided.

Keywords: multidimensional fixed point; s-distance space; (s,q)-distance space; balanced distance space

MSC: 47H09; 47H10; 54H25

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1. Introduction

For the consideration of *multiple fixed points*, we begin our survey with on the topic of coupled fixed points. Opoitsev introduced and studied the concept of the *coupled fixed point* and published numerous articles in the period 1975–1986 (see [1–4]). Opoitsev was inspired by some tangible problems arising in the dynamic of collective behavior in mathematical economics and used coupled fixed points for mixed monotone nonlinear operators that satisfy certain non-expansive-type conditions. Later, in 1987, Guo and Lakshmikantham [5], studied the concept of coupled fixed points in connection with coupled quasi-solutions of an initial value problem for ordinary differential Equations (see also [6]) from [5]. In the same year, Chang and Ma [7] discussed some fixed point results and an iterative approximation in order to obtain coupled fixed points with mixed monotone condensing set-valued operators. Next, Chang, Cho, and Huang [8] obtained coupled fixed point results of $\frac{1}{2}$-α-contractive with some generalized condensing mixed monotone operators, where α denotes Kuratowski's measure of non-compactness.

In [9], Bhasker and Lakshmikantham obtained coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces in the presence of Banach contraction-type conditions. Essentially, the results by Bhaskar and Lakshmikantham in [9] incorporate coupled fixed point theorems into the context of bivariable mixed monotone mappings.

In 2010, Samet and Vetro [10] considered a fixed point of $\mathtt{r}$-order to extend the concept of coupled fixed points. After one year, Berinde and Borcut [11] introduced the concept of tripled fixed points and proved the existence and uniqueness of triple fixed-point results using three-variable mixed monotone mappings. Additionally, in 2012, Karapinar and Berinde [12] generalized the triple fixed points with quadruple fixed points and studied them in ordered metric spaces.

After these preliminary papers, a substantial number of articles were dedicated to the study of triple fixed points, quadruple fixed points, as well as multidimensional fixed points, also called fixed points of $m-$order or $m-$tuple fixed points.

In 2016, Choban [13] introduced a new concept of distance spaces, and Berinde and Choban [14,15] further studied ordered distance spaces satisfying certain contraction conditions for the multidimensional fixed points. Ansari et al. introduced the concept of C- and inverse C-class distance in [16,17]. More recently, Rashid et al. [18,19] proved some multidimensional fixed point results for more generalized contractions in C-distance spaces and also presented an application of the main result. The key objective of this article is to study the multiple fixed points in the presence of ordered distance spaces with generalized contractions. Inspired by [18–23], an application of our result that demonstrates the existence of the solution of the system of integral equations is also provided.

2. Preliminaries

Let us include some basic concepts of [14].

A function $d : X \times X \to \mathbb{R}$ is called a distance on a nonempty set X, if for all $\xi, \eta \in X$:

1. $d(\xi, \eta) \geq 0$,
2. If $d(\xi, \eta) + d(\eta, \xi) = 0$ then $\xi = \eta$,
3. If $\xi = \eta$ then $d(\xi, \eta) = 0$.

A sequence $\{\xi_\nu : \nu \in X\}$ in a distance space (X, d) is:

1. convergent and converges toξ if and only if $\lim_{\nu \to \infty}[d(\xi, \xi_\nu) + d(\xi_\nu, \xi)] = 0$,
2. a Cauchy sequence if $\lim_{\mathtt{r}, \nu \to \infty}[d(\xi_\nu, \xi_{\mathtt{r}}) + d(\xi_{\mathtt{r}}, \xi_\nu)] = 0$.

A distance space (X, d) is complete if every Cauchy sequence in X converges to some fixed point $\xi \in X$.

A distance function d on a nonempty set X is called a C-distance if every Cauchy sequence that converges has a unique limit point.

A distance function d on a nonempty set X is symmetric if $d(\xi, \eta) = d(\eta, \xi)$ for all $\xi, \eta \in X$. Let (X, d) be a distance space and $\mathtt{r} \in \mathbb{N}$. Consider

$$d^{\mathtt{r}}((\xi_1, \ldots, \xi_{\mathtt{r}}), (\eta_1, \ldots, \eta_{\mathtt{r}})) = \sup\{d(\xi_\imath, \eta_\imath) : 1 \leq \imath \leq \mathtt{r}\}.$$

Clearly, $d^{\mathtt{r}}$ is a distance on $X^{\mathtt{r}}$.

Proposition 1 ([14])**.** *If (X, d) is a distance space, then $(X^{\mathtt{r}}, d^{\mathtt{r}})$ inherits all properties of (X, d).*

Let $\mathtt{r}$ be any natural number and $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_{\mathtt{r}})$ be a collection of mappings where each $\Upsilon_\imath$ is defined as

$$\{\Upsilon_\imath : \{1, 2, \ldots, \mathtt{r}\} \to \{1, 2, \ldots, \mathtt{r}\} : \ 1 \leq \imath \leq \mathtt{r}\}.$$

Let (X, d) be a distance space and $\Omega : X^{\mathtt{r}} \to X$ be a mapping. The operators Ω and Υ generate another mapping $\Upsilon\Omega : X^{\mathtt{r}} \to X^{\mathtt{r}}$, where $\Upsilon\Omega(\xi_1, \ldots, \xi_{\mathtt{r}}) = (\eta_1, \ldots, \eta_{\mathtt{r}})$ and each

$$\eta_\imath = \Omega(\xi_{\Upsilon_\imath(1)}, \ldots, \xi_{\Upsilon_\imath(\mathtt{r})}),$$

for any $(\xi_1, \ldots, \xi_{\mathtt{r}}) \in X^{\mathtt{r}}$ and $\imath \in \{1, 2, \ldots, \mathtt{r}\}$. A point $\tau = (\tau_1, \ldots, \tau_{\mathtt{r}}) \in X^{\mathtt{r}}$ is called a *multiple fixed point* of Ω with respect to Υ if it becomes a fixed point of $\Upsilon\Omega$, i.e.,

$$\tau_\imath = \Omega(\tau_{\Upsilon_\imath(1)}, \ldots, \tau_{\Upsilon_\imath(\mathtt{r})}) \text{ for any } \imath \in \{1, 2, \ldots, \mathtt{r}\} \ \text{ if } \tau = \Upsilon\Omega(\tau).$$

For any $\tau = (\tau_1, \ldots, \tau_{\mathtt{r}}) \in X^{\mathtt{r}}, \tau(1) = \Upsilon\Omega(\tau)$ and $\tau(\nu + 1) = \Upsilon\Omega(\tau(\nu))$, for each $\nu \in \mathbb{N}$. The sequence $O(\Omega, \Upsilon, \tau) = \{\tau(\nu) : \nu \in \mathbb{N}\}$ is Picard sequence at the point τ corresponding to the operator $\Upsilon\Omega$.

An operator $\Omega : X^{\mathtt{r}} \to X$ is said to be:

(i) Υ-*contractive* if

$$d^{\mathtt{r}}(\Upsilon\Omega(\xi), \Upsilon\Omega(\eta)) < d^{\mathtt{r}}(\xi, \eta), \text{ for all } \xi, \eta \in X^{\mathtt{r}}$$

with $d^{\mathtt{r}}(\xi,\eta) > 0$;

(ii) Y-*contraction* if there exists a number $\kappa \in [0,1)$ such that

$$d(\Omega(\xi_1,\ldots,\xi_{\mathtt{r}}),\Omega(\eta_1,\ldots,\eta_{\mathtt{r}})) \leq \kappa \sup\{d(\xi_{\imath},\eta_{\imath}) : \imath \leq \mathtt{r}\},$$

for all $(\xi_1,\ldots.,\xi_{\mathtt{r}}),(\eta_1,\ldots.,\eta_{\mathtt{r}}) \in X^{\mathtt{r}}$.

Proposition 2 ([18]). *A mapping $\Omega : X^{\mathtt{r}} \to X$ is said to be:*

(i) Y-*Kannan-type contraction if there is some $\kappa \in [0,\frac{1}{2})$ such that*

$$d(\Omega(\tau_1,\ldots,\tau_{\mathtt{r}}),\Omega(\gamma_1,\ldots,\gamma_{\mathtt{r}})) \leq \kappa \sup_{1\leq\imath\leq\mathtt{r}} \begin{bmatrix} d(\tau_{\imath},\Omega(\tau_1,\ldots,\tau_{\mathtt{r}}))+ \\ d(\gamma_{\imath},\Omega(\gamma_1,\ldots,\gamma_{\mathtt{r}})) \end{bmatrix},$$

for any $(\tau_1,\ldots,\tau_{\mathtt{r}}),(\gamma_1,\ldots,\gamma_{\mathtt{r}}) \in X^{\mathtt{r}}$;

(ii) Y-*Chatterjea-type contraction if there is some $\kappa \in [0,\frac{1}{2})$ such that*

$$d(\Omega(\tau_1,\ldots,\tau_{\mathtt{r}}),\Omega(\gamma_1,\ldots,\gamma_{\mathtt{r}})) \leq \kappa \sup_{1\leq\imath\leq\mathtt{r}} \begin{bmatrix} d(\tau_{\imath},\Omega(\gamma_1,\ldots,\gamma_{\mathtt{r}}))+ \\ d(\gamma_{\imath},\Omega(\tau_1,\ldots,\tau_{\mathtt{r}})) \end{bmatrix},$$

for any $(\tau_1,\ldots,\tau_{\mathtt{r}}),(\gamma_1,\ldots,\gamma_{\mathtt{r}}) \in X^{\mathtt{r}}$.

3. *s*-Distance Space

Definition 1. *A function $d : X \times X \to \mathbb{R}$ is called a b-metric space on a nonempty set X if for all $\xi,\eta,\zeta \in X$, d satisfies the following axioms:*

1. $d(\xi,\eta) \geq 0$;
2. $d(\xi,\eta) = 0$ *if and only if* $\xi = \eta$;
3. $d(\xi,\eta) = d(\eta,\xi)$;
4. *for* $b \geq 1$, $d(\xi,\eta) \leq b[d(\xi,\zeta) + d(\zeta,\eta)]$.

The pair (X,d) is called a b-metric space.

Definition 2 ([13]). *A function $d : X \times X \to \mathbb{R}$ is called s-distance on a nonempty set X if for all $\xi,\eta,\zeta \in X$, d satisfies the following axioms:*

1. $d(\xi,\eta) \geq 0$;
2. $d(\xi,\eta) + d(\eta,\xi) = 0$ *if and only if* $\xi = \eta$;
3. *for* $s > 0$, $d(\xi,\eta) \leq s[d(\xi,\zeta) + d(\zeta,\eta)]$.

The pair (X,d) is called s-distance space.

If $d(\xi,\eta) = d(\eta,\xi)$ for all$\xi,\eta \in X$, then d is said to be symmetric s-distance.

Remark 1. *(1) Every b-metric space is ans-distance space but notconversely. (2) In s-distance space, if $\xi_\nu \to \xi$ and $\eta_\nu \to \eta$, then $d(\xi_\nu,\eta_\nu) \nrightarrow d(\xi,\eta)$, indicates that d is not continuous.*

Next, we give an example for clarification of the above remark.

Example 1. *Let $X = \{\alpha_1,\alpha_2,\alpha_3\}$ and $d : X \times X \to [0,\infty)$ be defined by*

$$d(\alpha_1,\alpha_2) = \frac{1}{4},\ d(\alpha_2,\alpha_1) = \frac{1}{2},$$
$$d(\alpha_1,\alpha_3) = d(\alpha_2,\alpha_3) = 1,\ d(\alpha_{\imath},\alpha_{\imath}) = 0,\ \textit{for all } \imath \in \{1,2,3\},$$
$$d(\alpha_3,\alpha_1) = d(\alpha_3,\alpha_2) = \frac{1}{3}.$$

Then, $d(\alpha_\imath, \alpha_j) \geq 0$ for all $\imath, j \in \{1,2,3\}$ and $d(\alpha_\imath, \alpha_j) + d(\alpha_j, \alpha_\imath) = 0$ if and only if $\alpha_\imath = \alpha_j$ for all $\imath, j \in \{1,2,3\}$. Furthermore,

$$\begin{aligned}
d(\alpha_1, \alpha_2) &< [d(\alpha_1, \alpha_3) + d(\alpha_3, \alpha_2)], \\
d(\alpha_1, \alpha_3) &< [d(\alpha_1, \alpha_2) + d(\alpha_2, \alpha_3)], \\
d(\alpha_2, \alpha_3) &< [d(\alpha_2, \alpha_1) + d(\alpha_1, \alpha_3)], \\
d(\alpha_2, \alpha_1) &< [d(\alpha_2, \alpha_3) + d(\alpha_3, \alpha_1)], \\
d(\alpha_3, \alpha_1) &< [d(\alpha_3, \alpha_2) + d(\alpha_2, \alpha_1)], \\
d(\alpha_2, \alpha_3) &< [d(\alpha_2, \alpha_1) + d(\alpha_1, \alpha_3)].
\end{aligned}$$

Hence, (X, d) is an s-distance space for any $s \geq 1$. However, it is not a b-metric space.

Theorem 1. *Let $\Omega : X^{\mathfrak{r}} \to X$ be a mapping on a complete symmetric s-distance space (X, d). If Ω is an Υ-Kannan-type contraction with $s\beta < 1, \beta = \frac{\kappa}{1-\kappa}, \kappa \in [0, \frac{1}{2})$ and for $\tau \in X^{\mathfrak{r}}$, the sequence $O(\Omega, \Upsilon, \tau) = \{\tau(\nu) : \nu \in \mathbb{N}\}$ is Picard, then $\{\tau(\nu) : \nu \in \mathbb{N}\}$ is Cauchy and Ω has a unique multiple fixed point.*

Proof. Let $\tau \in X^{\mathfrak{r}}$ and consider the Picard sequence $\tau(1) = \Upsilon\Omega(\tau), \tau(\nu+1) = \Upsilon\Omega(\tau(\nu))$. Firstly, we have to show $\{\tau(\nu)\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence. For this, consider

$$d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)) = d^{\mathfrak{r}}(\Upsilon\Omega(\tau(\nu-2)), \Upsilon\Omega(\tau(\nu-1))).$$

Then, there exists $\kappa \in [0, \frac{1}{2})$ such that

$$\begin{aligned}
d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)) &\leq \kappa[d^{\mathfrak{r}}(\tau(\nu-2), \Upsilon\Omega(\tau(\nu-2)) + d^{\mathfrak{r}}(\tau(\nu-1), \Upsilon\Omega(\tau(\nu-1))] \\
&\leq \kappa[d^{\mathfrak{r}}(\tau(\nu-2), \tau(\nu-1)) + d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu))] \\
&\leq \frac{\kappa}{1-\kappa} d^{\mathfrak{r}}(\tau(\nu-2), \tau(\nu-1)) \\
&\vdots \\
&\leq (\frac{\kappa}{1-\kappa})^{\nu-2} d^{\mathfrak{r}}(\tau(1), \tau(2)) \\
&= \beta^{\nu-2} d^{\mathfrak{r}}(\tau(1), \tau(2)),
\end{aligned}$$

where $\beta = \frac{\kappa}{1-\kappa}$, $0 < \beta < 1$. By applying limit $\nu \to \infty$, we obtain

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)) = 0.$$

Now, for $l > \nu$

$$\begin{aligned}
d^{\mathfrak{r}}(\tau(\nu), \tau(l)) &\leq s[d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) + d^{\mathfrak{r}}(\tau(\nu+1), \tau(l))] \\
&\leq sd^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) + s^2 d^{\mathfrak{r}}(\tau(\nu+1), \tau(\nu+2)) \\
&\quad + s^3 d^{\mathfrak{r}}(\tau(\nu+2), \tau(\nu+3)) \\
&\quad + \cdots + s^{l-\nu-1}[d^{\mathfrak{r}}(\tau(l-2), \tau(l-1)) + d^{\mathfrak{r}}(\tau(l-1), \tau(l))] \\
&\leq s\beta^{\nu-1} d^{\mathfrak{r}}(\tau(1), \tau(2)) + s^2\beta^{\nu} d^{\mathfrak{r}}(\tau(1), \tau(2)) + s^3\beta^{\nu+1} d^{\mathfrak{r}}(\tau(1), \tau(2)) \\
&\quad + \cdots + s^{l-\nu-1}\beta^{l-3} d^{\mathfrak{r}}(\tau(1), \tau(2)) + s^{l-\nu}\beta^{l-2} d^{\mathfrak{r}}(\tau(1), \tau(2)) \\
&\leq s\beta^{\nu-1} d^{\mathfrak{r}}(\tau(1), \tau(2))[1 + s\beta + s^2\beta^2 + \cdots + s^{l-\nu-1}\beta^{l-\nu-1}] \\
&\leq s\beta^{\nu-1} d^{\mathfrak{r}}(\tau(1), \tau(2)) \left(\frac{1 - (s\beta)^{l-\nu}}{1 - s\beta} \right).
\end{aligned}$$

Letting $l, \nu \to \infty$, the above expression converges to 0.

Hence, $\{\tau(\nu)\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence, and because the space is complete, there exists $\mu \in X^{\mathtt{r}}$ such that $\tau(\nu) \to \mu$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_{\mathtt{r}})$. We need to show that $\mu = \Upsilon\Omega(\mu)$.
For this, consider

$$\begin{aligned} d^{\mathtt{r}}(\Upsilon\Omega(\mu), \mu) &\leq s[d^{\mathtt{r}}(\Upsilon\Omega(\mu), \tau(\nu)) + d^{\mathtt{r}}(\tau(\nu), \mu)] \\ &\leq s[d^{\mathtt{r}}(\Upsilon\Omega(\mu), \Upsilon\Omega(\tau(\nu-1)) + d^{\mathtt{r}}(\tau(\nu), \mu)] \\ &\leq s[\kappa(d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) + d^{\mathtt{r}}(\tau(\nu-1), \Upsilon\Omega(\tau(\nu-1)))) + d^{\mathtt{r}}(\tau(\nu), \mu)] \\ &\leq s[\kappa(d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) + d^{\mathtt{r}}(\tau(\nu-1), \tau(\nu))) + d^{\mathtt{r}}(\tau(\nu), \mu)], \end{aligned}$$

which implies that

$$(1 - s\kappa)d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) \leq s\kappa d^{\mathtt{r}}(\tau(\nu-1), \tau(\nu)) + sd^{\mathtt{r}}(\tau(\nu), \mu).$$

Because $\lim_{\nu\to\infty} d^{\mathtt{r}}(\tau(\nu), \tau(\nu+1)) = 0$ and $\tau(\nu) \to \mu$, then $\lim_{\nu\to\infty} d^{\mathtt{r}}(\tau(\nu), \mu) = 0$. Now, by applying limit $\nu \to \infty$ to the above inequality and then using these conditions in the resulting expression, it follows that

$$(1 - s\kappa)d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) \leq 0.$$

Given $s\beta < 1$, which further implies $s\kappa < 1$, $d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) = 0$, and hence, $\mu = \Upsilon\Omega(\mu)$. Therefore,

$$\mu_{\imath} = \Omega(\mu_{\Upsilon_{\imath}(1)}, \mu_{\Upsilon_{\imath}(2)}, \ldots, \mu_{\Upsilon_{\imath}(\mathtt{r})}),$$

i.e., Ω has a multidimensional fixed point.

To prove the uniqueness of the fixed point of $\Upsilon\Omega$, suppose on contrary that $v \in X^{\mathtt{r}}$ is another fixed point of $\Upsilon\Omega$ with $v \neq \mu$. Consider

$$\begin{aligned} d^{\mathtt{r}}(\mu, v) &= d^{\mathtt{r}}(\Upsilon\Omega(\mu), \Upsilon\Omega(v)) \\ &\leq \kappa[d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) + d^{\mathtt{r}}(v, \Upsilon\Omega(v))] \\ &\leq \kappa[d^{\mathtt{r}}(\mu, \mu) + d^{\mathtt{r}}(v, v)] = 0. \end{aligned}$$

So, $\mu = v$. Thus, the proof is completed. □

Example 2. *Let* $X = \left\{\frac{1}{\nu} : \nu \in \mathbb{N}\right\} \cup \{0\}$. *Define for all* $\nu, m \in \mathbb{N}$

$$d\left(0, \frac{1}{\nu}\right) = d\left(\frac{1}{\nu}, 0\right) = \frac{1}{5\nu}, d(x_\nu, x_m) = |x_\nu - x_m|.$$

Then, (X, d) *is a symmetric s-distance space. Now,*

$$X \times X = \{(x, y),\ x, y \in X\},$$

and

$$d^2(x, y) = \sup_{i\leq 2}\{d(x_i, y_i)\}.$$

Then, (X^2, d^2) *is also a symmetric s-distance space.*

Now, define a mapping $\Omega : X^2 \to X$ *such that*

$$\Omega(x_1, x_2) = \frac{x_1}{4} \text{ for all } (x_1, x_2) \in X^2,$$

and a mapping $\Upsilon : X \to X^2$ *such that*

$$\Upsilon(x) = (\Upsilon_1(x), \Upsilon_2(x)),$$

where $\Upsilon_i : \{1,2\} \to \{1,2\}$ *are defined as*

$$\begin{pmatrix} \Upsilon_1(1) & \Upsilon_1(2) \\ \Upsilon_2(1) & \Upsilon_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Define $\Upsilon\Omega : X^2 \to X^2$, *which is a composition of* Ω *and* Υ, *as*

$$\Upsilon\Omega(x_1, x_2) = \left(\Omega\left(x_{\Upsilon_1(1)}, x_{\Upsilon_2(2)}\right), \Omega\left(x_{\Upsilon_2(1)}, x_{\Upsilon_2(2)}\right)\right) = \left(\frac{x_1}{4}, \frac{x_2}{4}\right).$$

Consider

$$d(\Omega(x_1, x_2), \Omega(y_1, y_2)) = d\left(\frac{x_1}{4}, \frac{y_1}{4}\right).$$

We need to show that Ω *is an* Υ*-Kannan-type contraction that is*

$$d(\Omega(x_1, x_2), \Omega(y_1, y_2)) \le k\left[\sup_{i\le 2}\{d(x_i, \Omega(x_1, x_2))\} + \sup_{i\le 2}\{d(y_i, \Omega(y_1, y_2))\}\right] \tag{1}$$

where $k \in \left[0, \frac{1}{2}\right)$.

If $x = \left(\frac{1}{\nu_1}, 0\right)$ *and* $y = \left(\frac{1}{\nu_2}, 0\right)$ *then*

$$\begin{aligned} d\left(\Omega\left(\frac{1}{\nu_1}, 0\right), \Omega\left(\frac{1}{\nu_2}, 0\right)\right) &= d\left(\frac{1}{4\nu_1}, \frac{1}{4\nu_2}\right) = \left|\frac{1}{4\nu_1} - \frac{1}{4\nu_2}\right|. \\ &\le \frac{1}{4\nu_1} + \frac{1}{4\nu_2} \end{aligned} \tag{2}$$

Now consider

$$\begin{aligned} &\sup_{i\le 2}\{d(x_i, \Omega(x_1, x_2))\} + \sup_{i\le 2}\{d(y_i, \Omega(y_1, y_2))\} \\ =\ &\sup_{i\le 2}\left\{d\left(x_i, \frac{x_1}{4}\right)\right\} + \sup_{i\le 2}\left\{d\left(y_i, \frac{y_1}{4}\right)\right\} \\ =\ &\sup\left\{d\left(x_1, \frac{x_1}{4}\right), d\left(x_2, \frac{x_1}{4}\right)\right\} + \sup\left\{d\left(y_1, \frac{y_1}{4}\right), d\left(y_2, \frac{y_1}{4}\right)\right\} \\ =\ &\sup\left\{d\left(\frac{1}{\nu_1}, \frac{1}{4\nu_1}\right), d\left(0, \frac{1}{4\nu_1}\right)\right\} + \sup\left\{d\left(\frac{1}{\nu_2}, \frac{1}{4\nu_2}\right), d\left(0, \frac{1}{4\nu_2}\right)\right\} \\ =\ &\sup\left\{\frac{3}{4\nu_1}, \frac{1}{20\nu_1}\right\} + \sup\left\{\frac{3}{4\nu_2}, \frac{1}{20\nu_2}\right\} \\ =\ &\frac{3}{4\nu_1} + \frac{3}{4\nu_2}. \end{aligned} \tag{3}$$

From (2) *and* (3) *we obtain*

$$\frac{1}{4\nu_1} + \frac{1}{4\nu_2} \le \frac{1}{3}\left(\frac{3}{4\nu_1} + \frac{3}{4\nu_2}\right),$$

that is,

$$d\left(\Omega\left(0, \frac{1}{\nu_1}\right), \Omega\left(\frac{1}{\nu_2}, 0\right)\right) < \frac{1}{3}\left[\sup_{i\le 2}\left\{d\left(x_i, \Omega\left(0, \frac{1}{\nu_1}\right)\right)\right\} + \sup_{i\le 2}\left\{d\left(y_i, \Omega\left(\frac{1}{\nu_2}, 0\right)\right)\right\}\right].$$

Similarly for the other values of x *and* y, *condition* 1 *is easily verified, so* Ω *is an* Υ*-Kannan-type contraction. Then, the mapping* $\Upsilon\Omega$ *is Kannan contraction and it has a fixed point.*

Now, $x^{\nu} = \Upsilon\Omega\big(x^{\nu-1}\big)$ *for* $x \in X^2$*. Choose* $x = \big(x_1^0, x_2^0\big)$

$$\begin{aligned} \Big(x_1^1, x_2^1\Big) &= \Upsilon\Omega\Big(x_1^0, x_2^0\Big) = \left(\frac{x_1^0}{3}, \frac{x_2^0}{3}\right), \\ \Big(x_1^2, x_2^2\Big) &= \Upsilon\Omega\Big(x_1^1, x_2^1\Big) = \left(\frac{x_1^0}{3^2}, \frac{x_2^0}{3^2}\right), \\ &\vdots \\ (x_1^{\nu}, x_2^{\nu}) &= \Upsilon\Omega\Big(x_1^{\nu-1}, x_2^{\nu-1}\Big) = \left(\frac{x_1^0}{3^{\nu}}, \frac{x_2^0}{3^{\nu}}\right). \end{aligned}$$

Applying limit $\nu \to \infty$*, we obtain*

$$\lim_{\nu\to\infty}(x_1^{\nu}, x_2^{\nu}) = (0,0) = (O_1, O_2),$$

which is a unique fixed point for $\Upsilon\Omega$ *and a unique multidimensional fixed point for* Ω*, i.e.,*

$$O_i = \Omega\Big(O_{\Upsilon_i(1)}, O_{\Upsilon_i(2)}\Big).$$

Corollary 1. *Let* (X,d) *be a complete symmetric s-distance space and* $\Omega : X^{\mathtt{r}} \to X$ *be a given mapping. If* Ω *is an* Υ*-contraction with* $s\beta < 1, \beta = \frac{\kappa}{1-\kappa}, \kappa \in [0, \frac{1}{2})$ *and for* $\tau \in X^{\mathtt{r}}$*, the sequence* $(\tau(\nu))_{\nu\in\gneqq}$ *is Picard, then* $(\tau(\nu))_{\nu\in\gneqq}$ *is Cauchy with a unique multiple fixed point of* Ω*.*

Corollary 2. *Let* (X,d) *be a complete b-metric space and* $\Omega : X^{\mathtt{r}} \to X$ *be a given mapping. If* Ω *is an* Υ*-Kannan-type contraction with* $b\beta < 1, \beta = \frac{\kappa}{1-\kappa}, \kappa \in [0, \frac{1}{2})$*, then the Picard sequence of the mapping* $\Upsilon\Omega$ *is Cauchy and* Ω *has a unique multiple fixed point.*

Proof. It can be proven easily along similar lines to the above theorem by inputting $s \geq 1$. □

Theorem 2. *Let a mapping* $\Omega : X^{\mathtt{r}} \to X$ *on a complete s-distance space an* Υ*-Chatterjea-type contraction with* $b\beta < 1, \beta = \frac{\kappa}{1-\kappa}, \kappa \in [0, \frac{1}{2})$*. Then, the Picard sequence* $(\tau(\nu))_{\nu\in\gneqq}$ *is Cauchy and* Ω *has a unique multiple fixed point.*

4. (s,q)-Distance Space

Definition 3 ([13])**.** *A function* $d : X \times X \to \mathbb{R}$ *on a nonempty set* X *is called* (s,q)*-distance if for all* $\xi, \eta, \zeta \in X$*,* d *satisfies the following axioms:*

1. $d(\xi,\eta) \geq 0$;
2. *If* $d(\xi,\eta) + d(\eta,\xi) = 0$ *then* $\xi = \eta$;
3. *If* $\xi = \eta$ *then* $d(\xi,\eta) = 0$;
4. $d(\xi,\eta) \leq qd(\eta,\xi)$*, for* $q > 0$;
5. *For* $s > 0$, $d(\xi,\eta) \leq s[d(\xi,\zeta) + d(\zeta,\eta)]$.

Remark 2. *(1) The class of s-distance spaces generalizes the class of* (s,q)*-distance spaces, that is, every* (s,q)*-distance space is an s-distance space but not conversely.*
(2) An (s,q)*-distance* d *is not necessarily a continuous function.*

Now, we have an example of an (s,q)-distance space.

Example 3. *Let* $X = \{\alpha_1, \alpha_2, \alpha_3\}$*and* $d : X \times X \to [0, \infty)$ *be defined as:*

$$\begin{aligned} d(\alpha_1, \alpha_2) &= \frac{1}{2}, d(\alpha_2, \alpha_1) = \frac{1}{4}, \\ d(\alpha_1, \alpha_3) &= d(\alpha_2, \alpha_3) = 1, d(\alpha_\imath, \alpha_\imath) = 0, \textit{ for all } i \in \{1, 2, 3\}, \\ d(\alpha_3, \alpha_1) &= d(\alpha_3, \alpha_2) = \frac{1}{2}. \end{aligned}$$

Clearly, $d(\alpha_\imath, \alpha_j) \geq 0$, $d(\alpha_\imath, \alpha_j) + d(\alpha_j, \alpha_\imath) = 0$ *if and only if* $\alpha_\imath = \alpha_j$ *and* $d(\alpha_j, \alpha_\imath) = \frac{1}{2}\, d(\alpha_\imath, \alpha_j)$ *for all* $\imath, j \in \{1, 2, 3\}$.

Furthermore,

$$\begin{aligned} d(\alpha_1, \alpha_2) &< \delta[d(\alpha_1, \alpha_3) + d(\alpha_3, \alpha_2)], \\ d(\alpha_1, \alpha_3) &< \delta[d(\alpha_1, \alpha_2) + d(\alpha_2, \alpha_3)], \\ d(\alpha_2, \alpha_3) &< \delta[d(\alpha_2, \alpha_1) + d(\alpha_1, \alpha_3)], \\ d(\alpha_2, \alpha_1) &< \delta[d(\alpha_2, \alpha_3) + d(\alpha_3, \alpha_1)], \\ d(\alpha_3, \alpha_1) &< \delta[d(\alpha_3, \alpha_2) + d(\alpha_2, \alpha_1)], \\ d(\alpha_3, \alpha_2) &< \delta[d(\alpha_3, \alpha_1) + d(\alpha_1, \alpha_2)]. \end{aligned}$$

Hence, d *is an* (s, q)*-distance on* X *with* $q = \frac{1}{2}$ *and* $s = \delta \geq 1$.

Example 4. *Let* $X = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. *A function* $\rho : X \times X \to \mathbb{R}$ *is defined as:*

$$\rho(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y} \\ 2(|y_1| + |y_2|), & \mathbf{x} = (1, 1) \\ 1, & \textit{otherwise}, \end{cases}$$

is clearly a distance function. Furthermore, the above function satisfies

$$\rho(\mathbf{x}, \mathbf{y}) \leq q\rho(\mathbf{y}, \mathbf{x}), \textit{ for all } \mathbf{x}, \mathbf{y} \in \mathbf{X} \textit{ and } q \geq 4.$$

Because

$$\rho(\mathbf{x}, \mathbf{y}) = \begin{cases} 4 & \mathbf{x} = (1, 1) \\ 1 & \textit{otherwise} \end{cases}$$

therefore

$$\rho((1, 1), (x_1, x_2)) < s[\rho((1, 1), (y_1, y_2)) + \rho((y_1, y_2), (x_1, x_2))], \textit{for any } s \geq 2.$$

Hence, ρ *is an* (s, q)*-distance on* X.

Theorem 3. *Consider a mapping* $\Omega : X^{\mathfrak{r}} \to X$ *on a complete* (s, q)*-distance space. If* Ω *is an* Υ*-contraction with* $s\kappa < 1$*, then the Picard sequence* $(\tau(\nu))_{\nu \in \gneqq}$ *is Cauchy, and as a result,* Ω *possesses a unique multiple fixed point.*

Proof. Consider the Picard sequence of the operator $\Upsilon\Omega$. Let $\tau \in X^{\mathfrak{r}}$ and $\tau(1) = \Upsilon\Omega(\tau)$, $\tau(\nu + 1) = \Upsilon\Omega(\tau(\nu))$. We need to show that $d^{\mathfrak{r}}(\tau(\nu), \tau(l)) \longrightarrow 0$. Consider

$$\begin{aligned} d^{\mathfrak{r}}(\tau(\nu), \tau(\nu + 1)) &= d^{\mathfrak{r}}(\Upsilon\Omega(\tau(\nu - 1)), \Upsilon\Omega(\tau(\nu))) \\ &\leq \kappa d^{\mathfrak{r}}(\tau(\nu - 1), \tau(\nu)) \\ &\vdots \\ &\leq \kappa^{\nu} d^{\mathfrak{r}}(\tau, \tau(1)). \end{aligned}$$

Thus,

$$\lim_{\nu \to \infty} d^{\mathfrak{r}}(\tau(\nu), \tau(\nu + 1)) = 0.$$

Because

$$d^{\mathfrak{r}}(\tau(\nu+1),\tau(\nu)) \leq qd^{\mathfrak{r}}(\tau(\nu),\tau(\nu+1)),$$

we obtain

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu+1),\tau(\nu)) = 0.$$

Now consider for $l > \nu$

$$\begin{aligned} d^{\mathfrak{r}}(\tau(\nu),\tau(l)) &\leq sd^{\mathfrak{r}}(\tau(\nu),\tau(\nu+1)) + sd^{\mathfrak{r}}(\tau(\nu+1),\tau(l)) \\ &\leq sd^{\mathfrak{r}}(\tau(\nu),\tau(\nu+1)) + s^2 d^{\mathfrak{r}}(\tau(\nu+1),\tau(\nu+2)) \\ &\quad + \cdots + s^{l-\nu-1}[d^{\mathfrak{r}}(\tau(l-2),\tau(l-1)) \\ &\quad + d^{\mathfrak{r}}(\tau(l-1),\tau(l))] \\ &\leq s\kappa^{\nu} d^{\mathfrak{r}}(\tau,\tau(1)) + s^2\kappa^{\nu+1} d^{\mathfrak{r}}(\tau,\tau(1)) \\ &\quad + \cdots + s^{l-\nu}\kappa^{l-1} d^{\mathfrak{r}}(\tau,\tau(1)) \\ &\leq s\kappa^{\nu} d^{\mathfrak{r}}(\tau,\tau(1)) \Big[1 + s\kappa + s^2\kappa^2 + \cdots + s^{l-\nu-1}\kappa^{l-\nu-1}\Big] \\ &\leq s\kappa^{\nu}\left(\frac{(1-(s\kappa)^{l-\nu})}{1-s\kappa}\right) d^{\mathfrak{r}}(\tau,\tau(1)). \end{aligned}$$

When applying the limit $\nu, l \to \infty$ over the above expression, it converges to 0 and because

$$d^{\mathfrak{r}}(\tau(l),\tau(v)) \leq qd^{\mathfrak{r}}(\tau(\nu),\tau(l)),$$

we have

$$\lim_{\nu,l\to\infty} d^{\mathfrak{r}}(\tau(l),\tau(v)) = 0.$$

Hence,

$$\lim_{\nu,l\to\infty} [d^{\mathfrak{r}}(\tau(l),\tau(v)) + d^{\mathfrak{r}}(\tau(\nu),\tau(l))] = 0,$$

that is, $\{\tau(\nu)\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence and because the space is complete, there exists $\mu \in X^{\mathfrak{r}}$ such that $\tau(\nu) \to \mu$, which implies that

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu),\mu) = 0 \text{ and } \lim_{\nu\to\infty} d^{\mathfrak{r}}(\mu,\tau(\nu)) = 0.$$

Now, to show that μ is a fixed point of $\Upsilon\Omega$, consider

$$\begin{aligned} d^{\mathfrak{r}}(\mu,\Upsilon\Omega(\mu)) &\leq s[d^{\mathfrak{r}}(\mu,\tau(\nu)) + d^{\mathfrak{r}}(\tau(\nu),\Upsilon\Omega(\mu))] \\ &\leq s[d^{\mathfrak{r}}(\mu,\tau(\nu)) + \kappa d^{\mathfrak{r}}(\tau(\nu-1),\mu)] \\ &\leq sd^{\mathfrak{r}}(\mu,\tau(\nu)) + s\kappa d^{\mathfrak{r}}(\tau(\nu-1),\mu). \end{aligned}$$

Taking the limit $\nu \to \infty$, we have

$$d^{\mathfrak{r}}(\mu,\Upsilon\Omega(\mu)) = 0.$$

As $d^{\mathfrak{r}}(\Upsilon\Omega(\mu),\mu) \leq qd^{\mathfrak{r}}(\mu,\Upsilon\Omega(\mu)) = 0$, thus $d^{\mathfrak{r}}(\Upsilon\Omega(\mu),\mu) = 0$, which further implies that

$$d^{\mathfrak{r}}(\Upsilon\Omega(\mu),\mu) + d^{\mathfrak{r}}(\mu,\Upsilon\Omega(\mu)) = 0,$$

and hence $\mu = \Upsilon\Omega(\mu)$.

Suppose that v is another fixed point of $\Upsilon\Omega$ different from μ. Consider

$$\begin{aligned} d^{\mathfrak{r}}(\mu,v) + d^{\mathfrak{r}}(v,\mu) &= d^{\mathfrak{r}}(\Upsilon\Omega(\mu),\Upsilon\Omega(v)) + d^{\mathfrak{r}}(\Upsilon\Omega(v),\Upsilon\Omega(\mu)) \\ &\leq \kappa d^{\mathfrak{r}}(\mu,v) + \kappa d^{\mathfrak{r}}(v,\mu). \end{aligned}$$

This implies $(1-\kappa)[d^{\mathfrak{r}}(\mu, v) + d^{\mathfrak{r}}(v, \mu)] \leq 0$. As $\kappa \in [0,1)$, then $d^{\mathfrak{r}}(\mu, v) + d^{\mathfrak{r}}(v, \mu) = 0$. That is, $\mu = v$, and the proof is completed. □

Theorem 4. *Let (X, d) be a complete (s, q)-distance space and $\Omega : X^{\mathfrak{r}} \to X$. If Ω is an Υ-Kannan-type contraction with $s\beta < 1$, avd $\beta = \frac{\kappa}{1-\kappa}$, then any Picard sequence of $\Upsilon\Omega$ is Cauchy and Ω has a unique multiple fixed point.*

Proof. Let $\tau \in X^{\mathfrak{r}}$ and $\tau(1) = \Upsilon\Omega(\tau), \tau(\nu+1) = \Upsilon\Omega(\tau(\nu))$. Consider

$$\begin{aligned} d^{\mathfrak{r}}(\tau(\nu), (\nu+1)) &= d^{\mathfrak{r}}(\Upsilon\Omega(\tau(\nu-1)), \Upsilon\Omega(\tau(\nu))) \\ &\leq \kappa\begin{bmatrix} d^{\mathfrak{r}}(\tau(\nu-1), \Upsilon\Omega(\tau(\nu-1))) \\ +d^{\mathfrak{r}}(\tau(\nu), \Upsilon\Omega(\tau(\nu))) \end{bmatrix} \\ &\leq \kappa[d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)) + d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1))], \end{aligned}$$

which implies

$$\begin{aligned} d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) &\leq \frac{\kappa}{1-\kappa} d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)) \\ &\vdots \\ &\leq \left(\frac{\kappa}{1-\kappa}\right)^{\nu-1} d^{\mathfrak{r}}(\tau(1), \tau(2)), \end{aligned}$$

and

$$\lim_{v\to\infty} d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) = 0.$$

Now consider, for $l > \nu$

$$\begin{aligned} d^{\mathfrak{r}}(\tau(\nu), \tau(l)) &= d^{\mathfrak{r}}(\Upsilon\Omega(\tau(\nu-1)), \Upsilon\Omega(\tau(l-1))) \\ &\leq \kappa[d^{\mathfrak{r}}(\tau(\nu-1), \Upsilon\Omega(\tau(\nu-1))) + d^{\mathfrak{r}}(\tau(l-1), \Upsilon\Omega(\tau(l-1)))] \\ &\leq \kappa[d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)) + d^{\mathfrak{r}}(\tau(l-1), \tau(l))] \\ &\leq \kappa\begin{bmatrix} \left(\frac{\kappa}{1-\kappa}\right)^{\nu-2} d^{\mathfrak{r}}(\tau(1), \tau(2)) + \\ \left(\frac{\kappa}{1-\kappa}\right)^{l-\nu+2} d^{\mathfrak{r}}(\tau(l-\nu+1), \tau(l-\nu)) \end{bmatrix}. \end{aligned}$$

Applying limit $l, \nu \to \infty$ over the above, we obtain

$$\lim_{l,\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu), \tau(l)) = 0.$$

Similar steps to those of the above theorems can be used to prove

$$\lim_{\nu,l\to\infty} [d^{\mathfrak{r}}(\tau(l), \tau(v)) + d^{\mathfrak{r}}(\tau(\nu), \tau(l))] = 0.$$

Thus, $\{\tau(\nu)\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence.

Because $(X^{\mathfrak{r}}, d^{\mathfrak{r}})$ is complete, there exists $\mu \in X^{\mathfrak{r}}$ such that $\tau(\nu) \to \mu$, i.e.,

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu), \mu) = 0 \text{ and } \lim_{\nu\to\infty} d^{\mathfrak{r}}(\mu, \tau(\nu)) = 0.$$

Now, to show that μ is a fixed point of $\Upsilon\Omega$, consider

$$\begin{aligned} d^{\mathfrak{r}}(\mu, \Upsilon\Omega(\mu)) &\leq s[d^{\mathfrak{r}}(\mu, \tau(\nu)) + d^{\mathfrak{r}}(\tau(\nu), \Upsilon\Omega(\mu))] \\ &\leq s[d^{\mathfrak{r}}(\mu, \tau(\nu)) + d^{\mathfrak{r}}(\Upsilon\Omega(\tau(\nu-1)), \Upsilon\Omega(\mu))] \\ &\leq s\left[d^{\mathfrak{r}}(\mu, \tau(\nu)) + \kappa\begin{pmatrix} d^{\mathfrak{r}}(\tau(\nu-1), \Upsilon\Omega(\tau(\nu-1))) \\ +d^{\mathfrak{r}}(\mu, \Upsilon\Omega(\mu)) \end{pmatrix}\right] \\ &\leq s[d^{\mathfrak{r}}(\mu, \tau(\nu)) + \kappa(d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)) + d^{\mathfrak{r}}(\mu, \Upsilon\Omega(\mu)))]. \end{aligned}$$

Applying limit $\nu \to \infty$ over the above expression, we have

$$\begin{aligned}
&d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) \\
&\leq s\Big[\lim_{\nu\to\infty} d^{\mathtt{r}}(\mu, \tau(\nu)) + \kappa\Big(\lim_{\nu\to\infty} d^{\mathtt{r}}(\tau(\nu-1), \tau(\nu)) + \lim_{\nu\to\infty} d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu))\Big)\Big] \\
&\leq 0 + 0 + s\kappa d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)).
\end{aligned}$$

Thus,

$$(1 - s\kappa)d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) \leq 0.$$

Then, $(1 - s\kappa)$ cannot be less or equal to 0, so $d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) = 0$ and $d^{\mathtt{r}}(\Upsilon\Omega(\mu), \mu) \leq qd^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) = 0$, which implies $d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu))+ d^{\mathtt{r}}(\Upsilon\Omega(\mu), \mu) = 0$. Consequently, $\mu = \Upsilon\Omega(\mu)$, i.e., $\mu_{\imath} = \Omega(\mu_{\Upsilon_{\imath}(1)}, \mu_{\Upsilon_{\imath}(2)}, \ldots, \mu_{\Upsilon_{\imath}(\mathtt{r})})$.

Now, to show that μ is a unique fixedpoint of $\Upsilon\Omega$, suppose that $\mu \neq v$, and v is also a fixed point of $\Upsilon\Omega$. Then, $v = \Upsilon\Omega(v)$. Consider,

$$\begin{aligned}
d^{\mathtt{r}}(\mu, v) + d^{\mathtt{r}}(v, \mu) &\leq (1 + q)d^{\mathtt{r}}(\Upsilon\Omega(\mu), \Upsilon\Omega(v)) \\
&\leq (1 + q)\kappa[d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) + (v, \Upsilon\Omega(v))] \\
&\leq \kappa(1 + q)[d^{\mathtt{r}}(\mu, \mu) + d^{\mathtt{r}}(v, v)],
\end{aligned}$$

which implies $d^{\mathtt{r}}(\mu, v) + æ^{\mathtt{r}}(v, \mu) = 0$ and thus$\mu = v$. So, μ is the unique fixed point of $\Upsilon\Omega$, i.e., μ is a unique multiple fixed point of Ω. □

5. Balanced Distance Space

Definition 4 ([13]). *Consider a distance space (X, d). If for every Cauchy sequence $(\xi_\nu)_{\nu\in\mathbb{N}}$ which converges to some $\xi \in X$ and any point $\eta \in X$ it satisfies $d(\eta, \xi) = \lim_{\nu\to\infty} d(\eta, \xi_\nu)$, then (X, d) is called a balanced distance space.*

Now, let us give an example for the elaboration of the above class.

Example 5. *Consider $X = \{2^{-\nu} : \nu \in \mathbb{N}\} \cup \{0, \gamma\}$ where $\gamma \geq 1$ and define*

$$\begin{aligned}
&d(0, \gamma) = \gamma,\ d(\gamma, 0) = \gamma + 1, \\
&d\big(2^{-\nu}, 2^{-\mathtt{r}}\big) = |2^{-\nu} - 2^{-\mathtt{r}}| \ \textit{for all } \nu, \mathtt{r} \in \mathbb{N}, \\
&d\big(2^{-\nu}, \gamma\big) = \gamma,\ d\big(\gamma, 2^{-\nu}\big) = \gamma + 1, \\
&d\big(2^{-\nu}, 0\big) = 2^{-\nu} = d\big(0, 2^{-\nu}\big).
\end{aligned}$$

Clearly, $d(\xi, \eta) \geq 0$ and$d(\xi, \eta) + d(\eta, \xi) = 0$ if and only if $\xi = \eta$, for all $\xi, \eta \in X$. Because $(2^{-\nu})_{\nu\in\mathbb{N}}$ is a convergent Cauchy sequenceand $\gamma \in X$, so

$$\begin{aligned}
&\lim_{\nu\to\infty} d(2^{-\nu}, \gamma) = \gamma = d(0, \gamma), \\
&\lim_{\nu\to\infty} d(\gamma, 2^{-\nu}) = \gamma + 1 = d(\gamma, 0), \\
&\lim_{\nu\to\infty} d\big(2^{-\nu}, 0\big) = 0 = d(0, 0) = \lim_{\nu\to\infty} d\big(0, 2^{-\nu}\big), \\
&\lim_{\nu\to\infty} d\big(2^{-\nu}, 2^{-\mathtt{r}}\big) = 2^{-\mathtt{r}} = d\big(0, 2^{-\mathtt{r}}\big), \\
&\lim_{\nu\to\infty} d\big(2^{-\mathtt{r}}, 2^{-\nu}\big) = 2^{-\mathtt{r}} = d\big(2^{-\mathtt{r}}, 0\big).
\end{aligned}$$

Thus, all conditions of balanced distance space hold.

Remark 3. *(1) A C-distance space may not be a balanced distance space but the converse is true.*
(2) A balanced distance d is always a continuous function.

Theorem 5. *Consider a mapping* $\Upsilon\Omega : X^{\mathtt{r}} \to X^{\mathtt{r}}$ *on a complete balanced distance space* (X, d) *with the following property:*
there exists $\kappa > 0$ *such that*

$$d^{\mathtt{r}}(\Upsilon\Omega(\xi), \Upsilon\Omega(\eta)) \le \kappa d^{\mathtt{r}}(\xi, \eta), \textit{for all } \xi, \eta \in X^{\mathtt{r}}.$$

If $\tau \in X^{\mathtt{r}}$ *and a Picard sequence* $\{\tau(\nu) \in X^{\mathtt{r}} : \nu \in \mathbb{N}\}$ *is Cauchy, then the set Fix*(Ω) *of multiple fixed points of* Ω *is nonempty. Moreover, if* Ω *is* Υ*-contractive, then* Ω *has a unique multiple fixed point.*

Proof. Because $\{\tau(\nu)\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence and (X, d) is complete balanced distance space, there exists $\mu \in X^{\mathtt{r}}$ such that $\tau(\nu) \to \mu$, i.e.,

$$\lim_{\nu\to\infty} d^{\mathtt{r}}(\tau(\nu), \mu) = 0 \text{ and } \lim_{\nu\to\infty} d^{\mathtt{r}}(\mu, \tau(\nu)) = 0.$$

Suppose $\mu \neq \Upsilon\Omega(\mu)$, then $d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) > 0$ and $d^{\mathtt{r}}(\Upsilon\Omega(\mu), \mu) > 0$. From the definition of balanced distance space,

$$\begin{aligned} \lim_{\nu\to\infty} d^{\mathtt{r}}(\tau(\nu+1), \Upsilon\Omega(\mu)) &= d^{\mathtt{r}}(\mu, \Upsilon\Omega(\mu)) > 0, \\ \lim_{\nu\to\infty} d^{\mathtt{r}}(\Upsilon\Omega(\mu), \tau(\nu+1)) &= d^{\mathtt{r}}(\Upsilon\Omega(\mu), \mu) > 0. \end{aligned}$$

The continuity of $\Upsilon\Omega$ implies

$$\lim_{\nu\to\infty} d^{\mathtt{r}}(\Upsilon\Omega(\mu), \Upsilon\Omega(\tau(\nu))) = 0.$$

Consider,

$$\lim_{\nu\to\infty} d^{\mathtt{r}}(\Upsilon\Omega(\mu), \tau(\nu+1)) = \lim_{\nu\to\infty} d^{\mathtt{r}}(\Upsilon\Omega(\mu), \Upsilon\Omega(\tau(\nu))) = 0,$$

which is a contradiction. Hence,

$$\begin{aligned} \mu &= \Upsilon\Omega(\mu) \\ \mu_{\mathtt{i}} &= \Omega\Big(\mu_{\Upsilon_{\mathtt{i}}(1)}, \mu_{\Upsilon_{\mathtt{i}}(2)}, \ldots, \mu_{\Upsilon_{\mathtt{i}}(\mathtt{r})}\Big). \end{aligned}$$

Hence, μ is a multidimensional fixed point of Ω.

If Ω is Υ-contractive, then

$$d^{\mathtt{r}}(\Upsilon\Omega(\xi), \Upsilon\Omega(\eta)) < d^{\mathtt{r}}(\xi, \eta), \text{for all} \xi, \eta \in X^{\mathtt{r}}.$$

Suppose on the contrary that v is another fixed point of $\Upsilon\Omega$. Then,

$$\begin{aligned} d^{\mathtt{r}}(\mu, v) + d^{\mathtt{r}}(v, \mu) &= d^{\mathtt{r}}(\Upsilon\Omega(\mu), \Upsilon\Omega(v)) + d^{\mathtt{r}}(\Upsilon\Omega(v), \Upsilon\Omega(\mu)) \\ &< d^{\mathtt{r}}(\mu, v) + d^{\mathtt{r}}(v, \mu), \end{aligned}$$

which is a contradiction. Therefore, the multidimensional fixed point of Ω is unique. □

6. (ψ, ϕ, θ)-Contractions in Distance Spaces

Let $\mathcal{S}$ denote the class of all nondecreasing continuous functions $\psi : [0, \infty) \to [0, \infty)$ with $\psi(\varkappa) = 0$ if and only if $\varkappa = 0$.

Theorem 6. *Let* $\Omega : X^{\mathtt{r}} \to X$ *be a mapping on complete* (s, q)*-distance space* (X, d) *with* $\mathtt{r} \in \mathbb{N}$. *For each* $\tau = (\tau_1, \ldots, \tau_{\mathtt{r}}), \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{\mathtt{r}}) \in X^{\mathtt{r}}, s, q > 0$, *if* $\Upsilon\Omega$ *satisfies:*

$$\psi((1+q)d^{\mathtt{r}}(\Upsilon\Omega(\tau), \Upsilon\Omega(\gamma))) \le \theta\bigg(\frac{d^{\mathtt{r}}(\tau, \gamma)}{(s+1)(q+1)}\bigg) + \phi\bigg(\frac{d^{\mathtt{r}}(\tau, \gamma)}{(s+1)(q+1)}\bigg),$$

where $\psi, \theta, \phi \in \mathcal{S}$ with $\psi(\xi) > \theta(\xi) + \phi(\xi)$ for $\xi > 0$. Then, Ω has a multiple fixed point.

Proof. Let $\tau \in X^{\mathfrak{r}}$, $\tau(1) = \Upsilon\Omega(\tau)$, $\tau(\nu+1) = \Upsilon\Omega(\tau(\nu))$. If $\tau(\nu) = \tau(\nu+1)$, then Ω has a fixed point. Suppose $\tau(\nu) \neq \tau(\nu+1)$. Then,

$$\begin{aligned}
&\psi(d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) + d^{\mathfrak{r}}(\tau(\nu+1), \tau(\nu))) \\
&\leq \psi((1+q)d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1))) \\
&= \psi((1+q)d^{\mathfrak{r}}(\Upsilon\Omega(\tau(\nu-1)), \Upsilon\Omega(\tau(\nu)))) \\
&\leq \theta\left(\frac{d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu))}{(s+1)(q+1)}\right) + \phi\left(\frac{d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu))}{(s+1)(q+1)}\right).
\end{aligned}$$

Because θ and ϕ are non-decreasing functions,

$$\begin{aligned}
\psi((1+q)d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1))) &\leq \theta(d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu))) + \phi(d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu))) \\
&< \psi(d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu))).
\end{aligned} \tag{4}$$

Furthermore, because ψ is non-decreasing, we obtain

$$(1+q)d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) < d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)).$$

The fact $(1+q) > 1$ implies

$$d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) \leq d^{\mathfrak{r}}(\tau(\nu-1), \tau(\nu)),$$

and thus, $\{d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1))\}$ is adecreasing sequence. Hence, there exists $s \geq 0$ such that

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) = s.$$

If $s > 0$, applying limit $\nu \to \infty$ to the condition (4), it follows

$$\begin{aligned}
\psi(s(1+q)) &\leq \theta(s) + \phi(s) \\
&\leq \theta(s(1+q)) + \phi(s(1+q)),
\end{aligned}$$

which is a contradiction. Then, $s = 0$ and hence

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu), \tau(\nu+1)) = 0.$$

In a similar manner, one can prove

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu+1), \tau(\nu)) = 0.$$

To show $\{\tau(\nu)\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence, suppose on contrary that $\{\tau(\nu)\}_{\nu\in\gneqq}$ is not Cauchy. Then, for every $\epsilon > 0$, there exist subsequences $\{\tau(\nu_\varpi)\}$ and $\{\tau(l_\varpi)\}$ of $\{\tau(\nu)\}$ with $\nu_\varpi > l_\varpi > \varpi$ such that

$$d^{\mathfrak{r}}(\tau(\nu_\varpi), \tau(l_\varpi)) + d^{\mathfrak{r}}(\tau(l_\varpi), \tau(\nu_\varpi)) \geq \epsilon.$$

Suppose that ν_ϖ is the smallest positive integer such that

$$(1+q)d^{\mathfrak{r}}(\tau(l_\varpi), \tau(\nu_\varpi)) \geq \epsilon$$

and

$$d^{\mathfrak{r}}(\tau(\nu_\varpi - 1), \tau(l_\varpi)) + d^{\mathfrak{r}}(\tau(l_\varpi), \tau(\nu_\varpi - 1)) < \epsilon.$$

Now, we have

$$\begin{aligned}
\psi(\epsilon) &\leq \psi[d^{\mathfrak{r}}(\tau(\nu_{\varpi}),\tau(l_{\varpi}))+d^{\mathfrak{r}}(\tau(l_{\varpi}),\tau(\nu_{\varpi}))] \\
&\leq \psi[(1+q)d^{\mathfrak{r}}(\tau(l_{\varpi}),\tau(\nu_{\varpi}))] \\
&= \psi[(1+q)d^{\mathfrak{r}}(\Upsilon\Omega(\tau(l_{\varpi}-1)),\Upsilon\Omega(\tau(\nu_{\varpi}-1)))] \\
&\leq \theta\left(\frac{d^{\mathfrak{r}}(\tau(l_{\varpi}-1),\tau(\nu_{\varpi}-1))}{(q+1)(s+1)}\right)+\phi\left(\frac{d^{\mathfrak{r}}(\tau(l_{\varpi}-1),\tau(\nu_{\varpi}-1))}{(q+1)(s+1)}\right) \\
&\leq \theta\left(\frac{\epsilon}{(q+1)^2(s+1)}\right)+\phi\left(\frac{\epsilon}{(q+1)^2(s+1)}\right) \\
&\leq \theta(\epsilon)+\phi(\epsilon),
\end{aligned}$$

which is a contradiction. Hence, $\{\tau(\nu)\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence. Because, the space is complete, there exists $v\in X^{\mathfrak{r}}$ such that

$$\limsup_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu),v)=0 \text{ and } \limsup_{\nu\to\infty} d^{\mathfrak{r}}(v,\tau(\nu))=0.$$

Consider,

$$\begin{aligned}
&\lim_{\nu\to\infty}\psi((1+q)d^{\mathfrak{r}}(\tau(\nu),\Upsilon\Omega(v))) \\
&=\lim_{\nu\to\infty}\psi((1+q)d^{\mathfrak{r}}(\Upsilon\Omega(\tau(\nu-1),\Upsilon\Omega(v))) \\
&\leq \lim_{\nu\to\infty}\theta\left(\frac{d^{\mathfrak{r}}(\tau(\nu-1),v)}{(q+1)(s+1)}\right)+\lim_{\nu\to\infty}\phi\left(\frac{d^{\mathfrak{r}}(\tau(\nu-1),v)}{(q+1)(s+1)}\right) \\
&\leq \lim_{\nu\to\infty}\theta(d^{\mathfrak{r}}(\tau(\nu-1),v))+\lim_{\nu\to\infty}\phi(d^{\mathfrak{r}}(\tau(\nu-1),v)) \\
&\leq \theta(0)+\phi(0)=0,
\end{aligned}$$

which implies

$$\lim_{\nu\to\infty}\psi((1+q)d^{\mathfrak{r}}(\tau(\nu),\Upsilon\Omega(v)))=0.$$

By applying the properties of ψ, we deduce

$$\lim_{\nu\to\infty} d^{\mathfrak{r}}(\tau(\nu),\Upsilon\Omega(v))=0.$$

Now,

$$\begin{aligned}
d^{\mathfrak{r}}(v,\Upsilon\Omega(v))+d^{\mathfrak{r}}(\Upsilon\Omega(v),v) &\leq (1+q)d^{\mathfrak{r}}(v,\Upsilon\Omega(v)) \\
&\leq (1+q)s[d^{\mathfrak{r}}(v,\tau(\nu))+d^{\mathfrak{r}}(\tau(\nu),\Upsilon\Omega(v))].
\end{aligned}$$

Applying limit $\nu\to\infty$ to both sides, we obtain

$$d^{\mathfrak{r}}(v,\Upsilon\Omega(v))+d^{\mathfrak{r}}(\Upsilon\Omega(v),v)=0,$$

and hence, $v=\Upsilon\Omega(v)$. □

Corollary 3. *Let (X,d) be a complete (s,q)-distance space and $\Upsilon\Omega: X^{\mathfrak{r}}\to X$ be a given mapping. If there exists $\kappa\in(0,1)$ such that*

$$d^{\mathfrak{r}}(\Upsilon\Omega(\xi),\Upsilon\Omega(\eta))\leq\kappa d^{\mathfrak{r}}(\xi,\eta), \textit{ for all } \xi,\eta\in X^{\mathfrak{r}},$$

then Ω has a unique multiple fixed point.

Proof. Putting $\psi(\xi)=\frac{\xi}{1+q}$, $\theta(\xi)=(1+q)(1+s)\kappa\xi$ and $\phi(\xi)=0$, for all $\xi\in[0,\infty)$ in the above theorem, it can be easily proven. □

7. Application

This section deals with the application of our result proven in Section 3 for s-distance spaces. Here, we are going to investigate the solution of integral equations by utilizing the concept of multiple fixed points.

Let $\tau, \gamma \in R$ with $\tau < \gamma$, and let $\acute{I} = [\tau, \gamma]$. Consider X to be a set of all real valued and continuous functions defined on $\acute{I}$; then, d is a complete s-distance on X where

$$d(\alpha, \beta) = \max_{\varkappa \in [\tau,\gamma]} (|\alpha(\varkappa)| - |\beta(\varkappa)|)^{2\varpi}, \Omega \, for \; all \; \alpha, \beta \in X \text{ and } \varpi \geq 1.$$

Consider the following integral system:

$$\begin{array}{rcl} \eta_1(\varkappa) & = & \omega + \int_\tau^\varkappa L(\eta_1(\mu), \eta_2(\mu), \ldots, \eta_{\mathtt{r}}(\mu)) \mathrm{d}\mu \\ \eta_\imath(\varkappa) & = & \omega + \int_\tau^\varkappa L(\eta_\imath(\mu), \eta_{\imath+1}(\mu), \ldots, \eta_{\mathtt{r}}(\mu), \eta_1(\mu), \ldots, \eta_{\imath-1}(\mu)) \mathrm{d}\mu \end{array} \tag{5}$$

for $\imath = 1, 2 \ldots, \mathtt{r}$, $\eta = (\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}}) \in X^{\mathtt{r}}$, $\mu \in \acute{I}$ and a mapping $L : \mathbb{R}^{\mathtt{r}} \to \mathbb{R}$ is such that

(i) L is continuous;

(ii) Ω for all$(\xi_1, \xi_2, \ldots, \xi_{\mathtt{r}}), (\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}}) \in \mathbb{R}^{\mathtt{r}}$,

$$|L(\xi_1, \xi_2, \ldots, \xi_{\mathtt{r}})| - |L(\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}})| \leq \kappa(\max_{1 \leq \imath \leq \mathtt{r}} \chi_\imath (|\xi_\imath| - |\eta_\imath|)^{2\varpi})^{\frac{1}{2\varpi}}.$$

A mapping $\Omega : X^{\mathtt{r}} \to X$ for all $\eta = (\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}}) \in X^{\mathtt{r}}$ and $\omega \in \mathbb{R}$ defined by

$$\Omega(\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}})(\varkappa) = \omega + \int_\tau^\varkappa L(\eta_1(\mu), \eta_2(\mu), \ldots, \eta_{\mathtt{r}}(\mu)) \mathrm{d}\mu.$$

Obviously, $\Omega \in C(\acute{I})$. Now, for the solution of system (5), we take the points $(\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}}), (\xi_1, \xi_2, \ldots, \xi_{\mathtt{r}}) \in X^{\mathtt{r}}$, and consider

$$\begin{aligned}
& d(\Omega(\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}}), \Omega(\xi_1, \xi_2, \ldots, \xi_{\mathtt{r}})) \\
& = \max_{\varkappa \in \acute{I}} (|\Omega(\eta_1, \eta_2, \ldots, \eta_{\mathtt{r}})(\varkappa)| - |\Omega(\xi_1, \xi_2, \ldots, \xi_{\mathtt{r}})(\varkappa)|)^{2\varpi} \\
& = \max_{\varkappa \in \acute{I}} \left\{ \begin{array}{c} |\omega + \int_\tau^\varkappa L(\eta_1(\mu), \eta_2(\mu), \ldots, \eta_{\mathtt{r}}(\mu)) \mathrm{d}\mu| - \\ |\omega + \int_\tau^\varkappa L(\xi_1(\mu), \xi_2(\mu), \ldots, \xi_{\mathtt{r}}(\mu)) \mathrm{d}\mu| \end{array} \right\}^{2\varpi} \\
& \leq \max_{\varkappa \in \acute{I}} \left\{ \begin{array}{c} |\omega| + |\int_\tau^\varkappa (L(\eta_1(\mu), \eta_2(\mu), \ldots, \eta_{\mathtt{r}}(\mu)) \mathrm{d}\mu| - \\ |\omega| - |\int_\tau^\varkappa (L(\xi_1(\mu), \xi_2(\mu), \ldots, \xi_{\mathtt{r}}(\mu)) \mathrm{d}\mu| \end{array} \right\}^{2\varpi} \\
& \leq \max_{\varkappa \in \acute{I}} \left\{ \begin{array}{c} \int_\tau^\varkappa |(L(\eta_1(\mu), \eta_2(\mu), \ldots, \eta_{\mathtt{r}}(\mu))| \mathrm{d}\mu - \\ \int_\tau^\varkappa |(L(\xi_1(\mu), \xi_2(\mu), \ldots, \xi_{\mathtt{r}}(\mu))| \mathrm{d}\mu \end{array} \right\}^{2\varpi} \\
& \leq \max_{\varkappa \in \acute{I}} \left\{ \int_\tau^\varkappa (|(L(\eta_1(\mu), \eta_2(\mu), \ldots, \eta_{\mathtt{r}}(\mu))| - |L(\xi_1(\mu), \xi_2(\mu), \ldots, \xi_{\mathtt{r}}(\mu)|) \mathrm{d}\mu \right\}^{2\varpi} \\
& \leq \max_{\varkappa \in \acute{I}} \left\{ \int_\tau^\varkappa \kappa(\max_{1 \leq \imath \leq \mathtt{r}} \chi_\imath \max_{\mu \in \acute{I}} (|\eta_\imath(\mu)| - |\xi_\imath(\mu)|)^{2\varpi})^{\frac{1}{2\varpi}} \mathrm{d}\mu \right\}^{2\varpi} \\
& \leq \kappa\chi \max_{\varkappa \in \acute{I}} (\varkappa - \tau)^{2\varpi} d(\eta_\imath, \xi_\imath) \\
& \leq \kappa\chi(\gamma - \tau)^{2\varpi} d(\eta_\imath, \xi_\imath) \\
& \leq \kappa\chi(\gamma - \tau)^{2\varpi} \sup\{d(\eta_\imath, \xi_\imath) : \imath \leq \mathtt{r}\} \\
& \leq \kappa\chi(\gamma - \tau)^{2\varpi} d^{\mathtt{r}}(\eta, \xi)
\end{aligned}$$

with $\chi = \max_{1 \leq \imath \leq \mathtt{r}} \chi_\imath$. Thus, all the assumptions of Corollary 1 are satisfied. Hence, the integral system (5) has a unique solution.

8. Conclusions

The main goal of this paper was to generalize most of the results in the literature dedicated to coupled, tripled, and quadruple fixed point theorems by taking particular values of $\mathtt{r} \in \mathbb{N}$. We discussed s-distance spaces, (s, q)-distance spaces, and balanced distance spaces with different contractive conditions, which are generalized structures as compared to the well-knownstructure of metric spaces.The results have been immediately applied toobtain the solution of a systemof integral equations.

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