

Article

An Algebraic Model for Quantum Unstable States

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Abstract: In this review, we present a rigorous construction of an algebraic method for quantum unstable states, also called Gamow states. A traditional picture associates these states to vectors states called Gamow vectors. However, this has some difficulties. In particular, there is no consistent definition of mean values of observables on Gamow vectors. In this work, we present Gamow states as functionals on algebras in a consistent way. We show that Gamow states are not pure states, in spite of their representation as Gamow vectors. We propose a possible way out to the construction of averages of observables on Gamow states. The formalism is intended to be presented with sufficient mathematical rigor.

Keywords: Gamow states; algebras of observables; time evolution of states

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1. Introduction

The conceptual development of non-relativistic quantum mechanics includes two points of view concerning the mathematical definitions of the basic notions of states and observables. In the former, a pure state is a normalized vector in a linear space, usually a Hilbert space, while mixed states and observables are represented by self-adjoint operators on a Hilbert space. In the latter, observables are certain distinguished members of an algebra, while states are linear functionals over this algebra with some additional conditions.

There exists a kind of states that somehow escape to this classification: those states that describe *unstable quantum states*. Traditionally, these unstable states have been described by vectors in a linear space, called Gamow vectors. In order to describe the meaning of these Gamow vectors, one uses a representation of unstable quantum states in terms of scattering resonances. A scattering resonance is characterized by a potential V perturbing an otherwise free dynamics H_0 . Thus, one has a Hamiltonian pair $\{H_0, H = H_0 + V\}$. Assume that a particle that evolves freely under the action of H_0 enters in the interacting region, where the potential V acts. If the time that the particle stays in the interacting region is much larger than the time that one expects it to stay if the potential V does not exist, we say that it is on a quasi-stable state or resonance. Resonances are the most intuitive representation of unstable quantum states [1–9].

Resonances have a simple characterization in terms of the S -matrix in the energy representation. Each of the resonances appear as a pair of complex conjugate poles of the analytic continuation of $S(E)$ into a two sheeted Riemann surface. Each pair is located on the second sheet and is of the form $E_R \pm i\Gamma/2$, where $E_R > 0$ is the resonant energy

and $1/\Gamma > 0$ is the mean life. For more formal details see [10], and for applications to decoherence phenomena see [11–18].

Quantum unstable states decay (approximately) exponentially for most times, with the exception of very short (Zeno time) [19] or very long (Khalfin time) times [20]. The last one is very weak and difficult to detect experimentally [21]. Since Zeno times are very short and Khalfin times appear after a very long time of exponential decay [22–25], a mathematical object showing exponential decay at all times, $t > 0$, would be a good approximation for the description of an unstable quantum state.

From this point of view, Nakanishi [26] proposed that this mathematical object should be an eigenvector ψ_D of the total Hamiltonian $H = H_0 + V$ with an eigenvalue $E_R - i\Gamma/2$, so that $H\psi_D = (E_R - i\Gamma/2)\psi_D$. In this case, for $t > 0$, one has the following formal time evolution of ψ_D (D stands for decay)

$$e^{-itH}\psi_D = e^{-itE_R}e^{-t\Gamma/2}\psi_D, \quad t > 0.$$

Since $\Gamma > 0$, this decay is exponential for all positive times. Equivalently, the energy distribution of a decaying state should have a Breit–Wigner shape

$$\frac{\Gamma}{2\pi} \frac{1}{(E - E_R)^2 + \Gamma^2/4}.$$

This picture has serious flaws. First of all, a self-adjoint Hamiltonian cannot have complex eigenvalues. In addition, physical Hamiltonians in non-relativistic quantum mechanics are semibounded, which is incompatible with the existence of a state with non-zero energy distribution for all values $-\infty < E < +\infty$.

Nevertheless, Nakanishi's picture of a Gamow state looks attractive. A rigorous definition of ψ_D is possible, provided that we extend the Hilbert space $\mathcal{H}$ supporting the Hamiltonian pair $\{H_0, H = H_0 + V\}$ into a larger space where ψ_D can be defined and has the desired properties. This can be done in several, albeit related, ways. One possibility is that ψ_D is a vector in a Hilbert space and it has the desired Breit–Wigner energy representation, although it is outside of the domain of H . Another equivalent possibility is that ψ_D cannot be normalized. The extension of the Hilbert space consists in immerse it into a larger structure called *rigged Hilbert space* or *Gelfand triplet*. Below in Section 2, we describe this structure, since it is important for our later discussion.

The vector state ψ_D is called the *Gamow vector* or more precisely, *the decaying Gamow vector* for the resonance characterized by the parameters E_R and Γ . It describes the decaying part of a unstable quantum state, which is the dominant part for most of values of time. We should say that Zeno and Khalfin times are not easily observable [21,27–33].

Although the picture is totally consistent, it is not free of some problems. For instance, averages of observables on Gamow vectors are difficult to define. See some attempts in [34–36].

This is just a part of the story concerning the representation of unstable quantum states as vector states. Indeed, this formalism has been applied in issues such as nuclear physics and irreversible phenomena in quantum mechanics [37–42]. This representation suggest that Gamow states are pure states, as being described by a single vector. However, these vectors are not ordinary Hilbert space vectors and do not represent stable states; therefore, one may suspect that Gamow states are not pure states. In fact, they are not and this will be discussed along the paper. However, this is not evident with the solely representation of a Gamow state as a Gamow vector.

Then, we need a more complete representation of Gamow states that preserves all good properties of Gamow vectors, yet permits us to define averages of observables on Gamow states and provides clear evidence that Gamow states are not pure states. This opens other kind of discussions such that in what sense we may define a non-zero entropy for a Gamow state. Although this latter problem has been already discussed [43,44], it is far from being closed.

Thus, the main objective of this paper is to propose an algebraic context in which Gamow states are functionals over an algebra of operators containing the relevant observables of the system under study. In this context, a state is a functional (linear mapping), f , between a complex topological algebra $\mathcal{A}$ (with unit, I , and an involution $A \mapsto A^\dagger$, for all $A \in \mathcal{A}$) and the field of complex numbers, $f : \mathcal{A} \mapsto \mathbb{C}$, with the following properties:

- (i) $f(I) = 1$;
- (ii) for any $A \in \mathcal{A}$, $f(A^\dagger A) \geq 0$ (positivity);
- (iii) continuity with respect to the topologies on $\mathcal{A}$ and $\mathbb{C}$.

We shall construct a Gamow state satisfying all these properties. Most of results have been discussed elsewhere, so that we have written this paper to provide a consistent review of these results.

This paper is organized as follows: In Section 2, we discuss the properties of rigged Hilbert spaces. They are an important pillar for our construction of the algebras of operators. In Section 3, we construct the algebras and define the Gamow functionals. Mathematical properties are given in Section 4. We finish the paper with some concluding remarks and an appendix on Hardy functions on a half plane, which play an essential role in our construction.

2. Rigged Hilbert Spaces: An Overview

Along this section, we review some important facts that concern either to the construction of Gamow states as well as the correct mathematical presentation of the Dirac formulation of quantum mechanics [45].

Definition 1. A rigged Hilbert space or Gelfand triplet is a tern of spaces,

$$\Phi \subset \mathcal{H} \subset \Phi^\times, \quad (1)$$

where:

- (i) $\mathcal{H}$ is a separable Hilbert space of infinite dimension.
- (ii) Φ is a dense subspace of $\mathcal{H}$ having its own topology under the condition that the canonical injection $i : \Phi \mapsto \mathcal{H}$, $i(\varphi) = \varphi$, $\forall \varphi \in \Phi$, be continuous.
- (iii) The space $\Phi^\times$ is the antidual space of Φ , which is the set of all continuous antilinear functionals on Φ .

If $F \in \Phi^\times$, $\varphi, \psi \in \Phi$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\langle \alpha \varphi + \beta \psi | F \rangle \equiv F(\alpha \varphi + \beta \psi) = \alpha^* F(\varphi) + \beta^* F(\psi) \equiv \alpha^* \langle \varphi | F \rangle + \beta^* \langle \psi | F \rangle, \quad (2)$$

where the star stands for complex conjugation. Antilinearity is chosen instead of linearity in order to fit with the Dirac notation of quantum mechanics, where brackets are linear to the right and antilinear to the left. The space $\Phi^\times$ is linear over the field of complex numbers and may have its own topology compatible with the topology on Φ [46–56]. We discuss here neither definitions nor use of these topologies.

Definition 2. Let us assume that A is a (densely defined) linear operator on the Hilbert space $\mathcal{H}$ and $A^\dagger$ its adjoint, such that for any φ in Φ , its image by $A^\dagger$, $A^\dagger \varphi$, is also in Φ . We denote this property by $A^\dagger \Phi \subset \Phi$. Then, we may extend A into the antidual space $\Phi^\times$ by using the duality formula:

$$\langle A^\dagger \varphi | F \rangle = \langle \varphi | AF \rangle, \quad \forall \varphi \in \Phi, \quad \forall F \in \Phi^\times. \quad (3)$$

For simplicity, we have use the same notation, A , equally valid for the operator A on $\mathcal{H}$ and for its extension on $\Phi^\times$. In addition, if $A^\dagger$ would have been continuous on Φ , then A should be continuous on $\Phi^\times$ with whatever topology on $\Phi^\times$ compatible with the topology

on Φ . Note that if A is symmetric (Hermitian) and its domain, $\mathcal{D}(A)$, has the property that $\Phi \subset \mathcal{D}(A)$, with $A\Phi \subset \Phi$, then (3) is valid if we replace A^+ by A .

It is interesting to discuss some properties that we shall use later and that we include here in order to make this paper self-contained and easier to read. We shall not use the following results in full generality as is not necessary to follow our ideas. More details can be found in the bibliography [46,48–54,57]. The next result helps to fix ideas.

Theorem 1 (The Gelfand–Maurin Theorem). *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and A a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$ and having a simple (not degenerate) absolutely continuous spectrum $\mathbb{R}^+ \equiv [0, \infty)$. Then [46,57], there exists a RHS, $\Phi \subset \mathcal{H} \subset \Phi^\times$, as in (1), such that:*

- (i) *The space $\Phi \subset \mathcal{D}(A)$, $A\Phi \subset \Phi$ (which means that for any $\varphi \in \Phi$, $A\varphi \in \Phi$) and A is continuous on Φ with the own topology on Φ .*
- (ii) *For almost all $\lambda \in \mathbb{R}^+$ (with respect to the Lebesgue measure on $\mathbb{R}^+$), there exists a functional $|\lambda\rangle \in \Phi^\times$ such that $A|\lambda\rangle = \lambda|\lambda\rangle$, where A has been extended to $\Phi^\times$ using the duality Formula (3). Functionals in the set $\{|\lambda\rangle\}_{\lambda \in [0, \infty)}$ are often called generalized eigenvectors of A .*
- (iii) *For any $\varphi, \psi \in \Phi$, one has*

$$\langle \psi | A\varphi \rangle = \int_0^\infty \lambda \langle \psi | \lambda \rangle \langle \lambda | \varphi \rangle d\lambda, \quad (4)$$

where $\langle \psi | \lambda \rangle$ is the action of the functional $|\lambda\rangle \in \Phi^\times$ on the vector $\psi \in \Phi$ and $\langle \lambda | \varphi \rangle = \langle \varphi | \lambda \rangle^*$.

- (iv) *Each function $\langle \lambda | \varphi \rangle$, with $\lambda \in [0, \infty)$, is square integrable, and therefore, belongs to the Hilbert space $L^2(\mathbb{R}^+)$.*
- (v) *The mapping $U : \mathcal{H} \mapsto L^2(\mathbb{R}^+)$ that assigns to each $\varphi \in \Phi$ a $\langle \lambda | \varphi \rangle \in L^2(\mathbb{R}^+)$ is unitary, so that the norms of the vector φ in $\mathcal{H}$ and $\langle \lambda | \varphi \rangle$ in $L^2(\mathbb{R}^+)$ are identical.*

Remark 1. *If we omit the arbitrary vectors $\varphi, \psi \in \Phi$, we may write (4) as*

$$A = \int_0^\infty \lambda |\lambda\rangle \langle \lambda| d\lambda. \quad (5)$$

The property $A\Phi \subset \Phi$, with continuity, implies that $A^n\Phi \subset \Phi$, with continuity. In addition, if $n \in \mathbb{N}_0$, we have

$$A^n = \int_0^\infty \lambda^n |\lambda\rangle \langle \lambda| d\lambda, \quad (6)$$

where the meaning of (6) is analogous to the meaning of (5). For $n = 0$, the above equation shows that

$$I = \int_0^\infty |\lambda\rangle \langle \lambda| d\lambda. \quad (7)$$

It is quite interesting to deepen on the meaning of the above formulas. Let us apply (7) to an arbitrary $\varphi \in \Phi$. Since I is an identity, we must have that

$$\varphi = I\varphi = \int_0^\infty |\lambda\rangle \langle \lambda | \varphi \rangle d\lambda =: F_\varphi. \quad (8)$$

This formula is quite interesting as shows that any $\varphi \in \Phi$ admits a span in terms of the generalized eigenvectors of A with complex coefficients given by $\langle \lambda | \varphi \rangle$. This span is very similar to the span of a vector in terms of an orthonormal basis, where the series has been replaced by an integral. This is why one refers to the set of eigen-functionals of A given by $\{|\lambda\rangle\}_{\lambda \in \mathbb{R}^+}$ as a *continuous basis* for Φ . Relations between continuous and orthonormal basis are well known [55].

The form of the integral in (8) suggests that it may represent a functional in $\Phi^\times$. In fact, if we define its action to the left on a $\psi \in \Phi$ as

$$\langle \psi | F_\varphi \rangle := \int_0^\infty \langle \psi | \lambda \rangle \langle \lambda | \varphi \rangle d\lambda, \quad (9)$$

we obtain the scalar product $\langle \psi | \varphi \rangle$. Therefore, the scalar product of two vectors in Φ admits an expansion in terms of the generalized eigenvectors $\{|\lambda\rangle\}_{\lambda \in \mathbb{R}^+}$. At the same time, it may be expressed as a scalar product on the Hilbert space $L^2(\mathbb{R}^+)$. The antilinearity of F_φ is obvious after (9). Let us prove that it is also continuous on Φ . Linearity and continuity of F_φ imply that $F_\varphi \in \Phi^\times$ as we want to show.

The topology on any locally convex space such as Φ is given by *seminorms*. A seminorm is like a norm in which we drop the requirement that the only vector with seminorm zero must be the zero vector. All other properties are identical to the properties of norms. In particular, a norm is a seminorm.

Let us assume that the topology on Φ is given by a family of seminorms $\{p_i\}_{i \in I}$. This family must contain the norm on $\mathcal{H}$, since the canonical injection $i : \Phi \rightarrow \mathcal{H}$ must be continuous.

Then, let us give a couple of continuity criteria for linear mappings on Φ [58].

Theorem 2. A linear mapping $F : \Phi \rightarrow \mathbb{C}$ is continuous on Φ if and only if for each $\varphi \in \Phi$, there exist a positive constant $C > 0$ and a finite number of the seminorms that define the topology on Φ , say $\{p_{i_1}, p_{i_2}, \dots, p_{i_k}\}$ such that

$$|\langle \varphi | F \rangle| \leq C \{p_{i_1}(\varphi) + p_{i_2}(\varphi) + \dots + p_{i_k}(\varphi)\}. \quad (10)$$

The action of F on φ is $\langle \varphi | F \rangle$. The constant C and the seminorms $\{p_{i_1}, p_{i_2}, \dots, p_{i_k}\}$ are the same for all $\varphi \in \Phi$.

Theorem 3. An operator (we assume that operators are always linear) $A : \Phi \rightarrow \Psi$, where Φ is another locally convex space, is continuous if and only if for each seminorm q_j on Ψ , there exists a constant $C_j > 0$ and a finite number of seminorms $\{p_{j_1}, p_{j_2}, \dots, p_{j_k}\}$ on Φ , such that for all $\varphi \in \Phi$, one has

$$q_j(A\varphi) \leq C_j \{p_{j_1}(\varphi) + p_{j_2}(\varphi) + \dots + p_{j_k}(\varphi)\}. \quad (11)$$

The seminorms $\{p_{j_1}, p_{j_2}, \dots, p_{j_k}\}$ as well as their number k may be different for another seminorm q_n on Ψ , in the left hand side of (11).

We are now in the position of proving the continuity of F_φ .

Proposition 1. The functional F_φ as defined in (9) is continuous.

To prove our claim, we just need to note that, due to the Schwarz inequality,

$$|\langle \psi | F_\varphi \rangle| \leq \int_0^\infty |\langle \psi | \lambda \rangle| |\langle \lambda | \varphi \rangle| d\lambda \leq \|\psi\| \|\varphi\|, \quad (12)$$

where we have used the fact that the norms of the function $\langle \lambda | \varphi \rangle$ and the vector φ are equal. Then, use (10) with $C \equiv \|\varphi\|$ and $p_i(\psi) := \|\psi\|$. Here, we need only one norm. In addition, the correspondence $\varphi \mapsto F_\varphi$ is one to one. Thus, it establishes a one to one mapping $\Phi \rightarrow \Phi^\times$. This mapping is nothing else than the canonical injection from Φ into $\Phi^\times$ and it is given by (7).

Remark 2. Let us go back to (6). Due to the properties of Φ in relation to A , this expression means that, for any pair $\varphi, \psi \in \Phi$ and for any $n \in \mathbb{N}_0$, the integral

$$\langle \psi | A^n \varphi \rangle = \int_0^\infty \lambda^n \langle \psi | \lambda \rangle \langle \lambda | \varphi \rangle d\lambda \quad (13)$$

converges absolutely. Mimicking the previous arguments, one may show that for each fixed $\varphi \in \Phi$, the functional $F_{\varphi,n}$ defined for all $\psi \in \Phi$ as

$$\langle \psi | F_{\varphi,n} \rangle = \int_0^\infty \lambda^n \langle \psi | \lambda \rangle \langle \lambda | \varphi \rangle d\lambda \quad (14)$$

is linear and continuous on Φ and, therefore, is an element of $\Phi^\times$. Therefore, the left hand side of (6) may be viewed as a mapping from Φ to $\Phi^\times$. For each $n \in \mathbb{N}$, its mapping is $\varphi \mapsto F_{\varphi,n}$.

It is important to note that (6) also implies that for any $n \in \mathbb{N}_0$ and all $\varphi \in \Phi$, the following integral converges:

$$\int_0^\infty \lambda^n |\langle \varphi | \lambda \rangle|^2 d\lambda. \quad (15)$$

On the topology on $\Phi^\times$

Are the mappings given in the right hand side of (6) continuous? First of all, we have to endow $\Phi^\times$ with a topology. For simplicity, let us use the weak topology. It is given by the following family of seminorms: For each fixed $\psi \in \Phi$, let us define the seminorm p_ψ on $\Phi^\times$ as

$$p_\psi(F) := \langle \psi | F \rangle, \quad \forall F \in \Phi^\times. \quad (16)$$

Then, for any $\varphi \in \Phi$, we have

$$\begin{aligned} |p_\psi(A^n \varphi)| &= |\langle \psi | F_{\varphi,n} \rangle| \leq \int_0^\infty \lambda^n |\langle \psi | \lambda \rangle| |\langle \lambda | \varphi \rangle| d\lambda \\ &\leq \left(\int_0^\infty \lambda^{2n} |\langle \psi | \lambda \rangle|^2 d\lambda \right)^{1/2} \left(\int_0^\infty |\langle \lambda | \varphi \rangle|^2 d\lambda \right)^{1/2} = C_n \|\varphi\|, \end{aligned} \quad (17)$$

where the second inequality in (17) is the Schwarz inequality, C_n is the constant given by the first parenthesis in the second row in (17) and the norm of φ is the Hilbert space norm. As the Hilbert space norm is one of the seminorms on Φ and p_φ is an arbitrary seminorm on $\Phi^\times$ with the weak topology, the continuity of the right hand side in (6) follows.

On the meaning of the brackets $\langle \lambda | \lambda' \rangle$

An interesting relation comes formally from the identity $I\varphi = \varphi$, where I is the identity (7) and $\varphi \in \Phi$, as follows:

$$\begin{aligned} \varphi &= \int_0^\infty |\lambda \rangle \langle \lambda | \varphi \rangle d\lambda = \int_0^\infty |\lambda \rangle \langle \lambda | d\lambda \int_0^\infty |\lambda' \rangle \langle \lambda' | \varphi \rangle d\lambda' \\ &= \int_0^\infty d\lambda \int_0^\infty d\lambda' |\lambda \rangle \langle \lambda | \lambda' \rangle \langle \lambda' | \varphi \rangle, \end{aligned} \quad (18)$$

then one must have

$$\langle \lambda | \lambda' \rangle = \delta(\lambda - \lambda'), \quad (19)$$

a relation widely used in the sequel. Note that $\langle \lambda | \lambda' \rangle$ is not a generalization of the scalar product. Furthermore, the existence of a general relation of the type $\langle F | G \rangle$ for arbitrary $F, G \in \Phi^\times$ is not known. A partial discussion on this kind of brackets goes beyond of the scope of the present paper, but it can be seen in [52–54].

From all the above comments, it should be obvious that an object such as $|\lambda \rangle \langle \lambda|$, with $\lambda \in [0, \infty)$, is a mapping $|\lambda \rangle \langle \lambda| : \Phi \mapsto \Phi^\times$, such that for all $\varphi \in \Phi$, $|\lambda \rangle \langle \lambda|(\varphi) = (\langle \lambda | \varphi \rangle) |\lambda \rangle \in \Phi^\times$. Same for $|\lambda \rangle \langle \lambda'|$ with $\lambda \neq \lambda'$.

Recalling that $|\lambda\rangle$ is a continuous antilinear functional on Φ and that $\langle\lambda|\varphi\rangle = \langle\varphi|\lambda\rangle^*$, it is pretty obvious that $\langle\lambda|$ is a continuous *linear* functional on Φ that transforms any $\varphi \in \Phi$ into the function $\langle\lambda|\varphi\rangle \in L^2(\mathbb{R}^+)$. The spaces of continuous linear and antilinear functionals are identical algebraically and topologically and have the same properties. We use the space of antilinear functionals to preserve the Dirac notation.

Another interesting point is the relation between each $\varphi \in \Phi$ and the function $\langle\lambda|\varphi\rangle \in L^2(\mathbb{R}^+)$, $\lambda \in \mathbb{R}^+$. Sometimes, it is more intuitive use the notation $\varphi(\lambda)$ instead of $\langle\lambda|\varphi\rangle$, although we shall use both in the sequel. The function in $L^2(\mathbb{R}^+)$ corresponding to the vector $A\varphi$ is

$$\begin{aligned}(A\varphi)(\lambda) &= \langle\lambda|A\varphi\rangle = \langle\lambda|\int_0^\infty \lambda' |\lambda'\rangle\langle\lambda'|\varphi\rangle d\lambda' = \int_0^\infty \lambda' \delta(\lambda - \lambda') \langle\lambda'|\varphi\rangle d\lambda' \\ &= \lambda \langle\lambda|\varphi\rangle = \lambda \varphi(\lambda).\end{aligned}\quad (20)$$

A comment on the relation of an abstract RHS and a concrete realization of its on terms of functions

The spectral theorem [58] guarantees that, under the conditions imposed on A , there exists a unitary operator $U : \mathcal{H} \mapsto L^2(\mathbb{R}^+)$ such that UAU^{-1} is the multiplication operator on $L^2(\mathbb{R}^+)$. It is said that U *diagonalizes* the operator A . This is exactly what is shown in (20). Thus, we must conclude that if U is the unitary operator that diagonalizes A , then, $U\varphi = \langle\lambda|\varphi\rangle$ for all $\varphi \in \Phi$.

This has an interesting consequence: We may represent the abstract Hilbert space (1) into a RHS of functions, $\mathcal{G} \subset L^2(\mathbb{R}^+) \subset \mathcal{G}^\times$, where $\mathcal{G}$ is the image of Φ by U . Along the structure of vector space, we may also transport by U the topology from Φ to $\mathcal{G}$, so that U and U^{-1} are diffeomorphisms [56,59]. The extension of U to a diffeomorphism $U : \Phi^\times \mapsto \mathcal{G}^\times$ comes readily after the duality formula:

$$\langle U\varphi|UF\rangle = \langle\varphi|F\rangle, \quad \forall \varphi \in \Phi, \quad \forall F \in \Phi^\times. \quad (21)$$

It comes that the extension $U : \Phi^\times \mapsto \mathcal{G}^\times$ is one to one and onto. It transports the weak topology on $\Phi^\times$ to a topology on $\mathcal{G}^\times$ that coincides with the weak topology produced by $\mathcal{G}$ [56]. We summarize this construction with the following diagram:

$$\begin{array}{ccccc}\Phi & \subset & \mathcal{H} & \subset & \Phi^\times \\ U \downarrow & & U \downarrow & & U \downarrow \\ \mathcal{G} & \subset & L^2(\mathbb{R}^+) & \subset & \mathcal{G}^\times\end{array} \quad (22)$$

This combinations of an abstract RHS and a realization of it by a RHS of functions (and generalized functions in $\mathcal{G}^\times$) is usual for many purposes [56]. It is clear from the present discussion that $\lambda \varphi(\lambda) \in \mathcal{G}$ for all $\varphi(\lambda) \in \mathcal{G}$.

3. The Model and Gamow States

To make this paper as self-contained as possible, we give next the structure of the model with some details, no matter if they have been already published in many places [60–63]. We need to introduce resonance phenomena within the context of non-relativistic quantum mechanics. Therefore, we have a two Hamiltonian structure, $\{H_0, H = H_0 + V\}$, where H_0 is a non-perturbed Hamiltonian and H is the total Hamiltonian, while V is the potential producing the resonance and other interesting quantum phenomena. These Hamiltonians have the minimal properties valid for the discussion under our interest. On a first approach, let us assume that H_0 has an absolutely continuous non-degenerate spectrum equal to the positive semi-axis of the real line, so that it admits the following spectral decomposition:

$$H_0 = \int_0^\infty E |E\rangle\langle E| dE, \quad (23)$$

where $|E\rangle$, $E \in \mathbb{R}^+$, are the generalized eigenvectors of H_0 with eigenvalue E , so that (23) has the form of (6). Clearly, this assumption is not universal, although it may be easily generalized for the study of some models such as the Lee-Friedrichs model and its generalizations [64–67]. As we have seen in the precedent section, this implies that we have constructed a RHS as in (1), such that $H_0\Phi \subset \Phi$ and H_0 is continuous on Φ . This is possible after the Gelfand–Maurin theorem [46,57].

An operator O is said to be compatible with H_0 if it admits a spectral decomposition of the following form:

$$O = \int_0^\infty dE O_E |E\rangle\langle E| + \int_0^\infty dE \int_0^\infty dE' O_{EE'} |E\rangle\langle E'|, \quad (24)$$

where O_E is a function on the variable E , while $O_{EE'}$ is a function on the variables E and E' . These functions satisfy some regularity conditions to be discussed in the next section.

Let us assume that resonances are produced on a resonance scattering with a Hamiltonian pair $\{H_0, H\}$ [10,68,69] and that the Møller wave operators (The formal definition of the Møller wave operators is the following: Let H_0 be a “free” Hamiltonian and $H = H_0 + V$ a “perturbed” or “total” Hamiltonian. First, let us assume that H_0 has a purely absolutely continuous spectrum. Then, for any $\varphi \in \mathcal{H}$, we define

$$\Omega_\pm \varphi = \lim_{t \rightarrow \pm\infty} e^{iHt} e^{-itH_0} \varphi,$$

where t denotes time and the limit is taken in the sense of the norm in $\mathcal{H}$. If the spectrum of H_0 were not absolutely continuous, for instance if H_0 had eigenvalues, let P be the orthogonal projection to the absolutely continuous subspace of H_0 . Then, in the above definition $\forall \varphi \in \mathcal{H}$ should be replaced by $\forall \varphi \in P\mathcal{H}$, $\Omega_\pm$ exists and have nice properties (asymptotic completeness, etc.) [69,70]. The operators $\Omega_\pm$ are unitary mappings from $\mathcal{H}$ onto the absolutely continuous space of H . Nevertheless, and for the sake of simplicity, let us avoid these technicalities and assume that H has also a simple absolutely continuous spectrum, which is $[0, \infty)$. In this case, $\Omega_\pm$ are unitary operators on $\mathcal{H}$ and $H = \Omega_\pm H_0 \Omega_\pm$.

Let us define the spaces $\Phi_\pm := \Omega_\pm \Phi$. Since now $\Omega_\pm$ are unitary, following the ideas in the previous section, we can construct a new pair of RHS, according to the following diagram

$$\begin{array}{ccccc} \Phi & \subset & \mathcal{H} & \subset & \Phi^\times \\ \Omega_\pm \downarrow & & \Omega_\pm \downarrow & & \Omega_\pm \downarrow \\ \Phi_\pm & \subset & \mathcal{H} & \subset & \Phi_\pm^\times \end{array} \quad (25)$$

where the extension of $\Omega_\pm$ to the duals, $\Phi_\pm^\times$, is defined via the duality formula

$$\langle \varphi | F \rangle = \langle \Omega_\pm \varphi | \Omega_\pm F \rangle, \quad (26)$$

for all $\varphi \in \Phi$ and $F \in \Phi^\times$. Following this extension, we may define the “perturbed” kets $|E_\pm\rangle$ as $\Omega_\pm |E\rangle = |E_\pm\rangle$ and, equivalently, $\langle E_\pm| = \langle E| \Omega_\pm^\dagger$. Since the absolutely continuous spectrum of H_0 is $[0, \infty)$, the values of E in the latter relation covers the whole positive semi-axis.

Thus, we see that these objects are well defined [59,71]. Then, we may use the following operators that we define through their spectral decomposition:

$$O_\pm := \Omega_\pm O \Omega_\pm^\dagger = \int_0^\infty dE O_E |E_\pm\rangle\langle E_\pm| + \int_0^\infty dE \int_0^\infty dE' O_{EE'} |E_\pm\rangle\langle E'_\pm|. \quad (27)$$

The operators $O_{\pm}$ act on $\Phi_{\pm}$ and have the same topological properties than O , as discussed in the next section. O_E and $O_{EE'}$ must have regularity conditions, so that if O and R were operators of the form (27), we may define the following two products,

$$O_{\pm}R_{\pm} := \int_0^{\infty} dE O_E R_E |E_{\pm}\rangle\langle E_{\pm}| + \int_0^{\infty} dE \int_0^{\infty} dE' O_{EE'} R_{EE'} |E_{\pm}\rangle\langle E'_{\pm}|, \quad (28)$$

where R_E and $R_{EE'}$ are the functions that correspond to the operator $R_{\pm}$ as in (27) for $O_{\pm}$. This definition is quite natural provided that we use the convention $\langle E_{\pm}|E'_{\pm}\rangle = \langle E|\Omega_{\pm}^{\dagger}\Omega_{\pm}|E'\rangle = \langle E|E'\rangle = \delta(E - E')$ [61].

Thus, we have defined a pair of independent algebras $\mathcal{A}_{\pm}$ of operators using the above product. Note that the same is true for the “free algebra” $\mathcal{A}$ of the operators of the form (24). In addition, we may introduce an involution into these algebras defined as follows: if $O_{\pm} \in \mathcal{A}_{\pm}$, then,

$$O_{\pm} \mapsto O_{\pm}^{\dagger} := \int_0^{\infty} dE O_E^* |E_{\pm}\rangle\langle E_{\pm}| + \int_0^{\infty} dE \int_0^{\infty} dE' O_{EE'}^* |E_{\pm}\rangle\langle E'_{\pm}|. \quad (29)$$

Then, the sets of operators $\mathcal{A}_{\pm}$ can be endowed with the structure of involutive algebras and the same is true for $\mathcal{A}$. This finally depends on the properties of the functions O_E and $O_{EE'}$.

Operators with the property that $O_{\pm} = O_{\pm}^{\dagger}$ are called Hermitian and also observables. It is customary to abridge the above notation by using $|E_{\pm}) := |E_{\pm})\langle E_{\pm}|$ and $|EE'_{\pm}) := |E_{\pm})\langle E'_{\pm}|$, so that

$$O_{\pm} = \int_0^{\infty} dE O_E |E_{\pm}) + \int_0^{\infty} dE \int_0^{\infty} dE' O_{EE'} |EE'_{\pm}). \quad (30)$$

The algebras $\mathcal{A}_{\pm}$ may possess respective identities, $I_{\pm}$, which should have the form:

$$I_{\pm} = \int_0^{\infty} dE \Omega_{\pm} |E\rangle\langle E| \Omega_{\pm}^{\dagger} = \int_0^{\infty} dE |E_{\pm})\langle E_{\pm}| = \int_0^{\infty} dE |E_{\pm}). \quad (31)$$

Observe that the existence of the identities on $\mathcal{A}$ and $\mathcal{A}_{\pm}$ depends on the space of test functions that we have chosen for functions O_E . For instance, if O_E belong to the Schwartz space of infinitely differentiable functions having zero limit at infinite, then, $\mathcal{A}_{\pm}$ do not have identity. On the contrary, if the space of test functions contains continuous bounded functions, these identities do exist. This latter space contains the Schwartz space as a subspace. In any case, $I_{\pm}$ are the canonical injections $I_{\pm} : \Phi_{\pm} \mapsto \Phi_{\pm}^{\times}$, no matter if they belong to $\mathcal{A}_{\pm}$ or not.

Note that the objects $|E_{\pm})$ and $|EE'_{\pm})$ introduced above in (30) or (31) are mappings from $\Phi_{\pm}$ into $\Phi_{\pm}^{\times}$ and they should no be confused with bras. We restrict the use of this notation to the situation for which it was designed and it is quite convenient to denote the action of some kind of important linear mappings on the algebras $\mathcal{A}_{\pm}$ as we shall see in the sequel.

For reasons of convenience, as the space of test functions O_E we may consider the space of complex continuous bounded functions on the real line, while for the space including the $O_{EE'}$ functions we may use tensor products of analytic functions on the variables E and E' . We shall discuss these subtleties and other issues concerning topologies and continuity of functionals on Appendix A.

By *functionals* we understand here linear mappings $\rho_{\pm} : \mathcal{A}_{\pm} \mapsto \mathbb{C}$ from each of the algebras $\mathcal{A}_{\pm}$ into the field of complex numbers $\mathbb{C}$. We also may add a continuity condition to $\rho_{\pm}$. The action of a functional $\rho_{\pm}$ on $O_{\pm} \in \mathcal{A}_{\pm}$ is denoted as $(\rho_{\pm}|O_{\pm})$. Let us define the following functionals $(\omega_{\pm}|$ and $(\omega\omega'_{\pm}|$, for fixed values of $\omega, \omega' \in [0, \infty)$, over the algebras $\mathcal{A}_{\pm}$ as follows [61]:

$$(\omega_{\pm}|O_{\pm}) := O_{\omega}, \quad (\omega\omega'_{\pm}|O_{\pm}) := O_{\omega\omega'}. \quad (32)$$

Taking into account (30), this implies that

$$(E_{\pm}|E'_{\pm}) = \delta(E - E'), \quad (EE'|\omega\omega') = \delta(E - \omega)\delta(E' - \omega'). \quad (33)$$

The most general form of the functionals $\rho_{\pm}$ on the algebras $\mathcal{A}_{\pm}$ should have the form

$$\rho_{\pm} = \int_0^{\infty} dE \rho_E (E_{\pm}| + \int_0^{\infty} dE \int_0^{\infty} dE' \rho_{EE'} (EE'_{\pm}|, \quad (34)$$

where ρ_E is a generalized function or distribution on the space of O_E functions and $\rho_{EE'}$ a generalized function or distribution on the space of the $O_{EE'}$ functions. Then after (33), the bracket $(\rho_{\pm}|O_{\pm})$ can be written as

$$(\rho_{\pm}|O_{\pm}) = \int_0^{\infty} dE \rho_E O_E + \int_0^{\infty} dE \int_0^{\infty} dE' \rho_{EE'} O_{EE'}. \quad (35)$$

Once we have defined the *observables* as the Hermitian elements of the algebras $\mathcal{A}_{\pm}$, we may define the *states* as the functionals $\rho_{\pm}$ with the following properties: (i) positivity, $(\rho_{\pm}|O_{\pm}^{\dagger}O_{\pm}) \geq 0$, for all $O_{\pm} \in \mathcal{A}_{\pm}$, and (ii) normalization, $(\rho_{\pm}|I_{\pm}) = 1$. Observe that normalization requires the existence of units in both algebras.

Within this formalism, Antoniou et al. [60] classify the states into three different types:

- *Pure states*
The state $\rho_{\pm}$ is a pure state if there exists a function $\psi(E)$ such that $\rho_E \equiv |\psi(E)|^2$ and $\rho_{EE'} \equiv \psi^*(E)\psi(E')$. If this were the case, then, $\rho_{EE} \equiv \rho_E$.
- *Mixtures*
A state $\rho_{\pm}$ is a mixture if it is not a pure state, but yet $\rho_{EE} \equiv \rho_E$. This is the typical situation that arises with quantum mixed states represented as trace class operators.
- *Singular diagonal states*
A state $\rho_{\pm}$ is singular diagonal if and only if $\rho_E \neq \rho_{EE}$. These states are quantum states far from equilibrium [72,73]. It is worthy to mention that the origin of this formalism comes from a paper that intended to accommodate these states within a standard quantum formalism [60].

Gamow States

The next point is to define a functional for the Gamow vectors. As discussed in many previous papers, for each resonance there are two Gamow vectors and, therefore, we must have a couple of Gamow functionals for each resonance. These functionals have already been defined in [61,62]. Each of these functionals correspond to each of the poles of the S -matrix or the extended resolvent that determines one of the resonances. Previous discussions suggest that one of the Gamow states should be a functional on the algebra of *in* observables, $\mathcal{A}_{-}$, and the other on the algebra of *out* observables, $\mathcal{A}_{+}$. Each of these algebras is the image of the other through the time operator T [61]. The respective resonance poles lie at the points $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$, respectively, with $E_R, \Gamma > 0$.

In this situation, it is necessary to fix the spaces of test functions for O_E and $O_{EE'}$, which we shall do in the next section. As we have remarked before, we need that the spaces of the O_E and the $O_{EE'}$ functions contain bounded continuous functions and analytic functions on a strip (or entire analytic functions), respectively. Once we have made this choice, we may define the *decaying Gamow functional*, ρ_D [74] as

$$\rho_D := \int_0^{\infty} dE \delta_{E_R} (E_{+}| + \int_0^{\infty} dE \int_0^{\infty} dE' \delta_{z_R} \otimes \delta_{z_R^*} (EE'_{+}|, \quad (36)$$

where δ_{E_R} is the Dirac delta at the point E_R . Its action on a function O_E gives O_{E_R} , which is the value of O_E at E_R . The action of the functional $\delta_{z_R} \otimes \delta_{z_R^*}$ on the function $O_{EE'}$ is given by $O_{z_R z_R^*}$, which is the value of the function at (z_R, z_R^*) .

The growing Gamow functional is defined as

$$\rho_G := \int_0^\infty dE \delta_{E_R} (E_-) + \int_0^\infty dE \int_0^\infty dE' \delta_{z_R^*} \otimes \delta_{z_R} (EE'_-). \quad (37)$$

Let us see why, in our opinion, this is the correct choice for these functionals. First of all, ρ_G and ρ_D represent states on the algebras $\mathcal{A}_-$ and $\mathcal{A}_+$, respectively. Take for instance ρ_D and show that it fulfils the properties of states:

1. *Positivity*: It means that for any $O_+ \in \mathcal{A}_+$, $(\rho_D|O_+^\dagger O_+) \geq 0$. This property is indeed satisfied. Taking into account (11) and (14), one has that

$$(\rho_D|O_+^\dagger O_+) = |O_{E_R}|^2 + |O_{z_R z_R^*}|^2 \geq 0. \quad (38)$$

2. *Normalization*. It means that $(\rho_D|I_+) = 1$. Proving this fact is trivial.
3. *Continuity*. Although this requirement is not essential, we may endowed the algebras $\mathcal{A}_\pm$ with locally convex topologies, so that this property is satisfied for ρ_D . See Section 4.

The functional ρ_G on $\mathcal{A}_-$ is also a state and the proof of this statement is similar to what we did for ρ_D .

The functionals ρ_D and ρ_G have some additional properties. First of all, we need to add to the space of test functions O_E the space of polynomials, so that we may calculate the mean value of the energy for the Gamow functionals. After (23), (27) and $H = \Omega_\pm H_0 \Omega_\pm^\dagger$, we have that

$$H = \int_0^\infty dE E |E_\pm\rangle, \quad (39)$$

so that the total Hamiltonian belongs to both algebras $\mathcal{A}_\pm$. See the construction of the topologies on $\mathcal{A}_\pm$ in Section 4. Then, after some simple algebra,

$$(\rho_D|H) = E_R, \quad (\rho_G|H) = E_R. \quad (40)$$

This result coincides with the result proposed in [36] and it is perhaps the most reasonable by the reasons exposed in [36]. Note that $(\rho_D|H^n) = (\rho_G|H^n) = E_R^n$, for any $n \in \mathbb{N}_0$. As customary, the averaged values of any observable $O_\pm$ on the state $\rho_\pm$ is given as in (35) and these formulas must be also applied to the states ρ_D and ρ_G .

Concerning time evolution, observe that both ρ_D and ρ_G are the sum of two contributions. Let us call regular to the first one and singular to the other, so that $\rho_D = \rho_R + \rho_S$, where R and S stand for *regular* and *singular*, respectively. We define the time evolution for functionals using the usual *duality formula* as:

$$(\rho_\pm(t)|O_\pm) := (\rho_\pm|O_\pm(t)) = (\rho_\pm|e^{-itH} O_\pm e^{-itH}). \quad (41)$$

Since,

$$e^{-itH} = \int_0^\infty dE e^{-itE} |E_\pm\rangle = \int_0^\infty dE e^{-itE} |E_\pm\rangle \langle E_\pm|, \quad (42)$$

after using

$$|E_\pm\rangle \langle E_\pm| E'_\pm \rangle \langle E'_\pm| = \delta(E - E') |E_\pm\rangle \langle E'_\pm| = \delta(E - E') |EE'_\pm\rangle \quad (43)$$

and

$$|E_\pm\rangle \langle E'_\pm| \omega_\pm \rangle \langle \omega'_\pm| = \delta(E' - \omega) |E_\pm\rangle \langle \omega'_\pm| = \delta(E' - \omega) |E\omega'_\pm\rangle, \quad (44)$$

we obtain after some algebra,

$$e^{-itH} O_\pm e^{itH} = \int_0^\infty dE O_E |E_\pm\rangle + \int_0^\infty dE \int_0^\infty dE' e^{-it(E-E')} O_{EE'} |EE'_\pm\rangle. \quad (45)$$

Thus, after (36) and (41),

$$(\rho_D(t)|O_+) = O_{E_R} + e^{-it(z_R - z_R^*)} O_{z_R z_R^*}, \quad (46)$$

and a similar expression for $(\rho_G(t)|O_-)$, so that,

$$\rho_D(t) = \rho_R(0) + e^{-t\Gamma} \rho_S(0); \quad \rho_G(t) = \rho_R(0) + e^{t\Gamma} \rho_S(0), \quad (47)$$

where we have used ρ_R and ρ_S to denote both regular and singular parts of ρ_D and ρ_G , indistinctly. Observe that the regular part of the Gamow states does not evolve with time and the singular part decays exponentially to the future for the decaying Gamow functional ρ_D and to the past for the growing Gamow functional. With a proper choice of the analytic extension of $O_{EE'}$, we can construct a picture for which the time evolution for ρ_D and ρ_G makes sense for $t > 0$ only and $t < 0$ only, respectively (see next section).

In some precedent formalisms, the Gamow state is represented as a vector. This may suggest that a Gamow state may be a pure state. However, unlike the pure states, the Gamow vectors are not normalized. In addition, they represent unstable states and not stable ones. The question whether Gamow states are pure or not was open until we have introduced the precedent formalism. The form (36) and (37) of the Gamow states shows that they are singular diagonal states and neither pure states nor mixtures. This opens the door of further investigations, such as, for instance, if Gamow states have a well defined entropy, which should not be zero. One possible solution was given in [43]. However, this solution assumes a instantaneous picture of the Gamow state and not a kinematical one, considering the time evolution of this entropy. This is still an open question that should be investigated.

As a matter of fact, an article of the Brussels school [75] in 1979 opens the possibility of defining an entropy operator, M , for unstable quantum systems. This entropy operator should act on the space of states and the entropy of a state is then the average of M on this state. A possible choice of M is a positive monotonic function of the time operator T , which in this case, should be properly defined on this context.

4. Mathematical Details

We need to discuss the exact nature of the algebras of operators introduced in the precedent section, and its topological properties.

To begin with, let us choose the space of functions O_E in (24). In principle, we do not need properties of differentiability for these functions, although this condition may be added. Let us assume that they are continuous on $[0, \infty)$. However, boundedness is not a property they should have, as we may see from (23). Thus, $O_E \in \mathcal{C}(0, \infty)$, where $\mathcal{C}(A)$ means the set of continuous complex functions on the open set A . For each compact set $K \subset (0, \infty)$, we define the seminorm

$$p_K(O_E) := \sup_{E \in K} |O_E|, \quad \forall O_E \in \mathcal{C}(0, \infty). \quad (48)$$

With this choice for the family of seminorms on $\mathcal{C}(0, \infty)$, this space is a Fréchet (metrizable and complete) locally convex space [76]. In addition, $\mathcal{C}(0, \infty)$ is an algebra with unit, I , which has the additional properties: (i) For all compact set $K \subset (0, \infty)$, $p_K(I) = 1$. (ii) For $O_E, R_E \in \mathcal{C}(0, \infty)$, and all compact set K , $p_K(O_E R_E) \leq p_K(O_E) p_K(R_E)$. The proof is trivial.

We may check that the Dirac delta $\delta(E - \omega)$, with $\omega \in (0, \infty)$ fixed, defined as

$$\langle \delta(E - \omega) | O_E \rangle := \int_0^\infty \delta(E - \omega) O_E dE = O_\omega, \quad (49)$$

is a continuous functional on $\mathcal{C}(0, \infty)$. The linearity is obvious. Let K any compact set with $\omega \in K$ (It could even be $K \equiv \{\omega\}$). We have

$$|\langle \delta(E - \omega) | O_E \rangle| = |O_\omega| \leq p_K(O_E), \quad (50)$$

which proves the continuity.

In order to introduce the Gamow functionals, we need that the function $O_{EE'}$ be analytically continuable for each variable. We propose the following construction:

Let us consider the space of Schwartz functions with compact support contained in $\mathbb{R}^+$, $\mathcal{S}(\mathbb{R}^+)$. The functions in the space of Fourier transforms, $\mathcal{F}[\mathcal{S}(\mathbb{R}^+)]$, have the following properties:

- (i) They are Schwartz functions with support on $\mathbb{R}$ [77].
- (ii) They are entire analytic functions, which implies that their support is the whole $\mathbb{R}$ [78].
- (iii) They are Hardy functions on the lower half plane [79] (see Appendix A).

Analogously, let us consider the space of Schwartz functions with compact support contained in $\mathbb{R}^- \equiv (-\infty, 0]$, $\mathcal{S}(\mathbb{R}^-)$. Let us consider the space of Fourier transforms of these functions, $\mathcal{F}[\mathcal{S}(\mathbb{R}^-)]$. Functions in $\mathcal{F}[\mathcal{S}(\mathbb{R}^-)]$ have the same properties (i)–(iii), except that they are Hardy functions on the upper half plane [79].

Finally, let us consider the space

$$\mathfrak{D}_D := \mathcal{F}[\mathcal{S}(\mathbb{R}^+)] \otimes \mathcal{F}[\mathcal{S}(\mathbb{R}^-)]. \quad (51)$$

The space (51) is a locally convex space which is a subspace of the two dimensional Schwartz space $\mathcal{S}(\mathbb{R}^2)$. There are several different sequences of seminorms (in fact norms) on $\mathcal{S}(\mathbb{R}^2)$ so that two different sequences provide the same topology. Here, we take into account that $\mathcal{S}(\mathbb{R}^2) \approx \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$ algebraically and topologically. Take the finite sum $\sum_n O_{E,n} R_{E',n} \in \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$ and define the seminorms [80]

$$\Pi_{r,s}^{m,q} \left(\sum_n O_{E,n} R_{E',n} \right) := \inf \left[\sum_n p_{r,s}(O_{E,n}) p_{m,q}(R_{E',n}) \right], \quad (52)$$

where the infimum refers to all possibilities of writing the two variable function $\sum_n O_{E,n} R_{E',n}$ as a tensor product. Here, $r, s, m, q \in \mathbb{N}_0$, so that this family of seminorms depends on four non-negative integer parameters. These seminorms are extended to the whole space $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$ by continuity. As norms on $\mathcal{S}(\mathbb{R})$, we may take

$$p_{r,s}(O_E) := \|E^r D^s O_E\|, \quad D^s := \frac{d^s}{dE^s}, \quad r, s \in \mathbb{N}_0, \quad (53)$$

where the norm in (53) is the Hilbert space norm on $L^2(\mathbb{R})$.

We may give to $\mathfrak{D}_D$ the structure of algebra if we define the product as follows: if $O_{EE'}, R_{EE'} \in \mathfrak{D}_D$, their product is given by the standard product between functions, $O_{EE'} R_{EE'}$. This product is obviously in $\mathfrak{D}_D$. On $\mathfrak{D}_D$ we take the topology inherited from $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$ and given by the family of seminorms (52).

Let us define the functionals $\delta_\omega \otimes \delta_{\omega'}$, $\omega, \omega' \in \mathbb{R}^+$, and $\delta_{z_R} \otimes \delta_{z_R}^*$ on each $O_{EE'} \in \mathfrak{D}_D$ as

$$\langle \delta_\omega \otimes \delta_{\omega'} | O_{EE'} \rangle = O_{\omega \omega'}, \quad \langle \delta_{z_R} \otimes \delta_{z_R}^* | O_{EE'} \rangle := O_{z_R z_R^*}. \quad (54)$$

This functionals are obviously linear. In addition, they are continuous on $\mathfrak{D}_D$ with the topology inherited from $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$. Let us prove the continuity of $\delta_{z_R} \otimes \delta_{z_R}^*$. First of all, let us take an element of $\mathfrak{D}_D$ of the form of the finite sum $\sum_n O_{E,n} R_{E',n}$, $O_{E,n} \in \mathcal{F}[\mathcal{S}(\mathbb{R}^+)]$ and $R_{E',n} \in \mathcal{F}[\mathcal{S}(\mathbb{R}^-)]$. Thus,

$$\langle \delta_{z_R} \otimes \delta_{z_R}^* | \sum_n O_{E,n} R_{E',n} \rangle = \sum_n O_{z_R,n} R_{z_R^*,n}. \quad (55)$$

Then, using the Titchmarsh theorem [79], for any $O_E \in \mathcal{F}[\mathcal{S}(\mathbb{R}^+)]$ and for any $R_{E'} \in \mathcal{F}[\mathcal{S}(\mathbb{R}^-)]$, we have

$$O_{z_R} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{O_E}{E - z_R} dE, \quad R_{z_R^*} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{R_{E'}}{E' - z_R^*} dE'. \quad (56)$$

Taking the modulus in the first equation of (56), we have

$$|O_{z_R}| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|O_E|}{|E - z_R|} dE \leq \left[\int_{-\infty}^{\infty} |O_E|^2 dE \right]^{1/2} \left[\int_{-\infty}^{\infty} \frac{dE}{|E - z_R|^2} \right]^{1/2} = C \|O_E\|. \quad (57)$$

The second inequality in (57) is nothing else than the Schwarz inequality for $L^2(\mathbb{R})$, so that the norm in the last term is just the Hilbert space norm. Consequently, (55) implies that

$$\left| \sum_n O_{z_R,n} R_{z_R^*,n} \right| \leq C^2 \sum_n \|O_{E,n}\| \|R_{E',n}\|. \quad (58)$$

This identity must go for any span of $\sum_n O_{E,n} R_{E',n} \in \mathfrak{D}_D$ as a tensor product, so that

$$\left| \sum_n O_{z_R,n} R_{z_R^*,n} \right| \leq C^2 \Pi_{0,0}^{0,0} \left(\sum_n O_{z_R,n} R_{z_R^*,n} \right), \quad (59)$$

which shows the continuity of $\delta_{z_R} \otimes \delta_{z_R^*}$. The proof of the continuity of $\delta_\omega \otimes \delta_{\omega'}$ is even simpler. If we now define

$$\mathfrak{D}_G := \mathcal{F}[\mathcal{S}(\mathbb{R}^-)] \otimes \mathcal{F}[\mathcal{S}(\mathbb{R}^+)], \quad (60)$$

we may show analogously that both $\delta_\omega \otimes \delta_{\omega'}$, $\omega, \omega' \in \mathbb{R}^+$, and $\delta_{z_R^*} \otimes \delta_{z_R}$ are linear and continuous on $\mathfrak{D}_G$.

With all these ingredients, we may define the topologies on the algebras $\mathcal{A}$ and $\mathcal{A}_\pm$ and establish the continuity of $\rho_\pm$ on $\mathcal{A}_\pm$.

We define the algebra $\mathcal{A}_+$ as the algebra of all objects, O_+ , of the form (19), where $O_{EE'} \in \mathfrak{D}_D$. Analogously, the algebra $\mathcal{A}_-$ is the algebra of the O_- as in (30), such that $O_{EE'} \in \mathfrak{D}_G$ (Note that after (27), we should have also two algebras of the type $\mathcal{A}$. However, this is not important for our purposes and we ignore this along this presentation). For any $O_\pm \in \mathcal{A}_\pm$, we define the following set of seminorms:

$$P_{K,r,s}^{m,q}(O_\pm) = p_K(O_E) + \Pi_{r,s}^{m,q}(O_{EE'}). \quad (61)$$

Before showing that (36) and (37) are continuous functionals on $\mathcal{A}_+$ and $\mathcal{A}_-$, let us consider some simple examples of linear continuous functionals on these algebras. Let us start with $(\omega_\pm|$ defined as

$$(\omega_\pm|O_\pm) = O_\omega \implies |O_\omega| \leq p_K(O_\pm), \quad (62)$$

where K is any compact set containing ω . Thus, $(\omega_\pm|$ is trivially linear and is continuous due to its linearity and the right hand side of (62). Note that the functionals $(\omega_\pm|$ may also be written as

$$(\omega_\pm| \equiv \int_0^\infty \delta_\omega(E_\pm) dE, \quad \delta_\omega \equiv \delta(E - \omega). \quad (63)$$

Compare (63) with the first term in (36) and (37).

In order to discuss the next example, we want to recall that the topology on the Schwartz space $\mathcal{S}(\mathbb{R})$ may be given by another sequence of seminorms different from (53). We may choose instead of (53) the following: If $O_E \in \mathcal{S}(\mathbb{R})$, define

$$q_{r,s}(O_E) := \sup_{E \in \mathbb{R}} |E^r D^s O_E|, \quad r, s \in \mathbb{N}_0. \quad (64)$$

Then, reconstruct (52) using this sequence of norms. We shall obtain the same topology on $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}) \approx \mathcal{S}(\mathbb{R}^2)$. This is equivalent to use on each $O_{EE'} \in \mathcal{S}(\mathbb{R}^2)$ the family of seminorms given by (Although the families (52) and (65) are different, we use the same notation for both)

$$\Pi_{r,s}^{m,q}(O_{EE'}) := \sup_{E,E' \in \mathbb{R}} \left| E^r E'^s \frac{\partial^{m+q} O_{EE'}}{\partial E^m \partial E'^q} \right|. \quad (65)$$

Now, let us define the functional $(\omega\omega'|$ on $O_{\pm} \in \mathcal{A}_{\pm}$ as

$$(\omega\omega'|O_{\pm}) := O_{\omega\omega'}, \quad \omega, \omega' \in \mathbb{R}^+. \quad (66)$$

This mapping is obviously linear. In addition,

$$|O_{\omega\omega'}| \leq \Pi_{0,0}^{0,0}(O_{\pm}), \quad (67)$$

which shows the continuity.

Let us consider ρ_D as in (36). From (59), (61) and (62), we have that, for any $O_+ \in \mathcal{A}_+$ (The real part of a resonance pole is positive, so that $E_R > 0$),

$$(\rho_D|O_+) = O_{E_R} + O_{z_R z_R^*} \implies |(\rho_D|O_+)| \leq C P_{K,0,0}^{0,0}(O_+), \quad (68)$$

where C is a positive constant and we have used (52) with (53). The continuity of ρ_D on $\mathcal{A}_+$ has been proven. The proof of the continuity of ρ_G on $\mathcal{A}_-$ is analogous.

Now, let us justify the choices (51) for $\mathfrak{D}_D$ and (60) for $\mathfrak{D}_G$. Functions in $\mathfrak{D}_D$ are tensor products of entire functions which are Hardy in the lower half plane times entire functions which are Hardy on the upper half plane (see Appendix A). To fix ideas assume that $O_{EE'} = f(E) \otimes g(E') \equiv f(E)g(E')$, the generalization to other functions of the tensor product should be obvious. Here, $f(E)$ and $g(E')$, which depend on different variables, admit analytic continuations to entire functions. While the continuation of $f(E)$ is Hardy on the lower half plane, the continuation of $g(E')$ is Hardy on the upper half plane.

Next, come back to the evolution Equation (45) for $\mathcal{A}_+$. Consider $e^{-it(E-E')} f(E)g(E')$, which appears under the integral sign in (45). Does this function belongs to $\mathfrak{D}_D$? Write $z = E - iy$ and $z' = E' + iw$, with $y, w > 0$. Then, z and z' lie on the lower and upper half planes, respectively. Thus,

$$e^{-it(z-z')} f(z)g(z') = e^{-itE} e^{-yt} f(z) e^{itE'} e^{-wt} g(z'). \quad (69)$$

Take $t > 0$. Since $y, w > 0$, (69) is bounded in modulus by $|f(z)| |g(z')|$. By the properties of Hardy functions on a half plane (see Appendix A), this means that $e^{-itz} f(z)$ is Hardy on the lower half plane and that $e^{itz'} g(z')$ is Hardy on the upper half plane. This happens only for $t > 0$ and is false for $t < 0$ [71]. In other words, we have chosen the space $\mathfrak{D}_D$ in such a way that the time evolution for $\rho_D(t)$ is defined only for $t > 0$. Analogously, time evolution for ρ_G is defined only for $t < 0$. This provides a construction of the Gamow functionals showing time asymmetry.

5. Discussion and Conclusions

The traditional picture of unstable quantum states in the basic non-relativistic quantum mechanics is given by the Gamow vectors. Physically, these Gamow vectors give the purely exponential part of a quantum decaying state. This makes sense, since deviations of this exponential decay law are difficult to observe and take place for very short and very long times only. From the mathematical point of view, they are eigenvectors of the total Hamiltonian with complex eigenvalues. These eigenvalues are given by poles of the scattering matrix in the energy representation. This point of view directly shows the exponential time evolution of the Gamow states, but it has some important flaws [10,81]. In particular, Hamiltonians are represented by self-adjoint operators, which do not have complex eigenvalues. This problem may be solve by extending the Hilbert space to the

dual space of a dense subspace, in Hilbert space, endowed with a finer topology than the topology inherited by the Hilbert space metric. It is in this dual where the Gamow vectors live. The total Hamiltonian as well as the evolution operator may be extended uniquely to the dual so that this weird mathematical properties of Gamow vectors make sense outside the Hilbert space.

Nevertheless, some other difficulties derived from using Gamow vectors to represent quantum unstable states still remain. For instance, there is no clear way to define averages of observables on Gamow vectors [34–36]. In addition, the representation of Gamow states as Gamow vectors suggests that these states are pure. This somehow contradicts intuition, since the decaying states should not have zero entropy. In any case, we should recall that Gamow vectors are not Hilbert space vectors in the domain of the Hamiltonian. Some attempts to define entropy for Gamow states have been realized and have given a non-zero result.

Thus, it is needed a representation of Gamow states beyond Gamow vectors. One possible solution of the above problems is the definition of Gamow states as functional over some topological algebras. A model thereof has been proposed in the present review article. It is shown that Gamow states are not pure states. We have defined rigorously energy averages on Gamow states, which opens the door for averages of other relevant observables. We leave the definition of the entropy for a future work. We believe that this definition should be based in the notion of *entropy operator* as proposed in [75]. This construction requires some further technicalities, as we need to define the corresponding Liouville image on the present context. In fact, one has to construct an operator M such that

$$[L, M] = -iD, \quad D \geq 0, \quad [M, D] = 0, \quad (70)$$

where L is the Liouvillian, defined on the tensor product of Hilbert spaces $\mathcal{H} \otimes \mathcal{H}$ as $L = H \otimes I - I \otimes H$, I is the identity operator, and D is some positive operator. One possible solution to (70) is $M = f(T)$, where $f(\lambda)$ is a non-negative monotonic function and T a time operator satisfying $[L, T] = -iI$. Obviously, these considerations show the need for some extension of the formalism described here.

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Appendix A. Hardy Functions on a Half Plane

Hardy functions on a half plane are complex analytic functions on an open half plane. Here, we shall consider two types of Hardy functions: Those which are analytic on the *upper half plane*, $\mathbb{C}^+ := \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$, or the *lower half plane*, $\mathbb{C}^- := \{z \in \mathbb{C}, \operatorname{Im} z < 0\}$, where $\operatorname{Im} z$ stands for the imaginary part of the complex variable $z = x + iy$.

Let $f_+(z)$ be a complex analytic function on $\mathbb{C}^+$. We say that $f_+(x)$ is a *Hardy function on the upper half plane* if it fulfils the following property:

$$\sup_{y>0} \int_{-\infty}^{\infty} |f_+(x + iy)|^2 dx < \infty. \quad (\text{A1})$$

Analogously, if $f_-(z)$ is a complex analytic function on $\mathbb{C}^-$, we say that $f_-(z)$ is a *Hardy function on the lower half plane* if

$$\sup_{y>0} \int_{-\infty}^{\infty} |f_-(x - iy)|^2 dx < \infty. \quad (\text{A2})$$

Hardy functions have the following properties:

1. The set of Hardy functions either on the upper half or on the lower half plane is a vector space over the field of complex numbers. Henceforth, we shall denote these spaces as $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively.
2. Let $f_+(z) \equiv f_+(x + iy)$ be a Hardy function on the upper half plane. Then, for almost all $x \in \mathbb{R}$, with respect to the Lebesgue measure (a.e.), the limit

$$f_+(x) := \lim_{y \rightarrow 0} f_+(x + iy), \quad (\text{A3})$$

exists. This function is called the *boundary value* of the Hardy function $f_+(z)$. In addition, this function $f_+(x)$ is square integrable:

$$\int_{-\infty}^{\infty} |f_+(x)|^2 dx < \infty. \quad (\text{A4})$$

The same result is valid for $f_-(z) \equiv f_-(x - iy)$. For any $f_{\pm}(z) \in \mathcal{H}_{\pm}$, their limit functions are a.e. unique.

3. Let $f_{\pm}(x)$ be the boundary value function of the Hardy function $f_{\pm}(z) \in \mathcal{H}_{\pm}$. This boundary value function can be used to obtain the values of $f_{\pm}(z)$, for all $z \in \mathbb{C}^{\pm}$, by means of the Titchmarsh formula:

$$f_{\pm}(z) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_{\pm}(x)}{x - z} dx. \quad (\text{A5})$$

In summary, given $f_{\pm}(z) \in \mathcal{H}_{\pm}$, we obtain its boundary value function, which is a complex function defined a.e. on the real line. This function is a.e. unique. Conversely, if we have the boundary value function of a function either in $\mathcal{H}_+$ or in $\mathcal{H}_-$, we can recover all values of this function on its half plane. Thus, the relation between a Hardy function and its boundary value function is one to one and onto, so that we may somehow identify the boundary value function with its Hardy function. We shall proceed with this identification in the sequel, unless otherwise stated.

4. If $f_{\pm}(x)$ is the boundary value function of a Hardy function $f_{\pm}(z) \in \mathcal{H}_{\pm}$, it is square integrable, i.e.,

$$\int_{-\infty}^{\infty} |f_{\pm}(x)|^2 dx < \infty. \quad (\text{A6})$$

After the identification of the boundary value function with the original Hardy function, we may say that all Hardy functions on a half plane are square integrable, so that $\mathcal{H}_{\pm} \subset L^2(\mathbb{R})$.

5. Now, the point is: Being given a square integrable function $f(x) \in L^2(\mathbb{R})$, how we may determine whether this function is the boundary value function of a Hardy function either on the upper or on the lower half planes? The answer is given by the Paley-Wiener theorem, which states the following:

The square integrable function $f_{\pm}(x)$ is in $\mathcal{H}_{\pm}$ if and only if its inverse Fourier transform is in $L^2(\mathbb{R}^{\mp})$, where $L^2(\mathbb{R}^+)$ is the Hilbert space of square integrable Lebesgue function on the half line $\mathbb{R}^+$. Similar definition for $L^2(\mathbb{R}^-)$. Moreover, if $\mathcal{F}$ represents the Fourier transform operation, one may conclude that

$$\mathcal{F}[L^2(\mathbb{R}^{\mp})] \equiv \mathcal{H}_{\pm}. \quad (\text{A7})$$

6. The Fourier transform is a unitary mapping on $L^2(\mathbb{R})$. Since $L^2(\mathbb{R}) = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^-)$, then Equation (A7) and the properties of the Fourier transform imply that

$$L^2(\mathbb{R}) = \mathcal{H}_+ \oplus \mathcal{H}_-. \quad (\text{A8})$$

This means that any Lebesgue square integrable function may be decomposed into an orthogonal sum of a Hardy function on the upper half plane plus a Hardy function on the lower half plane.

7. The Fourier transform of a Schwartz function is also a Schwartz function. Therefore, the Fourier transform of a Schwartz function supported on $\mathbb{R}^+$, which means zero outside $\mathbb{R}^+$, is a Hardy function on the upper half plane for which its boundary value function is a Schwartz function on the whole real line. Analogously, the Fourier transform of a Schwartz function supported on $\mathbb{R}^-$ is a Hardy function on the lower half plane for which the boundary value function is a Schwartz function supported on the whole $\mathbb{R}$.
8. Another Paley-Wiener theorem [78] states that the Fourier transform of a Schwartz function with compact support is entire analytic. Thus, the Fourier transforms of Schwartz functions supported either on $\mathbb{R}^+$ or in $\mathbb{R}^-$ are Hardy functions on the corresponding half plane and entire analytic.
9. Let $t > 0$, $f_+(z) \in \mathcal{H}_+$ and consider the function $e^{itz} f_+(z)$. Since $f_+(z)$ is analytic on the upper half plane, so is $e^{itz} f_+(z)$. Let us prove that (A1) holds for $t > 0$, so that $e^{itz} f_+(z)$ is in $\mathcal{H}_+$, for $t > 0$. In fact,

$$\int_{-\infty}^{\infty} |e^{itz} f_+(z)|^2 dx = \int_{-\infty}^{\infty} e^{-ty} |f_+(x + iy)|^2 dx \leq \int_{-\infty}^{\infty} |f_+(x + iy)|^2 dx, \quad (\text{A9})$$

and our claim has been proved for $t > 0$. However, for any $t_0 < 0$, there always exists a function $g_+(z) \in \mathcal{H}_+$ such that $e^{it_0 z} g_+(z) \notin \mathcal{H}_+$. Similar properties hold for $t < 0$ and $f_-(z) \in \mathcal{H}_-$.

These are the most relevant properties of Hardy functions on a half plane for our purposes. For other properties, see [71] or the general text [79].

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