## Article

# Unilateral Laplace Transforms on Time Scales 

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#### Abstract

We review the direct and inverse Laplace transforms on non-uniform time scales. We introduce full backward-compatible unilateral Laplace transforms and studied their properties. We also present the corresponding inverse integrals and some examples.


Keywords: discrete Laplace transform; time scale; delta derivative; nabla derivative; nabla exponential; delta exponential

MSC: 44A10

## 1. Introduction

Mathematical developments do not solely depend on internal dynamics. The motivational effects of problems in science and engineering on these development processes may be fundamental. The attempts to solve such problems have triggered many new concepts in mathematics. Whenever notions and tools are not sufficient/appropriate to solve a given problem or do not yield the desired/fruitful results, we need to search for new ones to proceed to the generalization of the existing one. However, when performing such a generalization, it must satisfy the backward compatibility with the previously existing methodology. Here, we present an important example.

The classical theory of a differential/difference equation involves maintaining a parallelism between the formulation, solution, and properties; this implies great similarity and allows the establishment of equivalent rules [1]. These types of systems have uniform domains. However, some situations arise where we may have discrete or mixed discrete/continuous irregular domains in which classic ARMA-type equations are not suitable. To overcome such a discrepancy and allow obtaining similar results, Aulbach and Hilger [2] introduced a framework that unifies and extends the continuous and discrete existing methodologies: the "Calculus on Time Scales".

A time scale, denoted by $\mathbb{T}$, is defined as an arbitrary non-empty closed subset of the real line $\mathbb{R}$ and can be classified as uniform or non-uniform (or, irregular). Usually, uniform time scales are used [3,4], but non-uniform ones stand out in many applications, as pointed out in $[5,6]$. To solve the differential-difference equations corresponding to applications with underlying non-uniform time scales; it is obvious that the classic Laplace and Z transforms are not suitable and must be redefined for such time scales. A first generalization of the one-sided Laplace transform (LT) was proposed by Bohner and Peterson [7-14]. A second approach to such a goal was pursued by Davis et al. [8,15], where the domain of the proposed generalized LT was taken as the global time scale (bilateral transforms). However, such papers deal with transforms that are not exactly backward compatible, because they do not recover the $Z$ transform when the time scale is discrete uniform. Moreover, they do not introduce an inverse transform. To overcome these difficulties, Ortigueira et al [4,6] proposed a different approach. They began by computing the exponentials, eigenfunctions of the nabla (causal) and delta (anti-causal) derivatives,
and, from them, defined an inverse Laplace transform that was backward-compatible with the Bromwich integral for the classic Laplace transform (LT), as well as the Cauchy integral for the $Z$ transform (ZT). Using the properties of the referred exponentials, they obtained nabla and delta bilateral LTs that were fully compatible with the classic LT and ZT.

In this work, following [6], we present unilateral nabla and delta Laplace transforms on non-uniform time scales. Unlike the unilateral delta LT introduced by Bohner and Peterson (see [16], page 118), both transforms will coincide with the $Z$ transform if the time scale is $\mathbb{Z}_{0}^{+}$. Moreover, we deduce the properties enjoyed by classical one-sided LT and ZT and present the fully compatible characteristics.

This new transform allows us to solve problems defined on irregular domains and implement control systems when the sampling is not uniform. Moreover, it will permit the introduction of fractional derivatives on irregular time scales by inverting the transform $s^{\alpha}$ and using the convolution theorem.

The study is organized as follows. Section 2 introduces basic definitions of time scales, including graininess, jump operators, nabla and delta derivatives/anti-derivatives, and corresponding nabla and delta exponentials. Section 3 presents unilateral nabla and delta Laplace transforms on non-uniform time scales and their properties, and show the referred compatibility. We present some examples. Finally, in the last section we present the conclusions.

## 2. On-Time Scale Calculus

### 2.1. Basic Definitions

Let $t \in \mathbb{T}$ be the current time instance. Let $\rho(t)$ and $\sigma(t)$ be positive real-valued functions representing the previous and next instances, respectively. With $n^{\text {th }}$, the power of the $\rho(t)$ and $\sigma(t)$, i.e., $\rho^{n}(t)$ and $\sigma^{n}(t)$, we refer to the $n^{t h}$ previous and $n^{t h}$ next instances from the current one $t$, respectively, defined by the recursions (see Figure 1).

$$
\rho^{0}(t):=t, \quad \rho^{1}(t):=\rho(t), \quad \rho^{n}(t):=\rho\left(\rho^{n-1}(t)\right) \quad \text { and } \quad \sigma^{0}(t):=t, \quad \sigma^{1}(t):=\sigma(t), \quad \sigma^{n}(t):=\sigma\left(\sigma^{n-1}(t)\right)
$$

where $n \geq 2$. These functions are useful for specifying the steps needed to go from one instance to another, allowing us to introduce the conceptual differences between two instances and later the translation used in the convolution.


Figure 1. Scheme for a time scale.
These functions allow us to introduce the graininess functions $v(t):=t-\rho(t)$ and $\mu(t):=\sigma(t)-t$. The $n^{\text {th }}$ power, i.e., $v^{n}(t)$ or $\mu^{n}(t)$, expresses the value of the difference between the current instance $t$ and the $n^{t h}$ previous one $\rho^{n}(t)$, as well as the difference between the current instance $t$ and the $n^{\text {th }}$ following one $\sigma^{n}(t)$, respectively. They can be recursively generated according to:

$$
\begin{equation*}
\mu^{n}(t)=\mu^{n-1}(t)+\mu\left(t+\mu^{n-1}(t)\right), \quad n=1,2, \ldots \tag{1}
\end{equation*}
$$

with $\mu^{0}(t):=0$, and

$$
\begin{equation*}
v^{n}(t)=v^{n-1}(t)+v\left(t-v^{n-1}(t)\right), \quad n=1,2, \ldots \tag{2}
\end{equation*}
$$

where $v^{0}(t):=0$. In the following, we can see an illustration of the recursions.

$$
\begin{aligned}
\sigma^{0}(t) & =t=t+\mu^{0}(t) \\
\sigma^{1}(t) & =\sigma(t)=t+\mu(t)=t+\mu^{1}(t) \\
\sigma^{2}(t) & =\sigma(\sigma(t))=\sigma(t)+\mu(\sigma(t))=t+\mu^{1}(t)+\mu^{1}(t)=t+\mu^{2}(t), \\
\cdot & \\
\cdot & \\
\cdot & \\
\sigma^{n}(t) & =t+\mu^{n}(t)
\end{aligned}
$$

moreover, analogously, the relation $\rho^{n}(t)=t-v^{n}(t)$ can be obtained. At this stage, one may wonder about the meaning of a difference, such as $t-\tau$ for $t, \tau \in \mathbb{T}$, and how to define it, so that it makes sense. Let $t>\tau$. This means that one can reach $t$ by moving from the instance $\tau$ into the future; otherwise, one can arrive at $\tau$ by moving from the instance $t$ into the past. In a more formal way, there exists $N \in \mathbb{N}$, such that $t=\sigma^{N}(\tau)$ or $\tau=\rho^{N}(t)$. Set

$$
v_{n}(t)=\rho^{n-1}(t)-\rho^{n}(t) \quad \text { and } \quad \mu_{n}(t)=\sigma^{n}(t)-\sigma^{n-1}(t)
$$

for $n \geq 1$. Then, one can write the difference $t-\tau$ in the following two ways:

$$
t-\tau=\sigma^{N}(\tau)-\tau=\mu^{N}(\tau)=\mu(\tau)+\mu(\tau+\mu(\tau))+\ldots=\sum_{n=1}^{N} \mu_{n}(\tau)
$$

or

$$
t-\tau=t-\rho^{N}(t)=v^{N}(t)=v(t)+v(t-v(t))+\ldots=\sum_{n=1}^{N} v_{n}(t)
$$

In this study, the time scales that we deal with are assumed as a union of two sets, i.e., isolated points and closed intervals. As we will see in Section 2.4, with the isolated points and boundary points of each interval, we construct a discrete time scale $\mathbb{T}=\left\{t_{n}: n \in \mathbb{Z}\right\}$. Each point $\mathbb{T}=\left\{t_{n}: n \in \mathbb{Z}\right\}$ is right dense, isolated, or left dense. For example, if $t_{k}$ is a left (or right) dense point, then it represents all points $t_{k-1}+h$ (or $t_{k+1}-h$ ), although we have to insert a limited computation when computing any derivative or integral. With this formulation, we can redefine direct and reverse graininess in a way that plays a vital role in representing the integral function in the summation forms. The direct graininess is given by $v_{n}=v\left(t_{n}\right)=t_{n}-t_{n-1}, \quad n \in \mathbb{Z}$, and the reverse graininess is defined by $\mu_{n}=\mu\left(t_{n}\right)=t_{n+1}-t_{n}, \quad n \in \mathbb{Z}$, by avoiding representing the reference instance $t_{0}$. In the following section, the nabla and delta exponentials will be formulated in terms of these two types of graininess.

### 2.2. Nabla and Delta Derivatives

Let $\mathbb{T}$ be a given time scale in which will introduce two derivatives.
Definition 1. The nabla derivative is a causal operator defined by

$$
f^{\nabla}(t)= \begin{cases}\frac{f(t)-f(\rho(t))}{v(t)} & \text { if } \quad v(t) \neq 0  \tag{3}\\ \lim _{h \rightarrow 0^{+}} \frac{f(t)-f(t-h)}{h} & \text { if } \quad v(t)=0\end{cases}
$$

while the delta derivative is an anti-causal operator given by

$$
f^{\Delta}(t)= \begin{cases}\frac{f(\sigma(t))-f(t)}{\mu(t)} & \text { if } \mu(t) \neq 0  \tag{4}\\ \lim _{h \rightarrow 0^{+}} \frac{f(t+h)-f(t)}{h} & \text { if } \mu(t)=0\end{cases}
$$

If we consider the time scales $\mathbb{T}=\left\{t_{n}: n \in \mathbb{Z}\right\}$ specialized in the previous section, consisting of the union of a numerable discrete set $\left\{t_{n}: n \in \mathbb{Z}\right\}$ and a sequence of closed intervals, then these above definitions assume slightly different forms

$$
f^{\nabla}\left(t_{n}\right)=\left\{\begin{array}{lll}
\frac{f\left(t_{n}\right)-f\left(t_{n-1}\right)}{v_{n}} & \text { for any } t_{n} \quad \text { that is not left dense }  \tag{5}\\
\lim _{h \rightarrow 0^{+}} \frac{f\left(t_{n}\right)-f\left(t_{n}-h\right)}{h} & \text { for any } & t_{n} \quad \text { that is left dense }
\end{array}\right.
$$

where $v_{n}=t_{n}-t_{n-1}$ and the delta derivative is given by

$$
f^{\Delta}\left(t_{n}\right)=\left\{\begin{array}{llll}
\frac{f\left(t_{n+1}\right)-f\left(t_{n}\right)}{\mu_{n}} & \text { for any } & t_{n} & \text { that is not right dense }  \tag{6}\\
\lim _{h \rightarrow 0^{+}} \frac{f\left(t_{n}+h\right)-f\left(t_{n}\right)}{h} & \text { for any } & t_{n} & \text { that is right dense }
\end{array}\right.
$$

where $\mu_{n}=t_{n+1}-t_{n}$.
Example 1. Consider a time scale defined by $\mathbb{T}$, such that $t_{n}=n+r(n), n=1,2, \ldots, L$. The sequence $r(n), n=1,2, \ldots, L$, is randomly uniformly distributed in the interval $(-0.5,0.5)$. With this sequence, we sampled a sinusoid to obtain a signal $f\left(t_{n}\right)=\cos \left(\frac{2 \pi}{T} t_{n}+\phi\right)$, where $\phi$ is a random initial phase, $L=250$, and $T=50$. In Figure 2, we depict the nabla derivative of $f\left(t_{n}\right)$ (to make the visualization easier, we used an interpolation on the plot).


Figure 2. Nabla derivative of a sinusoidal signal.
The delta derivative gives a similar function.

### 2.3. The Nabla and Delta Anti-Derivatives

The relations (3)-(4) and (5)-(6) can be inverted to give the corresponding antiderivatives [6]. The inverses of nabla and delta derivatives of (3) and (4) are given by

$$
\begin{equation*}
f^{\nabla^{-1}}(t)=\sum_{n=0}^{\infty} v_{n+1}(t) f\left(t-v^{n}(t)\right) \tag{7}
\end{equation*}
$$

where $v_{n}(t)$ was introduced in (2), and

$$
\begin{equation*}
f^{\Delta^{-1}}(t)=-\sum_{n=0}^{\infty} \mu_{n+1}(t) f\left(t+\mu^{n}(t)\right) \tag{8}
\end{equation*}
$$

where $\mu_{n}(t)$ was given in (1). Here, $f^{\nabla^{-1}}(-\infty)=f^{\Delta^{-1}}(+\infty)=0$ is assumed.
These anti-derivatives allow us to introduce two definite nabla and delta integrals over $[a, b]$, respectively, by

$$
\int_{a}^{b} f(t) \nabla t \quad \text { and } \quad \int_{a}^{b} f(t) \Delta t
$$

If $\mathbb{T}$ consists of the isolated points, then there exists a $j \in \mathbb{N}$, such that $b=\sigma^{j}(a)=$ $a+\mu^{j}(a)$ or $a=\rho^{j}(b)=b-\nu^{j}(b)$. By considering (7) we will write

$$
\begin{equation*}
\int_{a}^{b} f(t) \nabla t=\sum_{n=0}^{j-1} v_{n+1}(b) f\left(b-v^{n}(b)\right) \tag{9}
\end{equation*}
$$

The lower terminal point of the integral will be taken as $a$, even if the summation on the right-hand side does not involve a multiplier with $f(a)$ and involves $f(\sigma(a))$ in the last multiplier. When we pass from the integral to the summation, we always keep this remark in mind. If $f^{\nabla^{-1}}$ is the nabla inverse of the function $f$, then the Equation (9) can be written as follows

$$
\begin{equation*}
\int_{a}^{b} f(t) \nabla t=\sum_{n=0}^{j-1} v_{n+1}(b) f\left(b-v^{n}(b)\right)=f^{\nabla^{-1}}(b)-f^{\nabla^{-1}}(a), \tag{10}
\end{equation*}
$$

which expresses the nabla definite integral in a Barrow-like formula.
In the next section, we will define unilateral (right and left) $\nabla$-LTs by splitting the integral representing the bilateral LT at the reference point $t_{0}$. Moreover, we will explain the relationship between $\nabla$-LT and $\nabla$-ZT. Therefore, it will be more useful to formulate the nabla integral on a discrete set $\left\{a=t_{0}, t_{1}, \ldots, b=t_{j}\right\}$ as follows:

$$
\begin{equation*}
\int_{a}^{b} f(t) \nabla t=\int_{t_{0}}^{t_{j}} f(t) \nabla t=\sum_{n=0}^{j-1} v_{n+1} f\left(t_{n+1}\right)=\sum_{n=1}^{j} v_{n} f\left(t_{n}\right) \tag{11}
\end{equation*}
$$

where $v_{n}=t_{n}-t_{n-1}$ for $n=1, \ldots, j-1$. and the summation do not involve $v_{0} f\left(t_{0}\right)$.
Analogously, from (8), we will write

$$
\begin{equation*}
\int_{a}^{b} f(t) \Delta t=\sum_{n=0}^{j-1} \mu_{n+1}(a) f\left(a+\mu^{n}(a)\right) \tag{12}
\end{equation*}
$$

The upper terminal point of the integral will be assumed as $b$ even if the summation on the right-hand side does not involve a multiplier with $f(b)$. Nevertheless, the last sum in this summation contains $\mu_{j}(a) f(\rho(b))$ as a last multiplier. When we pass from the integral to the summation, we always recall this note. If $f^{\Delta^{-1}}$ is the delta inverse of the function $f$, then Equation (12) can be written as follows

$$
\begin{equation*}
\int_{a}^{b} f(t) \Delta t=\sum_{n=0}^{j-1} \mu_{n+1}(a) f\left(a+\mu^{n}(a)\right)=f^{\Delta^{-1}}(b)-f^{\Delta^{-1}}(a) \tag{13}
\end{equation*}
$$

that resembles again the classic Barrow formula.
For the reasons we explained above for the nabla case, the formulation (13) for the discrete set $\left\{a=t_{0}, t_{1}, \ldots, b=t_{j}\right\}$ can be given

$$
\begin{equation*}
\int_{a}^{b} f(t) \Delta t=\sum_{n=0}^{j-1} \mu\left(t_{n}\right) f\left(t_{n}\right)=\sum_{n=0}^{j-1} \mu_{n} f\left(t_{n}\right) \tag{14}
\end{equation*}
$$

where $\mu_{n}=\mu\left(t_{n}\right)=t_{n+1}-t_{n}$ for all $n=0,1, \ldots, j-1$. This sum already involves $\mu\left(t_{0}\right) f\left(t_{0}\right)$ as a parcel. Moreover, it is known that [16]

$$
f(t) g(t)=\left[f^{\nabla}(t) g(t)+f(\rho(t)) g^{\nabla}(t)\right]^{\nabla^{-1}}
$$

so that we can obtain the integration of the parts formula for the nabla derivative:
Theorem 1. Let $f$ and $g$ be $\nabla$-differentiable functions, then

$$
\begin{equation*}
\int_{a}^{b} f^{\nabla}(t) g(t) \nabla t=f(b) g(b)-f(a) g(a)-\int_{a}^{b} f(\rho(t)) g^{\nabla}(t) \nabla t \tag{15}
\end{equation*}
$$

is fulfilled.
Similarly, by using (13), one can obtain the integration of the parts rule for the delta derivative

$$
f(t) g(t)=\left[f^{\Delta}(t) g(t)+f(\sigma(t)) g^{\Delta}(t)\right]^{\Delta^{-1}}
$$

which leads us to:
Theorem 2. Let $f$ and $g$ be $\Delta$-differentiable functions, then

$$
\begin{equation*}
\int_{a}^{b} f^{\Delta}(t) g(t) \Delta t=f(b) g(b)-f(a) g(a)-\int_{a}^{b} f(\sigma(t)) g^{\Delta}(t) \Delta t \tag{16}
\end{equation*}
$$

is satisfied.

Relations (15) and (16) originate from the initial conditions in the nabla and delta Laplace transforms, respectively.

### 2.4. The Nabla and Delta Unit Step Functions

We define the nabla and delta unit step functions, respectively, as follows [4]:

$$
u_{\nabla}\left(t, t_{0}\right)=\left\{\begin{array}{lll}
1 & \text { if } & t=t_{n} \geq t_{0}  \tag{17}\\
0 & \text { if } & t=t_{n}<t_{0}
\end{array}\right.
$$

and

$$
u_{\Delta}\left(t, t_{0}\right)=\left\{\begin{array}{lll}
0 & \text { if } & t=t_{n}>t_{0}  \tag{18}\\
1 & \text { if } & t=t_{n} \leq t_{0}
\end{array}\right.
$$

It is obvious that $u_{\nabla}\left(t, t_{0}\right)=u_{\Delta}\left(-t, t_{0}\right)$. Moreover, the derivatives of these functions give the nabla and delta impulse functions:

$$
D_{\nabla}\left[u_{\nabla}\left(t, t_{0}\right)\right]=\delta_{\nabla}\left(t, t_{0}\right)=\left\{\begin{array}{ccc}
\frac{1}{v_{1}\left(t_{0}\right)}\left(=\frac{1}{v_{0}}\right) & \text { if } & t=t_{n}=t_{0}  \tag{19}\\
0 & \text { if } & t=t_{n} \neq t_{0}
\end{array}\right.
$$

and

$$
D_{\Delta}\left[u_{\Delta}\left(t, t_{0}\right)\right]=\delta_{\Delta}\left(t, t_{0}\right)=\left\{\begin{array}{cll}
-\frac{1}{\mu_{1}\left(t_{0}\right)}\left(=-\frac{1}{\mu_{0}}\right) & \text { if } \quad t=t_{n}=t_{0}  \tag{20}\\
0 & \text { if } \quad t=t_{n} \neq t_{0}
\end{array}\right.
$$

If $t \in h \mathbb{Z}$, then these two impulses coincide with the Kronecker impulses given by [4]. Moreover, if $t \in \mathbb{R}$, is a non-isolated point, we have

$$
\begin{equation*}
\delta_{\nabla}\left(t, t_{0}\right)=-\delta_{\Delta}\left(t, t_{0}\right)=\delta\left(t-t_{0}\right) \tag{21}
\end{equation*}
$$

where $\delta$ represents the classical Dirac distribution.

### 2.5. The Nabla and Delta General Exponentials

Consider a time scale as above. A recursive procedure allows us to obtain the eigenfunctions (exponentials) of nabla and delta derivatives [6].

### 2.5.1. The General Nabla Exponential

Let $t>t_{0}$. Supposing $f^{\nabla}(t)=s f(t)$,

$$
f(t)[1-s v(t)]=f(t-v(t))
$$

is obtained. If we write $t=t-v(t)$ in the last equation, then we have

$$
\begin{equation*}
f(t-v(t))[1-s v(t-v(t))]=f(t-v(t)-v(t-v(t)) \tag{22}
\end{equation*}
$$

Proceeding in the same way for $n$ times and using the relations $v^{n}(t)=v^{n-1}(t)-$ $v\left(t-v^{n-1}(t)\right)$ and $v(t)=v_{1}(t), v(t-v(t))=v_{2}(t), \ldots, v\left(t-v^{n-1}(t)\right)=v_{n}(t)$, we have

$$
f(t) \prod_{k=1}^{n}\left[1-s v_{k}(t)\right]=f\left(t-v^{n}(t)\right)=f\left(t_{0}\right)
$$

By using $f\left(t_{0}\right)=1, v_{n-k+1}=\mu_{k}\left(t_{0}\right)$ and $\nabla t_{k}=t_{k}-t_{k-1}$ for $k=1,2, \ldots, n$ in the last equation, we have the discrete and integral form of the nabla exponential

$$
\begin{equation*}
f(t)=\prod_{k=1}^{n}\left[1-s \mu_{k}\left(t_{0}\right)\right]^{-1}=e^{-\sum_{k=1}^{n} \frac{\ln \left(\left[1-s \mu_{k}\left(t_{0}\right)\right]\right)}{\mu_{k}\left(t_{0}\right)} \nabla t_{k}}=e^{-\int_{t_{0}}^{t} \frac{\ln (1-s \mu(\tau))}{\mu(\tau)} \nabla \tau} \tag{23}
\end{equation*}
$$

for $t>t_{0}$.
Now let $t<t_{0}$. By following the same way used for $t>t_{0}$ but by starting from the point $t_{0}$, we have

$$
\begin{equation*}
f\left(t_{0}\right) \prod_{k=1}^{n}\left[1-s v_{k}\left(t_{0}\right)\right]=f\left(t_{0}-v^{n}\left(t_{0}\right)\right)=f\left(t_{n}\right) \tag{24}
\end{equation*}
$$

where $f\left(t_{0}\right)=1$. Applying the relation $v_{n-k+1}\left(t_{0}\right)=\Delta t_{n-k+1}$ for $k=1,2, \ldots, n$, to the last equation, for $t<t_{0}$ we have

$$
\begin{equation*}
f\left(t_{n}\right)=\prod_{k=1}^{n}\left[1-s v_{n-k+1}\left(t_{0}\right)\right]=e^{\sum_{k=1}^{n} \frac{\ln \left(\left[1-s v_{n-k+1}\left(t_{0}\right)\right]\right)}{v_{n-k+1}\left(t_{0}\right)} \Delta t_{n-k+1}}=e^{\int_{t}^{t_{0}} \frac{\ln (1-s v(\tau))}{v(\tau)} \Delta \tau} . \tag{25}
\end{equation*}
$$

Consequently, from (23) and (24), we have

$$
e_{\nabla}\left(t, t_{0} ; s\right)= \begin{cases}\prod_{k=1}^{n}\left[1-s \mu_{k}\left(t_{0}\right)\right]^{-1} & \text { if } t>t_{0}  \tag{26}\\ 1 & \text { if } t=t_{0} \\ \prod_{k=1}^{n}\left[1-s v_{k}\left(t_{0}\right)\right] & \text { if } t<t_{0}\end{cases}
$$

where it is assumed that $\mu_{k} \neq 0$ and $v_{k} \neq 0$ for some $k$.
Moreover, taking into consideration $v_{n}\left(t_{0}\right)=\mu_{-n}=v_{-n+1}$ and $\mu_{n}\left(t_{0}\right)=v_{n}=\mu_{n-1}$ for $n \geq 1$, we have the equivalent representations of the above formulation

$$
e_{\nabla}\left(t_{n}, t_{0} ; s\right)=\left\{\begin{array}{lll}
\prod_{k=1}^{n}\left[1-s v_{k}\right]^{-1} & \text { if } & t_{n}>t_{0}  \tag{27}\\
1 & \text { if } & t_{n}=t_{0} \\
\prod_{k=0}^{-n-1}\left[1-s v_{-k}\right] & \text { if } & t_{n}<t_{0},
\end{array}=\left\{\begin{array}{lll}
\prod_{k=0}^{n-1}\left[1-s \mu_{k}\right]^{-1} & \text { if } & t_{n}>t_{0} \\
1 & \text { if } & t_{n}=t_{0} \\
\prod_{k=1}^{-n}\left[1-s \mu_{-k}\right] & \text { if } & t_{n}<t_{0}
\end{array}\right.\right.
$$

Furthermore, from (23) and (25), the generalized nabla exponential for any time scale is as follows

$$
e_{\nabla}\left(t, t_{0} ; s\right)= \begin{cases}\exp \left(-\int_{t_{0}}^{t} \frac{\ln (1-s \mu(\tau))}{\mu(\tau)} \nabla \tau\right) & \text { if } t>t_{0}  \tag{28}\\ 1 & \text { if } t=t_{0} \\ \exp \left(\int_{t}^{t_{0}} \frac{\ln (1-s v(\tau))}{v(\tau)} \Delta \tau\right) & \text { if } t<t_{0}\end{cases}
$$

### 2.5.2. The General Delta Exponential

For $t>t_{0}$, assuming $f^{\Delta}(t)=s f(t)$, we have

$$
f(t+\mu(t))=f(t)[1+s \mu(t)] .
$$

Set $t=t_{0}$, then from the above equation we have

$$
\begin{equation*}
f\left(t_{0}+\mu\left(t_{0}\right)\right)=f\left(t_{0}\right)\left[1+s \mu\left(t_{0}\right)\right] . \tag{29}
\end{equation*}
$$

Let $t_{0}=t_{0}+\mu\left(t_{0}\right)$. We have

$$
f\left(t_{0}+\mu\left(t_{0}\right)+\mu\left(t_{0}+\mu\left(t_{0}\right)\right)\right)=f\left(t_{0}+\mu\left(t_{0}\right)\right)\left[1+s \mu\left(t_{0}+\mu\left(t_{0}\right)\right)\right] .
$$

Repeating the above relation $n$ times and using that $f\left(t_{0}\right)=1$, we obtain

$$
f\left(t_{n}\right)=f\left(t_{0}+\mu^{n}\left(t_{0}\right)\right)=f\left(t_{0}\right) \prod_{k=1}^{n}\left[1+s \mu_{k}\left(t_{0}\right)\right]=\prod_{k=1}^{n}\left[1+s \mu_{k}\left(t_{0}\right)\right]
$$

where we used the relation $\mu^{n}\left(t_{0}\right)=\mu^{n-1}\left(t_{0}\right)+\mu\left(t_{0}+\mu^{n-1}\left(t_{0}\right)\right)$, and $\mu\left(t_{0}+\mu\left(t_{0}\right)=\right.$ $\mu_{2}\left(t_{0}\right), \ldots, \mu\left(t_{0}+\mu^{n-1}\left(t_{0}\right)=\mu_{n}\left(t_{0}\right)\right.$. By using $\mu_{k}\left(t_{0}\right)=\nabla t_{k}=t_{k}-t_{k-1}$, the last equation yields

$$
\begin{equation*}
f\left(t_{n}\right)=\prod_{k=1}^{n}\left[1+s \mu_{k}\left(t_{0}\right)\right]=e^{\sum_{k=1}^{n} \frac{\ln \left(\left[1+s \mu_{k}\left(t_{0}\right)\right]\right)}{\mu_{k}\left(t_{0}\right)} \mu_{k}\left(t_{0}\right)}=e^{\int_{t_{0}}^{t} \frac{\ln (1+s \mu(\tau))}{\mu(\tau)} \nabla \tau} \tag{30}
\end{equation*}
$$

for $t>t_{0}$.
Now we assume that $t<t_{0}$, i.e., $t_{0}=t+\mu^{m}(t)$. By following the same procedure applied for the case $t>t_{0}$, we have

$$
\begin{equation*}
f(t)=\prod_{k=1}^{m}\left[1+s \mu_{k}(t)\right]^{-1} . \tag{31}
\end{equation*}
$$

Since $\mu_{1}(t)=v_{m}\left(t_{0}\right), \mu_{2}(t)=v_{m-1}\left(t_{0}\right), \ldots, \mu_{m}(t)=v_{1}\left(t_{0}\right)$, and $v_{m-k+1}\left(t_{0}\right)=\Delta t_{m-k+1}$, from the last equation

$$
\begin{equation*}
f(t)=\prod_{k=1}^{m}\left[1+s v_{m-k+1}\left(t_{0}\right)\right]^{-1}=e^{-\sum_{k=1}^{m} \frac{\ln \left(\left[1+s v_{m-k+1}\left(t_{0}\right)\right]\right)}{v_{m-k+1}\left(t_{0}\right)} \Delta t_{m-k+1}}=e^{-\int_{t}^{t_{0}} \frac{\ln (1+s v(\tau))}{v(\tau)} \Delta \tau} \tag{32}
\end{equation*}
$$

is obtained.
From (30) and (31), the delta exponential in the discrete form can be given as in [6]

$$
e_{\Delta}\left(t, t_{0} ; s\right)= \begin{cases}\prod_{k=1}^{n}\left[1+s \mu_{k}\left(t_{0}\right)\right] & \text { if } t>t_{0}  \tag{33}\\ 1 & \text { if } t=t_{0} \\ \prod_{k=1}^{n}\left[1+s v_{k}\left(t_{0}\right)\right]^{-1} & \text { if } t<t_{0}\end{cases}
$$

where $t=t_{0}+\mu^{n}\left(t_{0}\right)$, when $t>t_{0}$ and $t=t_{0}-v^{n}\left(t_{0}\right)$, when $t<t_{0}$, and it is assumed that $\mu_{k} \neq 0$ and $v_{k} \neq 0$ for some $k$.

By using $v_{n}\left(t_{0}\right)=\mu_{-n}=v_{-n+1}$ and $\mu_{n}\left(t_{0}\right)=v_{n}=\mu_{n-1}$ for $n \geq 1$, we have the equivalent representations of the above formulation

$$
e_{\Delta}\left(t_{n}, t_{0} ; s\right)=\left\{\begin{array}{lll}
\prod_{k=0}^{n-1}\left[1+s \mu_{k}\right] & \text { if } t_{n}>t_{0}  \tag{34}\\
1 & \text { if } t_{n}=t_{0} \\
\prod_{k=1}^{-n}\left[1+s \mu_{-k}\right]^{-1} & \text { if } t_{n}<t_{0}
\end{array}= \begin{cases}\prod_{k=1}^{n}\left[1+s v_{k}\right] & \text { if } t_{n}>t_{0} \\
1 & \text { if } t_{n}=t_{0} \\
\prod_{k=0}^{-n-1}\left[1+s v_{-k}\right]^{-1} & \text { if } t_{n}<t_{0}\end{cases}\right.
$$

From (30) and (32), we have the integral representations for the general delta exponential as follows:

$$
e_{\Delta}\left(t, t_{0} ; s\right)= \begin{cases}\exp \left(\int_{t_{0}}^{t} \frac{\ln (1+s \mu(\tau))}{\mu(\tau)} \nabla \tau\right) & \text { if } t \geq t_{0}  \tag{35}\\ 1 & \text { if } t=t_{0} \\ \exp \left(-\int_{t}^{t_{0}} \frac{\ln (1+s v(\tau))}{v(\tau)} \Delta \tau\right) & \text { if } t<t_{0}\end{cases}
$$

### 2.5.3. Properties of the Exponentials

The above-introduced exponentials enjoy some properties that will be useful in the following section [6]. Here, we will consider those most interesting for our objectives. As it is easy to verify, there is a relation between both exponentials

$$
\begin{equation*}
e_{\Delta}\left(t, t_{0} ; s\right)=1 / e_{\nabla}\left(t, t_{0} ;-s\right) \tag{36}
\end{equation*}
$$

Therefore, we do not need two exponentials. So, we will use the nabla exponential only and remove the subscript $\nabla$ whenever it is not needed $\left(e\left(t, t_{0} ; s\right)\right)$.

In many situations, it is important to know how exponentials increase/decrease for $s \in \mathbb{C}$. As said, consider the nabla exponential case. We start by noting that each graininess $v_{k}$ defines a Hilger circle centered at its reciprocal, passing through $s=0$ and being located in the right-hand half-complex plane. With $\mu_{k}$, the situation is similar, but the Hilger circles will be located in the left half complex plane. If we assume that $h_{\max }=\max \left(v_{k}\right), k \in \mathbb{Z}$, $h_{\min }=\min \left(v_{k}\right)$, and $k \in \mathbb{Z}$, then the circle centered at $1 / h_{\min }$ will be the outermost one amongst all right Hilger circles while the circle centered at $1 / h_{\max }$ will be the innermost. Since we have infinite graininess values in the time scale, the number of corresponding circles is infinite. However, if the graininess is constant, then they reduce to one. Lastly, in the case of zero graininess, we will talk about the imaginary axis instead of circles. Considering all of these facts and the definition of the nabla exponentials $e_{\nabla}\left(t, t_{0} ; s\right)$, one can observe the following:

- It is a real-valued function for any $s \in \mathbb{R}$;
- It is a positive real-valued function for any $s \in \mathbb{R}$, such that $s<1 / h_{\max }$;
- oscillates for any $s \in \mathbb{R}$, such that $s>1 / h_{\text {min }}$;
- It is a bounded function for $s \in \mathbb{C}$ in the innermost Hilger circle $\left|1-s h_{\max }\right|=1$;
- It has an absolute value that increases as $|s|$ increases outside the outermost Hilger circle $\left|1-s h_{\min }\right|=1$, going to infinite as $|s| \rightarrow \infty$.
- Let $\mathbb{T}=h \mathbb{Z}, h>0$. We make $t_{0}=0$ and use $v(t)=\mu(t)=h, t=n h, n \in \mathbb{Z}$, leading to $e(t, 0 ; s)=[1-s h]^{-n}$. Using $z^{-1}$ for $[1-s h]$, we obtain the current discrete-time exponential, $z^{n}$.
- Let $\mathbb{T}=\mathbb{R}$. Return to the above case and use $h=\frac{t}{n}$. As

$$
\lim _{n \rightarrow \infty}\left(1-\frac{s t}{n}\right)^{-n}=e^{s t}
$$

we obtain $e_{\nabla}(t, 0)=e^{s t}$.
The main properties of the exponential read [6]

1. Interchanging the role of instances.

$$
e\left(t_{0}, t ; s\right)=1 / e\left(t, t_{0} ; s\right)
$$

2. Scale changing.

Let $a \in \mathbb{R}^{+}$. Then, the equality

$$
e\left(a t, t_{0} ; s\right)=e\left(t, t_{0} ; a s\right)
$$

holds.
To prove this, we first assume that $n>0$, i.e., $t>0$. Since, $v_{k}=v\left(t_{k}\right)=t_{k}-t_{k-1}$ for $k \in \mathbb{Z}$, we have $v\left(a t_{k}\right)=a t_{k}-a t_{k-1}=a\left(t_{k}-t_{k-1}\right)=a v\left(t_{k}\right)$, where $a t_{k} \in a \mathbb{T}$. From here, we have

$$
e\left(a t, t_{0} ; s\right)=\prod_{k=1}^{n}\left[1-s v\left(a t_{k}\right)\right]^{-1}=\prod_{k=1}^{n}\left[1-a s v\left(t_{k}\right)\right]^{-1}=e\left(t, t_{0} ; a s\right)
$$

where $e\left(t, t_{0} ; s\right)=\prod_{k=1}^{n}\left[1-s v\left(t_{k}\right)\right]^{-1}$ as in (27) with $v_{k}=v\left(t_{k}\right)$.
Let $n<0$, i.e., $t<0$. Since $v_{-k}=v\left(t_{-k}\right)=t_{-k}-t_{-k-1}$ for $k \in \mathbb{N}$, we have $v\left(a t_{-k}\right)=$ $a t_{-k}-a t_{-k-1}=a\left(t_{-k}-t_{-k-1}\right)=a v\left(t_{-k}\right)$. Then, we have

$$
e\left(a t, t_{0} ; s\right)=\prod_{k=1}^{-n-1}\left[1-s v\left(a t_{-k}\right)\right]^{-1}=\prod_{k=1}^{-n-1}\left[1-a s v\left(t_{-k}\right)\right]^{-1}=e\left(t, t_{0} ; a s\right)
$$

where $e\left(t, t_{0} ; s\right)=\prod_{k=0}^{-n-1}\left[1-s v_{-k}\right]$ for $t<0$ as in (27) with $v_{-k}=v\left(t_{-k}\right)$.
3. Product of exponentials.

The general products of exponentials may not be exponentials in the known sense, even if they are well-defined. However, the following relations are satisfied [6]:
(a) By using the first and second properties given just above, we have

$$
e\left(t, t_{0} ; s\right) / e\left(\tau, t_{0} ; s\right)=e(t, \tau ; s), \quad t \geq \tau
$$

(b) Let $t_{n}>t_{m}$. Then, we obtain

$$
e\left(t_{n}, t_{0} ; s\right)=e\left(t_{m}, t_{0} ; s\right) e\left(t_{n}, t_{m} ;-s\right)
$$

4. Shift Property.

Here, we deal with nonuniform time scales. Since $t_{n}-t_{m}$ may not be an element of the nonuniform time scale $\mathbb{T}$, there is no guarantee of an existing mean for $e\left(t_{n}-t_{m}, t_{j}, \pm s\right)$ with $t_{j} \in \mathbb{T}$ Instead of using $t_{n}-t_{m}$ to define the shift property, we use $t_{n-m}$, which is equal to $t_{n}-t_{m}=t-\tau$ when the time scales are uniform. Let $n>m$. From the definition of the (nabla) exponential in (27) we have

$$
\begin{align*}
e\left(t_{n-m}, t_{0} ; s\right) & =\prod_{k=1}^{n-m}\left[1-s v_{k}\right]^{-1}=\frac{\prod_{k=1}^{n}\left[1-s v_{k}\right]^{-1}}{\prod_{k=n-m+1}^{n}\left[1-s v_{k}\right]^{-1}} \\
& =\frac{e\left(t_{n}, t_{0} ; s\right)}{e\left(t_{n}, t_{n-m} ; s\right)}=e\left(t_{n}, t_{0} ; s\right) e\left(t_{n-m}, t_{n} ; s\right) \tag{37}
\end{align*}
$$

where we used property 1 . For $m>n$, one can have the same equality in (37).

## 3. Laplace Transforms on Time Scales

The nabla and delta exponentials introduced in the last section allow us to define with generality two Laplace transforms defined over any time scale. The nabla and delta two-sided Laplace transforms were introduced and studied before in [6]. However, in agreement with the considerations we conducted in the previous sub-section, we will revise both and consider the corresponding one-sided transforms as well. However, we will pay
attention to the $\nabla$-Laplace transform $(\nabla$-LT) to distinguish it from the usual LT defined on $\mathbb{R}$ [17].

### 3.1. Inverse and Bilateral $\nabla$-Laplace Transform

In [6], Ortigueira et al. first presented the inverse nabla-Laplace transform before defining the corresponding direction.

Definition 2. Let $f(t)$ be defined on a given time scale $\mathbb{T}$ and assume it has a $L T, F(s)=\mathcal{L} f(t)$. Then, $f(t)$ can be synthesized through a continuous, infinite set of elemental exponentials with differential amplitudes:

$$
\begin{equation*}
f(t)=\frac{1}{2 \pi i} \oint_{C} F(s) \frac{1}{e\left(t-v(t), t_{0} ;-s\right)} d s \tag{38}
\end{equation*}
$$

where the integration path $C$, is a simple-closed contour in a region of analyticity of the integrand and encircles the poles of the delta exponential. Relation (38) is also called the synthesis equation of the LT. For this reason, the direct transform is also called the analysis formula [18].

The coherence of (38) can be verified through some tests

- Let us take $F_{\nabla}(s)=1$ and calculate its inverse Laplace transform. By definition of $e_{\nabla}\left(t, t_{0} ; s\right)$ in (27), one can see that the integrand is analytic when $t>t_{0} ;$ it is in the form of $\frac{1}{P(s)}(P$ is a polynomial of a degree greater than 1$)$ when $t<t_{0}$. So, its integral over $C$ for both cases is null. When $t=t_{0}$, we have only one pole, and

$$
\begin{equation*}
f(t)=\frac{1}{2 \pi i} \oint_{C} F_{\nabla}(s) \frac{1}{e\left(t-v(t), t_{0} ;-s\right)} d s=\frac{1}{2 \pi i} \oint_{C} \frac{1}{1+s v_{1}\left(t_{0}\right)} d s=\frac{1}{v_{1}\left(t_{0}\right)}=\frac{1}{v_{0}} . \tag{39}
\end{equation*}
$$

So, the inverse Laplace transform of $F_{\nabla}(s)=1$ is equal to $\delta_{\nabla}\left(t_{n}, t_{0}\right)$ where $\delta_{\nabla}$ is a nabla impulse function given by (19). In view of (21), this result makes sense and is compatible with its continuous counterpart obtained for a non-isolated point $t \in \mathbb{R}$.

- Let $F(s)=e\left(t_{k}, t_{0} ;-s\right), k \in \mathbb{Z}$. Then

$$
\begin{equation*}
f\left(t_{k}\right)=\frac{1}{2 \pi i} \oint_{C} e\left(t_{k}, t_{0} ;-s\right) \frac{1}{e\left(t_{n-1}, t_{0} ;-s\right)} d s=\frac{1}{2 \pi i} \oint_{C} e\left(t_{k}, t_{n-1} ;-s\right) d s=\delta_{\nabla}\left(t_{k}, t_{n}\right) \tag{40}
\end{equation*}
$$

where the relationship between the nabla and delta exponentials [6]

$$
e\left(t, t_{0} ; s\right) \frac{1}{e\left(\tau, t_{0} ;-s\right)}=e(t, \tau ; s)
$$

was used together with

$$
\frac{1}{2 \pi i} \oint_{C} e\left(t_{k}, t_{n-1} ;-s\right) d s=\left\{\begin{array}{lll}
0 & \text { if } & k \neq n  \tag{41}\\
\frac{1}{v_{n}} & \text { if } & k=n
\end{array}\right.
$$

- Consider a uniform time scale $\mathbb{T}=h \mathbb{Z}, h \in \mathbb{R}^{+}$. In this case, $t=n h, n \mathbb{Z}, v(t)=h$, and $t_{0}=0$ so that $e\left(t-v(t), t_{0} ;-s\right)=(1-s h)^{n}$. With
- $\quad z^{-1}=(1-s h)$, we obtain the inverse $Z$ transform;
- $\quad$ Setting $h=t / n$ and letting $h \rightarrow 0$ we arrive to the Bromwich integral inverse of the usual continuous-time LT.
- Moreover, by $\nabla$-differentiating on both sides of (38), we have $\mathcal{L}_{\nabla}\left\{f^{\nabla}(t)\right\}(s)=$ $s \mathcal{L}_{\nabla}\{f(\rho(t))\}(s)$. This result will be confirmed later by direct transformation.

Definition 3. Attending to the relation (40), the nabla-Laplace transform of a function $f: \mathbb{T} \rightarrow \mathbb{C}$ is defined by

$$
\begin{equation*}
F_{\nabla}(s)=\mathcal{L}_{\nabla}\{f(t)\}(s)=\sum_{n=-\infty}^{\infty} v_{n} f\left(t_{n}\right) e\left(t_{n}, t_{0} ;-s\right) \tag{42}
\end{equation*}
$$

for those values $s \in \mathbb{C}$ for which the corresponding series converges.
Considering the relation between $\nabla$-integration and $\nabla$-summation given in (11), the above formulation can be formulated in terms of the integral, as follows:

$$
\begin{equation*}
\mathcal{L}_{\nabla}\{f(t)\}(s)=\int_{-\infty}^{+\infty} f(t) e\left(t, t_{0} ;-s\right) \nabla t \tag{43}
\end{equation*}
$$

Similar to the above procedure, we can test the coherence of the definition by:

- if $f(t)=\delta_{\nabla}\left(t_{n}, t_{0}\right)$, then from (42) and (43)

$$
\mathcal{L}_{\nabla}\left\{\delta_{\nabla}\left(t_{n}, t_{0}\right)\right\}(s)=\int_{-\infty}^{+\infty} \delta_{\nabla}\left(t, t_{0}\right) e\left(t, t_{0} ;-s\right) \nabla t=\sum_{n=-\infty}^{\infty} v_{n} \delta_{\nabla}\left(t_{n}, t_{0}\right) e\left(t_{n}, t_{0} ;-s\right)=1
$$

where we use $\delta_{\nabla}\left(t_{0}, t_{0}\right)=1 / v_{0}$. This reproduces the above result $\mathcal{L}_{\nabla}^{-1}\{1\}=\delta_{\nabla}\left(t_{n}, t_{0}\right)$.
Theorem 3 (Inverse $\nabla$-Laplace transform). Let $F_{\nabla}(s)$ be the Laplace transform of $f: \mathbb{T} \rightarrow \mathbb{C}$ defined by (42) Then the formula in (38) represents the inverse $\nabla$-Laplace transform.

Proof. Inserting the nabla-Laplace transform formulation of $f(t)$ given by (42) into (38), we have, attending to (40)

$$
\begin{align*}
\mathcal{L}_{\nabla}^{-1}\left\{F_{\nabla}(s)\right\}\left(t_{n}\right) & =\frac{1}{2 \pi i} \oint_{C} \sum_{k=-\infty}^{\infty} v_{k} f\left(t_{k}\right) e\left(t_{k}, t_{0} ;-s\right) \frac{1}{e\left(t_{n-1}, t_{0} ;-s\right)} d s \\
& =\sum_{k=-\infty}^{\infty} v_{k} f\left(t_{k}\right) \frac{1}{2 \pi i} \oint_{C} e\left(t_{k}, t_{0} ;-s\right) \frac{1}{e\left(t_{n-1}, t_{0} ;-s\right)} d s \\
& =\sum_{k=-\infty}^{\infty} v_{k} f\left(t_{k}\right) \frac{1}{2 \pi i} \oint_{C} e\left(t_{k}, t_{n-1} ;-s\right) d s \\
& =\sum_{k=-\infty}^{\infty} v_{k} f\left(t_{k}\right) \delta_{\nabla}\left(t_{k}, t_{n}\right)=f\left(t_{n}\right) \tag{44}
\end{align*}
$$

for all $n \in \mathbb{N}$, where we use the uniform convergence of the summation representing nabla-Laplace transform.

- Consider a uniform time scale $\mathbb{T}=h \mathbb{Z}, h \in \mathbb{R}^{+}$. In this case, $t=n h, n \mathbb{Z}, v(t)=h$, and $t_{0}=0$ so that $e(t-v(t), 0 ;-s)=(1-s h)^{n}$. With
- $\quad z^{-1}=(1-s h)$ we obtain the usual $Z$ transform:
- Setting $h=t / n$ and letting $h \rightarrow 0$ we obtain the continuous-time bilateral LT.


### 3.2. The Unilateral $\nabla$-Laplace Transform

Definition 4. We define unilateral (one-sided) $\nabla$-Laplace transform by:

$$
\begin{equation*}
\mathcal{L}_{\nabla}\{f(t)\}(s)=\sum_{n=-\infty}^{\infty} u_{\nabla}\left( \pm\left(t_{n}-t_{0}\right)\right) v_{n} f\left(t_{n}\right) e\left(t_{n}, t_{0} ;-s\right) \tag{45}
\end{equation*}
$$

where $s \in \mathbb{C}$, $t_{0}$ is the reference point used for defining the nabla exponential and $u$ is the nabla unit step function given by (17). The sign in the argument of $u$ in the formulation (45) defines the left $(-)$ and right $(+)$ transforms.

To show the integral forms of the unilateral (one-sided) $\nabla$-Laplace transforms, we need to consider the relationship between the $\nabla$-integration and $\nabla$-summation in Section 2.3. According to the remark noted there, because the summation in (45) involves the multiplier $v_{0} f\left(t_{0}\right)$ when $n \geq n_{0}$, the lower terminal point of the corresponding integral must be $\rho\left(t_{0}\right)$. So, the integral form of the right unilateral $\nabla$-Laplace transform is given by

$$
\begin{equation*}
\mathcal{L}_{\nabla}^{+}\{f(t)\}(s)=\int_{\rho\left(t_{0}\right)}^{+\infty} f(t) e\left(t, t_{0} ;-s\right) \nabla t . \tag{46}
\end{equation*}
$$

If $t \in \mathbb{R}$, then $\rho\left(t_{0}\right) \rightarrow t_{0}^{-}$and the above formula turns into the following

$$
\begin{equation*}
\mathcal{L}^{+}\{f(t)\}(s)=\int_{t_{0}^{-}}^{+\infty} f(t) e\left(t, t_{0} ;-s\right) \nabla t . \tag{47}
\end{equation*}
$$

It means that $t_{0}$ is fully involved. In this respect, the nabla situation is different from the corresponding delta, as we will see later. It is very interesting to note that (47) agrees with a modified Laplace transform used in the control [19].

It is obvious that the left unilateral $\nabla$-Laplace transform is in the integral form of

$$
\begin{equation*}
\mathcal{L}_{\nabla}^{-}\{f(t)\}(s)=\int_{-\infty}^{\rho\left(t_{0}\right)} f(t) e\left(t, t_{0} ;-s\right) \nabla t \tag{48}
\end{equation*}
$$

Theorem 4. Let $f(t)$ be a bounded function of the exponential type, i.e.,
(i) There exists $A \in \mathbb{R}^{+}$and $a \in \mathbb{R}$, such that

$$
\left|f\left(t_{n}\right)\right|<A / e\left(t_{n}, t_{0} ;-a\right) \quad\left(n<n_{1}<0\right)
$$

when $\left|1+a v_{M}^{*}\right|<\left|1+s v_{m}\right|$, where $v_{M}^{*}$ is one of $v_{k}$ so that $\left|1+a v_{M}^{*}\right|=\max _{v_{k}}\left|1+a v_{k}\right|, v_{m}=$ $\min _{k \in 1,2, \ldots, n} v_{k}$ with $n>0$, and
(ii) There exists $B \in \mathbb{R}^{+}$and $b \in \mathbb{R}$ such that

$$
\left|f\left(t_{n}\right)\right|<B / e\left(t_{n}, t_{0} ;-b\right) \quad\left(n>n_{2}>0\right)
$$

when $\left|1+s v_{M}\right|<\left|1+b v_{m}^{*}\right|$, where $v_{m}^{*}$ is one of $v_{-k}$ so that $\left|1+b v_{m}^{*}\right|=\min _{v_{-k}}\left|1+b v_{-k}\right|$, $v_{M}=\max _{k \in\{0,1,2, \ldots,-n-1\}} v_{-k}$ with $n<0$.

If $s \in \mathbb{C}$ is inside the intersection of the circles defined by $\left|1+s v_{M}\right|<\left|1+b v_{m}^{*}\right|$ and $\left|1+s v_{M}\right|<\left|1+b v_{m}^{*}\right|$, then the integral in (45) is convergent.

The set of values $s \in \mathbb{C}$ for which (45) exists and is finite is called the region of convergence (ROC).

Proof. By considering $n_{1}<0$ and $n_{2}>0$, we split the summation corresponding to the nabla-Laplace transform into three summations; using the assumptions on the function on $f$, we have

$$
\begin{align*}
\left|\sum_{n=-\infty}^{+\infty} v_{n} f\left(t_{n}\right) e\left(t, t_{0},-s\right)\right| & \leq V \sum_{n=n_{1}+1}^{n_{2}}\left|f\left(t_{n}\right) e\left(t, t_{0},-s\right)\right|+A V \sum_{n=-\infty}^{n_{1}} e\left(t, t_{0},-s\right) / e\left(t, t_{0},-a\right) \\
& +B V \sum_{n=n_{2}+1}^{+\infty} e\left(t, t_{0},-s\right) / e\left(t, t_{0},-b\right) \tag{49}
\end{align*}
$$

where $V=\sup _{n \in \mathbb{N}} v_{n}$.
It is obvious that the first summation on the right-hand side of the inequality is finite. The finiteness of the second summation can be obtained as follows

$$
\begin{align*}
\sum_{n=-\infty}^{n_{1}} e\left(t, t_{0},-s\right) / e\left(t, t_{0},-b\right) & \leq \sum_{n=-\infty}^{n_{1}} \frac{\prod_{k=0}^{-n-1}\left|1+s v_{-k}\right|}{\prod_{k=0}^{-n-1}\left|1+b v_{-k}\right|} \leq \sum_{n=-\infty}^{n_{1}} \frac{\prod_{k=0}^{-n-1}\left|1+s v_{M}\right|}{\prod_{k=0}^{-n-1}\left|1+b v_{m}^{*}\right|} \\
& \leq \sum_{n=-\infty}^{n_{1}}\left|\frac{1+s v_{M}}{1+b v_{m}^{*}}\right|^{n}<\infty \tag{50}
\end{align*}
$$

when $\left|1+s v_{M}\right|<\left|1+b v_{m}^{*}\right|$, where $v_{m}^{*}$ and $v_{M}$ are as supposed in assumption (i).
For the third summation, we have

$$
\begin{align*}
\sum_{n=n_{2}+1}^{+\infty} e\left(t, t_{0},-s\right) / e\left(t, t_{0},-a\right) & \leq \sum_{n=n_{2}+1}^{+\infty} \frac{\prod_{k=1}^{n}\left|1+a v_{k}\right|}{\prod_{k=1}^{n}\left|1+s v_{k}\right|} \leq \sum_{n=n_{2}+1}^{+\infty} \frac{\prod_{k=1}^{n}\left|1+a v_{M}^{*}\right|}{\prod_{k=1}^{n}\left|1+s v_{m}\right|} \\
& \leq \sum_{n=n_{2}+1}^{+\infty}\left|\frac{1+a v_{M}^{*}}{1+s v_{m}}\right|^{n}<\infty \tag{51}
\end{align*}
$$

when $\left|1+a v_{M}^{*}\right|<\left|1+s v_{m}\right|$, where $v_{M}$ and $v_{m}$ are as supposed in assumption (ii).
Consequently, (50) and (51) give the convergence of the series defined in (45).

It is clear that the verifications of the two inequalities in this theorem are sufficient conditions for the existence of the nabla bilateral LT.

Example 2. As an application of the LT just introduced, we computed the transform of the nabla derivative obtained in Example 1, using $N=500$ exponentials corresponding to equal values for $s$. Such values were generated according to the following formula: $s(k)=i \omega_{k}=i \frac{\pi}{N}(k-1), k=$ $1,2, \ldots N$. In Figure 3, we depict the absolute value of the transform as a function of $\omega_{k}$.


Figure 3. Transforms of the signals used in Example 1.
It is interesting to note the ringing at low frequencies due to the truncation of the signal and the similarity to the classic continuous-time case.

### 3.3. Backward Compatibility

We are going to show that one-sided $\nabla$-Laplace transforms given by (46) are backward compatible. Assume first that $t \geq t_{0}$, to obtain the right transform that enjoys the following properties:

- Let $\mathbb{T}=\mathbb{R}_{0}^{+}$and $t_{0}=0$. Then, we have $\rho\left(t_{0}\right)=0$ and $e(t, 0 ;-s)=e^{-s t}$. From (46), we obtain the modified classical one-sided LT [19]

$$
\begin{equation*}
\mathcal{L}_{\nabla}^{+}\{f(t)\}(s)=\int_{0^{-}}^{+\infty} e^{-s t} f(t) d t \tag{52}
\end{equation*}
$$

- Let $\mathbb{T}=\mathbb{Z}_{0}^{+}$, i.e $t=n$ and $t_{0}=0$. Then, we have $e(n, 0 ;-s)=(1+s)^{-n}$ for $n \in \mathbb{Z}_{0}^{+}$. By changing them with corresponding terms in (45) and using the relation between $\nabla$-integration and $\nabla$-summation (11), we have

$$
\begin{equation*}
\mathcal{L}_{\nabla}^{+}\{f(t)\}(s)=\int_{\rho(0)}^{+\infty} f(t) e(t, 0 ;-s) \nabla t=\sum_{n=0}^{\infty} f(n) z^{-n} \tag{53}
\end{equation*}
$$

where the transformation $1+s=z$ was used. The last summation in (53) is the well-known $Z$ transform [3].
Assume now that $t<t_{0}$. Then, we have $u_{\nabla}\left( \pm\left(t-t_{0}\right)\right)=u_{\nabla}\left(t_{0}-t\right)$, which leads to the left transform that verifies:

- Let $\mathbb{T}=\mathbb{R}_{0}^{-}$and $t_{0}=0$. Then, we have $\rho\left(t_{0}\right)=0$ and $e(t, 0 ;-s)=e^{-s t}$. By replacing these with corresponding terms in (48), we obtain the classical one-sided left LT

$$
\mathcal{L}_{\nabla}^{-}\{f(t)\}(s)=\int_{-\infty}^{0^{-}} e^{-s t} f(t) d t
$$

- Let $\mathbb{T}=\mathbb{Z}^{-}$, i.e., $t=n$ and $t_{0}=0$. Then, we have $\rho(0)=-1$ and $e(n, 0 ;-s)=$ $(1+s)^{-n}$. By applying these in (48), we have

$$
\mathcal{L}_{\nabla}^{-}\{f(t)\}(s)=\int_{-\infty}^{\rho(0)} f(t) e\left(t, t_{0} ;-s\right) \nabla t=\sum_{n=-\infty}^{-1} f(n) z^{-n}
$$

where the transformation $1+s=z$ is used. The last summation is the left $Z$ transform.
Remark 1. The left transforms are not very useful in applications. For this reason, we will stop considering it, in the following.
3.4. Some Properties of the Unilateral $\nabla$-Laplace Transform

The unilateral $\nabla$-LT enjoys some interesting properties:

- Linearity

The linearity of $\nabla$-LT is an obvious result of its definition (45).

- Transform of the $\nabla$-derivative.

We suppose that $\lim _{t \rightarrow+\infty} f(t) e\left(t, t_{0} ;-s\right)=0$ holds for a given function $f: \mathbb{T} \rightarrow \mathbb{R}$. By integrating the parts formula for the nabla derivative in (15) to the integral, we have:

$$
\begin{gather*}
\mathcal{L}_{\nabla}^{+}\left\{f^{\nabla}(t)\right\}(s)=\int_{\rho\left(t_{0}\right)}^{+\infty} f^{\nabla}(t) e\left(t, t_{0} ;-s\right) \nabla t=-f\left(\rho\left(t_{0}\right)\right) e\left(\rho\left(t_{0}\right), t_{0} ;-s\right)+s \int_{\rho\left(t_{0}\right)}^{+\infty} f(\rho(t)) e\left(t, t_{0} ;-s\right) \nabla t \\
=-f\left(\rho\left(t_{0}\right)\right) e\left(\rho\left(t_{0}\right), t_{0} ;-s\right)+s \mathcal{L}_{\nabla}^{+}\{f(\rho(t))\}(s) .  \tag{54}\\
\text { If } \mathbb{T}=\mathbb{R}_{0}^{+}, \text {then } \rho(t)=t, \text { and we have from (54) } \\
\mathcal{L}^{+}\left\{f^{\prime}(t)\right\}(s)=-f\left(t_{0}^{-}\right)+s \mathcal{L}^{+}\{f(t)\}(s) .
\end{gather*}
$$

which is the well-known result of the classical LT. It is important to mention that this is a property of the LT, not of any other operator or system.

- The transform of the $\nabla$-anti-derivative.

Let $F(t)=\int_{t_{0}}^{t} f(\tau) \nabla \tau$ with $\lim _{t \rightarrow+\infty} F(t) e\left(t, t_{0} ;-s\right)=0$. By the nabla derivative of $\nabla$ exponential $\left(e\left(t, t_{0} ;-s\right)\right)^{\nabla}=-s e\left(t, t_{0} ;-s\right)$ and the nabla derivative of the multiplication of two functions

$$
(F(t) G(t))^{\nabla}=F^{\nabla}(t) G(t)+F(\rho(t)) G^{\nabla}(t)
$$

we have

$$
\begin{align*}
\mathcal{L}_{\nabla}^{+}\{F(\rho(t))\}(s) & =\int_{\rho\left(t_{0}\right)}^{+\infty} F(\rho(t)) e\left(t, t_{0} ;-s\right) \nabla t \\
& \left.=-\frac{1}{s}\left[\lim _{t \rightarrow+\infty} F(t) e\left(t, t_{0} ;-s\right)-F\left(\rho\left(t_{0}\right)\right) e\left(\rho\left(t_{0}\right), t_{0} ;-s\right)\right)\right]+\frac{1}{s} \int_{\rho\left(t_{0}\right)}^{+\infty} f(t) e\left(t, t_{0} ;-s\right) \nabla t \\
& \left.=\frac{1}{s} F\left(\rho\left(t_{0}\right)\right) e\left(\rho\left(t_{0}\right), t_{0} ;-s\right)\right)+\frac{1}{s} \mathcal{L}_{\nabla}\{f(t)\}(s) . \tag{55}
\end{align*}
$$

If $t \in \mathbb{R}$, then this equality turns into the well-known relation

$$
\mathcal{L}^{+}\{F(t)\}(s)=\frac{1}{s} \mathcal{L}^{+}\{f(t)\}(s) .
$$

- Initial value.

Suppose that $f(t)$ has $\nabla$-LT. Using the property "Transform of the $\nabla$-derivative", we have

$$
\lim _{\Re(s) \rightarrow \infty}\left[s \mathcal{L}_{\nabla}^{+}\{f(\rho(t))\}(s)-f\left(\rho\left(t_{0}\right)\right) e\left(\rho\left(t_{0}\right), t_{0} ;-s\right)\right]=\lim _{\Re(s) \rightarrow+\infty} \mathcal{L}_{\nabla}^{+}\left\{f^{\nabla}(t)\right\}(s)=0
$$

since $\lim _{\Re(s) \rightarrow \infty} \mathcal{L}_{\nabla}^{+}\left\{f^{\nabla}(t)\right\}(s)=0$ is satisfied for a function $f$ whose nabla derivative exists and of exponential type II. Hence,

$$
\lim _{\Re(s) \rightarrow+\infty} s \mathcal{L}_{\nabla}^{+}\{f(\rho(t))\}(s)=f\left(\rho\left(t_{0}\right)\right) e\left(\rho\left(t_{0}\right), t_{0} ;-s\right)
$$

holds.

- Final value.

Let us suppose that $f(t)$ has $\nabla$-LT. From the property "Transform of the $\nabla$-derivative", one can write

$$
\begin{aligned}
\lim _{s \rightarrow 0} s \mathcal{L}_{\nabla}^{+}\{f(\rho(t))\}(s) & =\lim _{s \rightarrow 0}\left[\mathcal{L}_{\nabla}^{+}\left\{f^{\nabla}(t)\right\}(s)+f\left(\rho\left(t_{0}\right)\right) e\left(\rho\left(t_{0}\right), t_{0} ;-s\right)\right] \\
& =\lim _{s \rightarrow 0} \int_{\rho\left(t_{0}\right)}^{+\infty} f^{\nabla}(t) e\left(t, \rho\left(t_{0}\right) ;-s\right) \nabla t+f\left(\rho\left(t_{0}\right)\right) \\
& =\int_{\rho\left(t_{0}\right)}^{+\infty} f^{\nabla}(t) \nabla t+f\left(\rho\left(t_{0}\right)\right)=\lim _{t \rightarrow+\infty} f(t) .
\end{aligned}
$$

- Time scaling.

Let $\rho\left(t_{0}\right)=0$ and $a>0$. By changing the variable $a t=\tau$ and applying the property of the scale changing in [6] for the nabla exponential in the definition of $\nabla$-LT, we have

$$
\begin{aligned}
\mathcal{L}_{\nabla}^{+}\{f(a t)\}(s) & =\int_{0}^{+\infty} f(a t) e\left(t, t_{0} ;-s\right) \nabla t=\frac{1}{a} \int_{0}^{+\infty} f(\tau) e\left(\frac{\tau}{a}, t_{0} ;-s\right) \nabla \tau \\
& =\frac{1}{a} \int_{0}^{+\infty} f(\tau) e\left(\tau, t_{0} ;-\frac{s}{a}\right) \nabla \tau=\frac{1}{a} \mathcal{L}_{\nabla}^{+}\{f(t)\}\left(\frac{s}{a}\right)
\end{aligned}
$$

### 3.5. Inverse and Bilateral $\Delta$-Laplace Transform

As such, in $\nabla$-case, for revealing the definition of the $\Delta$-Laplace transform in the correct way, the inverse $\Delta$-Laplace transform will be first introduced. As the inverse Laplace transform of $F_{\Delta}(s)=1$ should equal the delta impulse given by (20), the formula for the inverse Laplace transform is obtained from the following formula

$$
\begin{equation*}
f\left(t_{n}\right)=\frac{1}{2 \pi i} \oint_{C} F_{\Delta}(s) e\left(t, t_{0} ; s\right) d s \tag{56}
\end{equation*}
$$

where the integration path, $C$, is a simple closed contour in a region of analyticity of the integrand and the poles of the (nabla) exponential.

The integrand of (56) is analytic for $t>t_{0}$ and has $n$ poles at $s=+\frac{1}{\mu_{k}\left(t_{0}\right)}(k=1,2, \ldots, n)$ for $t<t_{0}$ when we assume that $F_{\Delta}(s)=1$. For both cases, the integral is null. To obtain the expected result, a pole for $n=0$ (or $t=t_{0}$ ) has to be introduced. For this, it is obvious that the exponential existing in the integrand have to be translated to the future by $\mu(t)$. This leads the formulation in (56) to convert into the following

$$
\begin{equation*}
f(t)=-\frac{1}{2 \pi i} \oint_{C} F_{\Delta}(s) e\left(t+\mu(t), t_{0} ; s\right) d s \tag{57}
\end{equation*}
$$

which is the definition of the inverse $\Delta$-Laplace transform. In addition to this, for $n=0$,

$$
f(t)=-\frac{1}{2 \pi i} \oint_{C} F_{\Delta}(s) e\left(t+\mu(t), t_{0} ; s\right) d s=-\frac{1}{2 \pi i} \oint_{C} \frac{1}{1-s \mu_{1}\left(t_{0}\right)}=\frac{1}{\mu_{1}\left(t_{0}\right)}=\frac{1}{\mu_{0}} .
$$

So, the inverse $\Delta$-Laplace transform of $F_{\Delta}(s)=1$ is equal to $-\delta_{\Delta}\left(t, t_{0}\right)$ given by (20) as expected. Furthermore, by the $\Delta$-differentiating on both sides of (57), we have $\mathcal{L}_{\Delta}\left\{f^{\Delta}(t)\right\}(s)=$ $s \mathcal{L}_{\Delta}\{f(\sigma(t))\}(s)$. This result will be reformulated by the one-sided delta direct transform in Section 3.7.

Definition 5. The delta Laplace transform of a function $f: \mathbb{T} \rightarrow \mathbb{C}$ is defined by

$$
\begin{equation*}
F_{\Delta}(s)=\mathcal{L}_{\Delta}\{f(t)\}(s)=\sum_{n=-\infty}^{\infty} \mu_{n} f\left(t_{n}\right)\left[e\left(t_{n}, t_{0} ; s\right)\right]^{-1} \tag{58}
\end{equation*}
$$

for those values $s \in \mathbb{C}$ for which the corresponding series converges.
The relationship between $\Delta$-integral and $\Delta$-summation given in Section 2.3 enable us to give the formula in (58) in the integral form as follows

$$
\begin{equation*}
\mathcal{L}_{\Delta}\{f(t)\}(s)=\int_{-\infty}^{+\infty} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t \tag{59}
\end{equation*}
$$

Theorem 5 (Inverse $\Delta$-Laplace transform). Let $F_{\Delta}(s)$ be the Laplace transform of $f: \mathbb{T} \rightarrow \mathbb{C}$ defined by (58). Then the formula in (57) represents the inverse $\Delta$-Laplace transform.

Proof. Inserting the delta-Laplace transform expression of $f(t)$ given by (58) into (57), we have

$$
\begin{align*}
\mathcal{L}_{\Delta}^{-1}\left\{F_{\Delta}(s)\right\}\left(t_{n}\right) & =-\frac{1}{2 \pi i} \oint_{C} \sum_{k=-\infty}^{\infty} \mu_{k} f\left(t_{k}\right) \frac{1}{e\left(t_{k}, t_{0} ; s\right)} e\left(t_{n+1}, t_{0} ; s\right) d s \\
& =\sum_{k=-\infty}^{\infty} \mu_{k} f\left(t_{k}\right)\left[-\frac{1}{2 \pi i} \oint_{C} \frac{1}{e\left(t_{k}, t_{0} ; s\right)} e\left(t_{n+1}, t_{0} ; s\right) d s\right] \\
& =\sum_{k=-\infty}^{\infty} \mu_{k} f\left(t_{k}\right)\left[-\frac{1}{2 \pi i} \oint_{C} e\left(t_{n+1}, t_{k} ;+s\right) d s\right] \\
& =\sum_{k=-\infty}^{\infty} \mu_{k} f\left(t_{k}\right) \delta_{\Delta}\left(t_{k}, t_{n}\right)=f\left(t_{n}\right) \tag{60}
\end{align*}
$$

for all $n \in \mathbb{N}$, where we use the uniform convergence of the summation representing the delta Laplace transform, the relationship between nabla and delta exponentials given by [6] as follows:

$$
\frac{1}{e\left(t, t_{0} ; s\right)} e\left(\tau, t_{0} ;-s\right)=e(\tau, t ; s)
$$

and

$$
\frac{1}{2 \pi i} \oint_{C} e\left(t_{n+1}, t_{k} ;+s\right) d s=\left\{\begin{array}{lll}
0 & \text { if } & k \neq n  \tag{61}\\
\frac{1}{\mu_{n}} & \text { if } & k=n
\end{array}\right.
$$

### 3.6. The Unilateral $\Delta$-Laplace Transform

Definition 6. We define the unilateral $\Delta-L T$

$$
\begin{equation*}
\mathcal{L}_{\Delta}\{f(t)\}(s)=\sum_{n=-\infty}^{\infty} u_{\Delta}\left(\mp\left(t_{n}-t_{0}\right)\right) \mu_{n} f\left(t_{n}\right)\left[e\left(t_{n}, t_{0} ; s\right)\right]^{-1} \tag{62}
\end{equation*}
$$

where $s \in \mathbb{C}$, $t_{0}$ is the reference point used for defining the delta exponential and $u_{\Delta}$ is the delta unit step function given by (18). The sign of the argument of $u$ in formulation (45) defines the left ( - ) and right $(+)$ transforms.

From the definition of the $\Delta$-unit step function, the right unilateral Laplace transform has the summation form of

$$
\begin{equation*}
\mathcal{L}_{\Delta}\{f(t)\}(s)=\sum_{n=1}^{\infty} \mu_{n} f\left(t_{n}\right)\left[e\left(t_{n}, t_{0} ; s\right)\right]^{-1} \tag{63}
\end{equation*}
$$

which does not involve $\mu_{0} f\left(t_{0}\right)$. By considering the relationship between the $\Delta$-integration and $\Delta$-summation in Section 2.3, the integral form of the unilateral right $\Delta$-Laplace transform is given by

$$
\begin{equation*}
\mathcal{L}_{\Delta}^{+}\{f(t)\}(s)=\int_{\sigma\left(t_{0}\right)}^{+\infty} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t \tag{64}
\end{equation*}
$$

If $t \in \mathbb{R}$, then $\sigma\left(t_{0}\right) \rightarrow t_{0}^{+}$and the above formula turns into the following

$$
\begin{equation*}
\mathcal{L}^{+}\{f(t)\}(s)=\int_{t_{0}^{+}}^{+\infty} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t \tag{65}
\end{equation*}
$$

It means that $t_{0}$ is not fully involved.
It is obvious that the unilateral left $\Delta$-Laplace transform has the following integral form

$$
\begin{equation*}
\mathcal{L}_{\Delta}^{-}\{f(t)\}(s)=\sum_{n=-\infty}^{0} \mu_{n} f\left(t_{n}\right)\left[e\left(t_{n}, t_{0} ; s\right)\right]^{-1}=\int_{-\infty}^{\sigma\left(t_{0}\right)} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t . \tag{66}
\end{equation*}
$$

Now, let us discuss the existence and the region of convergence $D_{\Delta}(f)$. The proof of the following theorem will be omitted here since it can be proved by following Theorem 4.

Theorem 6. Let $f(t)$ be a bounded function of the exponential type, i.e.,
(i) There exists $A \in \mathbb{R}^{+}$and $a \in \mathbb{R}$, such that

$$
\left|f\left(t_{n}\right)\right|<A e\left(t_{n}, t_{0} ; a\right) \quad\left(n<n_{1}<0\right)
$$

when $\left|1-a \mu_{M}^{*}\right|<\left|1-s \mu_{m}\right|$, where $\mu_{M}^{*}$ is one of $\mu_{k}$, so that $\left|1-a \mu_{M}^{*}\right|=\max _{\mu_{-k}}\left|1-a \mu_{-k}\right|$, $\mu_{m}=\min _{k \in 1,2, \ldots,-n} \mu_{-k}$ with $n<0$, and
(ii) There exists $B \in \mathbb{R}^{+}$and $b \in \mathbb{R}$, such that

$$
\left|f\left(t_{n}\right)\right|<B e\left(t_{n}, t_{0} ; b\right) \quad\left(n>n_{2}>0\right)
$$

when $\left|1-s \mu_{M}\right|<\left|1-b \mu_{m}^{*}\right|$, where $\mu_{m}^{*}$ is one of $\mu_{-k}$, so that $\left|1-b \mu_{m}^{*}\right|=\min _{\mu_{k}}\left|1-b \mu_{k}\right|$, $\mu_{M}=\max _{k \in\{0,1,2, \ldots, n-1\}} \mu_{k}$ with $n>0$.

If $s \in \mathbb{C}$ is inside the intersection of the circles defined by $\left|1-a \mu_{M}^{*}\right|<\left|1-s \mu_{m}\right|$ and $\left|1-s \mu_{M}\right|<\left|1-b \mu_{m}^{*}\right|$, then the integral in (62) is convergent.

The set of values $s \in \mathbb{C}$ for which (62) exists and is finite is called the region of convergence (ROC).

Let us now prove that the one-sided $\Delta$-LT defined by (62) ensures backward compatibility. At first, we suppose that $t \geq t_{0}$. Then, from (63) and (64), we determine the following relations:

- Let $\mathbb{T}=\mathbb{R}_{0}^{+}$and $t_{0}=0$. Then, we have $\sigma\left(t_{0}\right)=0$ and $e(t, 0 ; s)=e^{s t}$. By interchanging these with the corresponding terms in (64), we obtain the classical one-sided Laplace transformation

$$
\mathcal{L}_{\Delta}^{+}\{f(t)\}(s)=\int_{0^{+}}^{+\infty} e^{-s t} f(t) d t
$$

- Let $\mathbb{T}=\mathbb{Z}_{0}^{+}$and $t_{0}=0$. Then, we have $e\left(t, t_{0} ; s\right)=(1-s)^{-n}$. By considering these in (63) and (64), we have

$$
\begin{equation*}
\mathcal{L}_{\Delta}^{+}\{f(t)\}(s)=\int_{\sigma(0)}^{+\infty} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t=\sum_{n=1}^{\infty} f(n)(1-s)^{n}=\sum_{n=1}^{\infty} f(n) z^{-n} \tag{67}
\end{equation*}
$$

where the transformation $1-s=z^{-1}$ is used. The summation in (67) is the wellknown right $Z$ transform without the first term $f(0)$. From this aspect, this result differs from the nabla case.
At the moment, we assume $t<t_{0}$. Then, we obtain the following relations:

- Let $\mathbb{T}=\mathbb{R}^{-}$and $t_{0}=0$. Then, we have $\sigma\left(t_{0}\right)=0$ and $e(t, 0 ; s)=e^{s t}$. By applying these to (66), it yields the classical one-sided left LT

$$
\begin{equation*}
\mathcal{L}_{\Delta}^{-}\{f(t)\}(s)=\int_{-\infty}^{0} e^{-s t} f(t) d t \tag{68}
\end{equation*}
$$

- Let $\mathbb{T}=\mathbb{Z}^{-}$and $t_{0}=0$. Then, we have $e\left(t, t_{0} ; s\right)=(1-s)^{-n}$. By changing these with corresponding terms in (62) and considering the relation between $\Delta$-integration and $\Delta$-summation in (14) we have

$$
\mathcal{L}_{\Delta}^{-}\{f(t)\}(s)=\int_{-\infty}^{\sigma(0)} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t=\sum_{n=-\infty}^{\rho(\sigma(0))} f(n)(1-s)^{n}=\sum_{n=-\infty}^{0} f(n) z^{-n}
$$

where the transformation $1-s=z^{-1}$ was used. The last summation consists of the sum of the well-known left $Z$ transform and the term $f(0)$.

### 3.7. Some Properties of the Unilateral $\Delta$-Laplace Transformation

As we have done before, here, we only take into account the properties of $\mathcal{L}_{\Delta}^{+}$. We will give some properties of the operator $\mathcal{L}_{\Delta}^{+}$in the following:

- Linearity.

The linearity of the $\Delta$-Laplace transformation can be easily obtained from its integral representation.

- Transform of the $\Delta$-derivative.

Let $\lim _{t \rightarrow+\infty} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1}=0$ for a given function $f: \mathbb{T} \rightarrow \mathbb{R}$. By integrating the parts formula for the delta derivative in (16) to (62), we have

$$
\begin{aligned}
\mathcal{L}_{\Delta}^{+}\left\{f^{\Delta}(t)\right\}(s) & =\int_{\sigma\left(t_{0}\right)}^{+\infty} f^{\Delta}(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t \\
& =\lim _{t \rightarrow+\infty} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1}-f\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1}-\int_{\sigma\left(t_{0}\right)}^{+\infty} f(\sigma(t))\left[e^{-1}\left(t, t_{0} ; s\right)\right]^{\Delta} \Delta t \\
& =-f\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1}+s \int_{\sigma\left(t_{0}\right)}^{+\infty} f(\sigma(t))\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t
\end{aligned}
$$

which yields

$$
\begin{equation*}
\mathcal{L}_{\Delta}^{+}\left\{f^{\Delta}(t)\right\}(s)=-f\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1}+s \mathcal{L}_{\Delta}^{+}\{f(\sigma(t))\}(s) \tag{69}
\end{equation*}
$$

It is remarkable that if $\mathbb{T}=\mathbb{R}_{0}^{+}$, then we have the well-known result of the classical LT

$$
\begin{equation*}
\mathcal{L}^{+}\left\{f^{\prime}(t)\right\}(s)=-f\left(t_{0}^{+}\right)+s \mathcal{L}^{+}\{f(t)\}(s) . \tag{70}
\end{equation*}
$$

- The transform of the $\Delta$-anti-derivative.

Let $F(t)=\int_{t_{0}}^{t} f(\tau) \Delta \tau$ with $\lim _{t \rightarrow+\infty} F(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1}=0$. By the delta derivative of $\Delta$-exponential $\left[e^{-1}\left(t, t_{0} ; s\right)\right]^{\Delta}=-s\left[e\left(t, t_{0} ; s\right)\right]^{-1}$ and the delta derivative of the multiplication of two functions $\left.(F(t) G(t))^{\Delta}=F^{\Delta}(t) G(t)+F(\sigma)\right) G^{\Delta}(t)$, we have

$$
\begin{aligned}
\mathcal{L}_{\Delta}\{F(\sigma(t))\}(s) & =\int_{\sigma\left(t_{0}\right)}^{+\infty} F(\sigma(t))\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t \\
& =-\frac{1}{s}\left[\lim _{t \rightarrow+\infty} F(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1}-F\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1}\right]+\frac{1}{s} \int_{\sigma\left(t_{0}\right)}^{+\infty} f(t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t \\
& =\frac{1}{s} F\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1}+\frac{1}{s} \mathcal{L}_{\Delta}\{f(t)\}(s) .
\end{aligned}
$$

- Initial value.

Assume that $f(t)$ has $\Delta$-LT. Taking into account the property "Transform of the $\Delta$ derivative" in (69), we have

$$
\lim _{\Re(s) \rightarrow+\infty}\left[s \mathcal{L}_{\Delta}^{+}\{f(\sigma(t))\}(s)+f\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1}\right]=\lim _{\Re(s) \rightarrow+\infty} \mathcal{L}_{\Delta}^{+}\left\{f^{\Delta}(t)\right\}(s)=0
$$

since $\lim _{\Re(s) \rightarrow+\infty} \mathcal{L}_{\Delta}^{+}\left\{f^{\Delta}(t)\right\}(s)=0$ is satisfied for a function $f$ whose delta derivative exists and is of exponential type II. Hence, we have

$$
\lim _{\Re(s) \rightarrow+\infty} s \mathcal{L}_{\Delta}^{+}\{f(\rho(t))\}(s)=f\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1} .
$$

- Final value.

Assume that $f(t)$ has $\Delta$-LT. From the property "Transform of the $\Delta$-derivative" in (69) we have

$$
\begin{aligned}
\lim _{s \rightarrow 0} s \mathcal{L}_{\Delta}^{+}\{f(\sigma(t))\}(s) & =\lim _{s \rightarrow 0}\left[\mathcal{L}_{\nabla}^{+}\left\{f^{\Delta}(t)\right\}(s)+f\left(\sigma\left(t_{0}\right)\right)\left[e\left(\sigma\left(t_{0}\right), t_{0} ; s\right)\right]^{-1}\right] \\
& =\lim _{s \rightarrow 0} \int_{\sigma\left(t_{0}\right)}^{+\infty} f^{\Delta}(\tau)\left[e\left(\tau, t_{0} ; s\right)\right]^{-1} \Delta \tau+f\left(\sigma\left(t_{0}\right)\right) \\
& =\int_{\sigma\left(t_{0}\right)}^{+\infty} f^{\Delta}(\tau) \Delta \tau+f\left(\sigma\left(t_{0}\right)\right)=\lim _{t \rightarrow+\infty} f(t)
\end{aligned}
$$

- Time scaling.

Let $t_{0}=0$ and $a>0$. By changing the variable $a t=\tau$ and using the property of the scale changing for the nabla exponential in the definition of $\Delta$-LT, we have

$$
\begin{aligned}
\mathcal{L}_{\Delta}^{+}\{f(a t)\}(s) & =\int_{0}^{+\infty} f(a t)\left[e\left(t, t_{0} ; s\right)\right]^{-1} \Delta t=\frac{1}{a} \int_{0}^{+\infty} f(\tau)\left[e\left(\frac{\tau}{a}, t_{0} ; s\right)\right]^{-1} \Delta \tau \\
& =\frac{1}{a} \int_{0}^{+\infty} f(\tau)\left[e\left(\tau, t_{0} ; \frac{s}{a}\right)\right]^{-1} \Delta \tau=\frac{1}{a} \mathcal{L}_{\Delta}^{+}\{f(t)\}\left(\frac{s}{a}\right) .
\end{aligned}
$$

## 4. Conclusions

We propose unilateral nabla and delta Laplace transforms on non-uniform time scales by following the methodology introduced in [6]. We showed that both coincide with the $Z$ transform if the time scale is $\mathbb{Z}_{0}^{+}$and with unilateral LT if the time scale is $\mathbb{R}_{0}^{+}$. Moreover, we obtained properties enjoyed by classical one-sided LT and ZT, showing the fully compatible character.

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