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Regularity for Quasi-Linear p-Laplacian Type Non-Homogeneous Equations in the Heisenberg Group

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Abstract: When $2 - 1/Q < p \leq 2$, we establish the $C^{0,1}_{\text{loc}}$ and $C^{1,\alpha}_{\text{loc}}$-regularities of weak solutions to quasi-linear p-Laplacian type non-homogeneous equations in the Heisenberg group $\mathbb{H}^n$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

Keywords: p-Laplacian type; non-homogeneous equations; Heisenberg group; regularities; Riesz potentials

MSC: 35H20; 35B65

1. Introduction

In this paper, we consider the equation

$$-\operatorname{div}_H a(x, Xu) = \mu \quad \text{in } \Omega \subset \mathbb{H}^n, \tag{1}$$

where Ω is a domain and μ is a Radon measure with $|\mu| < \infty$ and $\mu(\mathbb{H}^n \setminus \Omega) = 0$; hence the Equation (1) can be considered as defined in all of $\mathbb{H}^n$. Here $Xu = (X_1u, X_2u, \ldots, X_{2n}u)$ is denoted as the horizontal gradient of a function $u : \Omega \to \mathbb{R}$, see Section 2 for more details, and the continuous function $a : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is assumed to be C^1 in the gradient variable and satisfies the following structural conditions for every $x, y \in \Omega$ and $z, \xi \in \mathbb{R}^{2n}$,

$$(|z|^2 + s^2)^{\frac{p-2}{2}} |\xi|^2 \leq \langle D_z a(x,z)\xi, \xi \rangle \leq L(|z|^2 + s^2)^{\frac{p-2}{2}} |\xi|^2; \tag{2}$$

$$|a(x,z) - a(y,z)| \leq L'|z|(|z|^2 + s^2)^{\frac{p-2}{2}} |x - y|^\alpha, \tag{3}$$

where $L, L' \geq 1, s \geq 0, \alpha \in (0,1]$ and $D_z a(x,z)$ is a symmetric matrix for every $x \in \Omega$. Here we call the Equation (1) with a satisfying (2) and (3) as quasi-linear p-Laplacian type non-homogeneous equation.

A function $u \in HW^{1,p}_{\text{loc}}(\Omega)$ is called as a weak solution to (1) if

$$\int_\Omega \langle a(x, Xu), X\varphi \rangle dx = \int_\Omega \varphi d\mu,$$

where $HW^{1,p}_{\text{loc}}(\Omega)$ is the first order p-th integrable horizontal local Sobolev space, namely, all functions $u \in L^p_{\text{loc}}(\Omega)$ with their distributional horizontal gradients $Xu \in L^p_{\text{loc}}(\Omega)$. Given the typical example $a(x,z) = (|z|^2 + s^2)^{\frac{p-2}{2}} z$, the Equation (1) becomes the sub-elliptic non-degenerate p-Laplacian equation with measure data

$$-\operatorname{div}_H(|Xu|^2 + s^2)^{\frac{p-2}{2}} Xu = \mu \quad \text{if } s > 0,$$

and the sub-elliptic p-Laplacian equation with measure data

$$-\operatorname{div}_H |Xu|^{p-2} Xu = \mu \quad \text{if } s = 0. \tag{4}$$

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When measure $\mu = 0$, the Equation (4) becomes the sub-elliptic p-Laplacian equation

$$-\operatorname{div}_H |Xu|^{p-2} Xu = 0. \tag{5}$$

Particularly, we call weak solutions to the Equation (5) as p-harmonic functions in $\Omega \subset \mathbb{H}^n$.

For p-harmonic functions in Euclidean spaces $\mathbb{R}^n$, their $C^{1,\alpha}$-regularity has been established by [1–5]. For p-harmonic functions in the Heisenberg group $\mathbb{H}^n$, their $C^{0,1}$ and $C^{1,\alpha}$-regularities have been established by [6–12]. It is therefore natural to consider the case of regularity for the corresponding inhomogeneous equation. In Euclidean spaces $\mathbb{R}^n$, when $2 - 1/n < p < \infty$, Duzaar-Mingione [13,14] built up the $C^{0,1}$-regularity of solutions to the Equation (1) with measure $\mu \in L^1(\Omega)$. In the Heisenberg group $\mathbb{H}^n$, when $2 \le p < \infty$, Mukherjee-Sire [15] built up the $C^{1,\gamma}$-regularity of solutions to the Equation (1) with measure $\mu = f \in L^q(\Omega)$ for some $q > Q = 2n + 2$ and some $\gamma \in (0, 1)$. But when $1 < p < 2$, the $C^{0,1}$ and $C^{1,\gamma}$-regularities for the Equation (1) in the Heisenberg group $\mathbb{H}^n$ are unknown. This paper aims to establish the $C^{0,1}$ and $C^{1,\gamma}$-regularities in the case $1 < p < 2$.

It is known that the Heisenberg group $\mathbb{H}^n$ is a typical step two Carnot group, see Section 2 for more details. There is the vertical vector field T on $\mathbb{H}^n$, which brings great difficulties to study the existence and the regularity of solutions for the equations. Therefore it is of great significance to study the equations in $\mathbb{H}^n$. The study of the existence of solutions for some complex nonlinear equations including (p, q)-Laplacian equations and $p(\cdot)$-Laplacian equations et al. in $\mathbb{H}^n$ attracted a lot of attentions in past decades, see [16–21]. Recently, the existence of solutions for p-biharmonic problem and Neumann problem have been given by Safari-Razani [22–24], which provides the basis for studying the regularity of solutions.

Before stating our main results, let us recall that truncated linear Riesz potentials are defined as

$$\mathbf{I}^{\mu}_{\beta}(x_0, 2R) := \int_0^R \frac{\mu(B(x,\rho))}{\rho^{Q-\beta}} \frac{d\rho}{\rho}, \quad \beta \in (0, Q].$$

Theorem 1. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1) with $\mu \in L^1_{\mathrm{loc}}(\Omega)$. If $2 - 1/Q < p \le 2$ and $a : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies the structural conditions (2) and (3), then there exist constants $c = c(n, p, L) > 0$ and $\bar{R} = \bar{R}(n, p, L, L', \alpha, \operatorname{dist}(x_0, \partial\Omega)) > 0$, such that the pointwise estimate*

$$\begin{aligned} |Xu(x_0)| \le & c \fint_{B_{2R}} (|Xu| + s)dx + c\left[\frac{|\mu|(B_{2R})}{R^{Q-1}}\right]^{\frac{2}{p}} + c\left[\frac{|\mu|(B_{2R})}{R^{Q-1}}\right]^{\frac{3Q-Qp-2}{Q-p}} \\ & + c[\mathbf{I}^{|\mu|}_1(x_0, 2R)]^{\frac{2}{p}} + c[\mathbf{I}^{|\mu|}_1(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \end{aligned} \tag{6}$$

holds for any $x_0 \in \mathbb{H}^n$, whenever $B_{2R}(x_0) \subset \Omega$ and $0 < R \le \bar{R}$. Furthermore, if $a(x, z)$ is independent of x, then (6) holds for any $0 < R < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$. Here $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

Theorem 2. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1). Assume that $2 - 1/Q < p \le 2$ and $a : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies the structural conditions (2) and (3). If we have $\mu = f \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > Q$, then Xu is Hölder continuous and there exist constants $c = c(n, p, L) > 0$ and $\bar{R} = \bar{R}(n, p, L, L', \alpha, \operatorname{dist}(x_0, \partial\Omega)) > 0$, such that for any $x_0 \in \Omega$, $0 < R \le \bar{R}$ and $x, y \in B_R(x_0) \subset \Omega$, the estimate*

$$\begin{aligned} |Xu(x) - Xu(y)| \le & c d(x,y)^{\gamma} \Bigg\{ \fint_{B_R} (|Xu| + s)dx + \|f\|^{\frac{2}{p}}_{L^q(B_R)} + \|f\|^{\frac{3Q-Qp-2}{Q-p}}_{L^q(B_R)} \\ & + [\mathbf{I}^{|\mu|}_1(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}^{|\mu|}_1(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \Bigg\} \end{aligned} \tag{7}$$

holds for some $\gamma = \gamma(n, p, L, \alpha, q) \in (0, 1)$. *In particular, if* $a(x, z)$ *is independent of* x, *then* (7) *holds for* $\bar{R} = \bar{R}(n, p, L, L', \mathrm{dist}(x_0, \partial\Omega)) > 0$ *and* $\gamma = \gamma(n, p, L, q) \in (0, 1)$. *Here* $Q = 2n + 2$ *is the homogeneous dimension of* $\mathbb{H}^n$.

Theorems 1 and 2 give when $2 - 1/Q < p \le 2$ the $C^{0,1}_{\mathrm{loc}}$ and $C^{1,\alpha}_{\mathrm{loc}}$-regularities of weak solutions to quasi-linear p-Laplacian type non-homogeneous Equations (1) in the Heisenberg group $\mathbb{H}^n$, see [15] for $2 \le p < \infty$, where $Q = 2n + 2$. Compared with the Euclidean space $\mathbb{R}^n$, the range of p is optimal, see [13].

Ideas of the Proofs

We sketch the ideas to prove Theorems 1 and 2. The basic geometries and properties of the Heisenberg group used in this paper are stated in Section 2.

We will prove Theorem 1 in Section 4. The proof of Theorem 1 relies on novel techniques established by Duzaar-Mingione [13] based on sharp comparison estimates of homogeneous equations with frozen coefficients. In Section 3, we establish two comparison estimates, see Lemmas 1 and 2 for details. Basing on two comparison estimates, we establish the main estimate of the weak solution u to the Equation (1), see Lemma 3 for details. Compared with the Euclidean setting, there exists the extra term $\sup_{B_{\bar{R}}} |Xv|$ in (34), which comes from commutators of the horizontal vector fields, see Proposition 1 for details. We use Lemma 2 to estimate the extra term in Section 4. In Section 4, basing on Lemma 3, we use scientific induction to obtain Lemma 4. Finally, we use Lemma 4 to prove Theorem 1 in Section 4.

We will prove Theorem 2 in Section 5. The proof of Theorem 2 relies on a perturbation lemma established by Mukherjee-Sire [15], see Lemma 6 for details. In Section 5, we use Lemma 2 to establish the weaker integral decay estimate of the oscillation of the gradient of the weak solution u to the Equation (1), see Lemma 5 for details. Basing on Lemmas 6 and 5, we obtain Proposition 2 in Section 5. Finally, we use Lemma 7 and Proposition 2 to prove Theorem 2 in Section 5. Lemma 7 follows from (13) and Lemma 2 in Section 5.

2. Preliminaries

2.1. Notations

In this paper, for $s \ge 0$, we denote

$$V(z) := (|z|^2 + s^2)^{\frac{p-2}{4}} z, \quad z \in \mathbb{R}^{2n}. \tag{8}$$

By (Lemma 2.1, [25]), the inequality

$$c^{-1}(|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2}{2}} \le \frac{|V(z_2) - V(z_1)|^2}{|z_2 - z_1|^2} \le c(|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2}{2}} \tag{9}$$

holds for any $z_1, z_2 \in \mathbb{R}^{2n}$ and any $s \ge 0$, where $c = c(n, p) > 0$ is independent of s, also see ([13], (2.2)). Inequality (9) and the structure condition (2) imply

$$c^{-1}|V(z_2) - V(z_1)|^2 \le \langle a(x, z_2) - a(x, z_1), z_2 - z_1 \rangle. \tag{10}$$

2.2. The Heisenberg Group

For an integer $n \ge 1$, we denote by $\mathbb{H}^n$ the Heisenberg group, which is identified with the Euclidean space $\mathbb{R}^{2n+1}$. The group multiplication on $\mathbb{H}^n$ is given by

$$x \circ y := \left(x_1 + y_1, \ldots, x_{2n} + y_{2n}, t + s + \frac{1}{2} \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) \right)$$

for points $x = (x_1, \dots, x_{2n}, t), y = (y_1, \dots, y_{2n}, s) \in \mathbb{H}^n$. The left invariant vector fields corresponding to the canonical basis of the Lie algebra are

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t,$$

and the only non-trivial commutator $T = \partial_t$ for $1 \le i \le n$. For any $1 \le i < j \le 2n$, we have

$$[X_i, X_{n+i}] = T, \quad [X_i, X_j] = 0\ \forall j \neq n+i.$$

We call $X_1, \dots, X_{2n}$ as horizontal vector fields and T as the vertical vector field. We denote $Q = 2n+2$ as the homogeneous dimension of $\mathbb{H}^n$.

Let $\Omega \subset \mathbb{H}^n$ be any domain (open connected subset). For any scalar function $f \in C^1(\Omega)$, we denote $Xf = (X_1 f, \dots, X_{2n} f)$ as the horizontal gradient; for any scalar function $f \in C^2(\Omega)$, we denote $XXf = (X_i X_j f)_{2n\times 2n}$ as the second order horizontal derivative and $\Delta_H f = \sum_{j=1}^{2n} X_j X_j f$ as the sub-Laplacian operator. We write lengths of Xf and XXf as

$$|Xu| = \left(\sum_{i=1}^{2n} |X_i u|^2\right)^{1/2}, \quad |XXu| = \left(\sum_{i,j=1}^{2n} |X_i X_j u|^2\right)^{1/2}.$$

For any vector valued function $F = (f_1, \dots, f_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$, we denote $\mathrm{div}_H(F) = \sum_{i=1}^{2n} X_i f$ as the horizontal divergence. The Haar measure in $\mathbb{H}^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$. We denote $|E|$ as the Lebesgue measure of a measurable set $E \subset \mathbb{H}^n$ and $⨍_E f dx = \frac{1}{|E|}\int_E f dx$ as the average of an integrable function f over set E.

We denote d as the Carnot–Carathéodory metric (CC-metric) and $B_r(x) = B(x,r) := \{y \in \mathbb{H}^n : d(x,y) < r\}$ as the CC-metric balls with the center $x \in \mathbb{H}^n$ and the radius $r > 0$. Here the CC-metric d is defined as the length of the shortest horizontal curves connecting two points, see [26]. For any points $x, y \in \mathbb{H}^n$, the CC-metric $d(x,y)$ is equivalent to the homogeneous metric $d_{\mathbb{H}^n}(x,y) = \|y^{-1} \circ x\|_{\mathbb{H}^n}$. Here the homogeneous norm for $x = (x_1, \dots, x_{2n}, t) \in \mathbb{H}^n$ is defined as $\|x\|_{\mathbb{H}^n} := \left(\sum_{i=1}^{2n} x_i^2 + |t|\right)^{1/2}$. Since these two metrics are equivalent, all the CC-metric balls $B_r(x)$ throughout this paper can be restated to the homogeneous metric balls $K_\rho(x) := \{y \in \mathbb{H}^n : d_{\mathbb{H}^n}(y,x) < \rho\}$.

The horizontal Sobolev space $HW^{1,p}(\Omega)$ with $1 \le p < \infty$ is the collection of all functions $u \in L^p(\Omega)$ with $Xu \in L^p(\Omega, \mathbb{R}^{2n})$. $HW^{1,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega,\mathbb{R}^{2n})}.$$

For any $m \ge 2$, the m-order horizontal Sobolev space $HW^{m,p}(\Omega)$ is the collection of all functions u with $Xu \in HW^{m-1,p}(\Omega)$, and its norm is defined in a similar way. For any $m \ge 1$, we denote $HW^{m,p}_{\mathrm{loc}}(\Omega)$ as the collection of all functions $u : \Omega \to \mathbb{R}$ such that $u \in HW^{m,p}(U)$ for all $U \Subset \Omega$, and $HW^{m,p}_0(\Omega)$ as the completion of $C_c^\infty(\Omega)$ equipped with the $\|\cdot\|_{HW^{m,p}(\Omega)}$-norm.

In the rest of this section, we recall some regularities and apriori estimates of the homogeneous equation corresponding to the Equation (1) with freezing of the coefficients. For any $x_0 \in \Omega$, we consider the equation

$$\mathrm{div}_H a(x_0, Xu) = 0 \quad \text{in } \Omega. \tag{11}$$

The following regularity theorem follows from (Theorem 1.1, [12]) and (Theorem 1.3, [10]), also see (Theorem 2.3, [15]).

Theorem 3. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (11). If $a(x_0, z)$ satisfies the condition (2) and $D_z a(x_0, z)$ is a symmetric matrix, then Xu is locally Hölder continuous.*

Moreover, there exist constants $c = c(n, p, L) > 0$ and $\beta = \beta(n, p, L) \in (0, 1)$ such that the followings hold,

$$\sup_{B_{R/2}} |Xu|^p \le c \fint_{B_R} (|Xu|^2 + s^2)^{\frac{p}{2}} dx; \tag{12}$$

$$\fint_{B_\rho} |Xu - (Xu)_{B_\rho}|^p dx \le c \Big(\frac{\rho}{R}\Big)^\beta \fint_{B_R} (|Xu|^2 + s^2)^{\frac{p}{2}} dx, \tag{13}$$

for every concentric $B_\rho \subset B_R \subset \Omega$ and $1 < p < \infty$.

Using Sobolev's inequality and Moser's iteration on the Caccioppoli type inequalities in [12], we have the following local estimate, for any $\sigma \in (0, 1)$ and $q > 0$,

$$\sup_{B_{\sigma R}} |Xu| \le c(1-\sigma)^{-\frac{Q}{q}} \bigg(\fint_{B_R} (|Xu|^2 + s^2)^{\frac{q}{2}} dx\bigg)^{\frac{1}{q}} \tag{14}$$

for some $c = c(n, p, L, q) > 0$, also see ((2.14), [15]), where $u \in C^{1,\beta}(\Omega)$ is a solution to the Equation (11) for some $\beta \in (0, 1)$, and $Q = 2n + 2$. Using (14) with $\sigma = 1/2$ and $q = 1$, for all $0 < r \le R/2$, we have

$$\int_{B_r} |Xu| dx \le c \Big(\frac{r}{R}\Big)^Q \int_{B_R} (|Xu| + s) dx, \tag{15}$$

for some $c = c(n, p, L) > 0$, also see ((2.16), [15]), where $u \in C^{1,\beta}(\Omega)$ is a solution to the Equation (11) for some $\beta \in (0, 1)$, and $Q = 2n + 2$.

The next result has been proved for the case $p \ge 2$ in (Proposition 3.1, [15]); the proof for the case $1 < p \le 2$ can be obtained with minor modifications. We omit the proof.

Proposition 1. *Let $B_{r_0} \subset \Omega$ and $u \in C^{1,\beta}(\Omega)$ be a solution to the Equation (11), with $\beta = \beta(n, p, L) \in (0, 1)$. Then there exists $c = c(n, p, L) > 0$ such that the inequality*

$$\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \le c \Big(\frac{\rho}{r}\Big)^\beta \bigg[\fint_{B_r} |Xu - (Xu)_{B_r}| dx + \chi r^\beta\bigg] \tag{16}$$

holds for all $0 < \rho < r < r_0$, where

$$\chi = \frac{1}{r_0^\beta}\bigg(s + \max_{1 \le i \le 2n} \sup_{B_{r_0}} |X_i u|\bigg).$$

3. Comparison Estimates

In this section, we fix $x_0 \in \Omega$ and denote $B_\rho = B(x_0, \rho)$ for every $\rho > 0$. For simplicity, we denote

$$M_\rho = \frac{|\mu|(B_\rho)}{\rho^{Q-1}}$$

for every $\rho > 0$, where $Q = 2n + 2$. Fix $R > 0$ such that $B_{2R} \subset \Omega$. We consider the Dirichlet problem

$$\begin{cases} \operatorname{div}_H a(x, Xw) = 0 & \text{in } B_{2R}; \\ w - u \in HW_0^{1,p}(B_{2R}). \end{cases} \tag{17}$$

Now we give the first comparison lemma.

Lemma 1. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1) and $2 - 1/Q < p \le 2$. Then the weak solution $w \in HW^{1,p}(B_{2R})$ to the Equation (17) satisfies the inequality*

$$\begin{aligned}\fint_{B_{2R}} |Xu - Xw|dx \le & cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{Q-p}} + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s)dx\right)^{\frac{2-p}{2}} \\ & + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}, \end{aligned} \tag{18}$$

where $c = c(n, p, L) > 0$ and $Q = 2n + 2$.

Proof of Lemma 1. For any integer $k \ge 0$, $R > 0$ and $\gamma > 0$, we define the truncation operators

$$T_k(t) := \max\left\{-\frac{k}{R^\gamma}, \min\left\{\frac{k}{R^\gamma}, t\right\}\right\}, \quad \Phi_k(t) := T_1(t - T_k(t)), \quad t \in \mathbb{R}.$$

Denote

$$C_k := \left\{x \in B_{2R} : \frac{k}{R^\gamma} < \frac{|u(x) - w(x)|}{m} \le \frac{k+1}{R^\gamma}\right\},$$

where $m > 0$, we will choose constants $\gamma > 0$ and $m > 0$ in the following. Since $w - u \in HW_0^{1,p}(B_{2R})$, we use $\phi = \Phi_k(\frac{u-w}{m})$ to test Equations (1) and (17), then we have

$$\int_{B_{2R}} \langle a(x, Xu) - a(x, Xw), X\phi\rangle dx = \int_{B_{2R}} \phi d\mu. \tag{19}$$

Note that

$$X_i\phi = \begin{cases} 0 & \text{in } B_{2R} \setminus C_k; \\ \frac{1}{m}(X_iu - X_iw) & \text{in } C_k. \end{cases}$$

This, together with (10) and (19), yields

$$\begin{aligned}\int_{C_k} |V(Xu) - V(Xw)|^2 dx \le & c\int_{C_k} \langle a(x, Xu) - a(x, Xw), Xu - Xw\rangle dx \\ = & cm\int_{B_{2R}} \langle a(x, Xu) - a(x, Xw), X\phi\rangle dx \\ = & cm\int_{B_{2R}} \Phi_k(\frac{u-w}{m})d\mu \\ \le & \frac{cm}{R^\gamma}|\mu|(B_{2R}).\end{aligned}$$

From this, by Hölder's inequality, we have

$$\begin{aligned}\int_{C_k} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \le & c|C_k|^{\frac{p-1}{p}}\left(\int_{C_k} |V(Xu) - V(Xw)|^2 dx\right)^{\frac{1}{p}} \\ \le & c|C_k|^{\frac{p-1}{p}}\left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}}[|\mu|(B_{2R})]^{\frac{1}{p}} \\ = & c\left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}}[|\mu|(B_{2R})]^{\frac{1}{p}}\left(\int_{C_k} 1dx\right)^{\frac{p-1}{p}} \\ \le & c\left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}}[|\mu|(B_{2R})]^{\frac{1}{p}}\left[\frac{1}{\left(\frac{mk}{R^\gamma}\right)^{\frac{Q}{Q-1}}}\int_{C_k} |u - w|^{\frac{Q}{Q-1}}\right]^{\frac{p-1}{p}}.\end{aligned} \tag{20}$$

Similarly, when $k = 0$, we have

$$\begin{aligned}\int_{C_0} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \leq & c|C_0|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \\ \leq & c|B_{2R}|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}}. \end{aligned} \tag{21}$$

Combining (20) and (21), we have

$$\begin{aligned} & \int_{B_{2R}} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \\ & = \int_{C_0} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx + \sum_{k=1}^{\infty} \int_{C_k} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \\ & \leq c|B_{2R}|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \\ & + c \sum_{k=1}^{\infty} \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \left[\frac{1}{\left(\frac{mk}{R^\gamma}\right)^{\frac{Q}{Q-1}}} \int_{C_k} |u - w|^{\frac{Q}{Q-1}} dx \right]^{\frac{p-1}{p}}. \end{aligned}$$

Note that

$$\begin{aligned} & \sum_{k=1}^{\infty} \left[\frac{1}{k^{\frac{Q}{Q-1}}} \int_{C_k} |u - w|^{\frac{Q}{Q-1}} dx \right]^{\frac{p-1}{p}} \\ & \leq \left(\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{\frac{Q(p-1)}{Q-1}} \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} \int_{C_k} |u - w|^{\frac{Q}{Q-1}} dx \right)^{\frac{p-1}{p}}. \end{aligned}$$

Since $2 - 1/Q < p \leq 2$ implies $Q(p-1)/(Q-1) > 1$, we have

$$\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{\frac{Q(p-1)}{Q-1}} \leq c.$$

Thus

$$\begin{aligned} & \int_{B_{2R}} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \\ & \leq c|B_{2R}|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \\ & \quad + c \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p} - \frac{Q(p-1)}{(Q-1)p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \left(\int_{B_{2R}} |u - w|^{\frac{Q}{Q-1}} dx \right)^{\frac{p-1}{p}}. \end{aligned}$$

By the Sobolev inequality, we have

$$\begin{aligned} & \int_{B_{2R}} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \\ & \leq c|B_{2R}|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \\ & \quad + c \left(\frac{m}{R^\gamma}\right)^{\frac{1}{p} - \frac{Q(p-1)}{(Q-1)p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \left(\int_{B_{2R}} |Xu - Xw| dx \right)^{\frac{Q(p-1)}{(Q-1)p}}. \end{aligned} \tag{22}$$

Noting that (9) implies

$$
\begin{aligned}
|Xu - Xw| =& \Big[(|Xu|^2 + |Xw|^2 + s^2)^{\frac{p-2}{2}} |Xu - Xw|^2\Big]^{\frac{1}{2}} (|Xu|^2 + |Xw|^2 + s^2)^{\frac{2-p}{4}} \\
\le& c|V(Xu) - V(Xw)|(|Xu|^2 + |Xw|^2 + s^2)^{\frac{2-p}{4}} \\
\le& c|V(Xu) - V(Xw)|[|Xu - Xw|^{\frac{2-p}{2}} + |Xu|^{\frac{2-p}{2}} + s^{\frac{2-p}{2}}].
\end{aligned}
\tag{23}
$$

By Young's inequality, we have

$$
|Xu - Xw| \le c|V(Xu) - V(Xw)|^{\frac{2}{p}} + \frac{1}{2}|Xu - Xw| + c|V(Xu) - V(Xw)|(|Xu| + s)^{\frac{2-p}{2}}.
$$

By Hölder's inequality, we have

$$
\begin{aligned}
\int_{B_{2R}} |Xu - Xw| dx \le& c\int_{B_{2R}} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \\
&+ c\left(\int_{B_{2R}} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx\right)^{\frac{p}{2}} \left(\int_{B_{2R}} (|Xu| + s) dx\right)^{\frac{2-p}{2}}.
\end{aligned}
\tag{24}
$$

Let $m = |\mu|(B_{2R})$ and $\gamma = Q - 2$. Then (22) becomes

$$
\fint_{B_{2R}} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \le cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{(Q-1)p}} \left(\fint_{B_{2R}} |Xu - Xw| dx\right)^{\frac{Q(p-1)}{(Q-1)p}},
$$

which, together with (24), yields

$$
\begin{aligned}
&\fint_{B_{2R}} |Xu - Xw| dx \\
&\le cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{(Q-1)p}} \left(\fint_{B_{2R}} |Xu - Xw| dx\right)^{\frac{Q(p-1)}{(Q-1)p}} \\
&\quad + c\left[M_{2R} + M_{2R}^{\frac{3Q-Qp-2}{(Q-1)2}} \left(\fint_{B_{2R}} |Xu - Xw| dx\right)^{\frac{Q(p-1)}{(Q-1)2}}\right] \left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{2-p}{2}}.
\end{aligned}
\tag{25}
$$

Finally, using Young's inequality to estimate the second and last terms in the right hand side of (25), we conclude (18). □

For the second comparison estimate, we require the Dirichlet problem with freezing of the coefficients. Let $w \in HW^{1,p}(B_{2R})$ be a weak solution to the Equation (17). We consider the Dirichlet problem

$$
\begin{cases}
\operatorname{div}_H a(x_0, Xv) = 0 & \text{in } B_R; \\
v - w \in HW_0^{1,p}(B_R).
\end{cases}
\tag{26}
$$

Now we give the second comparison lemma.

Lemma 2. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1) and let $w \in HW^{1,p}(B_{2R})$ be a weak solution to the Equation (17). Assume that $2 - 1/Q < p \leq 2$. Then the weak solution $v \in HW^{1,p}(B_R)$ to the Equation (26) satisfies*

$$\begin{aligned}\fint_{B_R} |Xu - Xv| dx \leq & cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{Q-p}} + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{2-p}{2}} \\ & + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \\ & + cR^{\alpha} \fint_{B_{2R}} (|Xu| + s) dx, \end{aligned} \tag{27}$$

where $c = c(n, p, L, L') > 0$ and $Q = 2n + 2$.

Proof of Lemma 2. By (Theorem 6.1, [27]) and the condition (2), we have

$$\int_{B_R} |Xv|^p dx \leq c_1 \int_{B_R} (|Xw| + s)^p dx, \tag{28}$$

where $c_1 = c_1(n, p, L) \geq 1$. Here in the proof of (Theorem 6.1, [27]), only the condition (2) and Sobolev inequality are used, and therefore (Theorem 6.1, [27]) can also be used in the Heisenberg group.

Using sub-elliptic reverse Hölder's inequality and Gehring's lemma, see (Section 3, [28]), we have

$$\left(\fint_{B_R} (|Xw| + s)^p dx\right)^{\frac{1}{p}} \leq c \fint_{B_{2R}} (|Xw| + s) dx. \tag{29}$$

Using (9) and (10), the fact that both v and w are weak solutions and $v - w \in HW_0^{1,p}(B_R)$, we have

$$\begin{aligned} & \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-2}{2}} |Xw - Xv|^2 dx \\ \leq & c \int_{B_R} |V(Xw) - V(Xv)|^2 dx \\ \leq & c \int_{B_R} \langle a(x_0, Xw) - a(x_0, Xv), Xw - Xv \rangle dx \\ = & c \int_{B_R} \langle a(x_0, Xw) - a(x, Xw), Xw - Xv \rangle dx, \end{aligned}$$

which, together with condition (3), yields

$$\begin{aligned} & \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-2}{2}} |Xw - Xv|^2 dx \\ \leq & cR^{\alpha} \int_{B_R} (|Xw|^2 + s^2)^{\frac{p-1}{2}} |Xw - Xv| dx \\ \leq & cR^{\alpha} \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-1}{2}} |Xw - Xv| dx. \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned} & \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-2}{2}} |Xw - Xv|^2 dx \\ \leq & cR^{2\alpha} \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p}{2}} |Xw - Xv|^2 dx. \end{aligned}$$

This and (9) imply

$$\int_{B_R} |V(Xw) - V(Xv)|^2 dx \le cR^{2\alpha} \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p}{2}} dx.$$

Combining this and (28), we have

$$\int_{B_R} |V(Xw) - V(Xv)|^2 dx \le cR^{2\alpha} \int_{B_R} (|Xw| + s)^p dx. \tag{30}$$

Similarly to (23), we have

$$|Xu - Xw|^p \le c|V(Xw) - V(Xv)|^p (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p(2-p)}{4}}.$$

From this, by Hölder's inequality, (28) and (30), we have

$$\begin{aligned}
&\fint_{B_R} |Xu - Xw|^p dx \\
&\le c\left(\fint_{B_R} |V(Xw) - V(Xv)|^2 dx\right)^{\frac{p}{2}} \left(\fint_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}} \\
&\le cR^{p\alpha} \fint_{B_R} (|Xw| + s)^p dx. \qquad (31)
\end{aligned}$$

By Hölder's inequality, (31) and (29), we have

$$\begin{aligned}
\fint_{B_R} |Xu - Xw| dx &\le c\left(\fint_{B_R} |Xu - Xw|^p dx\right)^{\frac{1}{p}} \\
&\le cR^{\alpha}\left(\fint_{B_R} (|Xw| + s)^p dx\right)^{\frac{1}{p}} \\
&\le cR^{\alpha} \fint_{B_{2R}} (|Xw| + s) dx. \qquad (32)
\end{aligned}$$

Using (18) in Lemma 1 and (32), we have

$$\begin{aligned}
\fint_{B_R} |Xu - Xv| dx =& \fint_{B_R} |Xu - Xw| dx + \fint_{B_R} |Xw - Xv| dx \\
\le& cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{Q-p}} + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{2-p}{2}} \\
&+ cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + cR^{\alpha} \fint_{B_{2R}} (|Xw| + s) dx. \qquad (33)
\end{aligned}$$

Noting that

$$\fint_{B_{2R}} (|Xw| + s) dx = \fint_{B_{2R}} |Xw - Xu| dx + \fint_{B_{2R}} (|Xu| + s) dx,$$

then using (18) in Lemma 1 to estimate the last integral in the hand side of (33), we conclude (27). Here we can choose R small enough such that $R^{\alpha} \le 1$. □

Now we give the main lemma.

Lemma 3. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1) and $2 - 1/Q < p \leq 2$ and let $v \in HW^{1,p}(B_{\tilde{R}})$ be a weak solution to the Equation (26) with $B_{\tilde{R}} \subset \Omega$. Then there exist $\beta = \beta(n,p,L) \in (0,1)$ and $c = c(n,p,L,L') > 0$ such that, for every $0 < \rho < R < \tilde{R}$, we have*

$$
\begin{aligned}
&\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\leq c\left(\frac{\rho}{R}\right)^\beta \fint_{B_{2R}} |Xu - (Xu)_{B_{2R}}| dx \\
&\quad + c\left(\frac{R}{\rho}\right)^Q \Bigg[M_{2R}^{\frac{2}{p}} + M_{2R}^{\frac{3Q-Qp-2}{Q-p}} + M_{2R}\left(\fint_{B_{2R}} (|Xu|+s) dx\right)^{\frac{2-p}{2}} \\
&\quad + M_{2R}\left(\fint_{B_{2R}} (|Xu|+s) dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \\
&\quad + R^\alpha \fint_{B_{2R}} (|Xu|+s) dx \Bigg] + c\left(\frac{\rho}{\tilde{R}}\right)^\beta \sup_{B_{\tilde{R}}} |Xv|,
\end{aligned} \tag{34}
$$

where $Q = 2n+2$.

Proof of Lemma 3. By Proposition 1 with $r = R$ and $r_0 = \tilde{R}$, we have

$$
\begin{aligned}
&\fint_{B_\rho} |Xv - (Xv)_{B_\rho}| dx \\
&\leq c\left(\frac{\rho}{R}\right)^\beta \left[\fint_{B_R} |Xv - (Xv)_{B_R}| dx + \sup_{B_{5R/4}} |Xv| \right] \\
&\leq c\left(\frac{\rho}{R}\right)^\beta \left[\fint_{B_R} |Xu - (Xu)_{B_R}| dx + 2\fint_{B_R} |Xu - Xv| dx \right] + c\left(\frac{\rho}{\tilde{R}}\right)^\beta \sup_{B_{\tilde{R}}} |Xv|.
\end{aligned}
$$

Noting that

$$
\fint_{B_\rho} |Xu - Xv| dx \leq c\left(\frac{R}{\rho}\right)^Q \fint_{B_R} |Xu - Xv| dx,
$$

we have

$$
\begin{aligned}
\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \leq & \fint_{B_\rho} |Xv - (Xv)_{B_\rho}| dx + 2\fint_{B_\rho} |Xu - Xv| dx \\
\leq & c\left(\frac{\rho}{R}\right)^\beta \fint_{B_R} |Xu - (Xu)_{B_R}| dx + c\left(\frac{R}{\rho}\right)^Q \fint_{B_R} |Xu - Xv| dx \\
& + c\left(\frac{\rho}{\tilde{R}}\right)^\beta \sup_{B_{\tilde{R}}} |Xv|.
\end{aligned}
$$

Finally, using the inequality

$$
\fint_{B_R} |Xu - (Xu)_{B_R}| dx \leq 2^{Q+1} \fint_{B_{2R}} |Xu - (Xu)_{B_{2R}}| dx
$$

and Lemma 2, we conclude (34). □

4. Proof of Theorem 1

In this section, we prove Theorem 1. Fix $x_0 \in \mathbb{H}^n$ and denote $B_R := B(x_0, R)$. Assume that $0 < R < \tilde{R} \leq \bar{R} = \bar{R}(n,p,L,L',\alpha,\mathrm{dist}(x_0,\partial\Omega))$. For any $H > \tilde{H} > 1$

and $i \in \{0,1,2,\dots\}$, we denote $R_i = R/(2H)^i$, $\tilde{R}_i = 5R/[4(2\tilde{H})^i]$, $B_i := B_{R_i}$, $k_i := |(Xu)_{B_i}|$, $A_i := \fint_{B_i} |Xu - (Xu)_{B_i}|dx$ and $M_i := M_{R_i}$. Then

$$\begin{aligned} k_{m+1} &= \sum_{i=0}^{m} (k_{i+1} - k_i) + k_0 \\ &\leq \sum_{i=0}^{m} \fint_{B_{i+1}} |Xu - (Xu)_{B_i}|dx + k_0 \\ &\leq (2H)^Q \sum_{i=0}^{m} A_i + k_0. \end{aligned} \tag{35}$$

Lemma 4. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1) and $2 - 1/Q < p \leq 2$, and let $v \in HW^{1,p}(B_R)$ be a weak solution to the Equation (26). Assume that there exists an integer $\tilde{m} \in \mathbb{N} \cup \{\infty\}$ such that $\tilde{m} \geq 1$ and*

$$\fint_{B_i} |Xu|dx \leq |Xu(x_0)| \tag{36}$$

holds whenever $0 \leq i \leq \tilde{m} - 1$. Then for every $\epsilon \in (0,1)$, there exists a constant $\tilde{c} = \tilde{c}(\epsilon) \geq 1$ such that

$$k_m \leq 2c_4 \mathcal{M} + 2c_3 \epsilon |Xu(x_0)| \tag{37}$$

holds whenever $m \leq \tilde{m} + 1$, where $c_3, c_4 \geq 1$, $Q = 2n + 2$ and

$$\begin{aligned} \mathcal{M} :=& \fint_{B_R} (|Xu| + s)dx + (1 + c_3\tilde{c}(\epsilon)) \left\{ [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right\} \\ &+ \sup_{B_{5R/8}} |Xv|. \end{aligned} \tag{38}$$

Proof of Lemma 4. By Lemma 3 with $0 < R/2H < R/2 < \tilde{R}/2$, we have

$$\begin{aligned} &\fint_{B_{R/2H}} |Xu - (Xu)_{B_{R/2H}}|dx \\ &\leq \frac{1}{4} \fint_{B_R} |Xu - (Xu)_{B_R}|dx \\ &\quad + cM_R^{\frac{2}{p}} + cM_R^{\frac{3Q-Qp-2}{Q-p}} + cM_R \left(\fint_{B_R} (|Xu| + s)dx \right)^{\frac{2-p}{2}} \\ &\quad + cM_R \left(\fint_{B_R} (|Xu| + s)dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \\ &\quad + cR^\alpha \fint_{B_R} (|Xu| + s)dx + \frac{1}{4} \left(\frac{R}{\tilde{R}} \right)^\beta \sup_{B_{\tilde{R}/2}} |Xv|. \end{aligned} \tag{39}$$

Here we choose $H = H(n, p, L) > 1$ large enough such that $c/H^\beta \leq 1/4$. Noting that

$$\fint_{B_R} |Xu|dx = \fint_{B_R} |Xu| - (Xu)_{B_R}dx + (Xu)_{B_R}$$

and choosing $\bar{R}$ small enough such that $c\bar{R} \leq 1/4$, we write (39) as

$$\begin{aligned}
&\fint_{B_{R/2H}} |Xu - (Xu)_{B_{R/2H}}| dx \\
&\leq \frac{1}{2} \fint_{B_R} |Xu - (Xu)_{B_R}| dx \\
&\quad + cM_R^{\frac{2}{p}} + cM_R^{\frac{3Q-Qp-2}{Q-p}} + cM_R \left(\fint_{B_R} (|Xu| + s) dx \right)^{\frac{2-p}{2}} \\
&\quad + cM_R \left(\fint_{B_R} (|Xu| + s) dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \\
&\quad + cR^{\alpha}((Xu)_{B_R} + s) + \frac{1}{4} \left(\frac{R}{\tilde{R}} \right)^{\beta} \sup_{B_{\tilde{R}/2}} |Xv|.
\end{aligned} \tag{40}$$

By (40) with $R = R_{i-1}$ and $\tilde{R} = \tilde{R}_{i-1}$, we have

$$\begin{aligned}
A_i \leq &\frac{1}{2} A_{i-1} + cM_{i-1}^{\frac{2}{p}} + cM_{i-1}^{\frac{3Q-Qp-2}{Q-p}} + cM_{i-1} \left(\fint_{B_{i-1}} (|Xu| + s) dx \right)^{\frac{2-p}{2}} \\
&+ cM_{i-1} \left(\fint_{B_{i-1}} (|Xu| + s) dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + cR_{i-1}^{\alpha}(k_{i-1} + s) \\
&+ \frac{1}{4} \left(\frac{\tilde{H}}{H} \right)^{\beta(i-1)} \sup_{B_{\tilde{R}_{i-1}/2}} |Xv|.
\end{aligned}$$

Summing up over $i \in \{1, \ldots, m\}$ the above inequality and letting $\tilde{H} = H/2^{1/\beta}$, and the fact

$$\sup_{B_{\tilde{R}_{i-1}/2}} |Xv| \leq \sup_{B_{5R/8}} |Xv|,$$

we have

$$\begin{aligned}
\sum_{i=1}^{m} A_i \leq &\frac{1}{2} \sum_{i=0}^{m-1} A_i + c \sum_{i=0}^{m-1} \left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i \left(\fint_{B_i} (|Xu| + s) dx \right)^{\frac{2-p}{2}} \right. \\
&\left. + M_i \left(\fint_{B_i} (|Xu| + s) dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \right] + c \sum_{i=0}^{m-1} R_i^{\alpha}(k_i + s) \\
&+ c \sup_{B_{5R/8}} |Xv|,
\end{aligned}$$

and therefore

$$\begin{aligned}
\sum_{i=1}^{m} A_i \leq &A_0 + 2c \sum_{i=0}^{m-1} \left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i \left(\fint_{B_i} (|Xu| + s) dx \right)^{\frac{2-p}{2}} \right. \\
&\left. + M_i \left(\fint_{B_i} (|Xu| + s) dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \right] + 2c \sum_{i=0}^{m-1} R_i^{\alpha}(k_i + s) \\
&+ 2c \sup_{B_{5R/8}} |Xv|.
\end{aligned} \tag{41}$$

Combining (35) and (41), we have

$$\begin{aligned} k_{m+1} \leq & cA_0 + k_0 + c\sum_{i=0}^{m-1}\left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\right. \\ & \left. + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right] + c\sum_{i=0}^{m-1}R_i^{\alpha}(k_i+s) \\ & + c\sup_{B_{5R/8}}|Xv|. \end{aligned} \tag{42}$$

By (36) and (42), whenever $1 \leq m \leq \tilde{m}$, we have

$$\begin{aligned} k_{m+1} \leq & c\left(A_0 + k_0 + \sum_{i=0}^{m-1}\left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}}\right]\right) \\ & + c\left(|Xu(x_0)|^{\frac{2-p}{2}} + s^{\frac{2-p}{2}} + |Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + s^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right)\sum_{i=0}^{m-1}M_i \\ & + c\sum_{i=0}^{m-1}R_i^{\alpha}(k_i+s) + c\sup_{B_{5R/8}}|Xv|. \end{aligned} \tag{43}$$

Note that

$$\begin{aligned} \sum_{i=0}^{\infty}M_i \leq & \sum_{i=0}^{\infty}\frac{|\mu|(B_i)}{R_i^{Q-1}} \\ \leq & \frac{2^Q-1}{\log 2}\int_R^{2R}\frac{|\mu|(B(x_0,\rho))}{\rho^{Q-1}}\frac{d\rho}{\rho} + \frac{(2H)^{Q-1}}{\log 2H}\sum_{i=0}^{\infty}\int_{R_{i+1}}^{R_i}\frac{|\mu|(B(x_0,\rho))}{\rho^{Q-1}}\frac{d\rho}{\rho} \\ \leq & c(H)\mathbf{I}_1^{|\mu|}(x_0,2R), \end{aligned}$$

the fact that $1 < p \leq 2$ implies $2/p \geq 1$ and $(3Q-Qp-2)/(Q-p) \geq 1$, and

$$\sum_{i=0}^{\infty}R_i^{\alpha} = R^{\alpha}\sum_{i=0}^{\infty}\frac{1}{(2H)^{\alpha i}} \leq \frac{R^{\alpha}}{1-1/(2H)^{\alpha}} \leq \frac{R^{\alpha}}{1-1/2^{\alpha}} =: d(R).$$

For $1 \leq m \leq \tilde{m}$, we write (43) as

$$\begin{aligned} k_{m+1} \leq & c\left(A_0 + k_0 + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\right) \\ & + c_3\left(|Xu(x_0)|^{\frac{2-p}{2}} + s^{\frac{2-p}{2}} + |Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + s^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right)\mathbf{I}_1^{|\mu|}(x_0,2R) \\ & + c\sum_{i=0}^{m-1}R_i^{\alpha}(k_i+s) + c\sup_{B_{5R/8}}|Xv|. \end{aligned} \tag{44}$$

By Young's inequality, we have

$$\mathbf{I}_1^{|\mu|}(x_0,2R)|Xu(x_0)|^{\frac{2-p}{2}} \leq \frac{\epsilon}{2}|Xu(x_0)| + \tilde{c}(\epsilon)[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}}$$

and

$$\mathbf{I}_1^{|\mu|}(x_0,2R)|Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \leq \frac{\epsilon}{2}|Xu(x_0)| + \tilde{c}(\epsilon)[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}},$$

which, together with (44), yield

$$k_{m+1} \leq c_4\mathcal{M} + c_5\sum_{i=0}^{m-1}R_i^{\alpha}k_i + c_3\epsilon|Xu(x_0)| \tag{45}$$

where $\mathcal{M}$ is as in (38). Here we choose $\bar{R}$ small enough such that $d(\bar{R}) \leq 1$.

Now we prove that the inequality

$$k_i \leq 2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)| \tag{46}$$

holds for every $0 \leq i \leq \tilde{m}+1$. When $i = 0$ and $i = 1$, we have

$$A_0 + k_0 + d(R)s \leq 3\fint_{B_R} (|Xu| + s)dx$$

and

$$k_1 \leq 2^Q H^Q \fint_{B_R} |Xu|dx.$$

When $1 \leq i \leq \tilde{m}+1$, we assume that (46) holds for every $i \leq m$ with $1 \leq m \leq \tilde{m}$, and prove it for $m+1$. By using (45) and the assumption (46) for $i \leq m-1$, we have

$$\begin{aligned}
k_{m+1} \leq & c_4\mathcal{M} + c_5 \sum_{i=0}^{m-1} R_i^\alpha (2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)|) + c_3\epsilon|Xu(x_0)| \\
= & [c_4 + 2c_4c_5 d(R)]\mathcal{M} + [2c_3c_5 d(R) + c_3]\epsilon|Xu(x_0)| \\
\leq & 2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)|.
\end{aligned}$$

Here we choose $\bar{R}$ small enough such that

$$d(\bar{R}) \leq \min\{1/(100c_3), 1/(100c_4), 1/(100c_5)\}.$$

We complete the proof. □

Now we prove Theorem 1.

Proof of Theorem 1. Define the set

$$\mathbb{S} := \left\{ i \in \mathbb{N} : |Xu(x_0)| \geq \fint_{B_i} |Xu|dx \right\},$$

and consider two cases: $\mathbb{S} = \mathbb{N}$ and $\mathbb{S} \neq \mathbb{N}$.

Case 1. When $\mathbb{S} = \mathbb{N}$, for every $i \in \mathbb{N}$, we have

$$\fint_{B_i} |Xu|dx \leq |Xu(x_0)|.$$

Using Lemma 4 with $\tilde{m} = \infty$, then letting $m \to \infty$, we have

$$|Xu(x_0)| = \lim_{m\to\infty} k_m \leq 2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)|. \tag{47}$$

Choosing $\epsilon = 1/(4c_3)$, we have

$$|Xu(x_0)| \leq 4c_4\mathcal{M}.$$

On the other hand, to estimate the last integral in $\mathcal{M}$, using (14) with $\sigma = 5/8$ and $q = 1$, we have

$$\begin{aligned}
\sup_{B_{5R/8}} |Xv| \leq & c\fint_{B_R} (|Xv| + s)dx \\
\leq & c\fint_{B_R} (|Xu| + s)dx + c\fint_{B_R} |Xu - Xv|dx,
\end{aligned}$$

from which, using Lemma 2 and Young's inequality, we have

$$\sup_{B_{5R/8}} |Xv| \leq cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{Q-p}} + c\fint_{B_{2R}} (|Xu| + s)dx. \tag{48}$$

Combining (47) and (48), we conclude (6) in the case.

Case 2. When $\mathbb{S} \neq \mathbb{N}$, we let $\tilde{m} := \min(\mathbb{N} \setminus \mathbb{S}) \geq 0$ and obtain

$$|Xu(x_0)| < \fint_{B_{\tilde{m}}} |Xu|dx, \tag{49}$$

and

$$\fint_{B_i} |Xu|dx < |Xu(x_0)| \tag{50}$$

for every $0 \leq i \leq \tilde{m} - 1$. When $\tilde{m} = 0$, we have $|Xu(x_0)| < (|Xu|)_{B_0}$, and therefore (6) holds true. When $\tilde{m} \geq 1$, the inequality (49) implies

$$|Xu(x_0)| < \fint_{B_{\tilde{m}}} |Xu|dx \leq \fint_{B_{\tilde{m}}} |Xu - (Xu)_{B_{\tilde{m}}}|dx + |(Xu)_{B_{\tilde{m}}}| = A_{\tilde{m}} + k_{\tilde{m}}. \tag{51}$$

Using (50) and Lemma 4, we have

$$k_{\tilde{m}} \leq 2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)|. \tag{52}$$

Since (50) satisfies the assumption (36), then combining (41) and (37), we have

$$\begin{aligned}
A_{\tilde{m}} \leq & A_0 + 2c\sum_{i=0}^{\tilde{m}-1}\left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\right. \\
& \left. + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right] + 2c\sum_{i=0}^{\tilde{m}-1} R_i^{\alpha}(2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)| + s) \\
& + 2c\sup_{B_{5R/8}} |Xv|,
\end{aligned}$$

from which, using (50) again, we have

$$\begin{aligned}
A_{\tilde{m}} \leq & c\fint_{B_R}(|Xu|+s)dx + c\left[[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\right] \\
& + \mathbf{I}_1^{|\mu|}(x_0,2R)\left[|Xu(x_0)|^{\frac{2-p}{2}} + s^{\frac{2-p}{2}} + |Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + s^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right] \\
& + cd(R)(2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)| + s) + c\sup_{B_{5R/8}} |Xv|.
\end{aligned} \tag{53}$$

Estimating (53) as in (43)–(46) in the proof of Lemma 4, we have

$$A_{\tilde{m}} \leq c\mathcal{M} + c\epsilon|Xu(x_0)|,$$

which, together with (51), yields

$$|Xu(x_0)| \leq c\mathcal{M} + c\epsilon|Xu(x_0)|.$$

Choosing $\epsilon = 1/(2c)$, we have

$$|Xu(x_0)| \leq 2c\mathcal{M}.$$

Combining this and (48), we conclude (6) in the case.

Finally, we note that if $a(x,z)$ is independent of x then we can assume $L' = 0$ and therefore all items containing R^α disappear. Thus the proof holds for any $R > 0$ whenever $B_{2R} \subset \Omega$. We complete the proof. □

5. Proof of Theorem 2

In this section, we prove Theorem 2. Fix $x_0 \in \mathbb{H}^n$ and denote $B_R := B(x_0, R)$. Assume that $0 < R < \bar{R} = \bar{R}(n, p, L, L', \alpha, \mathrm{dist}(x_0, \partial\Omega))$. To prove Threorem 2, we need the following lemmas.

Lemma 5. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1), $2 - 1/Q < p \le 2$ and $B_{\bar{R}} \subset \Omega$. Then there exist $c = c(n, p, L, L') > 0$ such that, for every $0 < \rho \le R \le \bar{R}/2$, we have*

$$\begin{aligned} \fint_{B_\rho} (|Xu| + s)dx \le & c\fint_{B_R} (|Xu| + s)dx \\ & + c\left(\frac{R}{\rho}\right)^Q \left[M_{2R}^{\frac{2}{p}} + M_{2R}^{\frac{3Q-Qp-2}{Q-p}} + M_{2R}\left(\fint_{B_{2R}} (|Xu| + s)dx\right)^{\frac{2-p}{2}} \right. \\ & \left. + M_{2R}\left(\fint_{B_{2R}} (|Xu| + s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + R^\alpha \fint_{B_{2R}} (|Xu| + s)dx \right], \end{aligned} \tag{54}$$

where $Q = 2n + 2$.

Proof of Lemma 5. Letting $v \in HW^{1,p}(B_R)$ be a weak solution to the Equation (26), we have

$$\int_{B_\rho} (|Xu| + s)dx \le \int_{B_\rho} (|Xv| + s)dx + \int_{B_\rho} |Xu - Xv|dx. \tag{55}$$

From (15), we have

$$\begin{aligned} \int_{B_\rho} (|Xv| + s)dx &\le c\left(\frac{\rho}{R}\right)^Q \int_{B_R} (|Xv| + s)dx \\ &\le c\left(\frac{\rho}{R}\right)^Q \int_{B_R} (|Xu| + s)dx + c\left(\frac{\rho}{R}\right)^Q \int_{B_R} |Xv - Xu|dx. \end{aligned} \tag{56}$$

Combining (55) and (56), then using Lemma 2 and the inequality

$$\fint_{B_\rho} |Xu - Xv|dx \le c\left(\frac{R}{\rho}\right)^Q \fint_{B_R} |Xu - Xv|dx,$$

we conclude (54). □

The following lemma is (Lemma 4.2, [15]).

Lemma 6. *Let $\phi : (0, \infty) \to [0, \infty)$ be a non-decreasing functions, $A > 1$ and $\epsilon \ge 0$ be fixed constants. Let $\psi, \Phi : (0, \infty) \to [0, \infty)$ be functions such that $\sum_{j=0}^{\infty} \psi(t^j r) \le \Phi(r)$ for any $0 < t < t_0 < 1$. Given any $a > 0$, suppose that*

$$\phi(\rho) \le A\left[\left(\frac{\rho}{r}\right)^a + \epsilon\right]\phi(r) + r^a\psi(r) \tag{57}$$

holds for any $0 < \rho < r \le R_0$, then there exists constants $\epsilon_0 = \epsilon_0(A, a) > 0$ and $c = c(A, a) > 0$ such that if $\epsilon \le \epsilon_0$, then for all $0 < \rho < r \le R_0$, we have

$$\phi(\rho) \le c\left[\left(\frac{\rho}{r}\right)^{a-\bar{\epsilon}} \phi(r) + \rho^{a-\bar{\epsilon}} r^{\bar{\epsilon}} \Phi(r)\right] \tag{58}$$

for any $0 < \bar{\epsilon} < a$.

Based on Lemmas 5 and 6, we obtain the following proposition.

Proposition 2. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1), $2 - 1/Q < p \leq 2$ and $B_{\bar{R}} \subset \Omega$. Then there exist $c = c(n, p, L, L') > 0$ such that, for any $0 < \bar{\epsilon} < Q$ and $0 < r < R \leq \bar{R}$, we have*

$$\int_{B_r} (|Xu| + s)dx \leq c\left(\frac{r}{R}\right)^{Q-\bar{\epsilon}} \left[\int_{B_R} (|Xu| + s)dx + R^Q \left\{ [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right\} \right], \tag{59}$$

where $Q = 2n + 2$.

Proof of Proposition 2. We fix $0 < r < R \leq \bar{R}$ and denote

$$\phi(r) := \int_{B_r} (|Xu| + s)dx.$$

By Lemma 5 with $\rho = r$ and $R \to R/2$, we have

$$\phi(r) \leq c\left(\frac{r}{R}\right)^Q \phi(R) + cR^Q \left[M_R^{\frac{2}{p}} + M_R^{\frac{3Q-Qp-2}{Q-p}} + M_R \left(\fint_{B_R} (|Xu| + s)dx \right)^{\frac{2-p}{2}} + M_R \left(\fint_{B_R} (|Xu| + s)dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \right] + cR^\alpha \phi(R),$$

which, together with Young's inequality, yields

$$\phi(r) \leq c\left[\left(\frac{r}{R}\right)^Q + R^\alpha + \epsilon_1 \right] \phi(R) + c\left(1 + \frac{1}{\epsilon_1}\right) R^Q \left[M_R^{\frac{2}{p}} + M_R^{\frac{3Q-Qp-2}{Q-p}} \right].$$

Note that

$$\sum_{j=0}^{\infty} M_{t^j R} \leq \mathbf{I}_1^{|\mu|}(x_0, 2R)$$

holds for any $t \in (0, 1)$ and $R > 0$. Using Lemma 6 with $a = Q$, choosing $\bar{R}$ small enough such that $\bar{R}^\alpha < \epsilon_0(n, p, L)/2$ and letting $\epsilon_1 = \epsilon_0(n, p, L)/2$, we have

$$\phi(r) \leq c\left[\left(\frac{r}{R}\right)^Q \phi(R) + r^{Q-\bar{\epsilon}} R^{\bar{\epsilon}} \left\{ [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right\} \right],$$

that is, (59). □

To obtain $C^{1,\gamma}_{\mathrm{loc}}$-regularity of u, we need the following lemma.

Lemma 7. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the Equation (1), $2 - 1/Q < p \leq 2$ and $B_{\bar{R}} \subset \Omega$. Then there exist $\beta = \beta(n, p, L) \in (0, 1)$ and $c = c(n, p, L, L') > 0$ such that, for every $0 < \rho < R < \bar{R}/2$, we have*

$$\begin{aligned}
&\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\leq c\left(\frac{\rho}{R}\right)^{\beta} \fint_{B_R} (|Xu| + s) dx \\
&\quad + c\left(\frac{R}{\rho}\right)^{Q} \Bigg[M_{2R}^{\frac{2}{p}} + M_{2R}^{\frac{3Q-Qp-2}{Q-p}} + M_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{2-p}{2}} \\
&\quad + M_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + R^{\alpha} \fint_{B_{2R}} (|Xu| + s) dx \Bigg],
\end{aligned} \tag{60}$$

where $Q = 2n + 2$.

Proof of Lemma 7. Letting $v \in HW^{1,p}(B_R)$ be a weak solution to the Equation (26), we have

$$\begin{aligned}
\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx &\leq 2 \fint_{B_\rho} |Xu - (Xv)_{B_\rho}| dx \\
&\leq 2 \fint_{B_\rho} |Xv - (Xv)_{B_\rho}| dx + 2 \fint_{B_\rho} |Xu - Xv| dx.
\end{aligned}$$

By (13), we have

$$\begin{aligned}
\fint_{B_\rho} |Xv - (Xv)_{B_\rho}| dx &\leq c\left(\frac{\rho}{R}\right)^{\beta} \fint_{B_R} (|Xv| + s) dx \\
&\leq c\left(\frac{\rho}{R}\right)^{\beta} \fint_{B_R} (|Xu| + s) dx + c\left(\frac{\rho}{R}\right)^{\beta} \fint_{B_R} |Xv - Xu| dx.
\end{aligned}$$

Combining the above two inequalities, then using the inequality

$$\fint_{B_\rho} |Xu - Xv| dx \leq c\left(\frac{R}{\rho}\right)^{Q} \fint_{B_R} |Xv - Xu| dx,$$

and Lemma 2, we conclude (60). □

Now we prove Theorem 2.

Proof of Theorem 2. Using Lemma 7 with $R = r/2$, we have

$$\begin{aligned}
&\int_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\leq c\left(\frac{\rho}{r}\right)^{Q+\beta} \int_{B_r} (|Xu| + s) dx \\
&\quad + c r^{Q} \Bigg[M_r^{\frac{2}{p}} + M_r^{\frac{3Q-Qp-2}{Q-p}} + M_r\left(\fint_{B_r} (|Xu| + s) dx\right)^{\frac{2-p}{2}} \\
&\quad + M_r\left(\fint_{B_r} (|Xu| + s) dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \Bigg] + r^{\alpha} \int_{B_r} (|Xu| + s) dx.
\end{aligned}$$

Using Young's inequality to estimate the second term in the hand side of the above inequality, then using Proposition 2, we have

$$\begin{aligned}
&\int_{B_\rho} |Xu-(Xu)_{B_\rho}|dx \\
&\le c\frac{\rho^{Q+\beta}R^{\bar\epsilon}}{r^{\beta+\bar\epsilon}R^Q}\left[\int_{B_R}(|Xu|+s)dx + R^Q\left\{[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\right\}\right] \\
&\quad + cr^Q\left[\left(1+\frac{1}{\epsilon_2}\right)\left(M_r^{\frac{2}{p}} + M_r^{\frac{3Q-Qp-2}{Q-p}}\right) + (\epsilon_2+r^\alpha)\fint_{B_r}(|Xu|+s)dx\right]
\end{aligned} \tag{61}$$

for every $0<\rho<r<R<\bar R$. Given $\mu = f\in L^q_{\mathrm{loc}}(\Omega)$ for some $q>Q$, then by Hölder's inequality, we have

$$\frac{|\mu|(B_r)}{r^{Q-1}} = \frac{1}{r^{Q-1}}\int_{B_r}|f|dx \le \frac{|B_r|^{1-1/q}}{r^{Q-1}}\left(\int_{B_r}|f|^q\right)^{1/q} \le cr^{1-Q/q}\|f\|_{L^q(B_r)}$$

and therefore,

$$M_r^{\frac{2}{p}} \le cr^{\left(1-\frac{Q}{q}\right)\frac{2}{p}}\|f\|^{\frac{2}{p}}_{L^q(B_r)}, \quad M_r^{\frac{3Q-Qp-2}{Q-p}} \le cr^{\left(1-\frac{Q}{q}\right)\frac{3Q-Qp-2}{Q-p}}\|f\|^{\frac{3Q-Qp-2}{Q-p}}_{L^q(B_r)}.$$

Thus, by Proposition 2 and (61), we have

$$\begin{aligned}
&\int_{B_\rho}|Xu-(Xu)_{B_\rho}|dx \\
&\le c\left[\frac{\rho^{Q+\beta}R^{\bar\epsilon}}{r^{\beta+\bar\epsilon}} + (r^\alpha+\epsilon_2)r^{Q-\bar\epsilon}R^{\bar\epsilon}\right] \\
&\quad\times\left[\fint_{B_R}(|Xu|+s)dx + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\right] \\
&\quad + c\left(1+\frac{1}{\epsilon_2}\right)r^{Q+\left(1-\frac{Q}{q}\right)\frac{2}{p}}\|f\|^{\frac{2}{p}}_{L^q(B_r)} + c\left(1+\frac{1}{\epsilon_2}\right)r^{Q+\left(1-\frac{Q}{q}\right)\frac{3Q-Qp-2}{Q-p}}\|f\|^{\frac{3Q-Qp-2}{Q-p}}_{L^q(B_r)}
\end{aligned}$$

for every $0<\rho<r<R<\bar R$. We choose $\delta,\bar\epsilon$ small enough such that

$$\delta+\bar\epsilon\le\alpha, \quad 2\delta+\bar\epsilon\le+\left(1-\frac{Q}{q}\right)\frac{2}{p}, \quad Q\bar\epsilon<\beta\delta$$

and therefore,

$$\alpha+Q-\bar\epsilon\ge Q+\delta, \quad Q-\bar\epsilon-\delta+\left(1-\frac{Q}{q}\right)\frac{2}{p}\ge Q+\delta, \quad \beta\delta-Q\bar\epsilon>0.$$

Here $1<p\le 2$ implies $\frac{3Q-Qp-2}{Q-p}\ge\frac{2}{p}$. Thus, letting $\epsilon_2 = r^{\delta+\bar\epsilon}$, we have

$$\begin{aligned}
&\int_{B_\rho}|Xu-(Xu)_{B_\rho}|dx \\
&\le c\left[\frac{\rho^{Q+\beta}}{r^{\beta+\bar\epsilon}} + r^{Q+\delta}\right]\left[\fint_{B_R}(|Xu|+s)dx + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\right. \\
&\quad\left. + \|f\|^{\frac{2}{p}}_{L^q(B_r)} + \|f\|^{\frac{3Q-Qp-2}{Q-p}}_{L^q(B_r)}\right]
\end{aligned}$$

for every $0 < \rho < r < R < \bar{R}$. Choosing $r = \rho^\kappa$ with some $\kappa \in (0,1)$, we rewrite the above inequality as

$$\begin{aligned}
&\int_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\le c\left[\rho^{Q+(1-\kappa)\beta-\kappa\bar{\epsilon}} + \rho^{\kappa(Q+\delta)}\right] \\
&\quad \times \left[⨍_{B_R} (|Xu| + s)dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right. \\
&\quad \left. + \|f\|_{L^q(B_r)}^{\frac{2}{p}} + \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}} \right] \\
&\le c\rho^{Q+\gamma}\left[⨍_{B_R} (|Xu| + s)dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right. \\
&\quad \left. + \|f\|_{L^q(B_r)}^{\frac{2}{p}} + \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}} \right],
\end{aligned}$$

where the second inequality follows when $Q + \gamma \le \min\{Q + (1-\kappa)\beta - \kappa\bar{\epsilon}, \kappa(Q+\delta)\}$. Here we can make sure that this is true with the choice of $\kappa = \kappa(\gamma)$ such that

$$\frac{Q+\gamma}{Q+\delta} \le \kappa \le \frac{\beta-\gamma}{\beta+\bar{\epsilon}}$$

for any $0 < \gamma \le (\beta\delta - Q\bar{\epsilon})/(Q + \beta + \delta + \bar{\epsilon})$. Also, note that if $\gamma, \bar{\epsilon}$ are small enough, $\kappa = \kappa(\gamma)$ can be chosen close enough to 1 and we can make sure $\rho^\kappa < R$, whenever $0 < \rho < R$. Thus, we obtain

$$\begin{aligned}
&⨍_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\le c\rho^{\gamma}\left[⨍_{B_R} (|Xu| + s)dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right. \\
&\quad \left. + \|f\|_{L^q(B_r)}^{\frac{2}{p}} + \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}} \right]
\end{aligned}$$

for every $0 < \rho < R < \bar{R}$. We complete the proof. □

6. Concluding Remarks

Recently, the $C^{0,1}$ and $C^{1,\alpha}$-regularities for the Equation (11) with Hörmander vector fields of Step two have been established by Citti-Mukherjee [29]. Here we call the vector fields $X_1, \ldots, X_m$ as Hörmander vector fields of step two if they satisfy the step two hypothesis of Nagel-Stein [30], that is,

$$X_1, \ldots, X_m \text{ and } \{[X_j, X_k]\}_{j,k\in\{1,\ldots,m\}} \text{ span the tangent space at each } x \in \Omega.$$

The proofs of Theorems 1 and 2 are based on some regularities and apriori estimates of the Equation (11), and therefore our methods and results can be extended to the Lie group with Hörmander vector fields of Step two.

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