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A Queueing Model for Traffic Flow Control in the Road Intersection

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Abstract: In this paper, we consider a simple road intersection with traffic light control and suggest a queueing model for the traffic flow in the intersection. The suggested model implements the well-known queue with state-dependent departure rates. Using this model, we define optimal state-dependent scheduling of the traffic lights in the intersection and consider its properties. Activity of the model is illustrated by numerical simulations. It is demonstrated that in practical conditions the suggested scheduling of the traffic lights allows the prevention of traffic jams in the intersection and resolves vehicles queues with reasonable waiting times in the crossing lanes.

Keywords: traffic flow; traffic lights; road intersection; queue theory; state-dependent rate

MSC: 60K30; 90B20

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1. Introduction

The queueing theory is one of the basic mathematical tools for analysis of both interrupted and uninterrupted traffic flows. The studies of uninterrupted flows using this theory can be tracked back to the 1950s (for historical notes see [1]). A classical paper [2] suggests the widely accepted interpretation of the problems appearing in the studies of traffic in the terms of queueing theory, and contemporary textbooks (see e.g., [3]) follow this direction.

The studies of interrupted traffic flows consider both the flows interrupted by arbitrary events, such as road accidents [4], and the flows interrupted by traffic control means, such as stop signs and traffic lights [5]. Usually, the interrupted flows are modelled by the $M/M/1$ queues and their extensions or considered as bulk service systems modelled by the $M^X/M^Y/1$ queues. In the analysis of controlled intersections, each lane is specified as an independent queue, and the traffic lights govern an alternation between these queues [3,6,7]. For brief overview of the queueing models of the traffic flows see [8].

In addition to the usual queue's attributes, traffic flows have certain characteristics, such as startup delays and yellow utilization periods [9], which play an essential role in the functionality of intersections and must be concerned in the models. These variable parameters can be considered using the methods of the queues with varying rates [5].

Formally, the queues with varying rates are divided into several types: the queues with time-dependent rates (see the seminal work [10] and its successors, e.g., [11,12]), the queues with event-dependent rates (the queue with event-dependent arrival rate is studied in [13]), and the queues with state-dependent rates [14–16] (see also chapter 2.9 in the textbook [17]).

In the paper, we follow the last approach and suggest a new model for traffic flow in the controlled intersection in the form of alternating $M/M_n/1$ queues with state-dependent departure rates. Then, the traffic lights in the intersection are controlled using only the arrival rates in the lanes as inputs, without referring to additional external parameters that provide direct adaptation of the scheduling to the traffic flow and queues at the entrance of the intersection.

The presented model is the first attempt of applying the state-dependent queue to intersection scheduling, and we use it to define an optimal schedule for traffic lights in the simplest intersection. Activity of the intersection according to the suggested model is verified by numerical simulations.

2. Problem Formulation

Consider a simple road intersection controlled by traffic lights, as shown in Figure 1. The intersection includes two pairs of the lanes: from north to south and from south to north, and from east to west and from west to east. The vehicles follow the directions indicated by the arrows. Motion of the vehicles is regulated by four standard traffic lights placed on the corners of the intersection.

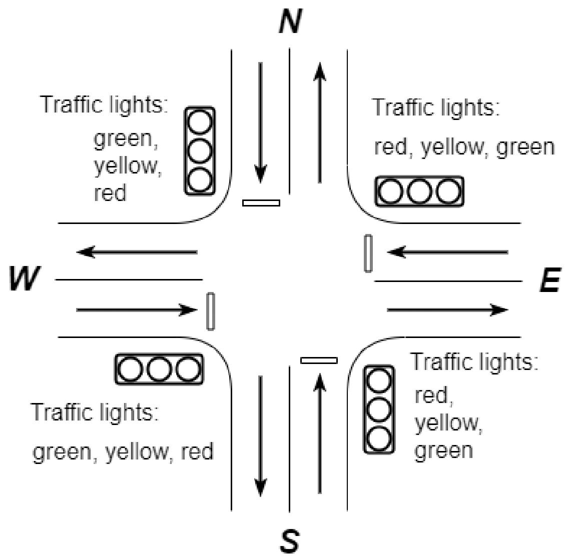

Figure 1. The road intersection with four lanes controlled by the standard red–yellow–green traffic lights. Directions of the vehicles' movements from north (N) to south (S) and backward, and from east (E) to west (W) and backward are denoted by the arrows. While arriving to the intersection, the vehicles may or may not stop before the stop lines ▭ at the entrances to the intersection.

The shown intersection represents the simplest situation, which allows intuitively clear modelling. The vehicles arrive to the entrances of the intersection and if green lights are switched on, they continue in their lanes via the intersection and then exit from it. Certainly, if green lights are switched on in the lanes from north to south and from south to north, then red lights in the lanes from west to east and from east to west must be switched off, and vice versa. In addition, to avoid accidents caused by physical and psychological reasons, between green and red lights switching on and off, in all lanes certain periods of yellow lights switching on are used. Then, the complete transportation circle in each lane is the following sequence of switches: green light, yellow light, red light, yellow light, and then again green light and so on.

In the simple road intersection shown in Figure 1, the lanes from north to south and from south to north act in parallel, so the switches of the traffic lights in these lanes are equivalent. The lanes from west to east and from east to west also act in parallel, but the switches in these lanes oppose the switches in the lanes from north to south and from south to north. As a result, while the vehicles in the lanes from north to south and from south to north follow through the intersection, the vehicles in the lanes from west to east and from east to west wait in the queues and vice versa.

The overall traffic in the intersection includes traffic flows in all the lanes with minimal waiting times that requires optimal alternating between the flows in the lanes. The alternating between the traffic flows is defined by the traffic lights scheduling that must, on one hand, provide maximal traffic in the intersection, and, on the other hand, prevent accidents between the vehicles from the crossing flows. Therefore, the problem is to define the scheduling which provides these results.

3. Suggested Model

The formulated problem consists of two parts: description of the traffic flows in the lanes of the interchange, and specification of the alternating between crossing traffic flows.

3.1. The Queueing Model of the Controlled Traffic Flow in the Lane

Since all four lanes in the considered intersection are equivalent, we can model any one of them and then distribute the results to the other. For example, let us consider the east–west lane shown in Figure 2.

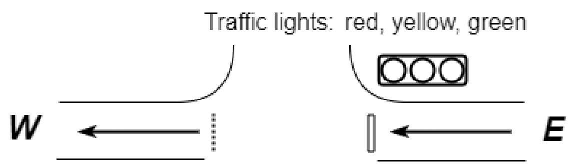

Figure 2. Single lane of the intersection controlled by traffic lights. Direction of the flow is from east (E) to west (W). An entrance to the intersection is denoted by the stop line ▭, and an exit from the intersection is denoted by the dotted line ⋯⋯.

The transportation circle in the lane is as follows [9,18]:

1. The green light switches on at time τ_0. After the startup delay Δ_1, the first vehicle enters the intersection (crosses the stop line) at time $\tau_1 = \tau_0 + \Delta_1$.
2. The rate of moving through the intersection (through the section between entrance to and exit from the intersection) increases approximately after the fourth vehicle reaches its saturation regime.
3. At the saturation regime the vehicles move through the intersection with an approximately constant rate and the headways between them are approximately equal. The flow in this regime is called saturation flow.
4. At time $\tau_2 > \tau_1$ green light switches off and yellow light switches on. Some vehicles continue their movement and exit the intersection, and the other slow down and stop.
5. At time τ_3 the last vehicle exits the intersection (passes the exit line). The period $\Delta_3 = \tau_3 - \tau_2$ is called the yellow utilization period.
6. Finally, at time $\tau_4 > \tau_3$ yellow light switches off and red light switches on, that ends the flow through the intersection.

Certainly, in real scenarios the presented flow can be disturbed by traffic jams, unpredictable behavior of the drivers, faults and disfunctions, and so on, but for ordinary situation the presented description is acceptable.

Let us consider the moments of switching green light. Assume that green light is switched off and that the arrival of the vehicles to the entrance of the intersection is governed by the Poisson distribution [17]

$$P_{arr}(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \tag{1}$$

with the arrival rate λ, which cannot be controlled and is considered as an external factor. The value $P_{arr}(n)$ is the probability that in the unit time interval the number of vehicles at the entrance of the intersection is n.

After switching the green light on, the vehicles begin to move through the interchange and cross the exit line with the departure rate $\mu_{1,k}$, which increases with the number k of vehicle up to a constant saturation value. By the conventional approach, we assume that the departures of the vehicles are governed by exponential distribution [17]

$$P_{dep}(k) = \mu_k e^{-\mu_k k}, \tag{2}$$

where $P_{dep}(k)$ is the probability that in the unit time interval the number of vehicles passed the exit line of the intersection is k.

To represent the dependence of the departure rate on the vehicle number j, $j = 1, 2, \ldots, k$, in the arriving flow, we define the departure rate $\mu_{1,j}$ as

$$\mu_{1,0} = const,\ \ \mu_{1,1} = \mu_{1,0} + ae^{-b},\ \ \mu_{1,2} = \mu_{1,1} + ae^{-b2},\ \ldots,\ \ \mu_{1,k} = \mu_{1,k-1} + ae^{-bk}, \tag{3}$$

or in the close form as

$$\mu_{1,k} = \mu_{1,0} + a\sum\nolimits_{j=1}^{k} e^{-bj} = \mu_{1,0} + a\frac{e^{-bk}\left(e^{bk} - 1\right)}{e^{b} - 1}, \tag{4}$$

where $a, b > 0$ are parameters representing specific physical conditions in the interchange like quality of the road, visibility, and so on. Dependence of the departure rate $\mu_{1,j}$ on number j is shown in Figure 3. In the figure, $\mu_{1,0} = 0$ and $a = b = 1$.

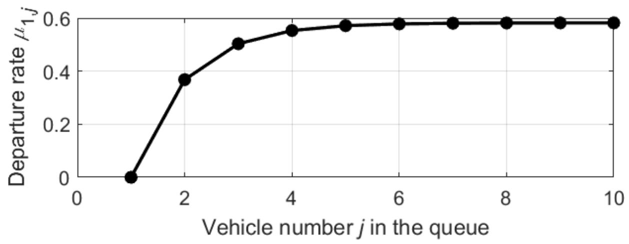

Figure 3. Dependence of the departure rate $\mu_{1,j}$ on the vehicle number j in the queue while the initial rate is $\mu_{1,0} = 0$ and the intersection parameters are $a = b = 1$.

For these parameters the departure rate converges and for $k = 4$ (the fourth vehicle) becomes close to the approximately constant value that coincides with the indicated above empirical observation.

Now consider the moment of switching the green light off. At this moment the departure rate begins to decrease and obtains a value of zero at the end of the yellow utilization period.

We assume that the departure rate after switching the green light off is inverse of the rate after switching the green light on. Let the number of vehicles in the intersection before the exit line including the vehicles before the entrance be k. Then the departure rates of the exiting vehicles are

$$\mu_{0,0} = const,\ \ \mu_{0,1} = \mu_{0,0} - ae^{-b},\ \ \mu_{0,2} = \mu_{0,1} - ae^{-b2},\ \ldots,\ \ \mu_{0,k} = \mu_{0,k-1} - ae^{-bk}, \tag{5}$$

or in the close form are

$$\mu_{0,k} = \mu_{0,0} - a\sum\nolimits_{j=1}^{k} e^{-bj} = \mu_{0,0} - a\frac{e^{-bk}\left(e^{bk} - 1\right)}{e^{b} - 1}, \tag{6}$$

Dependence of the departure rate $\mu_{0,j}$ on number j is shown in Figure 4. In the figure, $\mu_{0,0} = \mu_{1,k}$ and $a = b = 1$.

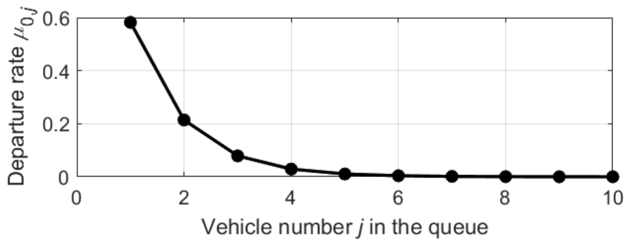

Figure 4. Dependence of the departure rate $\mu_{0,j}$ on vehicle number j in the queue while the initial rate is $\mu_{0,0} = \mu_{1,k}$ and the intersection parameters are $a = b = 1$.

Thus, the switches of green, yellow and red lights govern the alternating between two $M/M_n/1$ queues: one with the increasing and the other with the decreasing departure rates.

Note that the implemented dependences of the departure rates μ_1 and μ_0 on the vehicle number j in the queue are heuristic and can be substituted by other appropriate monotonically increasing and decreasing functions, respectively.

The queue $M/M_n/1$, in which arrival is governed by Poisson distribution with constant arrival rate λ and departure is governed by exponential distribution with state-dependent departure rate μ_n, was suggested by Harris in 1967 [14]. A simple explanation of the properties of this queue is presented in the textbook [17]. In 2016, Abouee-Mehrizi and Baron [16] extended this definition to the $M/G/1$ queue; however, for the considered problem we remain with the $M/M_n/1$ queue with the defined above rates.

Recall that in the $M/M/1$ queue with arrival rate λ and departure or service rate μ, the expected number of customers in service (also interpreted as the offered load or the utilization coefficient) is defined by the ratio [17]

$$\rho = \frac{\lambda}{\mu}, \tag{7}$$

and necessary and sufficient condition for the system to be in the steady state is

$$\rho < 1. \tag{8}$$

In the traffic flows analysis, the ratio ρ is called the traffic intensity [3], and condition (8) guarantees continuous flow of the traffic.

Denote by t_0 the period between the moments of the green light switching off and on and by t_1 the period between the moments of the green light switching on and off. During the period t_0 the yellow and red lights are switching on and off, and during the period t_1 only the green light is switched on. The diagram of these switches is shown in Figure 5.

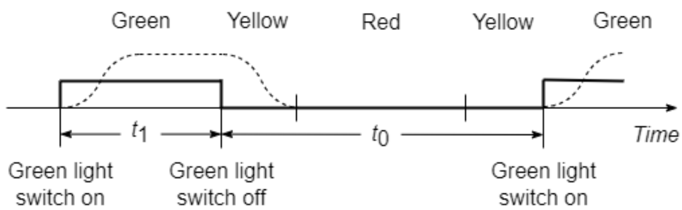

Figure 5. Diagram of the switches of the green light. The green light is switched on during the period t_1 and it is switched off during the period t_0. During the period t_0 yellow and red lights are also switched on and off. Dotted line schematically depicts the changes of the traffic in the intersection (usually called discharge rate [9]).

Let $T = t_1 + t_0$ be the time of complete transportation circle. Then, the expected number n_T of vehicles passed in the lane through the intersection during the transportation circle is

$$t_1\mu_1 + t_0\mu_0 = n_T, \tag{9}$$

and departure rate is

$$\frac{t_1\mu_1 + t_0\mu_0}{T} = \mu_T. \tag{10}$$

From condition (8) it follows that for steady traffic flow in the intersection we need to satisfy the following inequality

$$\lambda < \mu_T, \tag{11}$$

which for given λ means the requirement to maximize the departure rate μ_T.

Lemma 1. *The departure rate μ_T reaches its maximum if $t_1 = T$ and $t_0 = 0$.*

Proof. The decreasing of the traffic flow after switching the green light off begins from the number k of vehicles, which entered the intersection when the green light was switched on, that is, $\mu_{0,0} = \mu_{1,k}$. Then from Formulas (4) and (6) it directly follows that the number n_T reaches its maximum

$$n_T = T\mu_{1,k}, \tag{12}$$

for $t_1 = T$ and so $t_0 = T - t_1 = 0$. □

Intuitively the statement of the lemma is obvious: if the green light in the lane is always switched on, then the throughput of this lane is maximal. It means that $\mu_T = \mu_1$, which follows from Equation (12).

3.2. The Queueing Model of the Controlled Traffic Flow in the Simple Crossroad

Let us apply the model of traffic flow in the lane to description of the flow in the intersection, which is a simple crossroad with two lanes. The scheme of the intersection is shown in Figure 6.

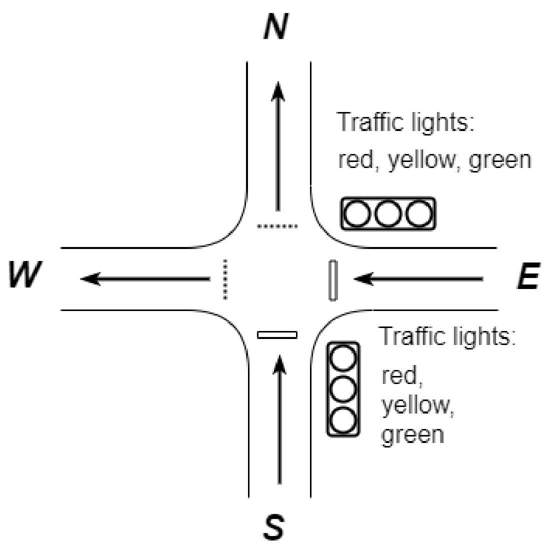

Figure 6. Simple intersection of two lanes controlled by standard red-yellow-green traffic lights. Directions of the vehicles movements from south (S) to north (N) and from east (E) to west (W) are denoted by the arrows. While arriving to the intersection, the vehicles may or may not stop before the stop lines ▭ at the entrances. Exits from the intersection are denoted by the dotted lines ⋯⋯.

In this intersection, which is a cross of two lanes, the vehicles follow from south to north and from east to west, and the flow in each lane is controlled by the standard red–yellow–green traffic lights.

The diagram of the traffic lights' switches is shown in Figure 7. In the figure, the upper indices define the lane. Similar to Figure 5, t_1^1 and t_1^2 are the periods during which the green light is switched on in the first and in the second lane, respectively. The periods t_0^1 and t_0^2, during which green light is switched off are divided into three parts:

$$t_0^l = t_{0,1}^l + t_{0,2}^l + t_{0,3}^l, \quad l = 1, 2, \tag{13}$$

during $t_{0,1}^l$ yellow light (switched after green) is switched on, during $t_{0,2}^l$ red light is switched on and during $t_{0,3}^l$ yellow light (switched after red) is switched on.

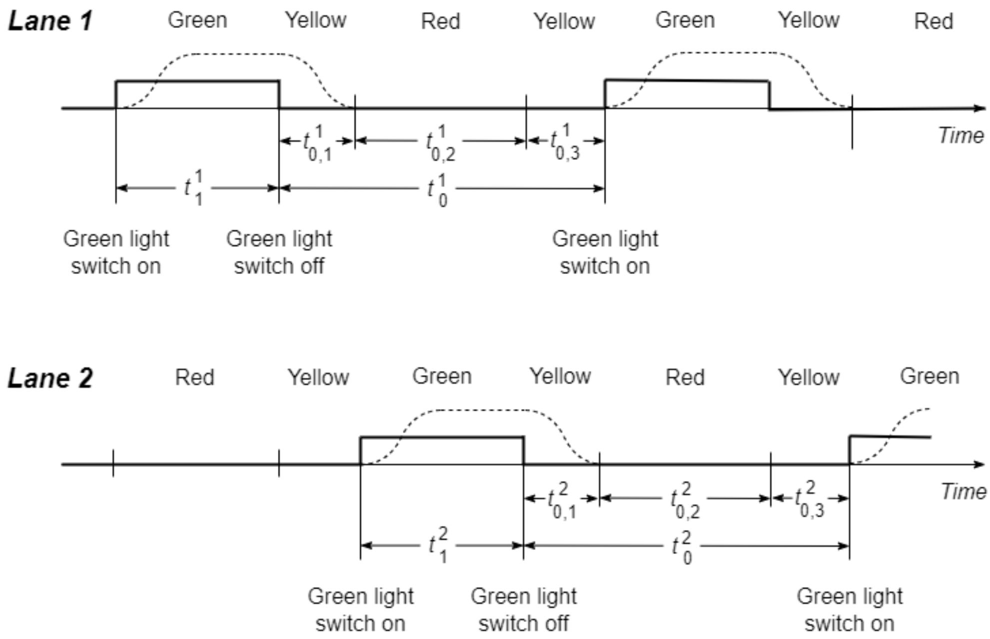

Figure 7. Diagram of the lights' switches in the lanes 1 and 2. The green light is switched on during the periods t_1^1 and t_1^2, and it is switched off during the periods t_0^1 and t_0^2. The periods t_0^l, $l = 1, 2$, consist of three parts $t_{0,1}^l$, $t_{0,2}^l$ and $t_{0,3}^l$ during which yellow and red lights are switched on.

Denote by λ^1 and μ^1 arrival and departure rates in the lane 1 and by λ^2 and μ^2 arrival and departure rates in the lane 2. In the notation of the departure rates, we will also use the bottom indices, which specify the considered period. As above, we assume that the traffic in each lane is described by the $M/M_n/1$ queue with the rates defined by Equations (4) and (6).

Consider lane 1 directed from south to north. Assume that the green light is switched off and the number of vehicles in the lane 1 of the intersection is k_1^1. Note that an expected number L_q of vehicles in the lane is defined by the time t_0^1 during which green light was switched off and the arrival rate λ^1

$$L_{q1}^1 = t_0^1 \lambda^1. \tag{14}$$

Then, if the number k_1^1 is unknown, the value L_{q1}^1 instead can be used.

Since the green light is switched off, the departure rate in the lane is $\mu_{1,0}^1 = 0$. In the moment the green light is switching on, the vehicles start to follow through the intersection, and departure rate of the first vehicle in the queue is $\mu_{1,1}^1$. The second vehicle follows through the intersection with the departure rate $\mu_{1,2}^1$ and so on up to the last vehicle, which follow the intersection with the departure rate $\mu_{1,k_1^1}^1$.

Denote the vehicle number in the queue by j, $j = 1, 2, \ldots, k_1^1$. The expected time required to the jth vehicle to pass the intersection after switching green light on is

$$\tau_{1,j}^1 = \frac{j}{\mu_{1,j}^1}. \tag{15}$$

The total time required to k_1^1 vehicles to pass the intersection is

$$T_{1,k_1^1}^1 = \tau_{1,k_1^1}^1 = \frac{k_1^1}{\mu_{1,k_1^1}^1}. \tag{16}$$

Dependence of the total time $T_{1,k_1^1}^1$ on the number k_1^1 of vehicles is shown in Figure 8, where, as above, the departure rates $\mu_{1,j}^1$ are defined by Equations (3) and (4) with $\mu_{1,0} = 0$ and $a = b = 1$.

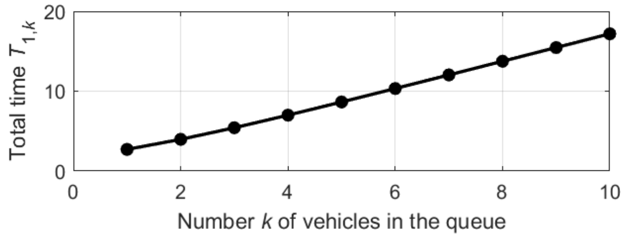

Figure 8. Dependence of the total time $T_{1,k_1^1}^1$ on the number k_1^1 of vehicles. The departure rates $\mu_{1,j}^1$ are defined by Equations (3) and (4) with the parameters $\mu_{1,0} = 0$ and $a = b = 1$.

Note again that the total time $T_{1,k_1^1}^1$ is equivalent to the time $\tau_{1,k_1^1}^1$ required to the jth vehicle to pass the intersection after switching green light on.

Assume that the period t_1^1 while green light in lane 1 is switched on is equal to the time $T_{1,k_1^1}^1$ required to k_1^1 vehicles to pass through the intersection:

$$t_1^1 = T_{1,k_1^1}^1. \tag{17}$$

Then, after the period t_1^1 green light switches off and yellow light switches on.

The yellow utilization period $t_{0,1}^l$ is defined as follows. At the end of the period t_1^1 the departure rate is $\mu_{1,k_1^1}^1$ and it is the departure rate $\mu_{0,0}^1 = \mu_{1,k_1^1}^1$ in the moment of switching green light off and yellow light on. Thus, at this moment traffic intensity is

$$\rho_0^1 = \frac{\lambda^1}{\mu_{0,0}^1}, \tag{18}$$

and an expected queue length [17], which is an expected number of vehicles in the queue [6], is

$$L_{q0}^1 = \rho_0^1 L_0^1 = \rho_0^1 \frac{\rho_0^1}{1 - \rho_0^1}, \tag{19}$$

where L_0^1 is an expected number of vehicles in the flow through the intersection at the moment of switching green light off and yellow light on.

Now similarly to Equations (15) and (16), we have

$$\tau_{0,j}^1 = \frac{j}{\mu_{0,j}^1}, \quad j = 1, 2, \ldots, L_{q0}^1, \tag{20}$$

and an expected total time required to L_{q0}^1 vehicles to utilize the yellow period is

$$T_{0,L_{q0}^1}^1 = \frac{L_{q0}^1}{\mu_{0,1}^1 - \mu_{1,L_{q0}^1}^1}. \tag{21}$$

Dependence of total time $T_{0,L_{q0}^1}^1$ on the number L_{q0}^1 of vehicles is shown in Figure 9.

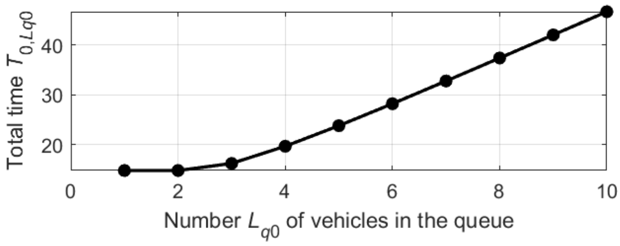

Figure 9. Dependence of the total time $T^1_{0,L^1_{q0}}$ on the number L^1_{q0} of vehicles. The departure rates $\mu^1_{0,j}$ are defined by Equations (5) and (6) with the parameters $\mu_{0,0} = \mu^1_{1,k^1_1}$ and $a = b = 1$.

Following general requirement that during the period $t^l_{0,1}$ when the yellow light is switched on the vehicles must exit the intersection, we assume that this period is equal to the time $T^1_{0,L^1_{q0}}$

$$t^1_{0,1} = T^1_{0,L^1_{q0}}. \tag{22}$$

By the presented reasoning for lane 1, we obtained two values: the period t^1_1 when the green light is switched on and the period $t^1_{0,1}$ when the yellow light is switched on, both with respect to the rates λ^1 and μ^1_1. By the same reasoning applied to the lane 2, we can also define the period t^2_1 and the period $t^2_{0,1}$, both with respect to the rates λ^2 and μ^2_1.

Then, an optimal scheduling of the traffic lights in the intersection is supplied by the following obvious fact.

Lemma 2. *Throughput in the intersection reaches its maximum if* $t^1_{0,2} = t^2_1$, $t^1_{0,3} = t^2_{0,1}$ *and* $t^2_{0,2} = t^1_1$, $t^2_{0,3} = t^1_{0,1}$.

Proof. Truthiness of the lemma follows directly from the observation (see Lemma 1) that for reaching a maximal departure rate during the transportation circle, for lane 1 the period when the green light is switched off must be as short as possible, and for lane 2 this period of lane 1 must be as long as possible, and vice versa. □

Figure 7 represents this synchronization of the switches.

Finally, let us define the criterion for changing the traffic lights. Assume that the green light in lane 1 is switched on and the system is in steady state. Then the traffic intensity in lane 1 is

$$\rho^1_1 = \frac{\lambda^1}{\mu^1_1}, \tag{23}$$

and an expected number of the vehicles passing the intersection in lane 1 is

$$L^1_1 = \frac{\rho^1_1}{1 - \rho^1_1}. \tag{24}$$

On the other hand, during the period $t^2_{0,2} = t^1_1$ the number of vehicles in the queue in lane 2 reaches the value

$$n^2_{0,2} = t^2_{0,2}\lambda^2. \tag{25}$$

Then, to satisfy Lemma 2 the green light in lane 1 must be switched off in the moment when number $n^2_{0,2}$ of vehicles in the queue in lane 2 becomes greater than the value L^1_1:

$$n^2_{0,2} > L^1_1, \tag{26}$$

The same criterion for lane 2 is written as

$$n^1_{0,2} > L^2_1, \tag{27}$$

where $n^1_{0,2}$ is a number of vehicles in the queue in lane 1 and L^2_1 is an expected number of vehicles passing the intersection in lane 2.

Thereby, we obtained the model which specifies the required times and criteria for scheduling traffic lights in a single lane and in the simplest cross of two lanes. In the next section, we use this model for analysis of the intersection declared in Section 2.

4. Solution of the Problem: Control of the Road Intersection with Four Lanes

We apply the suggested models of scheduling traffic lights in a single lane and in the intersection of two lanes to the intersection of four lanes.

4.1. Scheduling the Traffic Lights

The considered intersection of four lanes (see Figure 1) does not allow left and right turns; thus, it is the direct extension of the considered intersection of two lanes. As above, we refer to the lane from south to north as to lane 1 and to the lane from east to west as to lane 2. In addition, we denote the lane from north to south as lane 3 and the lane from west to east as lane 4.

The lanes 1 and 3 act in parallel and restrict the traffic in the lanes 2 and 4 which also work in parallel and restrict the traffic in the lanes 1 and 3. At the same time, lanes 1 and 3 are independent and do not restrict the traffic in each other, and the same holds for the lanes 2 and 4.

There are several ways to schedule the lights in this situation. For example, following the model presented in Section 3.2, we can independently schedule the lights for the intersections of the lanes 1 and 2 and of the lanes 3 and 4, and then synchronize these two intersections. Or, alternatively, we can combine the independent lanes 1 and 3 into one lane A and the independent lanes 2 and 4 into one lane B and then schedule the lights in the intersection of the lanes A and B and consider the intersection of the lanes A and B. Below we follow the second option. The intersection of the combined lanes A and B is shown in Figure 10.

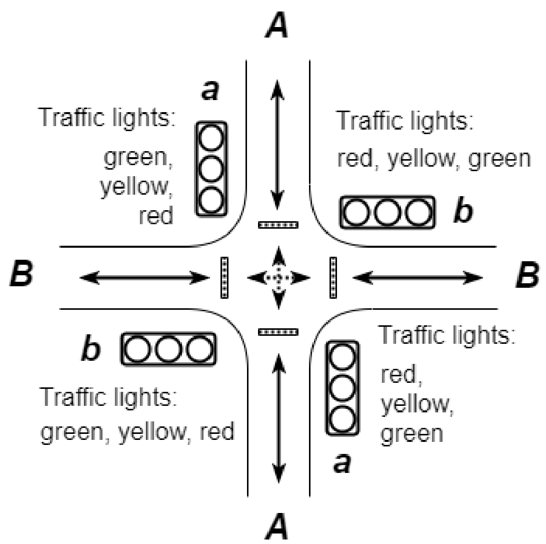

Figure 10. Intersection of combined lanes A and B controlled by standard red-yellow-green traffic lights a and b, respectively. The traffic lights denoted by the same letter are equal and mimic each other. The stop lines and the exits are denoted by the dotted lines in bars .

The intersection of the combined lanes is a simple intersection of two lanes which can be directly considered using the model suggested in Section 3.2. However, to apply this model, the arrival and departure rates and the numbers of the vehicles in the lanes must be defined.

To associate the variables and parameters with the lanes A and B we use the upper indices a and b, respectively. Thus, by λ^a and μ^a we denote arrival and departure rates in the lane A and by λ^b and μ^b arrival and departure rates in the lane B. Similarly, k^a and k^b denote the number of vehicles in the lanes A and B, respectively, t^a and t^b denote the periods of switching the lights on and off, and so on for the other variables. The bottom indices have the same meaning as above.

Parameters of the lanes 1 and 3 and of the lanes 2 and 4 can be combined in several ways. Following the meaning of the considered task, we define the rates λ^a and μ^a as follows. Let

$$\rho^i = \frac{\lambda^i}{\mu^i}, \quad i = 1, \ldots, 4, \tag{28}$$

be the traffic intensities of the lanes $1, \ldots, 4$, correspondingly. Then traffic intensities of the lanes A and B, are, respectively,

$$\rho^a = \max\left\{\rho^1, \rho^3\right\} \quad \text{and} \quad \rho^b = \max\left\{\rho^2, \rho^4\right\}. \tag{29}$$

Consequently, the rates λ^a, μ^a and λ^b, μ^b are the ratios of the lanes with maximal traffic intensity, that is

$$\lambda^a = \begin{cases} \lambda^1 \text{ if } \rho^1 > \rho^3, \\ \lambda^3 \text{ otherwise,} \end{cases} \quad \text{and} \quad \mu^a = \begin{cases} \mu^1 \text{ if } \rho^1 > \rho^3, \\ \mu^3 \text{ otherwise,} \end{cases} \tag{30}$$

and

$$\lambda^b = \begin{cases} \lambda^2 \text{ if } \rho^2 > \rho^4, \\ \lambda^4 \text{ otherwise,} \end{cases} \quad \text{and} \quad \mu^b = \begin{cases} \mu^2 \text{ if } \rho^2 > \rho^4, \\ \mu^4 \text{ otherwise.} \end{cases} \tag{31}$$

The further reasoning about the combined lanes A and B literally follows the considerations presented in Section 3.2 and applied to the rates λ^a, μ^a and λ^b, μ^b. As a result, we obtain the period t_1^a when in the lane A green light is switched on and the period t_{01}^a when in the lane A yellow light is switched on. Similarly, for the lane B we obtain the period t_1^b when the green light is switched on and the period t_{01}^b when the yellow light is switched on.

Then, by Lemma 2 we specify the periods $t_{0,2}^a = t_1^b$ and $t_{0,2}^b = t_1^a$ during which the red lights are switched on and the periods $t_{0,3}^a = t_{0,1}^b$ and $t_{0,3}^b = t_{0,1}^a$ during which the yellow lights are switched on.

Finally, we use the calculated periods for definition of the corresponding periods of switching on green, red, and yellow lights in the lanes 1 and 3 and the lanes 2 and 4. The periods in the lanes 1 and 3 are defined by the periods in the lane A and the periods in the lanes 2 and 4 are defined by the periods in the lane B.

From Lemma 2 and Equations (30) and (31), it directly follows that the presented steps result in maximal throughput in the considered intersection. In the next section we illustrate this statement and functionality of the intersection by numerical simulations.

4.2. Numerical Simulations

The simulation considers the activity of the intersection of two lanes. The goal of the simulation is to verify that the suggested method processes the reasonable inputs, in the considered situation random arrival rates λ^1 and λ^2, and that in spite of the growth of the pass and total times (see Figures 8 and 9), the resulting periods t_1^1, t_1^2 and t_0^1, t_0^2 do not diverge.

The simulation procedure directly implements the presented above formulas. The outline of the procedure is as follows.

```
Procedure (intersection activity)
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Input: periods t_0^1 and t_0^2 of green light switched off in lanes 1 and 2, number of transportation circles N, parameters $a > 0$, $b > 0$.

Output: N periods t_1^1, t_0^1 and t_1^2, t_0^2 of green light switched on and off in the lanes 1 and 2, respectively.

1. For $n = 1$ to N do:

Lane 1:

2. draw arrival rate λ^1 randomly
3. $k_1^1 = \lambda^1 t_0^1$; {*number of vehicles at the entrance*}
4. $\mu_{1,1}^1 = ae^{-b}$; {*equation (4) with* $\mu_{1,0} = 0$ *and* $k = 1$}
5. $\mu_{1,k_1^1}^1 = a\left(e^{-bk_1^1}\left(e^{bk_1^1} - 1\right)/\left(e^b - 1\right)\right)$; {*equation (4) with* $\mu_{1,0} = 0$ *and* $k = k_1^1$}
6. $k_0^1 = k_1^1 - k_1^1 \rho^1$; {*number of vehicles at the moment of switching green light off, where* $\rho^1 = \lambda^1/\mu_{1,k_1^1}^1$, *equation (7)*}
7. $\mu_{0,0}^1 = \mu_{1,k_1^1}^1$; {*initial departure rate, equation (6)*}
8. $\mu_{0,k_0^1}^1 = \mu_{0,0}^1 - a\left(e^{-bk_0^1}\left(e^{bk_0^1} - 1\right)/\left(e^b - 1\right)\right)$; {*equation (6) with* $k = k_0^1$}
9. $t_1^1 = k_1^1/\mu_{1,k_1^1}^1$; {*period of green light switched on, equation (16), (17)*}
10. $t_{0,1}^1 = k_0^1/\left(\mu_{0,0}^1 - \mu_{0,k_0^1}^1\right)$; {*period of yellow light switched on before red, equation (21) and (22), Figure 7*}
11. $t_{0,3}^1 = t_{0,1}^1$; {*period of yellow light switched on after red, Figure 7*}

Lane 2:

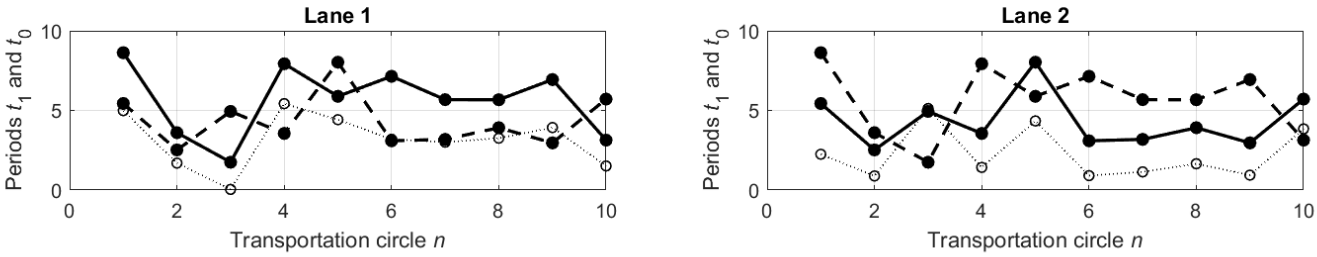
12. draw arrival rate λ^2 randomly
13. $k_1^2 = \lambda^2 t_0^2$; {*number of vehicles at the entrance*}
14. $\mu_{1,1}^2 = ae^{-b}$; {*equation (4) with* $\mu_{1,0} = 0$ *and* $k = 1$}
15. $\mu_{1,k_1^2}^2 = a\left(e^{-bk_1^2}\left(e^{bk_1^2} - 1\right)/\left(e^b - 1\right)\right)$; {*equation (4) with* $\mu_{1,0} = 0$ *and* $k = k_1^2$}
16. $k_0^2 = k_1^2 - k_1^2 \rho^2$; {*number of vehicles at the moment of switching green light off, where* $\rho^2 = \lambda^2/\mu_{1,k_1^2}^2$, *equation (7)*}
17. $\mu_{0,0}^2 = \mu_{1,k_1^2}^2$; {*initial departure rate, equation (6)*}
18. $\mu_{0,k_0^2}^2 = \mu_{0,0}^2 - a\left(e^{-bk_0^2}\left(e^{bk_0^2} - 1\right)/\left(e^b - 1\right)\right)$; {*equation (6) with* $k = k_0^2$}
19. $t_1^2 = k_1^2/\mu_{1,k_1^2}^2$; {*period of green light switched on, equations (16), (17)*}
20. $t_{0,1}^2 = k_0^2/\left(\mu_{0,0}^2 - \mu_{0,k_0^2}^2\right)$; {*period of yellow light switched on before red, equations (21) and (22), Figure 7*}
21. $t_{0,2}^2 = t_1^1$; {*period of red light switched on, Lemma 2*}
22. $t_{0,3}^2 = t_{0,1}^2$; {*period of yellow light switched on after red, Figure 7*}
23. $t_0^2 = t_{0,1}^2 + t_{0,2}^2 + t_{0,3}^2$; {*period of green light switched off*}

Lane 1:

24. $t_{0,2}^1 = t_1^2$; {*period of red light switched on, see Lemma 2*}
25. $t_0^1 = t_{0,1}^1 + t_{0,2}^1 + t_{0,3}^1$; {*period of green light switched off*}
26. End for.

In addition, in the procedure, the conditions $\lambda^1 < \mu_{1,k_1^1}^1$ and $\lambda^2 < \mu_{1,k_1^2}^2$ (see Equation (8)) were checked and preserved.

Example of the periods t_1^1, t_1^2 and t_0^1, t_0^2 during which in the lanes 1 and 2 green light is switched on and off is shown in Figure 11. In the figure the starting periods t_0^1 and t_0^2 when green light is switched off are $t_0^1 = t_0^2 = 10$ and parameters $a = b = 1$.



(**a**) (**b**)

Figure 11. Periods during which the green light is switched on (t_1) and off (t_0) in Lane 1 (**a**) and Lane 2 (**b**). In the figure, solid line depicts the periods t_1 when the green light is switched on and dashed line depicts the periods t_0 of the green light is switched off. The dotted line denotes the random values of the scaled arrival rates λ^1 (lane 1, (**a**)) and λ^2 (lane 2, (**b**)).

The periods t_1 and t_0, during which the green light is switched on and off, coincide with the arrival rates λ^1 and λ^0 and preserve finite values. As it was expected, the periods t_1 and t_0, on one hand, correlate with the rates λ^1 and λ^0 in the same lane, but on the other hand, depend on the arrival rate on the crossing lane.

5. Discussion

The queues theory is one of the basic tools used in traffic analysis and traffic light control, and its basic principles are considered in the textbooks on transportation engineering [3,9]. These techniques are used both to describe and analyze of the traffic on highways and to develop the control schemes of the traffic lights in the roads' intersections.

The suggested model continues this tradition of modelling traffic in the road intersections. However, in contrast to the existing models, which are based on $M/M/1$ queues, it considers the $M/M_n/1$ queue with the departure rates depending on the vehicle number in the queue. Such an approach allows scheduling the traffic lights using one input measurable parameter, the arrival rate to the intersection, while the departure rates and, consequently, the periods of switching the lights on and off, are calculated using parameters of the intersection and heuristic dependence of the departure rate on the vehicle number in the queue.

This heuristic dependence is the most questionable issue in the model. On one hand, it is clear that these dependences for green lights switching on and off are defined by monotonically decreasing and increasing functions, respectively, and that these functions must obtain the intersection properties as parameters. On the other hand, the form of the functions and their dependence on the intersection's parameters are arbitrary and must be defined with respect to the considered situation.

In the paper, we considered the simplest intersection without turns, pedestrian crossing, bus stops, and so on, and verified the model for this intersection. In addition, we had not considered the issue of accident avoidance. We assumed that during the yellow utilization period the vehicles will exit from the intersection and will free the intersection for the vehicles starting to enter from the crossing lane after red light switching off. In the next studies, and especially in the intersections with turns, this issue must be considered in detail. As it follows for numerical simulations, the model correctly processes the reasonable inputs and provides the lights' scheduling, which correctly reacts on the counted arrival rates.

The suggested model complements the existing queue theory models of scheduling traffic lights [6,7,18], and can be used as a generic basis for modeling specific intersections and controlling the traffic lights.

Implementation of the suggested model, even in its current simple form, to scheduling traffic lights may help to prevent traffic jams in intersections and resolve vehicle queues with reasonable waiting times in the crossing lanes.

6. Conclusions

In the paper, we presented a new model for scheduling traffic lights in the controlled intersection. Following the model, the intersection is described by alternating $M/M_n/1$ queues with state-dependent departure rates.

Given the parameters of the intersection, the departure rate of each vehicle is calculated with respect to the vehicle's number in the queue. Then, the periods during which the traffic lights are switched on and off are specified with respect to these departure rates and the arrival rates in the lanes, which are considered as input values.

The presented model is the first attempt of applying state-dependent queues to the problem of intersection scheduling. However, even in its current form, the model clearly describes the intersection activity and results in the reasonable scheduling.

The activity of the intersection and its scheduling according to the suggested model were validated by numerical simulations which justify the suggested approach.

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