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Optimizing Echo State Networks for Enhancing Large Prediction Horizons of Chaotic Time Series

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Abstract: Reservoir computing has shown promising results in predicting chaotic time series. However, the main challenges of time-series predictions are associated with reducing computational costs and increasing the prediction horizon. In this sense, we propose the optimization of Echo State Networks (ESN), where the main goal is to increase the prediction horizon using a lower count number of neurons compared with state-of-the-art models. In addition, we show that the application of the decimation technique allows us to emulate an increase in the prediction of up to 10,000 steps ahead. The optimization is performed by applying particle swarm optimization and considering two chaotic systems as case studies, namely the chaotic Hindmarsh–Rose neuron with slow dynamic behavior and the well-known Lorenz system. The results show that although similar works used from 200 to 5000 neurons in the reservoir of the ESN to predict from 120 to 700 steps ahead, our optimized ESN including decimation used 100 neurons in the reservoir, with a capability of predicting up to 10,000 steps ahead. The main conclusion is that we ensured larger prediction horizons compared to recent works, achieving an improvement of more than one order of magnitude, and the computational costs were greatly reduced.

Keywords: chaos; echo state network; Hindmarsh–Rose neuron; Lorenz system; time-series prediction; decimation; particle swarm optimization

MSC: 68T07; 68U01; 68W99



Citation: González-Zapata, A.M.; Tlelo-Cuautle, E.; Ovilla-Martinez, B.; De la Fraga, L.G. Optimizing Echo State Networks for Enhancing Large Prediction Horizons of Chaotic Time Series. *Mathematics* **2022**, *10*, 3886. <https://doi.org/10.3390/math10203886>

Academic Editors: Luca Pancioni, Giacomo Innocenti and Fernando Corinto

Received: 9 September 2022

Accepted: 17 October 2022

Published: 19 October 2022

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1. Introduction

Nowadays, the prediction of chaotic time series remains an open challenge. Some well-known applications are, for example, weather predictions [1–3], health-related pathologies [4], and financial procedures [5], among others. In particular, these cases exhibit chaotic behavior that is often modeled by sets of strongly coupled nonlinear differential equations [6] so that their solutions depend on the dynamic characteristics, which have high sensitivity to the initial conditions, i.e., the solutions of two identical chaotic systems with slight differences in the initial conditions will be significantly different after a brief evolution time [7]. Fortunately, some works provide guidelines to solve these types of chaotic systems by applying appropriate numerical methods [8].

Due to the difficulties in modeling chaotic systems, promising techniques such as machine learning (ML) as universal approximations of nonlinear functions are employed to learn the behavior of chaotic systems from experimental data, making predictions of the time series without specific knowledge of the model [9]. Some ML techniques that are applied to chaotic time series predictions are support vector machine (SVM) [10], artificial neural networks (ANN), recurrent neural networks with long short-term memory (RNN-LSTM) [6], feedforward neural networks [11,12], and reservoir computing (RC) [13–16], among others. The RC technique is one of the most commonly used for predicting chaotic

time series, as shown in [6,17,18], where the authors compared RC with ANN and RNN-LSTM, showing that RC predicted a more significant number of steps ahead. Some improvements to RC, such as including a knowledge-based model (hybrid model) [19], increased the prediction time almost twofold. In this sense, the main challenges are related to how to increase the prediction horizon and reduce computing costs. To cope with these issues, different alternatives were explored in [20], where the authors proposed a robust Echo State Network (ESN) by replacing the Gaussian likelihood function in the Bayesian regression with a Laplace likelihood function. In [21], the authors carried out a Bayesian optimization of the reservoir hyperparameters (leaking rate, spectral radius, etc.) since selecting them is difficult due to all the possible combinations. In [22], the authors also used Bayesian optimization to find a good set of hyperparameters for the ESN and proposed a cross-validation method to validate their results. In [23], one can see the application of a backtracking search optimization algorithm (BSA) and its variants to determine the most appropriate output weights of an ESN given that the optimization problem was complex. In [24], the authors optimized the ESN input matrix, which was randomly defined in the original proposal [25], using the selective opposition gray wolf algorithm. Recently, in [26], an ESN with improved topology was proposed for accurate and efficient time-series predictions using a smooth reservoir activation function. The prediction times increased in all of these cases but the study systems were still lacking in some aspects. For example, they did not include slow-behavior chaotic systems as artificial neurons, where the change in amplitude between the current point and the previous one was small, tending to be monotone or periodic.

In this work, we show the time-series predictions of two chaotic systems, namely the Lorenz system [27] and the Hindmarsh–Rose neuron (HRN), which has slow dynamic behavior [28]. We show the use of ESNs as an ML technique with the goal of improving the prediction horizon. The horizon is highly improved by decimating the time series without affecting its dynamical characteristics, such as the Lyapunov spectrum and Kaplan–Yorke dimension. The prediction horizon is also increased by optimizing the reservoir input matrix of the ESN $\mathbf{W}_{in}$, applying the particle swarm optimization (PSO) algorithm. It is highlighted that in all the experiments, our work considers a small reservoir size compared to similar works, making our proposal more efficient in terms of computational costs. The paper concludes by showing a comparison of our results and related studies, observing a similar or even a much better prediction horizon with very few neurons. The remainder of the document is structured as follows. Section 2, describes the topology of the ESN and PSO algorithms. Section 3 describes the datasets used for the times-series prediction. Section 4 presents the technique for the chaotic time-series prediction using ESN without optimization. Section 5 shows the prediction results of the optimized ESN with different decimation cases. Finally, the conclusions are summarized in Section 6.

2. ESN and PSO Algorithms

2.1. Echo State Network

In general, the basic structure of an ESN consists of an input layer, an internal reservoir network containing N nodes in the hidden layer, and an output layer [24]. One recommendation is that N is in the range $(\frac{T}{10} \leq N \leq \frac{T}{2})$, with T being the number of sample data [29]. For optimization purposes, the optimal value of N depends on the periodicity of the training data and learning task complexity. A reservoir size that is too small may cause model inaccuracy, whereas a reservoir size that is too large may lead to slow training and data overfitting [30]. The ESN topology is shown in Figure 1, where $\mathbf{W}_{in} \in \mathbb{R}^{N_x \times (1+N_u)}$ is the input weight matrix; $\mathbf{W} \in \mathbb{R}^{N_x \times N_x}$ is the recurrent weight matrix in the reservoir; $u(t) \in \mathbb{R}^{N_u}$ is the input signal; $y(t) \in \mathbb{R}^{N_y}$ is the network output; $\mathbf{x}(t) \in \mathbb{R}^{N_x}$ is the state vector of the reservoir neuron activations; and $t = 1, \dots, T$ denotes the discrete time.

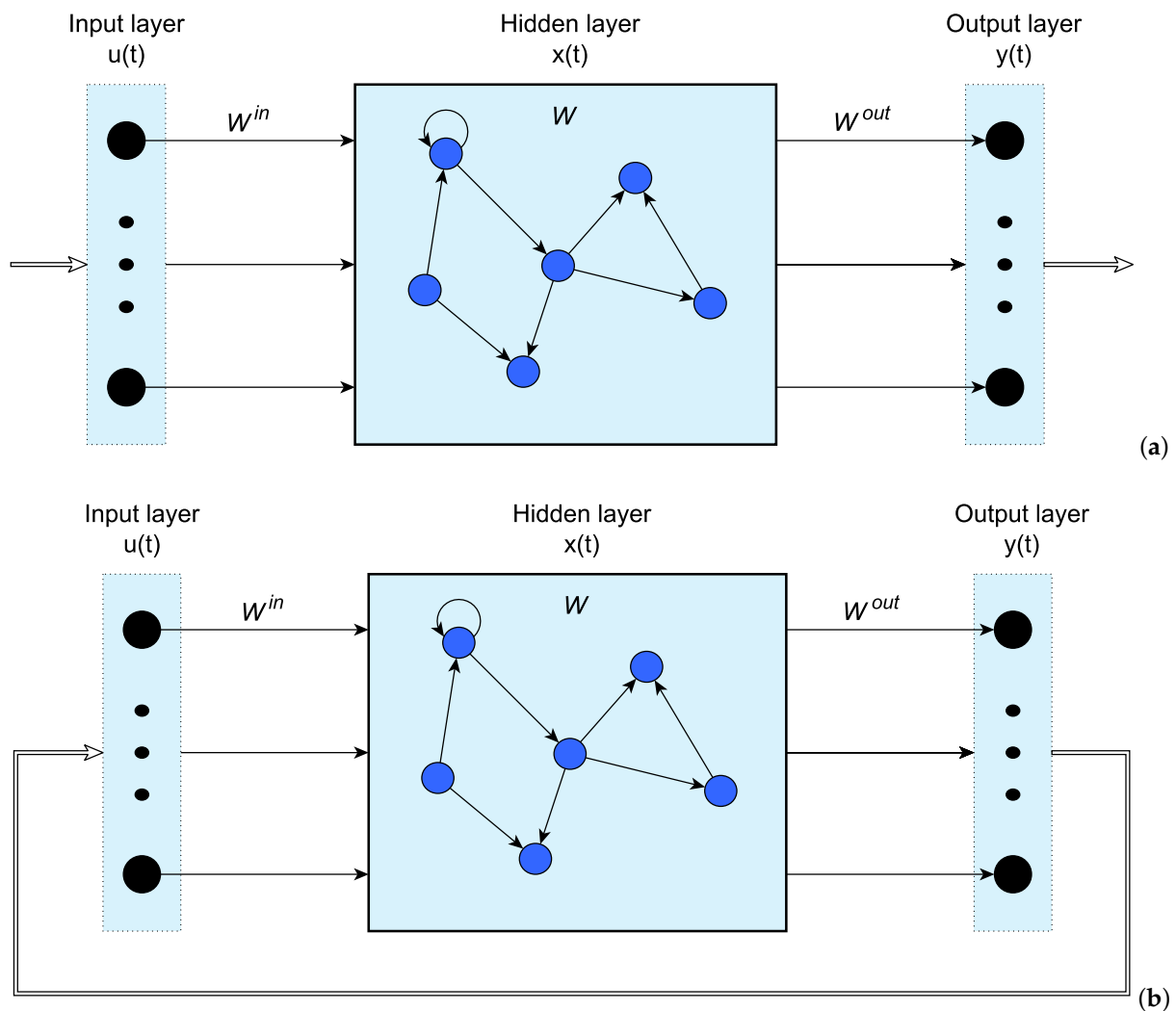


Figure 1. Generic topology of an Echo State Network. (a) Training stage, and (b) Prediction stage.

The leaky integrator ESN is the most frequently used variant. The state vector update equation is described by (1), where parameter $a \in (0, 1]$ is the leaking rate, and this parameter controls the update speed of the reservoir state vector $x(t)$. In [31], some considerations are given to select this parameter, where the authors suggest that the gain (γ) be 1.

$$x(t+1) = (1 - a\gamma)x(t) + \gamma f(Wx(t) + W_{in}[b_{in}; u(t)]) \quad (1)$$

The new state vector update is given in (2), where $b_{in} = 1$ is the input bias value, f is an activation function such as $\tanh(\cdot)$, and W_{in} and W are the weight matrices generated randomly from a uniform distribution in $[-1, 1]$ when the reservoir was initially established. The matrix W must be re-scaled to a sparse matrix with a spectral radius (ρ_r) less than or equal to 1 [21], according to (3) to guarantee the echo state property [32].

$$x(t+1) = (1 - a)x(t) + f(Wx(t) + W_{in}[b_{in}; u(t)]) \quad (2)$$

$$W_r = \frac{\rho_r}{\rho_r(W)} W \quad (3)$$

In the training stage shown in Figure 1a, the main goal is to minimize the error between $y(t)$ and the target output $y^{target}(t)$. The output weight matrix W_{out} is the parameter “trainable” and it is available just after the training stage. Equation (4) describes this matrix, where λ is the regularization coefficient of the reservoir, which is used to prevent

overfitting. $\mathbf{X} \in \mathbb{R}^{(1+N_u+N_x) \times T}$ is the matrix collecting all column-vectors $[1; u(t); \mathbf{x}(t)]$ by concatenating horizontally during the training period $t = 1, \dots, T$.

$$\mathbf{W}_{\text{out}} = \mathbf{y}^{\text{target}} \mathbf{X}^T (\mathbf{X} \mathbf{X}^T + \lambda \mathbf{I})^{-1} \quad (4)$$

In the prediction stage shown in Figure 1b, the current output is the feedback and is used as the input $u(t)$ to predict the next value using (5), where $b_{\text{out}} = 1$ is the output bias value; the rest of the elements are part of a vertical vector (or matrix) concatenation.

$$y(t+1) = \mathbf{W}_{\text{out}} [b_{\text{out}}; u(t); \mathbf{x}(t)] \quad (5)$$

2.2. Particle Swarm Optimization (PSO) Algorithm

In the basic PSO algorithm, the particle swarm consists of n particles, and the position of each particle stands for the potential solution in D-dimensional space [33]. The PSO algorithm is simple in concept, easy to implement, and computationally efficient. The original procedure for implementing the PSO is as follows:

1. Initialize a population of particles with random positions and velocities on D-dimensions in the problem space.
2. For each particle, evaluate the desired optimization fitness function in D-variables.
3. Compare the particle's fitness evaluation with its p_{best} . If the current value is better than p_{best} , then set p_{best} equal to the current value, and P_i equals the current location X_i in the D-dimensional space.
4. Identify the particle in the neighborhood with the best success so far and assign its index to the variable g .
5. Change the velocity and position of the particle according to the following equation:

$$\begin{aligned} v_{id} &= v_{id} + c_1 \text{rand}() (p_{id} - x_{id}) + c_2 \text{rand}() (p_{gd} - x_{id}) \\ x_{id} &= x_{id} + v_{id} \end{aligned} \quad (6)$$

6. Loop to step 2 until a criterion is met, usually a sufficiently good fitness or a maximum number of iterations [34].

3. Datasets of Lorenz System and HRN

3.1. Lorenz System

The Lorenz system is widely known for its chaotic behavior and is one of the most studied classical models. Equation (7) describes the Lorenz system, which generates chaotic behavior by setting $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$ [27].

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned} \quad (7)$$

Figure 2a shows that the dynamic range of the Lorenz system is $[-20, 20]$. However, for appropriate predictions using ML as the ESN with the activation functions as the hyperbolic tangent, it is recommended to use data within a dynamic range $[-1, 1]$, as shown in [18]. For this reason, the Lorenz system is scaled making the change of variables given in (8) into (7).

$$\begin{aligned} x &= k_1 u \\ y &= k_2 v \\ z &= k_3 w \end{aligned} \quad (8)$$

The scaled Lorenz system is described in (9), where $k_1 = k_2 = k_3 = 20$, and Figure 2b shows the behavior of the scaled chaotic time series.

$$\begin{aligned}\dot{u} &= \sigma \left(\frac{k_2}{k_1} v - u \right) \\ \dot{v} &= \frac{k_1}{k_2} u (\rho - k_3 w) - v \\ \dot{w} &= \frac{k_1 k_2}{k_3} uv - \beta w\end{aligned}\quad (9)$$

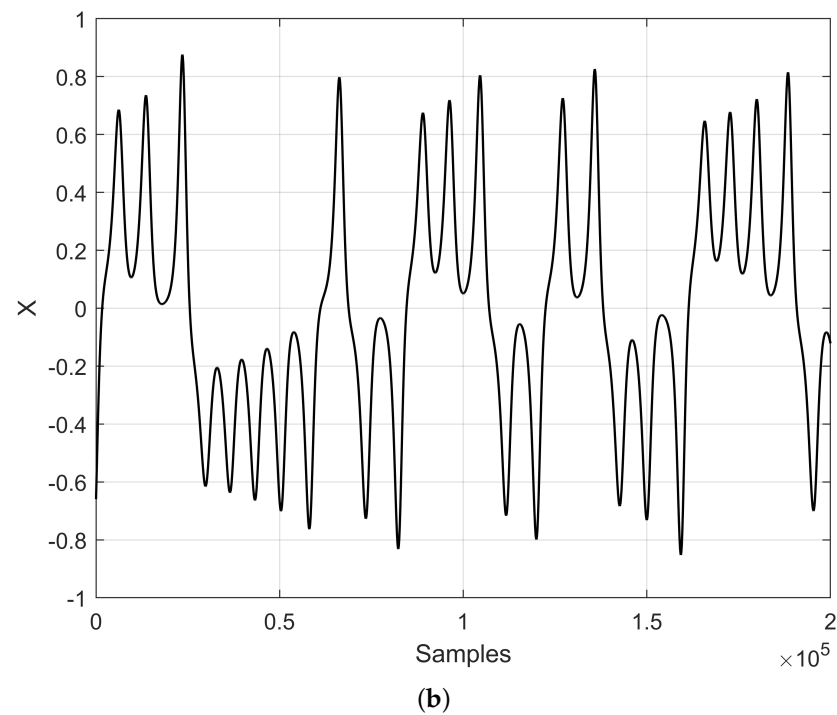
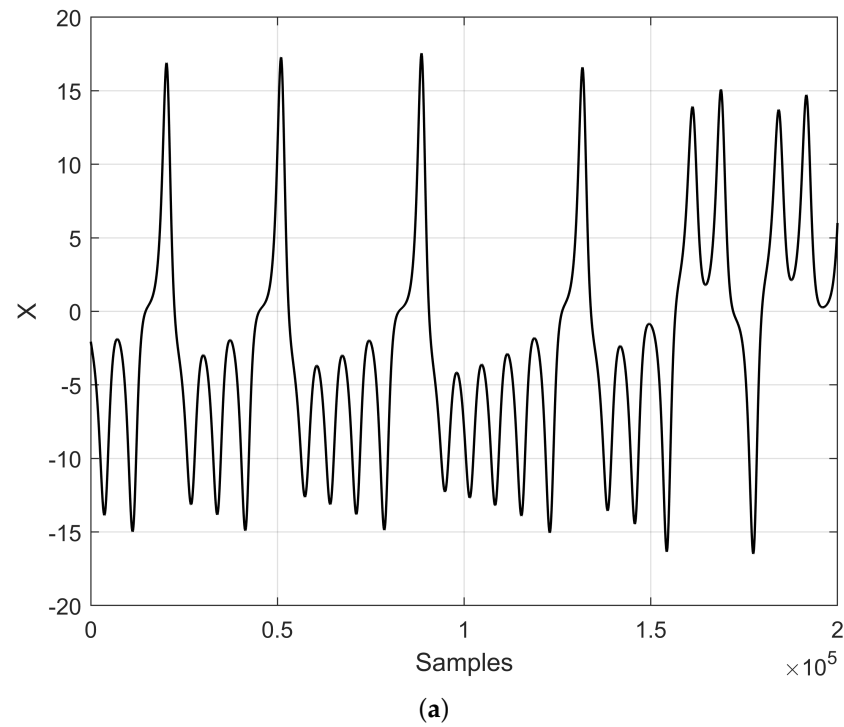


Figure 2. Lorenz chaotic time series solving (7) using a fourth-order Runge–Kutta, with $h = 0.0001$ and initial conditions $(x_0, y_0, z_0) = (0.1; 0.1; 0.1)$. (a) Original, (b) Scaled.

3.2. Hindmarsh–Rose Neuron

The HRN model represents snail neurons obtained by voltage-clamp experiments. It is simple and easy to implement and can effectively simulate a molluscan neuron's repetitive and irregular behavior. It also has many neuronal behavioral characteristics such as periodic peak discharge, periodic cluster discharge, chaotic peak discharge, and chaotic cluster discharge [35]. The HRN model is given in (10) [28], where x is the neuronal membrane voltage, y is the recovery variable associated with the current in the neuronal cell, z is the slow adaptability current, and the other parameters can have the following values to generate chaotic behavior: $a = 0.5$, $b = 1$, $a_1 = -0.1$, $b_1 = -0.045$, $k = 0.2$, $\varphi = 1$, $\varepsilon = 0.02$, and $s = -1.605$ [28]. Figure 3 shows the chaotic time series of the variable x of the HRN.

$$\begin{aligned}\dot{x} &= -s(-ax^3 + x^2) - y - bz \\ \dot{y} &= \varphi(x^2 - y) \\ \dot{z} &= \varepsilon(sa_1x + b_1 - kz)\end{aligned}\quad (10)$$

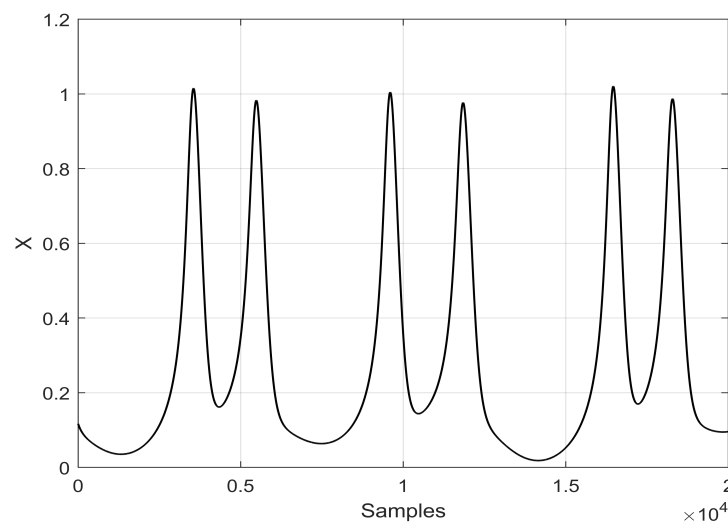


Figure 3. Chaotic time series of variable x of HRN by setting $h = 0.01$ and initial conditions $(x_0, y_0, z_0) = (0.1169; 0.0356; 0.0103)$.

3.3. Decimation of the Chaotic Time Series

To emulate an increase in the prediction horizon, we decided to decimate the time series before conducting the reservoir training, as shown in [18]. The decimation operation by an integer factor $D > 1$ on a sequence $\mathbf{x}[\mathbf{n}]$ consists of keeping every D th sample of $[\mathbf{x}(\mathbf{n})]$ and removing $D - 1$ in-between samples, generating an output sequence $\mathbf{y}[\mathbf{d}]$ [36]. One can only use this proposal if the main characteristics of the time series do not change, and therefore its Lyapunov exponents and Kaplan–Yorke dimension remain at around the same values without decimation.

This work used the time-series analysis (TISEAN) software to evaluate the Lyapunov exponents from the chaotic time series and the Kaplan–Yorke dimension [37], whose equation is given in (11) and includes the Lyapunov exponents' values.

$$D_{KY} = k + \frac{\sum_{i=1}^k LE_i}{|LE_{k+1}|} \quad (11)$$

4. Prediction Using ESNs

This section describes the prediction of both chaotic systems, the Lorenz and HRN, using ESNs.

4.1. Lorenz System

The prediction of the Lorenz chaotic time series is performed herein by applying decimation D . Table 1 shows the values of the positive Lyapunov exponent (LE+) and Kaplan–Yorke dimension (D_{KY}) without decimation (None) and for the three decimation cases with D equal to 10, 25, and 50. The corresponding chaotic time series from this table are shown in Figure 4. As can be seen, the dynamical behavior of the chaotic time series by applying decimation was quite similar to the original data and this helped to increase the prediction horizon, as shown in the remainder of this manuscript.

Table 1. Lyapunov exponent (LE+) and Kaplan–Yorke dimension (D_{KY}) of the Lorenz system without decimation and for decimation values of 10, 25, and 50.

D	None	10	25	50
LE+	0.0061	0.0016	0.0022	0.0038
D_{KY}	2.7995	2.2518	2.1265	2.1079

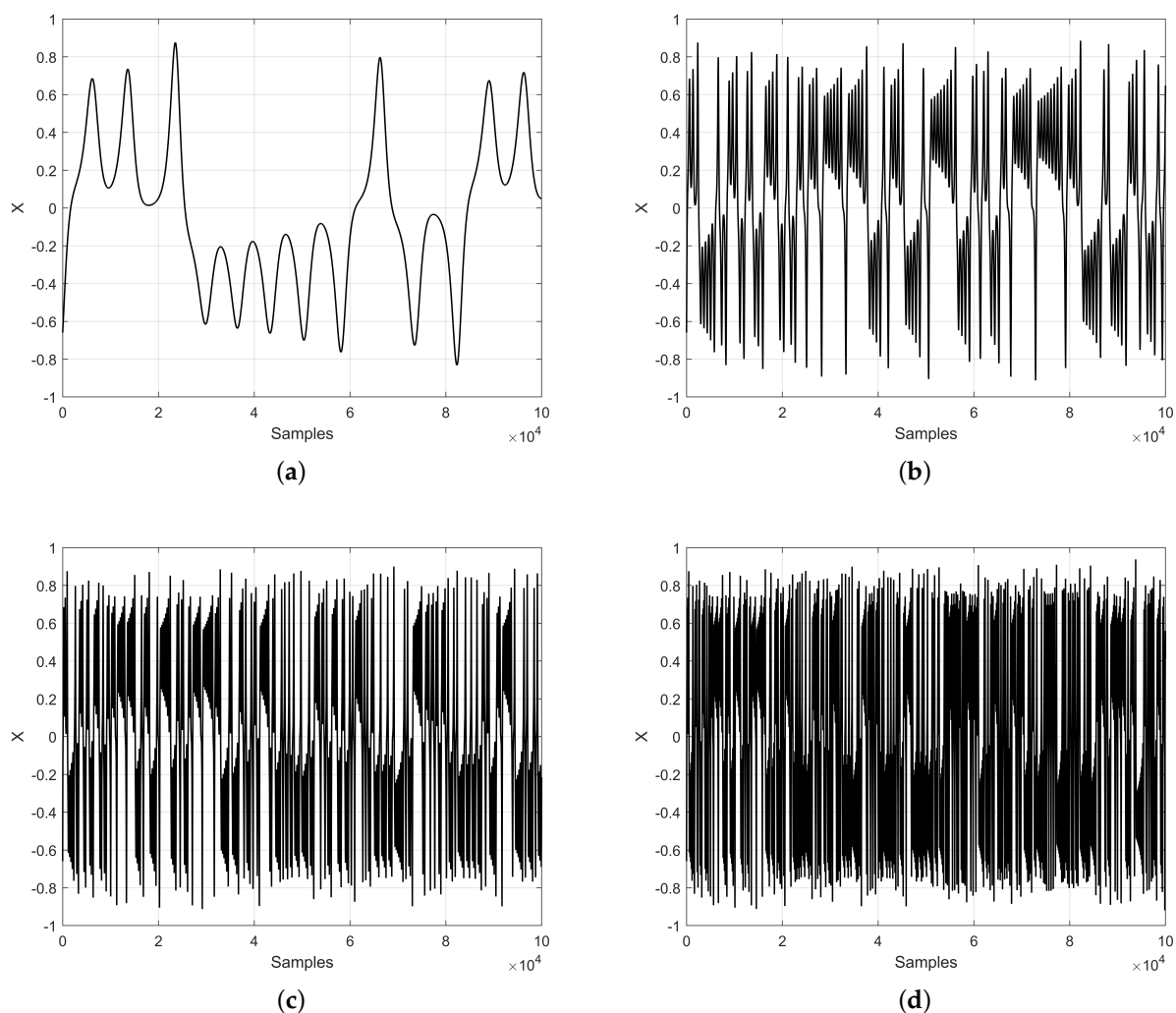


Figure 4. Chaotic time series of Lorenz system: (a) without decimation $D = 0$, and with (b) $D = 10$, (c) $D = 25$, and (d) $D = 50$.

For the selection of the ESN hyperparameters to predict the time series of the Lorenz system, we used the grid search method, varying the leaking rate and spectral radius, with 100 neurons ($N = 100$) in the reservoir. According to the recommendation mentioned in Section 2.1, there should be between 1000 and 5000 neurons for a training dataset

of 10,000 points; however, we sought to increase the prediction horizon with the least number of neurons possible. The matrices $\mathbf{W}$ and $\mathbf{W}_{in}$ were randomly generated from a range of $[-1, 1]$, and $\mathbf{W}$ was re-scaled according to the spectral radius to guarantee the echo state property and fully-connected topology. Figure 5 shows the results of the grid search, where it can be seen that for large values of the leaking rate and spectral radius, the root-mean-squared error (RMSE) tended to increase.

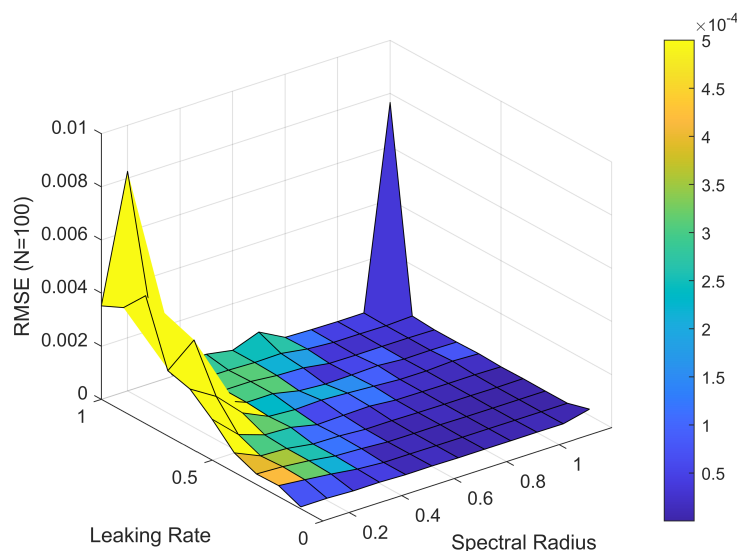


Figure 5. Grid search for the selection of ESN hyperparameters.

The prediction of the chaotic time series was conducted by generating 10,000 data points from the time series to perform the reservoir training, establishing a leaking rate of 0.5 and a spectral radius of 0.41, according to the results shown in Figure 5, and finally, a regularization parameter of the ridge regression of 1×10^{-8} . Table 2 shows the results of the prediction for the times series generated from the Lorenz system using reservoir computing with three decimation values. It can be seen that in all the decimation cases, the total prediction steps increased, multiplying the decimation value by the steps ahead. For example, for the case where $D = 25$, there was a prediction of 60 steps ahead and a total of 1500 steps ahead. To measure the root mean square error (RMSE) and correlation coefficient (CC), we made several predictions to obtain approximately 5000 points. In all cases, the error values were very close, allowing us to conclude that decimation enabled an increase in the number of predicted steps ahead without significantly changing the RMSE and CC.

Table 2. Results of the chaotic time-series prediction of Lorenz system using ESN with and without the three different decimation values.

D	Steps Ahead	Total Steps Ahead	Steps Predicted	RMSE	CC
None	250	250	5000	4.6020×10^{-6}	0.999960
10	80	800	5040	2.3270×10^{-6}	0.999992
25	60	1500	5040	6.4784×10^{-6}	0.999968
50	30	1500	5010	1.4494×10^{-6}	0.999995

4.2. Hindmarsh–Rose Neuron

As with the prediction of the chaotic time series for the Lorenz system, a similar process was performed for the HRN. First, the Lyapunov exponents and Kaplan–Yorke

dimension were calculated using different decimation values, as shown in Table 3. Figure 6 shows the results of the time series for the four cases listed in Table 3.

Table 3. Lyapunov exponent (LE+) and Kaplan–Yorke dimension (D_{KY}) of the Hindmarsh–Rose Neuron without decimation and for decimation values of 10, 25, and 50.

D	None	10	25	50
LE+	0.0012	0.0113	0.0283	0.0566
D_{KY}	2.3401	2.2754	2.2748	2.2736

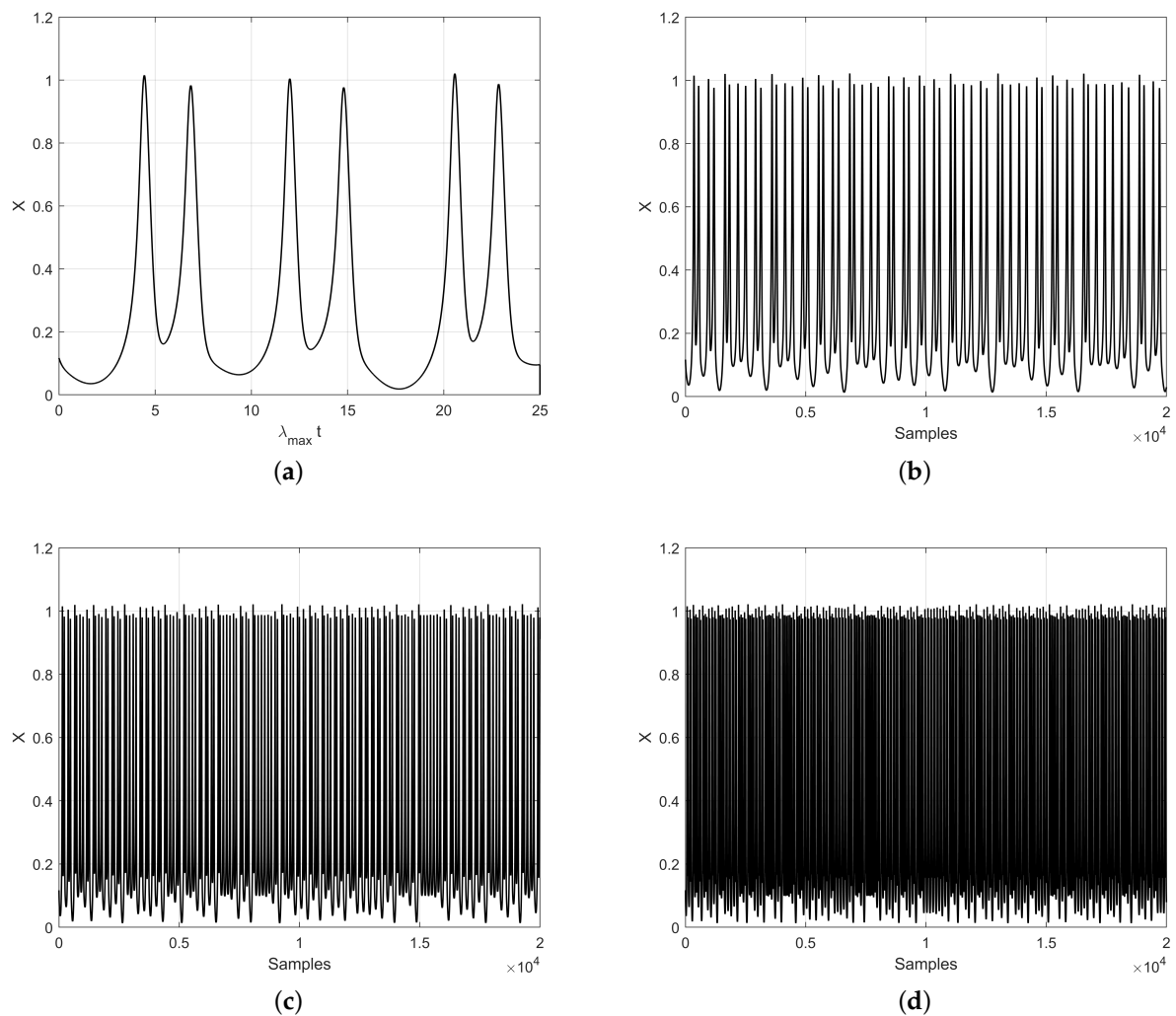


Figure 6. HRN chaotic time series: (a) without decimation, and with decimation (b) $D = 10$, (c) $D = 25$, and (d) $D = 50$.

The next step was to predict the different decimated time series using the ESN. For this, again, we generated 10,000 data points, as with the Lorenz system, to perform the training, setting a leaking rate of 0.5 and a spectral radius of 0.41. The matrices $\mathbf{W}$ and $\mathbf{W}_{in}$ were randomly generated and the reservoir comprised 100 neurons. Table 4 shows the obtained results.

Table 4. Prediction results of the HRN chaotic time series shown in Figure 6 and using the ESN with 100 neurons in the hidden layer.

D	Steps Ahead	Total Steps Ahead	Steps Predicted	RMSE	CC
None	350	350	10,150	4.5872×10^{-5}	0.999696
10	100	1000	10,000	4.1105×10^{-7}	0.999997
25	45	1125	10,035	1.5395×10^{-5}	0.999898
50	7	350	10,000	1.4543×10^{-4}	0.999045

Decimating the HRN time series increased the number of predicted steps ahead. However, when $D = 50$, it was impossible to improve the prediction, even though this decimation value did not change the system's dynamics. This indicates that the maximum decimation value that allows for an increase in the prediction steps depends on each chaotic system.

5. ESN Optimization Using PSO

Metaheuristics are iterative techniques and algorithms that use some level of randomness to discover optimal or near-optimal solutions to computationally hard problems [30]. Among the most commonly applied are differential evolution (DE) [38], genetic algorithms (GA) [39], and PSO [40]. This section shows the application of PSO to enhance time-series prediction [30]. We show the optimization of the ESN applying PSO and considering different decimation cases. The final results of predicting the chaotic time series of the Lorenz system and HRN show the superiority of our technique, allowing predictions of up to 10,000 steps ahead as detailed below.

5.1. Echo State Network Optimization

The $\mathbf{W}_{in}$ and $\mathbf{W}$ matrices were randomly generated. For this reason, we proposed to optimize some of them to increase the prediction horizon. The weight matrix $\mathbf{W}$ of the reservoir must be re-scaled to a sparse matrix with a spectral radius less than or equal to 1 to maintain the echo state property [41]. This condition complicates the optimization of the $\mathbf{W}$ matrix; therefore, we decided to optimize the $\mathbf{W}_{in}$ matrix.

The initial population of the PSO algorithm corresponds to the values of the $\mathbf{W}_{in}$ matrix used in Section 4 to increase the prediction. The fitness function is the RMSE between the target values (y_{target}) and the predicted values (y_{pred}) of the chaotic time series. This procedure is performed for the Lorenz system and HRN.

$$RMSE = \sqrt{\frac{1}{T} \sum_{i=1}^T (y_{target} - y_{pred})^2} \quad (12)$$

Figure 7 shows our proposed optimization process to predict the time series of the Lorenz system and the HRN.

5.2. Enhanced Prediction of Lorenz System Using Optimized ESN

From the optimization process of the ESN, we chose the two most optimistic cases from Table 2, which correspond to the chaotic time series with decimations of 25 and 50. The PSO algorithm was run with a population of 30 particles and 10 generations, and we decided to perform a scaled optimization. For example, when $D = 25$, the optimization was performed for 100 steps ahead, the $\mathbf{W}_{in}$ matrix was in the initial population for the subsequent optimization of 150 steps ahead, and so on, until it was impossible to increase the steps ahead with a small error. Table 5 shows the results of the different optimization runs for the two decimations, 25 and 50. Figure 8 shows the prediction results for $D = 25$ and Figure 9 for $D = 50$. In both cases, one can appreciate that the predicted steps ahead were increased and that the absolute error was low.

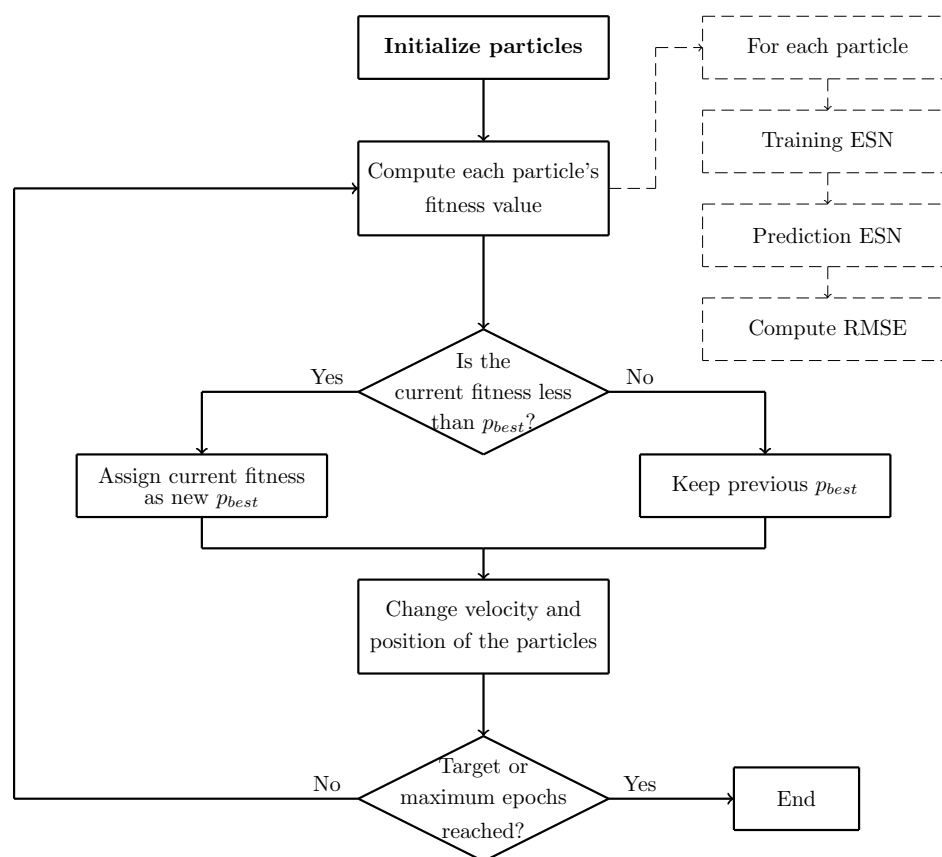


Figure 7. Proposed flow diagram of the ESN optimization process for predicting chaotic time series.

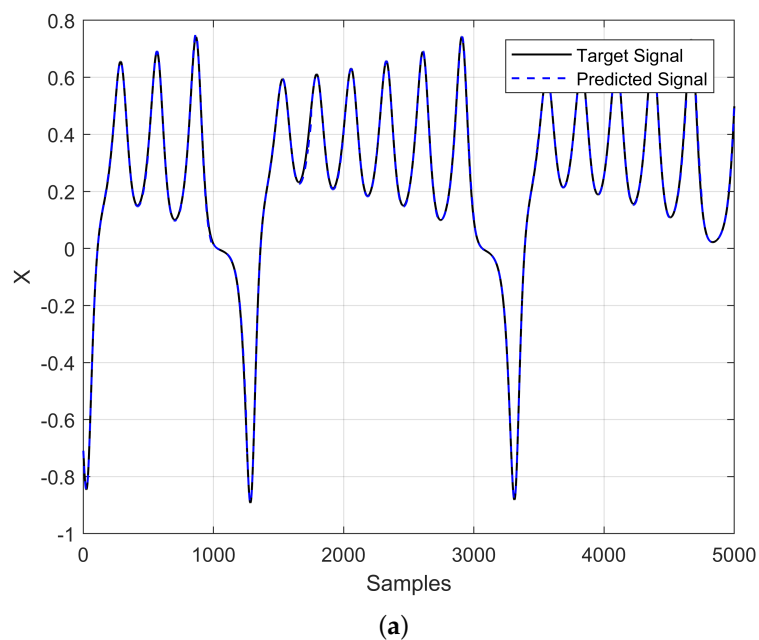


Figure 8. Cont.

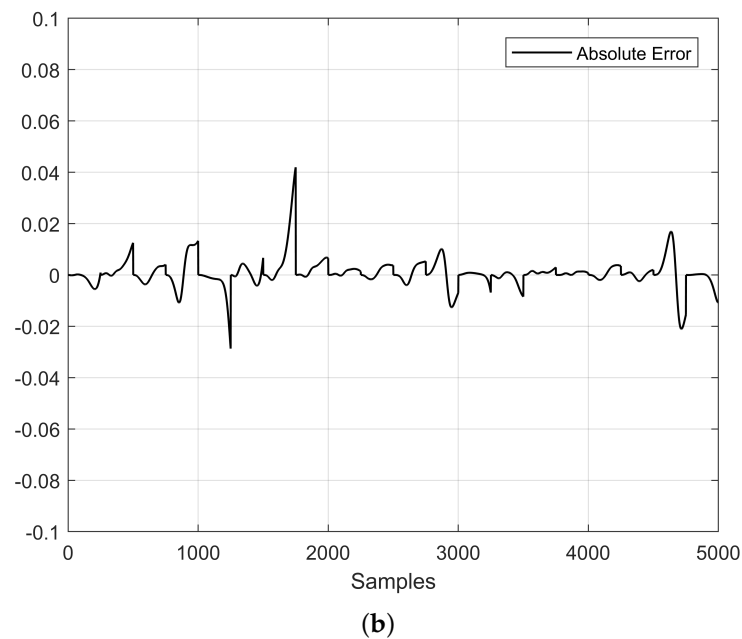


Figure 8. Time-series prediction (Lorenz system) using an optimized ESN with $D = 25$. (a) Predicted time series and target signal, and (b) Absolute error.

Table 5. Optimized ESN for times-series prediction of Lorenz system.

Run	D	Steps Ahead	Total Steps Ahead	Steps Predicted	RMSE
1	25	250	6250	5000	3.3315×10^{-5}
2					7.7769×10^{-6}
3					6.1854×10^{-5}
4					1.9145×10^{-5}
5					1.2868×10^{-5}
6					4.6597×10^{-5}
1	50	200	10,000	5000	0.5899×10^{-5}
2					8.8973×10^{-6}
3					3.6936×10^{-5}
4					5.1623×10^{-5}
5					1.3510×10^{-5}
6					7.2846×10^{-5}

5.3. Enhanced Prediction of Hindmarsh–Rose Neuron Using Optimized ESN

For the HRN, we performed the same scaled optimization procedure and selected the cases where $D = 10$ and $D = 25$, which obtained the best results, as seen in Table 4. Table 6 shows the optimization results.

From Table 6 and Figures 10 and 11, one can conclude that there was not a big difference between the number of total predicted steps ahead and that there was also no significant difference between the absolute error. However, when $D = 25$, one can observe more dynamics in the graph despite having a window of 5000 steps.

Table 6. Optimized ESN for times-series prediction of HRN.

Run	D	Steps Ahead	Total Steps Ahead	Steps Predicted	RMSE
1	10	700	7000	5600	2.5441×10^{-6}
2					1.3352×10^{-5}
3					6.2291×10^{-6}
4					1.2683×10^{-5}
5					1.5212×10^{-5}
6					3.2412×10^{-6}
1	25	300	7500	5100	3.4141×10^{-6}
2					3.8000×10^{-6}
3					6.8645×10^{-7}
4					1.0267×10^{-6}
5					1.8325×10^{-6}
6					1.3844×10^{-5}

5.4. Prediction Results and Comparison with Related Works

In this work, we took into account two main objectives; the first was focused on how to increase the prediction steps of chaotic time series and the second was to keep a reservoir with a small size. Table 7 shows the results obtained from the prediction using the optimized ESN with decimation cases for the Lorenz system and HRN, and they were compared with similar works. As one can appreciate, our prediction results were pretty competitive. For example, for the Lorenz system without decimation, the prediction was 250 steps ahead, which was a similar result to that reported in [25], where the authors used three times the number of neurons in the reservoir. In [6], the authors predicted 460 steps using 5000 neurons in the reservoir, which was significantly different from our work in that in all cases, the ESN consisted of only 100 neurons in the reservoir. On the other hand, we can highlight how the decimation process increased the number of predicted steps ahead; only with a decimation of 10 did we improve on the results presented in [42].

Table 7. Prediction results using optimized ESN without/with decimation (D) and comparison with related works.

Method	Lorenz				HRN			
	Steps Ahead	RMSE	Neurons	W_{elem}	Steps Ahead	RMSE	Neurons	W_{elem}
ESN	250	4.6020×10^{-6}	100	1×10^4	350	4.5872×10^{-5}	100	1×10^4
ESN + D10	800	2.3270×10^{-6}	100	1×10^4	1000	4.1105×10^{-7}	100	1×10^4
ESN + D25	1500	6.4784×10^{-6}	100	1×10^4	1125	1.5395×10^{-5}	100	1×10^4
ESN + D50	1500	1.4494×10^{-6}	100	1×10^4	350	1.4543×10^{-4}	100	1×10^4
ESN + PSO + D10	—	—	—	—	7000	2.5441×10^{-6}	100	1×10^4
ESN + PSO + D25	6250	3.3315×10^{-6}	100	1×10^4	7500	3.4141×10^{-6}	100	1×10^4
ESN + PSO + D50	10,000	5.8987×10^{-6}	100	1×10^4	—	—	—	—
ESN [25]	300	—	300	9×10^4	—	—	—	—
ESN [6]	460	—	5000	5×10^6	—	—	—	—
RNN-LSTM [6]	180	—	5000	5×10^6	—	—	—	—
ANN [6]	120	—	5000	5×10^6	—	—	—	—
ESN [42]	700	—	300	9×10^4	—	—	—	—
RESN [20]	500	0.2238	200	4×10^4	—	—	—	—

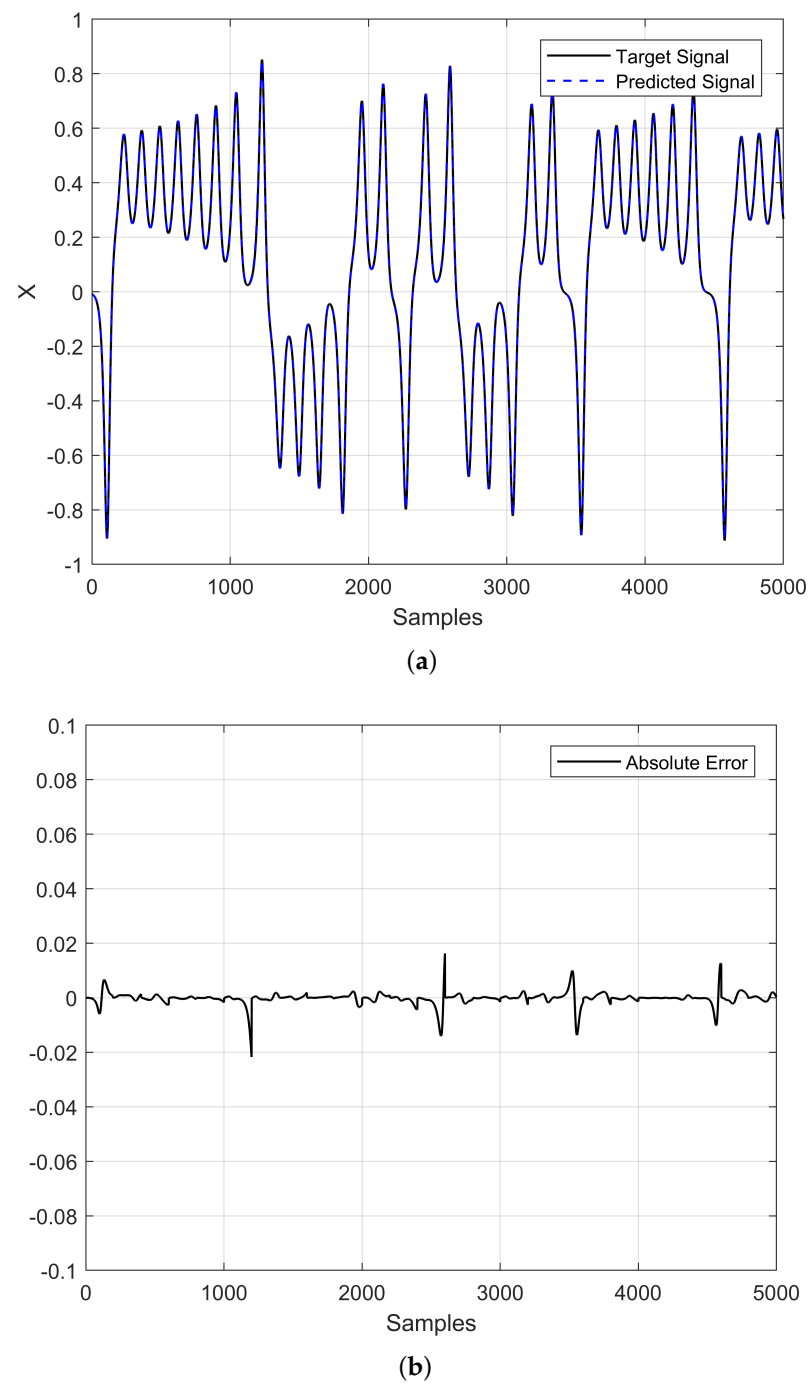


Figure 9. Time-series prediction (Lorenz system) using an optimized ESN with $D = 50$. (a) Predicted time series and target signal, and (b) Absolute error.

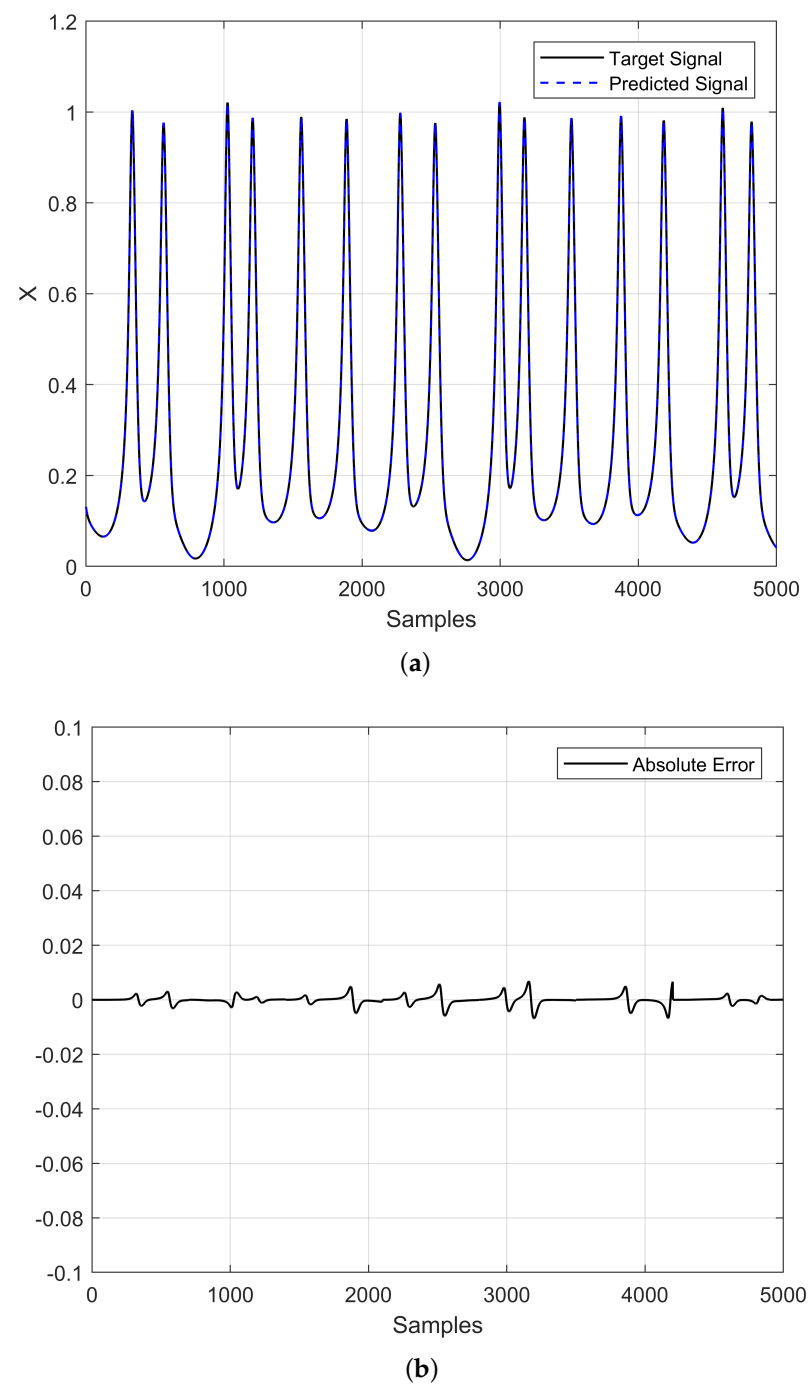


Figure 10. Time-series prediction (HRN) using an optimized ESN with $D = 10$. (a) Predicted time series and target signal, and (b) Absolute error.

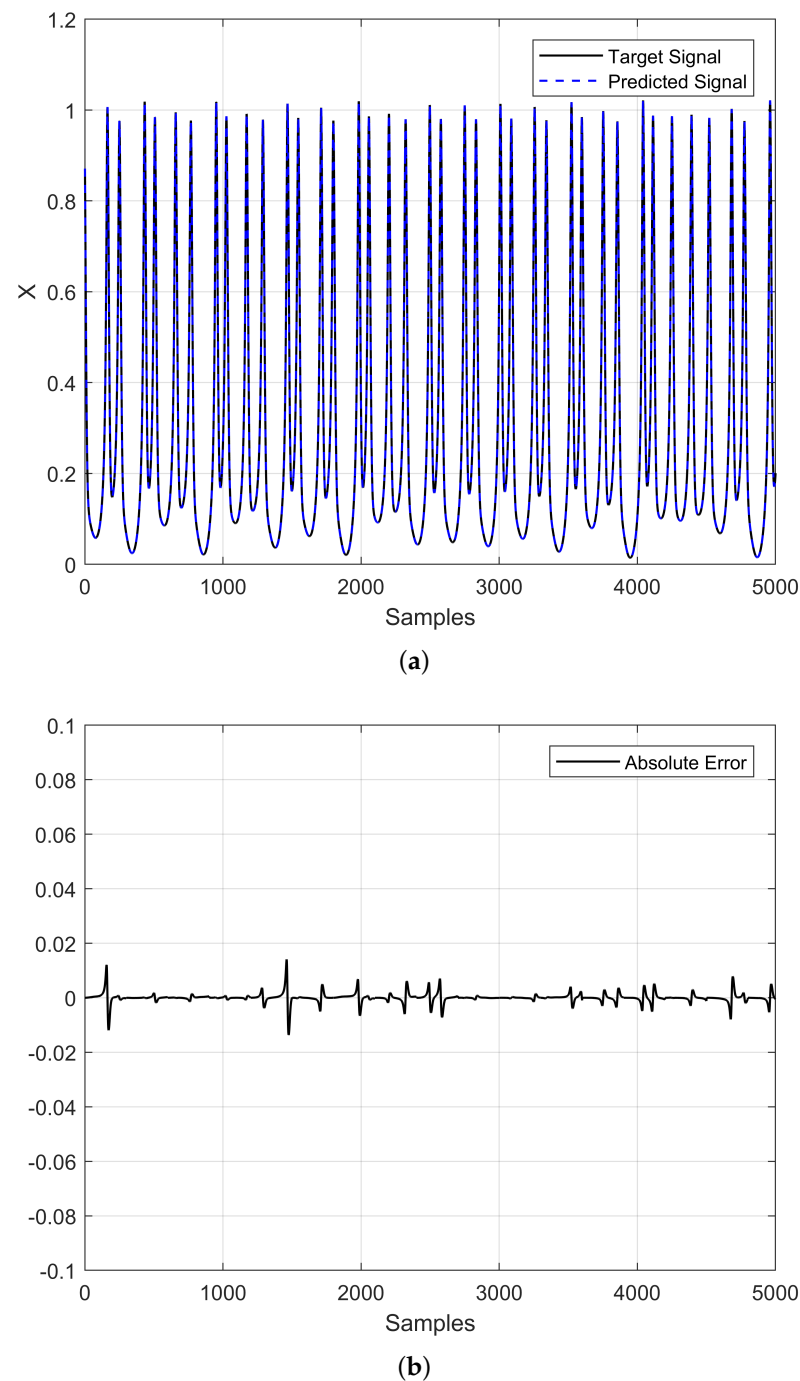


Figure 11. Times series prediction (HRN) using an optimized ESN with $D = 25$. (a) Predicted time series and target signal, and (b) Absolute Error.

Another interesting conclusion is that the optimization of the $\mathbf{W}_{in}$ input matrix resulted in an improvement of more than one order of magnitude in the best case. Finally, to the best of our knowledge, chaotic time-series prediction of neural behavior such as the HRN has not been performed before. This type of neuron has a slow behavior that can complicate the prediction task in long windows of time. One can also use other neuron models such as cellular neural networks [43], the Hopfield neuron [44], and the Huber–Braun neuron [45], as well as other systems related to health modeling [4].

6. Conclusions

This work aimed to increase the prediction horizon of chaotic time series using optimized ESNs, including the decimation of the time series generated from the Lorenz system and HRN. The first contribution was the strategy to decimate the time series. In this case, the HRN presented slow behavior, thus requiring special attention during the generation of time series, since by using a larger step size in the numerical method (h), the chaos can be diminished and one cannot apply decimation to guarantee a large prediction horizon. In this manner, the prediction results given in Table 7, show that our proposed method for optimizing ESN including decimation greatly increased the predicted steps ahead without affecting the time-series dynamics. The second contribution was the optimization of the W_{in} input matrix of the ESN, implementing a scaled optimization strategy. The optimization was carried out for a certain number of steps ahead and the resulting optimized matrix was used to initialize the population of the following test, guaranteeing the improvement of the prediction results with more steps ahead. This process was carried out until it was impossible to improve the prediction with an established minimum error value.

Another important conclusion is that our proposed method guaranteed larger prediction horizons with just 100 neurons, which is a very good size reduction of the reservoir compared to similar works and obtained quite competitive prediction results with lower computational costs.

Author Contributions: Investigation, A.M.G.-Z., I.C.-V., E.T.-C., B.O.-M. and L.G.D.I.F.; Writing—review and editing, A.M.G.-Z., I.C.-V. and E.T.-C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflicts of interest.

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