mathematics

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Solvability for a Class of Integro-Differential Inclusions Subject to Impulses on the Half-Line

Paola Rubbioni

Department of Mathematics and Computer Science, University of Perugia, 06123 Perugia, Italy; paola.rubbioni@unipg.it

Abstract: In this paper, we study a semilinear integro-differential inclusion in Banach spaces, under the action of infinitely many impulses. We provide the existence of mild solutions on a half-line by means of the so-called extension-with-memory technique, which consists of breaking down the problem in an iterate sequence of non-impulsive Cauchy problems, each of them originated by a solution of the previous one. The key that allows us to employ this method is the definition of suitable auxiliary set-valued functions that imitate the original set-valued nonlinearity at any step of the problem's iteration. As an example of application, we deduce the controllability of a population dynamics process with distributed delay and impulses. That is, we ensure the existence of a pair trajectory-control, meaning a possible evolution of a population and of a feedback control for a system that undergoes sudden changes caused by external forces and depends on its past with fading memory.

Keywords: semilinear differential inclusions; impulsive problems; feedback controls; distributed delay; population dynamics

MSC: Primary: 34G20; 34G25; 34A37; Secondary: 92D25; 93B52

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Citation: Rubbioni, P. Solvability for a Class of Integro-Differential Inclusions Subject to Impulses on the Half-Line. *Mathematics* **2022**, *10*, 224. https://doi.org/10.3390/math10020224

Academic Editors: Anca Croitoru, Radko Mesiar, Anna Rita Sambucini and Bianca Satco

Received: 10 December 2021
Accepted: 10 January 2022
Published: 12 January 2022

1. Introduction

In this paper, we study the existence of solutions to the semilinear integro-differential inclusion

$$y'(t) \in A(t)y(t) + F\left(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\right),\ t \geq t_0, \tag{1}$$

with the initial condition $y(t_0) = \mathbf{v} \in E$, where E is a real Banach space. Here, $\{A(t)\}_{t\geq 0}$ is a family of linear operators acting on the Banach space, $F : [t_0, +\infty[\times E \times E \multimap E$ is a given set-valued map, and k is a nonnegative real function.

Currently, many researchers around the world are investigating the semilinear integro-differential equations or inclusions, as witnessed, for example, by the recent articles [1–8]. One of the main reasons for this research is that these equations are well suited to serve as a model for real phenomena such as heat transfer or the spread of epidemics or population dynamics, in which it is significant to take into account the spatial diffusion of the phenomenon or the past of the phenomenon itself (e.g., [9,10]).

The need to introduce delays in models describing real phenomena has appeared clear since the beginning of the last century, due to the fact that some of the processes involved in the dynamics may depend on the past status of the population. Think, for example, of the study of a phenomenon in which only individuals of childbearing age are to be considered. Clearly, in this case, the time between birth and the moment when the individual is involved in the reproductive process is not irrelevant, leading to a non-negligible maturation delay influencing the evolution of the population over time. Among all the pioneers' works on delay equations, we wish to recall that of Volterra, from which a whole class of differential equations will take his name (see [11] for a historical review). In the decades following the 1950s, the delay equations have been studied either in the case of concentrated delay, distributed delay, or functional delay. We refer to the papers [12,13] for the first studies concerning semilinear differential equations with delay, to the monographs [14–16] for a more in depth treatment on delay differential equations, and to [17] for the use of delays in population models.

Along this line, we will apply the results of the investigation on the semilinear integro-differential inclusion (1) to the study of a population dynamics model described by the parametric integro-differential equation involving a distributed delay

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds\right) + \omega(t,x)\,, \tag{2}$$

subject to feedback controls given by

$$\omega(t,\cdot) \in W(u(t,\cdot))\,.$$

Here, $u, b, \omega : [t_0, +\infty[\times[0,1] \to \mathbb{R}$ and $g : [t_0, +\infty[\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are given functions, while $W : L^2([0,1]) \multimap L^2([0,1])$ is a multimap.

The real value $u(t,x)$ represents the population density at time t and place x, the removal coefficient $-b(t,x)$ the death rate and displacement of the population, and the nonlinearity $g : [t_0, +\infty[\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ the population development law.

The function g includes a Volterra integral, and this is what formalizes the distributed delay in the model, providing a spanning effect by means of the memory kernel

$$k(t,s) = \frac{e^{-(t-s)/T}}{T}.$$

This kernel is given by the exponential distribution of probability $\mathcal{k}(\tau) = \frac{e^{-\tau/T}}{T}$. Since $\mathcal{k}$ is decreasing, the two-variable function k assigns a greater weight to the most recent events, increasingly fading the influence of those further away in time. Note that this happens in a maximum range indicated by T. In fact, the positive number T provides the width of the action of the kernel: the larger T is, the more the system's memory is extended to past events affecting its present state. Thus, we can say that the value T shows the range of significance of the delay. We point out that, inasmuch as the process is set on the whole half-line, the number T can be chosen arbitrarily large. In other words, the relevance of the delay on the status of the solution trajectory can be chosen arbitrarily, thus making the model particularly versatile.

The set-valued function W provides the sets where the feedback controls can be taken. Feedback controls often appear in models from the life Sciences, especially in systems biology. For a detailed description of the topic see [18].

Further, we consider the presence of infinitely many impulses on the system. These are represented by given functions acting in correspondence to times t_m, where $\{t_m\}_{m\geq 1}$ is an increasing diverging sequence of positive numbers, and leading to jumps on the solutions' functions according to relations for all $m \geq 1$

$$\lim_{h\to 0^+} y(t_m + h) = y(t_m) + I_m(y(t_m)),$$

or

$$\lim_{h\to 0^+} u(t_m + h, x) = u(t_m, x) + \mathcal{I}_m(u(t_m, x)),$$

in case we deal with the general integro-differential inclusion (1) or the model's parametric differential Equation (2), respectively.

Problems involving instantaneous impulses have been extensively studied in the literature and are still a topic of considerable interest, as can be seen in recent articles [19–25]. For a first approach to the subject, we refer to the now classic monographs [26,27]. The reason lies in that impulse functions are needed in the modeling of a wide range of real phenomena whenever an external factor that extends for a very short period of time—to the point of being considered instantaneous—intervenes to disturb the system, causing sudden changes in the evolution of the trajectories that describe the evolutionary dynamics of the process. For example, but not limited to them, impulse functions can represent the administration of antibiotics on a bacterial population in the treatment of a disease, or abrupt changes of prices in economics, or the use of pesticides in pre-established times to keep a pest in a crop below a certain threshold (in biology, these functions are called "regulation functions").

This paper is organized as follows:

The most important notions necessary to place the topics covered in the manuscript are shortly collected in Section 2.

Then, in Section 3, the Cauchy problem driven by the semilinear integro-differential inclusion (1) is formally stated, and the existence of mild solutions is provided, both in and out of the presence of impulses. We achieve our main existence result by means of an "extension-with-memory" process, which generates an impulsive mild solution starting from the mild solutions of an ordered iterative sequence of non-impulsive Cauchy problems. As far as we know, this method was first used in [28] and in [29] without memory and with functional delay, respectively, but on a compact interval in both cases. More recently, it has been firstly used on the half-line in [25] to provide the existence of mild solutions of an impulsive Cauchy problem driven by the semilinear differential equation with functional delay

$$y'(t) = A(t)y(t) + f(t, y(t), y_t),\ t \geq t_0,$$

where $y_t(\theta) := y(t+\theta)$, $\theta \in [-\tau, 0]$, $t \in [t_0, +\infty[$. Contrary to what one might think at first glance, the analogous problem governed by the semilinear integro-differential equation

$$y'(t) = A(t)y(t) + f\left(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\right),\ t \geq t_0,$$

was still open, even on a compact interval. Indeed, the different nature between function $t \mapsto y_t$ and function $t \mapsto \int_{t_0}^{t} k(t,s)y(s)ds$ does not allow the same demonstration arguments to be used. In the present work, we provide an answer to this open problem, even in the multivalued case. The key of our procedure is given by the introduction of suitable auxiliary set-valued functions, which "imitate", satisfying its own properties (cf. next Lemma 1), the original set-valued nonlinearity F at any step of the problems' iteration. Furthermore, we would like to point out that the extension-with-memory method enables the existence of solutions with no hypothesis on the impulse functions, unlike other approaches adopted in the literature, for which those functions are supposed to be at least continuous.

In Section 4, we consider the system governed by the parametric integro-differential equation with distributed delay (2) under the action of feedback controls and impulses. We solve the feedback control problem by rewriting the model as an impulsive Cauchy problem driven by a semilinear integro-differential inclusion in the space $E = L^2([0,1])$. In this way, we can apply the general result obtained in Section 3 and lead to the existence of a pair, trajectory-control, providing the controllability of the population dynamics process.

2. Essential Preliminary Notions

We recall some basic notations and definitions.

Let X and Y be two topological spaces. A set-valued function (or "multivalued map", or, shortly, "multimap") $\mathcal{F} : X \multimap Y$ is *upper semicontinuous* at $x_0 \in X$ if, for every open $V \subset Y$ such that $\mathcal{F}(x_0) \subset V$, there exists a neighborhood U of x_0 such that $\mathcal{F}(x) \subset V$ for every $x \in U$. A multimap $\mathcal{F}$ is *upper semicontinuous* if it is upper semicontinuous at every $x_0 \in X$.

Let E be a real Banach space endowed with the norm $\|\cdot\|$. By the symbol $C(J, E)$, we denote the space of E-valued continuous functions on a closed, bounded interval $J \subset \mathbb{R}$, while by $L^p(J,E)$ we denote the space of all functions $v : J \to E$ such that their p-power is Bochner integrable endowed with the norm $\|v\|_{L^p(J,E)} = \left[\int_J \|v(z)\|^p \, dz\right]^{\frac{1}{p}}$ (shortly, $L^p(J)$ and $\|v\|_{L^p}$, respectively, if $E = \mathbb{R}$), $p \geq 1$. Moreover, for any $a \in \mathbb{R}$, by the symbol $L^1_{loc}([a,+\infty[,E)$, we mean the space of all functions $v : [a,+\infty[\to E$ such that $v \in L^1(J,E)$ for every compact $J \subset [a,+\infty[$ (shortly, $L^1_{loc}([a,+\infty[)$ if $E = \mathbb{R}$). Then, throughout the paper for a given function $y : [a,+\infty[\to E$ and a fixed $t \in [a,+\infty[$, we will use the symbol

$$y(t^+) := \lim_{h \to 0^+} y(t+h),$$

whenever the limit exists.

A family $\{T(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators on E is said to be a *(strongly continuous) evolution system* on the half-line (see, e.g., [30]) if

(T1) $T(s,s) = I, \quad T(t,r)T(r,s) = T(t,s)$ for $t \geq s \geq 0$; and

(T2) for every $x \in E$, the map $\xi_x : (t,s) \mapsto T(t,s)x$ is continuous.

Further, a family of linear operators $\{A(t)\}_{t \geq 0}$ generates an evolution system on the half-line $\{T(t,s)\}_{t \geq s \geq 0}$ (see, e.g., [31]) if

(T3) $$\frac{\partial T(t,s)}{\partial t} = A(t)T(t,s) \quad \text{and} \quad \frac{\partial T(t,s)}{\partial s} = -T(t,s)A(s), \quad t \geq s \geq 0.$$

We conclude this section recalling that the *Hausdorff measure of noncompactness* in E is the function χ on the family of nonempty subsets of E taking nonnegative real values defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 \; : \; \Omega \text{ has a finite } \varepsilon\text{-net}\}, \text{ for all bounded } \Omega \subset E.$$

The symbol χ_{L^2} will denote the Hausdorff MNC in $E = L^2([0,1])$. For the properties of the Hausdorff measure of noncompactness, we refer to [32].

3. Existence of Impulsive Mild Solutions on the Half-Line

Let E be a real Banach space, and $\{t_m\}_{m \in \mathbb{N}}$ a set of fixed real numbers such that $0 \leq t_0 < t_1 < t_2 < \ldots$ and $\lim_{m \to \infty} t_m = +\infty$. By the symbol $\mathcal{PC}([t_0,+\infty[,E)$, we denote the set of functions

$$\mathcal{PC}([t_0,+\infty[,E) := \left\{ y : [t_0,+\infty[\to E \; : \; \begin{array}{l} y_{|\,]t_{m-1},t_m]} \text{ is continuous, for all } m \in \mathbb{N}^+; \\ \exists \lim\limits_{h \to 0^+} y(t_m+h) \equiv y(t_m^+) \in E, \text{ for all } m \in \mathbb{N}. \end{array} \right\}$$

Let $\mathbf{v} \in E$ be fixed, and consider the corresponding initial value problem driven by a semilinear integro-differential inclusion subject to impulses $I_m : E \to E$, $m \in \mathbb{N}^+$ at the given times $\{t_m\}_{m \in \mathbb{N}^+}$

$$(P) \begin{cases} y'(t) \in A(t)y(t) + F\left(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\right), \; t \geq t_0, \; t \neq t_m, \; m \in \mathbb{N}^+, \\ y(t_0) = \mathbf{v}, \\ y(t_m^+) = y(t_m) + I_m(y(t_m)), \; m \in \mathbb{N}^+. \end{cases}$$

In this Section we suppose that:

(A) $\mathcal{A} := \{A(t)\}_{t \geq 0}$ is a family of linear operators, $A(t) : D(A) \subset E \to E$, $D(A)$ dense subset of E not depending on t, generating an evolution system on the half-line $\{T(t,s)\}_{t \geq s \geq 0}$;

(F) $F : [t_0,+\infty[\times E \times E \multimap E$ is a multimap satisfying the properties:

(F0) F takes compact and convex values;

(F1) for every $v, w \in E$, the multimap $F(\cdot, v, w)$ admits a strongly measurable selection;
(F2) for a.e. $t \in [t_0, +\infty[$, the multimap $F(t, \cdot, \cdot)$ is upper semicontinuous;
(F3) there exists a nonnegative function $\alpha \in L^1_{loc}([t_0, +\infty[)$ such that, for a.e. $t \geq t_0$ and all $v, w \in E$,

$$\|F(t, v, w)\| \leq \alpha(t)(1 + \|v\| + \|w\|) \,, \tag{3}$$

where $\|F(t, v, w)\| := \sup\{\|w\| : w \in F(t, v, w)\}$; and
(F4) there exists a nonnegative function $h \in L^1_{loc}([t_0, +\infty[)$ such that

$$\chi(F(t, \Omega_1, \Omega_2)) \leq h(t)[\chi(\Omega_1) + \chi(\Omega_2)] \,, \tag{4}$$

for a.e. $t \geq t_0$ and every bounded $\Omega_1, \Omega_2 \subset E$;

(k) $k : \Delta_\infty \to \mathbb{R}^+, \Delta_\infty = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq t_0\}$, is a continuous function.

We study the existence of mild solutions to (P), according to the following definition:

Definition 1. *A function $y \in \mathcal{PC}([t_0, +\infty[, E)$ is said to be a* mild solution *to (P) if*

$$y(t) = T(t, t_0)\mathbf{v} + \sum_{t_0 < t_m < t} T(t, t_m) I_m(y(t_m)) + \int_{t_0}^t T(t, s) f(s)\, ds, \ t \geq t_0, \tag{5}$$

where $f : [t_0, +\infty[\to E$ is a L^1_{loc}-function on $[t_0, +\infty[$ such that

$$f(s) \in F\left(s, y(s), \int_{t_0}^s k(s, \tau) y(\tau) d\tau\right) \text{ for a.e. } s \geq t_0,$$

with the agreement that $\sum_{t_0 < t_m < t} T(t, t_m) I_m(y(t_m)) = 0$ if $t \in [t_0, t_1]$.

Note that every mild solution also satisfies the conditions.

$y(t_0) = \mathbf{v}$;
$y(t_m^+) = y(t_m) + I_m(y(t_m))\,,\ m \in \mathbb{N}^+$.

Before stating the main theorem of this section, we provide a preliminary result. It will be a strategic and decisive tool in applications. Indeed, thanks to the property stated by Lemma 1, we can apply the extension-with-memory technique (later shown) to integro-differential equations or inclusions having a two-variables kernel inside the Volterra integral, which is new in the literature, as far as we know. As a consequence, our existence theorem will be allowed to operate in a much wider class of models than is possible with the current results, to our knowledge.

Lemma 1. *Let E be a real Banach space, and $\{t_m\}_{m \in \mathbb{N}}$ a sequence of real numbers such that $0 \leq t_0 < t_1 < t_2 < \ldots$ and $\lim_{m \to +\infty} t_m = +\infty$.*
Assume that $F : [t_0, +\infty[\times E \times E \multimap E$ and $k : \Delta_\infty \to \mathbb{R}^+$, respectively, satisfy (F) and (k).
Then, for every $m \in \mathbb{N}^+$ and every set of functions $\{y_i \in C([t_i, t_{i+1}], E) : i = 0, \ldots, m-1\}$, the multimap $F_m : [t_m, t_{m+1}] \times E \times E \multimap E$ defined by

$$F_m(t, v, w) := F\left(t, v, w + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(t, \tau) y_i(\tau) d\tau\right),\ t \in [t_m, t_{m+1}], v, w \in E \tag{6}$$

satisfies (F) in its $[t_m, t_{m+1}]$-restricted version.

Proof. Let $m > 0$ and $\{y_i \in C([t_i, t_{i+1}], E) : i = 0, \ldots, m-1\}$ be fixed. For the sake of simplicity, we denote the properties on $[t_m, t_{m+1}]$ by (F0), ..., (F4) , as the corresponding on $[t_0, +\infty[$. Clearly, property (F0) trivially holds.

Then, let us fix $v, w \in E$ and consider the multimap on $[t_m, t_{m+1}]$ (see (6))

$$F_m(\cdot, v, w) = F\left(\cdot, v, w + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(\cdot, \tau) y_i(\tau) d\tau\right).$$

We notice that it can be seen as the Nemytskii superposition operator $N : [t_m, t_{m+1}] \to E$ of the function $F_v : [t_m, t_{m+1}] \times E \multimap E$,

$$F_v(t, \eta) := F(t, v, \eta),\ t \in [t_m, t_{m+1}], \eta \in E$$

by the function $q_w : [t_m, t_{m+1}] \to E$,

$$q_w(t) := w + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(t, \tau) y_i(\tau) d\tau,\ t \in [t_m, t_{m+1}].$$

In fact, we get

$$N(t) := F_v(t, q_w(t)) = F(t, v, q_w(t)) = F_m(t, v, w),\ t \in [t_m, t_{m+1}].$$

Obviously, by (F0) of F we have that F_v takes compact values, by (F1) that $F_v(\cdot, \eta)$ has a strongly measurable selector for every $\eta \in E$, and by (F2) that $F_v(t, \cdot)$ is upper semicontinuous for a.e. $t \in [t_m, t_{m+1}]$.

Moreover, q_w is strongly measurable; indeed, the functions $f_i : [t_m, t_{m+1}] \times [t_i, t_{i+1}] \to E$, $i = 0, \ldots, m-1$, defined by

$$f_i(t, \tau) = k(t, \tau) y_i(\tau),\ (t, \tau) \in [t_m, t_{m+1}] \times [t_i, t_{i+1}]$$

are continuous on $[t_m, t_{m+1}] \times [t_i, t_{i+1}]$ as product of continuous functions (cf. (k)). Thus, q_w is in turn continuous on $[t_m, t_{m+1}]$ and hence strongly measurable.
It is therefore possible to apply Theorem 1.3.5 of [33] and claim that N has a strongly measurable selector. Thus F_m satisfies (F1).

Now, let us fix $t \in [t_m, t_{m+1}]$ such that $F(t, \cdot, \cdot)$ is upper semicontinuous, and consider the multimap on $E \times E$ (see (6) again)

$$F_m(t, \cdot, \cdot) = F\left(t, \cdot, \cdot + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(t, \tau) y_i(\tau) d\tau\right).$$

Since the vector

$$w_0 := \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(t, \tau) y_i(\tau) d\tau$$

is a fixed element in E, the map $w \mapsto w + w_0$ is just a translation function, thus $F_m(t, \cdot, \cdot)$ is the composition of a continuous single-valued function and an upper semicontinuous multimap. Thus, it is upper semicontinuous as well. Hence, property (F2) is satisfied by F_m.

In order to prove that F_m satisfies (F3), let us fix $v, w \in E$, and $t \in [t_m, t_{m+1}]$ such that F satisfies inequality (3). Then, according to (6), we have

$$\begin{aligned} \|F_m(t, v, w)\| &\leq \alpha(t)\left(1 + \|v\| + \left\|w + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(t, \tau) y_i(\tau) d\tau\right\|\right) \\ &\leq \alpha(t)(1 + \|v\| + \|w\|) + \alpha(t) \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \|k(t, \tau) y_i(\tau)\| d\tau. \end{aligned}$$

Recalling that k is a positive continuous function (see (k)), for every $i = 0, \ldots, m-1$ there exist

$$k_{m,i} := \max_{(t,\tau) \in [t_m, t_{m+1}] \times [t_i, t_{i+1}]} k(t, \tau).$$

Hence, we get

$$\begin{aligned}
\|F_m(t,v,w)\| &\leq \alpha(t)(1+\|v\|+\|w\|) + \underbrace{\alpha(t)\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} k_{m,i}\|y_i\|_{L^1([t_i,t_{i+1}],E)}}_{:=\beta_m(t)} \\
&\leq \alpha(t)(1+\|v\|+\|w\|) + \beta_m(t)(1+\|v\|+\|w\|) \\
&= \alpha_m(t)(1+\|v\|+\|w\|),
\end{aligned}$$

being $\alpha_m := \alpha + \beta_m \in L^1_+([t_m, t_{m+1}])$.

Finally, concerning (F4), let us fix two bounded sets $\Omega_1, \Omega_2 \subset E$, and $t \in [t_m, t_{m+1}]$ such that F satisfies inequality (4). By (6) and the properties of algebraic sub-additivity and nonsingularity of the Hausdorff measure of noncompactness, we have

$$\begin{aligned}
\chi(F_m(t,\Omega_1,\Omega_2)) &\leq h(t)\left[\chi(\Omega_1) + \chi\left(\Omega_2 + \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} k(t,\tau)y_i(\tau)d\tau\right)\right] \\
&\leq h(t)\left[\chi(\Omega_1) + \chi(\Omega_2) + \sum_{i=0}^{m-1}\chi\left(\left\{\int_{t_i}^{t_{i+1}} k(t,\tau)y_i(\tau)d\tau\right\}\right)\right] \\
&= h(t)[\chi(\Omega_1) + \chi(\Omega_2)],
\end{aligned}$$

showing the property. □

To obtain the existence of mild solutions to our impulsive Cauchy problem (P), we consider an ordered iterative sequence of non-impulsive Cauchy problems, whose mild solutions generate the solutions of the impulsive problem by means of an extension-with-memory process.

Theorem 1. *Let E be a real Banach space, $\mathbf{v} \in E$, $\{t_m\}_{m\in\mathbb{N}}$ with $0 \leq t_0 < t_1 < t_2 < \ldots$ and $\lim_{m\to+\infty} t_m = +\infty$, and $I_m : E \to E$ for $m \in \mathbb{N}^+$ be given. Suppose that $\mathcal{A}$, F and k, respectively, satisfy hypotheses (A), (F), and (k). Then, problem (P) has at least one mild solution on $[t_0, +\infty[$.*

Proof. In association to problem (P), let us consider an iterative sequence of Cauchy problems related to the intervals given by the increasing sequence $\{t_m\}_{m\in\mathbb{N}}$ as follows.

If $m = 0$, we consider the problem

$$(P_0) \quad \begin{cases} y'(t) \in A(t)y(t) + F\left(t, y(t), \int_{t_0}^t k(t,s)y(s)ds\right),\ t \in [t_0, t_1], \\ y(t_0) = \mathbf{v}. \end{cases}$$

If $m > 0$, we define a multimap $F_m : [t_m, t_{m+1}] \times E \times E \multimap E$ as

$$F_m(t,v,w) := F_{m-1}\left(t, v, w + \int_{t_{m-1}}^{t_m} k(t,\tau)\bar{y}_{m-1}(\tau)d\tau\right),\ t \in [t_m, t_{m+1}], v, w \in E \tag{7}$$

(of course, here we mean $F_0 = F$) and a vector in E as

$$\mathbf{v}_m := \bar{y}_{m-1}(t_m) + I_m(\bar{y}_{m-1}(t_m)), \tag{8}$$

and consider the problem

$$(P_m) \quad \begin{cases} y'(t) \in A(t)y(t) + F_m\left(t, y(t), \int_{t_m}^t k(t,s)y(s)ds\right),\ t \in [t_m, t_{m+1}], \\ y(t_m) = \mathbf{v}_m, \end{cases}$$

where $\bar{y}_{m-1} \in C([t_{m-1}, t_m], E)$ is a mild solution of problem (P_{m-1}).

We prove that these mild solutions really exist, by extension.

First, we consider $m = 0$. It is easy to check that, when restricted to $[t_0, t_1]$, the hypotheses on the family $\mathcal{A}$ and on the maps F and k come down to the hypotheses of Theorem 5.1 in [34]; actually, that theorem acts on an interval $[0, b]$, but it still holds in $[t_0, t_1] \subset [0, t_1]$ (recall that t_0 is fixed greater or equal to 0). Thus we can claim that (P_0) has at least one mild solution $\bar{y}_0 \in C([t_0, t_1], E)$, i.e., a continuous function on the interval $[t_0, t_1]$ such that

$$\bar{y}_0(t) = T(t, t_0)\mathbf{v} + \int_{t_0}^{t} T(t, s) f_0(s)\, ds, \text{ for every } t \in [t_0, t_1], \tag{9}$$

$$f_0 \in L^1([t_0, t_1], E),\ f_0(s) \in F\left(s, \bar{y}_0(s), \int_{t_0}^{s} k(s, \tau)\bar{y}_0(\tau)d\tau\right), \text{ for a.a. } s \in [t_0, t_1]. \tag{10}$$

Let us now fix $m > 0$. It is clear that in the interval $[t_m, t_{m+1}]$ we have for $\mathcal{A}$ and k the same situation as in $[t_0, t_1]$. Concerning F_m, notice that it can be rewritten as

$$F_m(t, v, w) = F\left(t, v, w + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(t, \tau)\bar{y}_i(\tau)d\tau\right),\ t \in [t_m, t_{m+1}], v, w \in E.$$

That is, for F_m Equation (6) holds. Hence, by Lemma 1 we can claim that F_m satisfies (F). Therefore, we can use [34] [Theorem 5.1] again and achieve the existence of a mild solution $\bar{y}_m$ to (P_m), i.e., a continuous function on the interval $[t_m, t_{m+1}]$ having the following representation:

$$\bar{y}_m(t) = T(t, t_m)\mathbf{v}_m + \int_{t_m}^{t} T(t, s) f_m(s)\, ds,\ t \in [t_m, t_{m+1}], \tag{11}$$

$$f_m \in L^1([t_m, t_{m+1}], E),$$

$$f_m(s) \in F_m\left(s, \bar{y}_m(s), \int_{t_m}^{s} k(s, \tau)\bar{y}_m(\tau)d\tau\right), \text{ a.a. } s \in [t_m, t_{m+1}]. \tag{12}$$

We wish to prove that the function $\bar{y} : [t_0, +\infty[\to E$ defined by

$$\bar{y}(t) := \begin{cases} \bar{y}_0(t), & t \in [t_0, t_1] \\ \bar{y}_m(t), & t \in]t_m, t_{m+1}], m > 0, \end{cases} \tag{13}$$

is a mild solution to (P).

To this aim, we firstly put (see (10), (12))

$$\bar{f}(t) := \begin{cases} f_0(t), & t \in [t_0, t_1] \\ f_m(t), & t \in]t_m, t_{m+1}], m > 0. \end{cases} \tag{14}$$

Thus, naturally, $\bar{f} \in L^1_{loc}([t_0, +\infty[, E)$. Further, we show that it is a selector of the multimap $F\left(\cdot, \bar{y}(\cdot), \int_{t_0}^{(\cdot)} k(\cdot, \tau)\bar{y}(\tau)d\tau\right)$ almost everywhere in $[t_0, +\infty]$. Indeed, note that by (10) we have

$$\bar{f}_{|[t_0, t_1]}(s) = f_0(s) \in F\left(s, \bar{y}_0(s), \int_{t_0}^{s} k(s, \tau)\bar{y}_0(\tau)d\tau\right), \text{ for a.a. } s \in [t_0, t_1].$$

Thus, by recalling (13) we obtain

$$\bar{f}(s) \in F\left(s, \bar{y}(s), \int_{t_0}^{s} k(s, \tau)\bar{y}(\tau)d\tau\right), \text{ for a.a. } s \in [t_0, t_1].$$

Let us now consider any $m > 0$. By (12), we get

$$\bar{f}_{|\,]t_m,t_{m+1}]}(s) = f_m(s) \in F_m\left(s, \bar{y}_m(s), \int_{t_m}^{s} k(s,\tau)\bar{y}_m(\tau)d\tau\right), \text{ for a.a. } s \in]t_m, t_{m+1}].$$

By using the expression of F_m (see (7)) and the definition of $\bar{y}$ (see (13)), we can therefore write

$$\begin{aligned} \bar{f}(s) \quad &\in \quad F\left(s, \bar{y}_m(s), \int_{t_m}^{s} k(s,\tau)\bar{y}_m(\tau)d\tau + \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} k(s,\tau)\bar{y}_i(\tau)d\tau\right) \\ &= F\left(s, \bar{y}(s), \int_{t_0}^{s} k(s,\tau)\bar{y}(\tau)d\tau\right), \text{ for a.a. } s \in]t_m, t_{m+1}]. \end{aligned}$$

Now, we can prove that $\bar{y}$ satisfies condition (5) (cf. Definition 1).

If $t \in [t_0, t_1]$, then by (13), (9), (14) we have

$$\bar{y}(t) \quad = \quad T(t,t_0)\mathbf{v} + \int_{t_0}^{t} T(t,s)f_0(s)\,ds = T(t,t_0)\mathbf{v} + \int_{t_0}^{t} T(t,s)\bar{f}(s)\,ds.$$

If $t \in]t_1, t_2]$, then by (13), (11), (8), we get

$$\begin{aligned} \bar{y}(t) \quad &= \quad T(t,t_1)\mathbf{v}_1 + \int_{t_1}^{t} T(t,s)f_1(s)\,ds \\ &= \quad T(t,t_1)[\bar{y}_0(t_1) + I_1(\bar{y}_0(t_1))] + \int_{t_1}^{t} T(t,s)f_1(s)\,ds. \end{aligned}$$

By (9), (T1), (14), (13) we obtain

$$\begin{aligned} \bar{y}(t) \quad &= \quad T(t,t_1)\left[T(t_1,t_0)\mathbf{v} + \int_{t_0}^{t_1} T(t_1,s)f_0(s)\,ds\right] + T(t,t_1)I_1(\bar{y}_0(t_1)) + \int_{t_1}^{t} T(t,s)f_1(s)\,ds \\ &= \quad T(t,t_0)\mathbf{v} + \int_{t_0}^{t_1} T(t,s)f_0(s)\,ds + T(t,t_1)I_1(\bar{y}_0(t_1)) + \int_{t_1}^{t} T(t,s)f_1(s)\,ds \\ &= \quad T(t,t_0)\mathbf{v} + T(t,t_1)I_1(\bar{y}(t_1)) + \int_{t_0}^{t} T(t,s)\bar{f}(s)\,ds. \end{aligned}$$

Thus, by the same arguments, we can say that if $t \in]t_m, t_{m+1}]$ for any $m > 0$ it holds that

$$\begin{aligned} \bar{y}(t) \quad &= \quad T(t,t_m)\mathbf{v}_m + \int_{t_m}^{t} T(t,s)f_m(s)\,ds \\ &= \quad T(t,t_0)\mathbf{v} + \sum_{i=1}^{m} T(t,t_i)I_i(\bar{y}(t_i)) + \int_{t_0}^{t} T(t,s)\bar{f}(s)\,ds \end{aligned}$$

and this concludes the proof. □

From careful reading of the proof, it appears that no hypotheses are needed on the impulse functions. Hence, they can be chosen arbitrarily. Thus, if we pick $I_m(v) = 0$ for every $v \in E$ and $m \in \mathbb{N}^+$, we immediately have the following existence result.

Corollary 1. *Let E be a real Banach space and $\mathbf{v} \in E$ be given. Suppose that $\mathcal{A}$, F, and k, respectively satisfy hypotheses (A), (F), and (k). Then, there exists at least one mild solution on $[t_0, +\infty]$ to the Cauchy problem*

$$\begin{cases} y'(t) \in A(t)y(t) + F\left(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\right), \ t \geq t_0, \\ y(t_0) = \mathbf{v}. \end{cases}$$

Of course, in this case, a mild solution is a function $y \in C([t_0, +\infty[, E)$ such that

$$y(t) = T(t, t_0)\mathbf{v} + \int_{t_0}^{t} T(t,s) f(s)\, ds, \ t \geq t_0,$$

with $f : [t_0, +\infty[\to E$ a L^1_{loc}-function on $[t_0, +\infty[$ and $f(s) \in F\Big(s, y(s), \int_{t_0}^{s} k(s,\tau)y(\tau)d\tau\Big)$ for a.e. $s \geq t_0$.

4. Example of Application: The Controllability of a Population Dynamics Process with Distributed Delay and Impulses

In this section, we apply Theorem 1 to the study of the following process with feedback controls described by a parametric integro-differential equation with distributed delay and subject to impulses.

Fixed $0 \leq t_0 < t_1 < t_2 < \dots$, with $\lim_{m\to+\infty} t_m = +\infty$, and $u_0 \in L^2([0,1])$, we consider the system

$$(FCP) \quad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \displaystyle\int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds\right) + \omega(t,x)\,, \\ \qquad\qquad t \geq t_0\,,\ t \neq t_m\,,\ m \in \mathbb{N}^+\,,\ a.e.\, x \in [0,1], \\ \omega(t,\cdot) \in W(u(t,\cdot))\,, t \geq t_0, \\ u(t_0,x) = u_0(x)\,,\ a.e.\, x \in [0,1], \\ u(t_m^+,x) = u(t_m,x) + \mathcal{I}_m(u(t_m,x))\,,\ \ m \in \mathbb{N}^+\,,\ a.e.\, x \in [0,1]. \end{cases}$$

In this model, the real value $u(t,x)$ represents the density of a population depending on time t and place x, considering the spatial range normalized to interval $[0,1]$, while the nonlinearity $g : [t_0, +\infty[\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ represents the population development law.

The dependence from the past state of the system is provided by the Volterra integral $\int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds$. Indeed, the positive number T gives the width of the action of the kernel, here given by the exponential distribution of probability $\mathcal{k}(\tau) = \frac{e^{-\tau/T}}{T}$. The larger T is, the more the system's memory is extended to past events affecting its present state. Hence, T establishes the width of the range of significance of the delay. Notice that, being the above problem set on the whole half-line, the value of T can be chosen arbitrarily large. This means that the relevance of the delay on the status of the solution trajectory can be chosen arbitrarily, leading to a particularly versatile model.

Moreover, the multimap $W : L^2([0,1]) \multimap L^2([0,1])$ gives the sets of controls, and the impulse functions $\mathcal{I}_m : \mathbb{R} \to \mathbb{R}$ and $m \in \mathbb{N}^+$ represent instantaneous external forces acting on the system.

Finally, the death rate and displacement of the population is given by the removal coefficient $-b(t,x)$.

We assume that the function $b : [0, +\infty[\times[0,1] \to \mathbb{R}^+$ satisfies the following conditions:

(b1) b is measurable;

(b2) there exists $s \in L^1_{loc}([0, +\infty[)$ such that

$$0 < b(t,x) \leq s(t)\,,$$

for every $t \geq 0$, a.e. $x \in [0,1]$; and

(b3) for every $x \in [0,1]$, the function $b(\cdot, x) : [0, +\infty[\to \mathbb{R}^+$ is continuous.

Consider the family of linear functions $A(t) : L^2([0,1]) \to L^2([0,1])$, $t \geq 0$, defined by

$$A(t)v(x) = -b(t,x)v(x),\ v \in L^2([0,1]), x \in [0,1]. \tag{15}$$

By [35] [Section 3.1] and [4] [Proposition 3.2 and Remark 3.1], we know that properties (b1)–(b3) imply that the family $\{A(t)\}_{t\geq 0}$ defined by (15) generates the noncompact evolution system

$$[T(t,s)v](x) = e^{\int_s^t -b(\sigma,x)d\sigma}v(x),\; t \geq s \geq 0,\; v \in L^2([0,1]),\; x \in [0,1]. \tag{16}$$

Hence, the next proposition holds.

Proposition 1. *Under assumptions (b.1)-(b.3), the family $\{A(t)\}_{t\geq 0}$ defined by (15) satisfies property (A).*

On the other functions appearing in the model, we assume the next hypotheses.
The function $g : [t_0, +\infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that

(g0) $g(t, v(\cdot), w(\cdot)) \in L^2([0,1])$, for every $t \geq t_0$, $v, w \in L^2([0,1])$;

(g1) for every $p, q \in \mathbb{R}$, the function $g(\cdot, p, q)$ is (strongly) measurable;

(g2) for a.e. $t \geq t_0$, the function $g(t, \cdot, \cdot)$ is continuous;

(g3) there exists a nonnegative function $\varphi \in L^1_{loc}([t_0, +\infty[)$ such that

$$|g(t, p, q)| \leq \varphi(t),$$

for a.e. $t \geq t_0$ and every $p, q \in \mathbb{R}$; and

(g4) there exists a nonnegative function $m \in L^1_{loc}([t_0, +\infty[)$ such that

$$\chi_{L^2}(g(t, \Omega_1(\cdot), \Omega_2(\cdot))) \leq m(t)[\chi_{L^2}(\Omega_1) + \chi_{L^2}(\Omega_2)]\,,$$

for a.e. $t \geq t_0$ and every bounded $\Omega_1, \Omega_2 \subset L^2([0,1])$.

The multimap $W : L^2([0,1]) \multimap L^2([0,1])$ satisfies the properties

(W0) W takes compact convex values;

(W1) W is upper semicontinuous;

(W2) there exists $R > 0$ such that, for every $v \in L^2([0,1])$,

$$\|W(v)\|_{L^2} \leq R(1 + \|v\|_{L^2}),$$

where $\|W(v)\|_{L^2} := \sup\{\|\eta\|_{L^2} : \eta \in W(v)\}$; and

(W3) there exists $Q > 0$ such that

$$\chi_{L^2}(W(\Omega)) \leq Q\chi_{L^2}(\Omega),$$

for every bounded $\Omega \subset L^2([0,1])$.

Now, we put:

$v, w : [t_0, +\infty[\to L^2([0,1])$ as

$$v(t)(x) = u(t,x) \text{ and } w(t)(x) = \omega(t,x),\; t \geq t_0,\; x \in [0,1]; \tag{17}$$

$f : [t_0, +\infty[\times L^2([0,1]) \times L^2([0,1]) \to L^2([0,1])$ as

$$f(t,v,w)(x) = g(t, v(x), w(x)), t \geq t_0,\; v, w \in L^2([0,1]),\; x \in [0,1]; \tag{18}$$

$I_m : L^2([0,1]) \to L^2([0,1])$, $m \in \mathbb{N}^+$, as

$$I_m(v)(x) = \mathcal{I}_m(v(x)), v \in L^2([0,1]),\; x \in [0,1]. \tag{19}$$

It is easy to check that by these positions we can write the feedback control problem (FCP) as the impulsive Cauchy problem with feedback controls driven by a semilinear integro-differential equation in $L^2([0,1])$.

$$\begin{cases} v'(t) = A(t)v(t) + f\Big(t, v(t), \int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T} v(s)\,ds\Big) + w(t)\,,\ t \geq t_0\,,\ t \neq t_m\,,\ m \in \mathbb{N}^+, \\ w(t) \in W(v(t))\,, t \geq t_0, \\ v(t_0) = v_0, \\ v(t_m^+) = v(t_m) + I_m(v(t_m))\,,\ \ m \in \mathbb{N}^+. \end{cases}$$

Finally, let us put:

$k : \Delta_\infty \to \mathbb{R}^+$ as

$$k(t,s) = \frac{e^{-(t-s)/T}}{T},\ t \geq s \geq t_0; \tag{20}$$

$F : [t_0, +\infty[\times L^2([0,1]) \times L^2([0,1]) \multimap L^2([0,1])$ as

$$F(t,v,w) = f(t,v,w) + W(v)\,,\ t \geq t_0,\ v,w \in L^2([0,1]). \tag{21}$$

Then, (FCP) can be further rewritten as

$$\begin{cases} v'(t) \in A(t)v(t) + F\Big(t, v(t), \int_{t_0}^{t} k(t,s)\, v(s)\,ds\Big)\,,\ t \geq t_0\,,\ t \neq t_m\,,\ m \in \mathbb{N}^+, \\ v(t_0) = v_0, \\ v(t_m^+) = v(t_m) + I_m(v(t_m))\,,\ \ m \in \mathbb{N}^+, \end{cases} \tag{22}$$

which is nothing more than a problem of the type (P) in the space $E = L^2([0,1])$.

In order to give the controllability of problem (FCP) we need the next result.

Proposition 2. *Under assumptions (g0)–(g4) and (W0)–(W3), the multimap F defined in* (21) *satisfies (F).*

Proof. First of all, we observe that by (g0), and recalling (18), one gets $f(t,v,w) \in L^2([0,1])$ for every $t \geq t_0$, $v,w \in L^2([0,1])$, so that F is well-defined. Moreover, hypothesis (W0) ensures that F satisfies (F0).

With regard to property (F1), it immediately follows from (g1). Indeed, fixed $v,w \in L^2([0,1])$, the function $t \mapsto f(t,v,w) = g(t,v(\cdot),w(\cdot))$ is (strongly) measurable. Hence, its translation given by $t \mapsto f(t,v,w) + \omega$, where ω is an arbitrary element of $W(v)$, is again measurable and represents a measurable selector of $F(\cdot,v,w)$.

Now, fixed $t \geq t_0$ for which (g2) and (g3) hold, the map $(v,w) \mapsto f(t,v,w)$ is continuous. In fact, for arbitrarily fixed $(v_0,w_0) \in L^2([0,1]) \times L^2([0,1])$, consider a sequence $(v_n,w_n) \to (v_0,w_0)$ in $L^2([0,1]) \times L^2([0,1])$. Then, by (g2) we can write

$$|g(t,v_n(x),w_n(x)) - g(t,v_0(x),w_0(x))|^2 \to 0,\ \text{for a.e. } x \in [0,1].$$

By (g3), we obtain the following estimate:

$$\begin{aligned} |g(t,v_n(x),w_n(x)) - g(t,v_0(x),w_0(x))|^2 &\leq \big[|g(t,v_n(x),w_n(x))| + |g(t,v_0(x),w_0(x))|\big]^2 \\ &\leq 4\varphi^2(t),\ \text{for a.e. } x \in [0,1]. \end{aligned}$$

We can therefore apply the Lebesgue dominated convergence theorem obtaining

$$\int_0^1 |g(t, v_n(x), w_n(x)) - g(t, v_0(x), w_0(x))|^2 dx \to 0,$$

and hence

$$\|f(t, v_n, w_n) - f(t, v_0, w_0)\|_{L^2} = \sqrt{\int_0^1 |g(t, v_n(x), w_n(x)) - g(t, v_0(x), w_0(x))|^2 dx} \to 0.$$

The arbitrariness of (v_0, w_0) leads to the continuity of $f(t, \cdot, \cdot)$. As a consequence, the multimap $F(t, \cdot, \cdot)$ is the sum of a continuous single-valued function and an upper semicontinuous multimap (see (21) and (W1)), so it is upper semicontinuous as well, i.e., F satisfies (F2).

Let us prove that F satisfies (F3). To this aim, we fix $t \geq t_0$ for which (g3) holds and any $v, w \in L^2([0,1])$. We clearly have

$$\|f(t, v, w)\|_{L^2}^2 = \int_0^1 [g(t, v(x), w(x))]^2 dx \leq \varphi^2(t),$$

so that

$$\|f(t, v, w)\|_{L^2} \leq \varphi(t)(1 + \|v\|_{L^2} + \|w\|_{L^2}).$$

Therefore, by using also (W2), we get

$$\begin{aligned} \|F(t, v, w)\|_{L^2} &\leq \|f(t, v, w)\|_{L^2} + \|W(v)\|_{L^2} \\ &\leq \alpha(t)(1 + \|v\|_{L^2} + \|w\|_{L^2}), \end{aligned}$$

where $\alpha(t) := \varphi(t) + R$. The function α belongs to $L^1_{loc}([t_0, +\infty[)$ since φ does.

Finally, let us fix $t \geq t_0$ for which (g4) holds and any bounded $\Omega_1, \Omega_2 \subset L^2([0,1])$. Then, by (21), (18), and (W3), we have

$$\begin{aligned} \chi_{L^2}(F(t, \Omega_1, \Omega_2)) &\leq \chi_{L^2}(g(t, \Omega_1(\cdot), \Omega_2(\cdot))) + \chi_{L^2}(W(\Omega_1)) \\ &\leq m(t)[\chi_{L^2}(\Omega_1) + \chi_{L^2}(\Omega_2)] + Q\chi_{L^2}(\Omega_1) \\ &\leq h(t)[\chi_{L^2}(\Omega_1) + \chi_{L^2}(\Omega_2)], \end{aligned}$$

where $h(t) := m(t) + Q$. As above, $m \in L^1_{loc}([t_0, +\infty[)$ implies $h \in L^1_{loc}([t_0, +\infty[)$, so that F satisfies (F4). □

We can now state the main result of this section. In this regard, we should bear in mind that problem (FCP) is *controllable* if there exists at least one *admissible pair for* (FCP), that is, a couple (u, ω) of functions $u, \omega : [t_0, +\infty[\times [0,1] \to \mathbb{R}$ such that: $u(t, \cdot) \in L^2([0,1])$ for every $t \geq t_0$; $u(\cdot, x) \in \mathcal{PC}([t_0, +\infty[, \mathbb{R})$, for all $x \in [0,1]$; u satisfies the identity

$$\begin{aligned} u(t,x) \quad &= \quad e^{\int_{t_0}^t -b(\sigma,x)d\sigma} u_0(x) + \sum_{t_0 < t_m < t} e^{\int_{t_m}^t -b(\sigma,x)d\sigma} \mathcal{I}_m(u(t_m, x)) + \\ &+ \int_{t_0}^t e^{\int_s^t -b(\sigma,x)d\sigma} \left[g\left(s, u(s,x), \int_{t_0}^s \frac{e^{-(s-\tau)/T}}{T} u(\tau, x)\, d\tau \right) + \omega(s,x) \right] ds, \end{aligned}$$

for every $t \in [t_0, +\infty[$, $x \in [0,1]$, where $\omega(s, \cdot) \in W(u(s, \cdot))$, a.e. $s \in [t_0, +\infty[$.

Theorem 2. *Under assumptions (b1)–(b3), (g0)–(g4), and (W0)–(W3), the problem* (FCP) *is controllable.*

Proof. The hypotheses of the theorem ensure that we can use Proposition 1 and then deduce that the family of linear operators $\{A(t)\}_{t \geq 0}$ defined in (15) satisfies property (A).

On the other hand, we can also use Proposition 2 and thus infer that the multimap F defined in (21) satisfies property (F). Therefore, we can apply Theorem 1 to problem (22) and obtain the existence of a function $v \in \mathcal{PC}\big([t_0, +\infty[, L^2([0,1])\big)$ such that

$$v(t) = T(t,t_0)v_0 + \sum_{t_0<t_m<t} T(t,t_m)I_m(v(t_m)) + \int_{t_0}^{t} T(t,s)f(s)\,ds,\ t \geq t_0, \tag{23}$$

where $f : [t_0, +\infty[\to L^2([0,1])$ is a locally summable selector of $F\Big(\cdot, v(\cdot), \int_{t_0}^{(\cdot)} k(\cdot,\tau)v(\tau)d\tau\Big)$ on $[t_0, +\infty[$.

Recalling now conditions (16)–(21), we can rewrite (23) as

$$\begin{aligned} u(t,x) &= e^{\int_{t_0}^{t} -b(\sigma,x)d\sigma} u_0(x) + \sum_{t_0<t_m<t} e^{\int_{t_m}^{t} -b(\sigma,x)d\sigma} \mathcal{I}_m(u(t_m,x)) \\ &\quad + \int_{t_0}^{t} e^{\int_{s}^{t} -b(\sigma,x)d\sigma} f(s)(x)\,ds,\ t \geq t_0, x \in [0,1] \end{aligned}$$

where $f(s) = g\Big(s, u(s,x), \int_{t_0}^{s} \frac{e^{-(s-\tau)/T}}{T} u(\tau,x)d\tau\Big) + \omega(s,x)$ and $\omega(s,\cdot) \in W(u(s,\cdot))$, for $x \in [0,1]$ and a.e. $s \geq t_0$, concluding the proof. $\square$

We conclude the section providing an example of nonlinearity g which satisfies properties (g0)–(g4).

Example 1. *Let us consider the function $g : [t_0, +\infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by*

$$g(t,p,q) = \frac{t}{1+|q|},\ t \geq 0,\ p,q \in \mathbb{R}. \tag{24}$$

It is easy to check that (g0)-(g3) are satisfied by g. As for (g4), we recall that in the space $L^2([0,1])$ the Hausdorff measure of noncompactness is equivalent to the measure of noncompactness

$$\chi^*_{L^2}(\Omega) = \lim_{h\to 0} \sup_{\theta\in\Omega} \sqrt{\int_0^1 [\theta(x+h) - \theta(x)]^2 dx} \tag{25}$$

for every bounded $\Omega \subset L^2([0,1])$, according to the relation

$$\chi_{L^2}(\Omega) \leq \chi^*_{L^2}(\Omega) \leq 2\chi_{L^2}(\Omega) \tag{26}$$

(see, e.g., [36]).

Then, let us fix $t \geq t_0$ and Ω_1, Ω_2 bounded subsets of $L^2([0,1])$. We have that

$$\begin{aligned} &\sup_{\theta\in g(t,\Omega_1,\Omega_2)} \sqrt{\int_0^1 [\theta(x+h) - \theta(x)]^2 dx} = \sup_{w\in\Omega_2} \sqrt{\int_0^1 \left[\frac{t}{1+|w(x+h)|} - \frac{t}{1+|w(x)|}\right]^2 dx} \\ &= t \sup_{w\in\Omega_2} \sqrt{\int_0^1 \left[\frac{|w(x)| - |w(x+h)|}{(1+|w(x+h)|)(1+|w(x)|)}\right]^2 dx} \leq t \sup_{w\in\Omega_2} \sqrt{\int_0^1 \big[|w(x)| - |w(x+h)|\big]^2 dx} \\ &\leq t \sup_{w\in\Omega_2} \sqrt{\int_0^1 2\big[w(x) - w(x+h)\big]^2 dx} = \sqrt{2}\, t \sup_{w\in\Omega_2} \sqrt{\int_0^1 \big[w(x) - w(x+h)\big]^2 dx}. \end{aligned}$$

Thus, bearing in mind (25) and (26), we deduce the next estimate

$$\begin{aligned}
\chi_{L^2}(g(t,\Omega_1,\Omega_2)) &\leq \chi^*_{L^2}(g(t,\Omega_1,\Omega_2)) = \lim_{h\to 0} \sup_{\theta\in g(t,\Omega_1,\Omega_2)} \sqrt{\int_0^1 [\theta(x+h)-\theta(x)]^2 dx} \\
&\leq \sqrt{2}\,t \lim_{h\to 0} \sup_{w\in\Omega_2} \sqrt{\int_0^1 \left[w(x)-w(x+h)\right]^2 dx} \\
&= \sqrt{2}\,t\,\chi^*_{L^2}(\Omega_2) \leq 2\sqrt{2}\,t\,\chi_{L^2}(\Omega_2) \\
&\leq 2\sqrt{2}\,t\,[\chi_{L^2}(\Omega_1)+\chi_{L^2}(\Omega_2)].
\end{aligned}$$

Hence, (g4) is fulfilled by taking $m(t) = 2\sqrt{2}\,t$, $t \geq 0$.

Lastly, we observe that Example 1 can be easily generalized to a function

$$g(t,p,q) = \frac{\psi(t)}{1+|q|},\ t\geq 0,\ p,q\in\mathbb{R}$$

where ψ is any L^1_{loc}-function.

5. Conclusions

This study reveals the existence of mild solutions on the half-line to the impulsive problem (P) under upper-Carathèodory assumptions on the multivalued nonlinearity (cf. properties (F1), (F2)). The extension-with-memory technique here adopted allows impulse functions to be used with no hypotheses, unlike other methods used in the literature, for which those functions are supposed to be at least continuous. Moreover, by means of positions (15)–(21), it is possible to deduce the controllability of the system (FCP) (see Section 4), which is a model for a population dynamics process and that can serve for a wide variety of real phenomena involving a distributed delay.

The methods and some of the results obtained in this paper are expected to be used to analyze problem (P) when the nonlinearity presents a lower semicontinuity type property, and to study the topological properties of the solution set on the half-line under upper or lower semicontinuity type assumptions, alongside all the consequences that results in abstract spaces reflect on models of spread of diseases as well as heat transfer or population dynamics.

Funding: This research has been accomplished within the national group GNAMPA of INdAM and the UMI Group T.A.A. "Approximation Theory and Applications". It has been supported by the INdAM-GNAMPA Project 2020 "Metodi topologici per problemi al contorno associati a certe classi di PDE" and the FRB 2019 "Integrazione, approssimazione, analisi nonlineare e loro applicazioni" by the University of Perugia.

Conflicts of Interest: The author declares no conflict of interest.

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