

# Article Local H∞ Control for Continuous-Time T-S Fuzzy Systems via Generalized Non-Quadratic Lyapunov Functions

Guolin Hu, Jian Zhang and Zhiguo Yan \*

School of Information and Automation, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China

\* Correspondence: yanzg@qlu.edu.cn

**Abstract:** This paper further develops a relaxed method to reduce conservatism in  $H\infty$  feedback control for continuous-time T-S fuzzy systems via a generalized non-quadratic Lyapunov function. Different from the results of some exisiting works, the generalized  $H\infty$  state feedback controller is designed. The relaxed stabilization conditions are obtained by applying Finsler's lemma with the homogenous polynomial multipliers, and the  $H\infty$  performance is acquired by solving an optimization problem. In addition, the proposed method could be expanded to handle other control problems for fuzzy systems. Two examples are given to show the validity of the proposed results.

**Keywords:** generalized non-quadratic Lyapunov function; T-S fuzzy system; feedback controller;  $H\infty$  performance

MSC: 93B36; 93B52



Citation: Hu, G.; Zhang, J.; Yan, Z. Local *H*∞ Control for Continuous-Time T-S Fuzzy Systems via Generalized Non-Quadratic Lyapunov Functions. *Mathematics* 2022, *10*, 3438. https://doi.org/ 10.3390/math10193438

Academic Editor: Vasyl Martsenyuk

Received: 30 July 2022 Accepted: 18 September 2022 Published: 21 September 2022

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/).

# 1. Introduction

Owing to its better approximation properties, the Takagi–Sugeno (T-S) system [1] has attracted much attention from different communities. The system comprises a set of linear models and normalized membership functions (MFs).

The analysis and synthesis of fuzzy systems have been widely studied, such as stability analysis [2–5], observer design [6,7], filter [8–10], etc. Due to some limitations of the proposed methods in the existing results, researchers are seeking new methods to obtain better results, such as larger domain of attraction or stability region, and better  $H\infty$  or  $H_2$  performance. To reduce conservatism, many methods usually focus on the form of Lyapunov functions (LFs), the structure of slack variables and analysis of MFs and its derivatives.

Considering the drawbacks of common quadratic LFs, the complex LFs such as the fuzzy Lyapunov functions (FLFs) [11], the non-quadratic Lyapunov functions (NQLFs) [12], the line-integral FLFs [13], and homogeneous polynomial Lyapunov functions (HPLFs) [14], the homogeneous polynomial non-quadratic Lyapunov functions (HPNQLFs) [15] were proposed successively to further reduce the conservatism of stability conditions. For instance, Ref. [12] proposed new relaxed linear matrix inequality (LMI) conditions based on NQLFs. Ref. [14] first provided local asymptotic stability conditions and obtained different estimations of the attraction domain by using HPLFs. Ref. [15] gave the asymptotically necessary and sufficient stability conditions for discrete-time fuzzy systems via non-parallel distributed compensation law (NPDCL) with HPNQLFs. In addition, Ref. [12] designed an NPDCL to outperform previous results. By using the multi-indexed matrix approach, a homogeneous polynomial nonparallel distribution compensator (HPNQDC) was designed in [16], but the Lyapunov matrix used is linearly dependent on MFs. Ref. [17] further generalized previous HPNQDC and enlarged the stabilization region. Based on congruence transformation and Polya's theorem, inner and outer slack variables were introduced in [18–21] to obtain a less



conservative conclusion. Therefore, the design of a generalized control law for the local  $H\infty$  performance of continuous-time T-S fuzzy system is worthy of investigation.

On the other hand, due to the general dependence of LFs on MFs, the time derivative of MFs (TDMFs) must be discussed in the proof process. Usually, there are several ways to deal with the TDMFs: (1) Assuming the upper or lower bounds of TDMF; (2) Finding LMI conditions to guarantee the upper bounds of TDMF hold; (3) Designing the line-integral FLFs; (4) Using a switching idea for the TDMFs. By exploring the upper bounds and properties of TDMFs, Ref. [22] proposed relaxed stability conditions. Considering the lower bounds of TDMF, Ref. [3] provided sufficient conditions to minimize the peak-topeak level performance of the T-S fuzzy model. However, they overlooked the fact that the TDMFs involved  $\dot{x}$  including the control law to be solved, which might be difficult to support in practice. Refs. [23–25] proposed the LMI conditions that guaranteed that the boundary of TDMF hold, but ignored disturbance. The line-integral FLFs designed in [13] could abstain from TDMFs, but the Lyapnov matrix structure had to be limited, which might bring about conservatism. Ref. [26] proposed a switching idea by discussing the property of TDMFs, while designing robust methods to commendably incorporate the boundary information of MFs is complex. Therefore, effective methods to manage the TDMFs remain to be explored.

From the above discussions, this paper considers the design of a generalized control law for local  $H\infty$  control of continuous-time T-S fuzzy systems to further reduce the conservatism of existing results. The contributions of this paper are as follows:

- (1) The generalized NQLFs and NPCL depending on multi-index MFs are designed, including that found in [27] and double-fuzzy-summation in [20] as a special case, and more variables are introduced to reduce the conservatism.
- (2) The new LMIs conditions, which reduce the adjusted parameters to be calculated [7] and avoid redundant restrictions such as LFs matrices, slack variables in [23,28], are obtained to bound the time derivatives of MFs with disturbance.
- (3) The extended stabilization conditions for *H*∞ performance are obtained by polynomial technology. As *q* increases, conservatism of obtained conditions will reduce, and the proposed method can be generalized to handle other cases, such as output feedback controller design [5], finite-time annular domain stability [29], mean-square strong stability [30].

## 2. Problem Statement and Preliminaries

## 2.1. The T-S Fuzzy System

The T-S fuzzy system (1) in  $\mathbb{C}_x$  ( $\mathbb{C}_x = \{x : |x_i| \le \theta_{xi}, i = 1, 2, \dots, n\}$ ) can be obtained by applying the local approximation method.

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(x(t))(A_ix(t) + B_{1i}\omega(t) + B_{2i}u(t))$$

$$= A_hx(t) + B_{1h}\omega(t) + B_{2h}u(t),$$

$$z(t) = \sum_{i=1}^{r} h_i(x(t))(C_{1i}x(t) + D_{1i}\omega(t) + D_{2i}u(t))$$

$$= C_{1h}x(t) + D_{1h}\omega(t) + D_{2h}u(t),$$

$$y(t) = \sum_{i=1}^{r} h_i(x(t))C_{2i}x(t) = C_{2h}x(t).$$
(1)

where the state  $x(t) \in \mathbb{R}^{n \times 1}$ , the measured output  $y(t) \in \mathbb{R}^{n_y \times 1}$ , the external disturbance  $\omega(t) \in \mathbb{R}^{n_w \times 1}$ , the control input  $u(t) \in \mathbb{R}^{n_u \times 1}$  and the controlled output  $z(t) \in \mathbb{R}^{n_z \times 1}$ .  $A_i \in \mathbb{R}^{n \times n}$ ,  $B_{1i} \in \mathbb{R}^{n \times n_\omega}$ ,  $B_{2i} \in \mathbb{R}^{n \times n_u}$ ,  $C_{1i} \in \mathbb{R}^{n_z \times n}$ ,  $C_{2i} \in \mathbb{R}^{n_y \times n}$ ,  $D_{1i} \in \mathbb{R}^{n_z \times n_\omega}$ ,  $D_{2i} \in \mathbb{R}^{n_z \times n_u}$  and  $D_{3i} \in \mathbb{R}^{n_y \times n_\omega}$ .  $h_i(x)$  is the normalised MF satisfying  $\sum_{i=1}^r h_i(x(t)) = 1$ .

The objective of this paper is to design a controller u(t) such that the system (1) with  $\omega = 0$  is locally asymptotically stable, and guarantees  $\int_0^t z(t)^T z(t) dt \le \gamma^2 \int_0^t \omega(t)^T \omega(t) dt$  (t > 0) under zero initial conditions.

#### 2.2. Notations and Properties

N denotes the natural numbers set. n! denotes factorial, i.e.,  $n! = n(n-1)\cdots(2)(1)$ for  $n \in \mathbb{N}$  with 0! = 1. For any  $\Phi \in \mathbb{R}^{n \times n}$ ,  $He(\Phi) = \Phi + \Phi^T$ . x(t), h(x(t)) are replaced, respectively, by x, h for convenience.  $\Omega_h = \sum_{i=1}^r h_i \Omega_i$ ,  $\Omega_{hh} = \sum_{i=1}^r \sum_{j=1}^r h_i h_j \Omega_{ij}$ .

$$diag\{a_1,\ldots,a_n\} = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix}.$$

Based on the multi-index notations proposed in [19], that is,  $\mathbb{I}_q = {\mathbf{i}_q = (i_1, i_2, \dots, i_q) \in \mathbb{N}^q \mid 1 \le i_k \le r}, \mathbb{I}_q^+ = {\mathbf{i}_q \in \mathbb{I}_q \mid i_1 \le i_2 \le \dots \le i_q}.$ 

The notation of  $\mathbf{i}_q / i_k$  ( $\mathbf{i}_q \in \mathbb{I}_q^+, 1 \le k \le r$ ) in [31] is recommended as follows:

$$\mathbf{i}_q/i_k = \begin{cases} i_1 \dots i_{k-1} i_{k+1} \dots i_p, & \text{if } i_k \in \mathbf{i}_q; \\ 0, & \text{otherwise.} \end{cases}$$
(2)

For instance, if  $i_1i_2i_2 \in \mathbb{I}_3^+$ ,  $k \in \{1, 2, 3, 4\}$ ,  $i_1, i_2 \neq i_3$ ,  $i_1, i_2 \neq i_4$ , then  $i_1i_2i_2/i_1 = i_2i_2$ ,  $i_1i_2i_2/i_2 = i_1i_2$ ,  $i_1i_2i_2/i_3 = 0$ ,  $i_1i_2i_2/i_4 = 0$ .

The form of MFs for a product is expressed:

$$h_{\mathbf{i}_{q}} = \prod_{l=1}^{q} h_{i_{l}} = h_{i_{1}} h_{i_{2}} \cdots h_{i_{q}} = h_{1}^{q_{1}} h_{2}^{q_{2}} \cdots h_{r}^{q_{r}},$$
(3)

where  $\sum_{i=1}^{r} q_i = q$ .

According to the convex sum property of MFs, one has

$$1 = \sum_{i=1}^{r} h_i = (\sum_{i=1}^{r} h_i)^q = \sum_{\mathbf{i}_q \in \mathbb{I}_q^+} g_{\mathbf{i}_q} h_{\mathbf{i}_q},$$
(4)

where  $g_{i_q} = \frac{q!}{q_1!q_2!\cdots q_r!}$ ,  $\sum_{i=1}^r q_i = q$ . For example, as r = 2, q = 4, if  $i_1 = i_2 = 1$ ,  $i_3 = i_4 = 2$ ,  $h_1h_1h_2h_2 = h_1^2h_2^2$  or  $i_1 = 1$ ,  $i_2 = i_3 = i_4 = 2$ ,  $h_1h_2h_2h_2 = h_1h_2^3$ , one gets  $g_{1122} = \frac{4!}{2!2!} = 6$  ( $q_1 = 2$ ,  $q_2 = 2$ ) or  $g_{1222} = \frac{4!}{1!3!} = 4$  ( $q_1 = 1$ ,  $q_2 = 3$ ).

The *q*-dimensional fuzzy summations of matrices are defined as follows:

$$\Omega_{\mathbf{h}} = \sum_{\mathbf{i}_q \in \mathbb{I}_q} h_{\mathbf{i}_q} \Omega_{\mathbf{i}_q} = \sum_{i_1=1}^r \cdots \sum_{i_q=1}^r h_{i_1} \cdots h_{i_q} \Omega_{i_1 \dots i_q}$$
$$\Omega_{\mathbf{h}^+} = \sum_{\mathbf{i}_q \in \mathbb{I}_q^+} h_{\mathbf{i}_q} \Omega_{\mathbf{i}_q} = \sum_{i_1=1}^r \cdots \sum_{i_q=i_{q-1}}^r h_{i_1} \cdots h_{i_q} \Omega_{i_1 \dots i_q}$$

To facilitate the formula derivation, the following lemma and several properties are cited.

**Lemma 1 (Finsler's Lemma [18]).** Let  $\zeta \in \mathbb{R}^n$ ,  $\Theta \in \mathbb{R}^{n \times n}$ ,  $\Phi \in \mathbb{R}^{m \times n}$ ,  $rank(\Phi) < n$ , then  $\zeta^T \Theta \xi < 0$ ,  $\forall \zeta \neq 0$ , such that  $\Phi \zeta = 0$ , if and only if one of the following conditions holds:

- (1)  $\Phi_{\perp}^{T}\Theta\Phi_{\perp} < 0$ , where  $\Phi_{\perp}$  satisfies  $\Phi\Phi_{\perp} = 0$ , and  $\Phi\Phi^{T} + \Phi_{\perp}^{T}\Phi_{\perp} > 0$ ;
- (2)  $\exists \mu \in \mathbb{R}: \Theta \mu \Phi^T \Phi < 0;$
- (3)  $\exists V \in \mathbb{R}^{m \times n}$ :  $\Theta + V\Phi + \Phi^T V^T < 0$ .

**Property 1** ([18]). *The following inequalities are equivalent:* 

(1) Find symmetric matrix  $X = X^T$ , such that  $T + A^T X^T + XA < 0$ .

(2) Find symmetric matrix  $X = X^T$ ,  $R_1$ ,  $R_2$ , such that

$$\begin{bmatrix} T + A^T R_1^T + R_1 A & (*) \\ X - R_1^T + X^T A & -R_2 - R_2^T \end{bmatrix} < 0.$$
 (5)

**Property 2** ([32]). Considering matrices  $\Phi = \Phi^T$  and  $\Psi = \Psi^T$  with suitable dimensions, if  $\Phi + \mu \Psi < 0$  and  $\Phi - \mu \Psi < 0$ , then  $\Phi + \dot{h}\Psi < 0$  holds for any  $\dot{h}$  and  $\mu$  satisfying  $|\dot{h}| \le \mu(\mu > 0)$ .

**Property 3** ([19]). Let  $\Omega_{\mathbf{i}_q}$ ,  $\mathbf{i}_q \in \mathbb{I}_q^+$  be the matrices of proper dimensions. Then,  $\Omega_{\mathbf{h}_q} < 0$  holds if

$$\Omega_{\mathbf{i}_{g}} < 0. \tag{6}$$

# 3. Main Results

**Theorem 1.** If given  $\mu_j > 0$ ,  $\|\frac{\partial h_j}{\partial x}\| \le \sigma_j$ ,  $j \in \mathbb{I}_1$ ,  $\|\omega\| \le \delta$ , there exist symmetric matrices  $P_{\mathbf{i}_q} > 0$ ,  $(\mathbf{i}_q \in \mathbb{I}_q)$  and  $M_{\mathbf{i}_{q+1}}$  ( $\mathbf{i}_{q+1} \in \mathbb{I}_{q+1}^+$ ), matrices  $F_{\mathbf{i}_q}$ ,  $L_{\mathbf{i}_q}^{l_1 l_2}$  and  $G_{\mathbf{i}_q}^{l_1 l_2}$  ( $\mathbf{i}_q \in \mathbb{I}_q^+$ ) ( $l_1, l_2 = 1, 2, 3$ ), such that

$$\Omega_{\mathbf{i}_{q+1}}^{(1)} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26} \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} \\ * & * & * & \Omega_{44} & \Omega_{45} & \Omega_{46} \\ * & * & * & * & \Omega_{55} & \Omega_{56} \\ * & * & * & * & * & \Omega_{66} \end{bmatrix} < 0,$$
(7)

$$\Omega_{\mathbf{i}_{q+1}}^{(2)} = \begin{bmatrix} \sum_{i=1}^{r} P_{\mathcal{P}(\mathbf{i}_{q+1}/i_k)} & 0 & (*) \\ 0 & I & (*) \\ \sum_{i_k=1}^{q} \left( \varepsilon_x A_{i_k} P_{\mathbf{i}} + \varepsilon_x B_{2i_k} F_{\mathbf{i}} \right) & B_{1i_k} & g_{\mathbf{i}_{q+1}} \frac{\alpha^2 \mu_j^2}{\rho \sigma_j^2} \end{bmatrix} > 0,$$
(8)

where  $\mathbf{\overline{i}} = \mathbf{i}_{q+1}/i_k$ ,

$$\begin{split} \Omega_{11} &= \sum_{i_k=1}^{q} He(L_{\mathbf{i}}^{11}A_{i_k}^{T} + B_{2i_k}F_{\mathbf{i}}) \pm \sum_{j=1}^{r} \mu_j \bar{\Omega}_{i_{q+1}}^{j}, \\ \bar{\Omega}_{i_{q+1}}^{j} &= \sum_{j=1}^{r} \sum_{j_q \in \mathcal{P}(\mathbf{i}_{q+1}/j)} (P_{j\dots j_q} + \dots + P_{j_1\dots j}) + M_{\mathbf{i}_{q+1}}, \\ \Omega_{12} &= \sum_{i_k=1}^{q} (g_{\mathbf{i}}B_{1i_k} + A_{i_k}L_{\mathbf{i}}^{21^{T}}), \\ \Omega_{13} &= \sum_{i_k=1}^{q} (F_{\mathbf{i}}^{T}D_{2i_k}^{T} + L_{\mathbf{i}}^{11}C_{1i_k}^{T} + A_{i_k}L_{\mathbf{i}}^{31^{T}}), \\ \Omega_{14} &= \sum_{i_k=1}^{q} (P_{\mathcal{P}(\mathbf{i})} - L_{\mathbf{i}}^{11} + A_{i_k}G_{\mathbf{i}}^{11}), \\ \Omega_{15} &= \sum_{i_k=1}^{q} (-L_{\mathbf{i}}^{12} + A_{i_k}G_{\mathbf{i}}^{12}), \Omega_{16} = \sum_{i_k=1}^{q} (-L_{\mathbf{i}}^{13} + A_{i_k}G_{\mathbf{i}}^{13}), \\ \Omega_{22} &= -g_{\mathbf{i}_{q+1}}\gamma^2, \Omega_{23} = \sum_{i_k=1}^{q} (D_{1i_k}^{T} + L_{\mathbf{i}}^{21}C_{1i_k}^{T}), \\ \Omega_{24} &= -\sum_{i_k=1}^{q} L_{\mathbf{i}}^{21}, \Omega_{25} = g_{\mathbf{i}_{q+1}}I - \sum_{i_k=1}^{q} L_{\mathbf{i}}^{22}, \\ \Omega_{26} &= -\sum_{i_k=1}^{q} L_{\mathbf{i}}^{23}, \Omega_{33} = \sum_{i_k=1}^{q} He(L_{\mathbf{i}}^{31}C_{1i_k}^{T}) - g_{\mathbf{i}_{q+1}}I, \\ \Omega_{34} &= \sum_{i_k=1}^{q} (-L_{\mathbf{i}}^{31} + C_{1i_k}G_{\mathbf{i}}^{11}), \\ \Omega_{35} &= \sum_{i_k=1}^{q} (-L_{\mathbf{i}}^{32} + C_{1i_k}G_{\mathbf{i}}^{12}), \\ \Omega_{36} &= g_{\mathbf{i}_{q+1}}I - \sum_{i_k=1}^{q} (L_{\mathbf{i}}^{33} + C_{1i_k}G_{\mathbf{i}}^{13}), \\ \Omega_{44} &= \sum_{i_k=1}^{q} He(-G_{\mathbf{i}}^{11}), \Omega_{45} = \sum_{i_k=1}^{q} (-G_{\mathbf{i}}^{12} - G_{\mathbf{i}}^{21^{T}}), \\ \Omega_{46} &= \sum_{i_k=1}^{q} (-G_{\mathbf{i}}^{33} - G_{\mathbf{i}}^{31^{T}}), \Omega_{55} = \sum_{i_k=1}^{q} He(-G_{\mathbf{i}}^{22}), \\ \Omega_{56} &= \sum_{i_k=1}^{q} (-G_{\mathbf{i}}^{33} - G_{\mathbf{i}}^{32^{T}}), \Omega_{66} = \sum_{i_k=1}^{q} He(-G_{\mathbf{i}}^{33}). \end{split}$$

then, the closed-loop system (1) in  $\Omega_x$  (:= { $x : x^T P_{\mathbf{h}_q}^{-1} x \leq \rho$ }) is locally asymptotically stable with a disturbance attenuation level  $\gamma$  under the controller (9)

$$u(t) = F_{\mathbf{h}_q^+} P_{\mathbf{h}_q}^{-1} \mathbf{x}.$$
(9)

Proof. Designing the generalized NQLF candidate as

$$V(x) = x^T P_{\mathbf{h}_q}^{-1} x, \tag{10}$$

and letting  $\Gamma = \dot{V}(x) + z^T z - \gamma^2 \omega^T \omega$ , we get

$$\Gamma = \eta^{T} \begin{bmatrix} \Lambda_{1} & P^{-1}B_{1h} + \bar{C}^{T}D_{1h} \\ (*) & -\gamma^{2} + D_{1h}^{T}D_{1h} \end{bmatrix} \eta,$$
(11)

where  $\Lambda_1 = He(P_{\mathbf{h}_q}^{-1}\bar{A}) + \dot{P}_{\mathbf{h}_q}^{-1} + \bar{C}^T\bar{C}, \ \eta^T = \begin{bmatrix} x^T & \omega^T \end{bmatrix}, \ \bar{A} = A_h + B_{2h}F_{\mathbf{h}_q}^+ P_{\mathbf{h}_q}^{-1}, \ \bar{C} = C_{1h} + D_{2h}F_{\mathbf{h}_q}^+ P_{\mathbf{h}_q}^{-1}. \ \Box$ 

On one hand,  $\Gamma \leq 0$  is guaranteed by the following formula:

$$\begin{bmatrix} He(P_{\mathbf{h}_{q}}^{-1}\bar{A}) + \dot{P}_{\mathbf{h}_{q}}^{-1} & P_{\mathbf{h}_{q}}^{-1}B_{1h} & \bar{C} \\ B_{1h}^{T}P_{\mathbf{h}_{q}}^{-1} & -\gamma^{2} & D_{1h}^{T} \\ \bar{C} & D_{1h} & -I \end{bmatrix} < 0.$$
(12)

Multiplying left and right by  $diag\{P_{\mathbf{h}_q}, I, I\}$  and using the relation  $\dot{P}_{\mathbf{h}_q}^{-1} = -P_{\mathbf{h}_q}^{-1}\dot{P}_{\mathbf{h}_q}P_{\mathbf{h}_q}^{-1}$ , Equation (12) is transformed into

$$\begin{bmatrix} He(\bar{A}P_{\mathbf{h}_{q}}) - \dot{P}_{\mathbf{h}_{q}} & B_{1h} & P_{\mathbf{h}_{q}}\bar{C} \\ B_{1h}^{T} & -\gamma^{2} & D_{1h}^{T} \\ \bar{C}P_{\mathbf{h}_{q}} & D_{1h} & -I \end{bmatrix} < 0.$$
(13)

Starting from the characteristics where  $\sum_{j=1}^{r} \dot{h}_j = 0$ , thus  $\sum_{j=1}^{r} \dot{h}_j M_{\mathbf{h}_{q+1}^+} = 0$  for any symmetric matrix  $M_{\mathbf{h}_{q+1}^+}$  with suitable size, adding  $\sum_{j=1}^{r} \dot{h}_j M_{\mathbf{h}_{q+1}^+}$  to (13) and using Property 2, we get

$$\begin{bmatrix} He(\bar{A}P_{\mathbf{h}_{q}}) \pm \Omega^{j} & B_{1h} & P_{\mathbf{h}_{q}}\bar{C}^{T} \\ B_{1h}^{T} & -\gamma^{2} & D_{1h}^{T} \\ \bar{C}P_{\mathbf{h}_{q}} & D_{1h} & -I \end{bmatrix} < 0,$$
(14)

where  $\Omega^j = \sum_{j=1}^r \dot{h}_j (P^j_{\mathbf{h}_q} + M_{\mathbf{h}_{q+1}^+}).$ 

Then, Equation (14) is transformed in to

$$Q + \Lambda_2^T \hat{P} + \hat{P} \Lambda_2 < 0 \tag{15}$$

where

$$Q = \begin{bmatrix} He(B_{2h}F_{\mathbf{h}_{q}^{+}}) - \Omega & B_{1h} & (F_{\mathbf{h}_{q}^{+}})^{T}D_{2h}^{T} \\ * & -\gamma^{2} & D_{1h}^{T} \\ * & * & -I \end{bmatrix},$$
$$\hat{P} = \begin{bmatrix} P_{\mathbf{h}_{q}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \Lambda_{2}^{T} = \begin{bmatrix} A_{h} & 0 & 0 \\ 0 & 0 & 0 \\ C_{1h} & 0 & 0 \end{bmatrix}.$$

Applying **Property 1** to (15), one gets

$$\Omega_{\mathbf{h}_{q+1}} = \begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} & \bar{\Omega}_{13} & \bar{\Omega}_{14} & \bar{\Omega}_{15} & \bar{\Omega}_{16} \\ * & -\gamma^2 & \bar{\Omega}_{23} & \bar{\Omega}_{24} & \bar{\Omega}_{25} & \bar{\Omega}_{26} \\ * & * & \bar{\Omega}_{33} & \bar{\Omega}_{34} & \bar{\Omega}_{35} & \bar{\Omega}_{36} \\ * & * & * & \bar{\Omega}_{44} & \bar{\Omega}_{45} & \bar{\Omega}_{46} \\ * & * & * & * & \bar{\Omega}_{55} & \bar{\Omega}_{56} \\ * & * & * & * & * & \bar{\Omega}_{66} \end{bmatrix} < 0,$$
(16)

where

$$\begin{split} \bar{\Omega}_{11} &= He(L_{\mathbf{h}_{q}^{+}}^{11}A_{h}^{T} + B_{2h}F_{\mathbf{h}_{q}^{+}}) \pm \Omega^{j}, \bar{\Omega}_{12} = B_{1h} + A_{h}L_{\mathbf{h}_{q}^{+}}^{21^{1}}, \\ \bar{\Omega}_{13} &= F_{\mathbf{h}_{q}^{+}}^{T}D_{2h}^{T} + L_{\mathbf{h}_{q}^{+}}^{11}C_{1h}^{T} + A_{h}L_{\mathbf{h}_{q}^{+}}^{31^{T}}, \bar{\Omega}_{14} = P_{\mathbf{h}_{q}} - L_{\mathbf{h}_{q}^{+}}^{11} + A_{h}G_{\mathbf{h}_{q}^{+}}^{11}, \\ \bar{\Omega}_{15} &= -L_{\mathbf{h}_{q}^{+}}^{12} + A_{h}G_{\mathbf{h}_{q}^{+}}^{12}, \bar{\Omega}_{16} = -L_{\mathbf{h}_{q}^{+}}^{13} + A_{h}G_{\mathbf{h}_{q}^{1}}^{13}, \\ \bar{\Omega}_{23} &= D_{1h}^{T} + L_{\mathbf{h}_{q}^{+}}^{21}C_{1h}^{T}, \bar{\Omega}_{24} = -L_{\mathbf{h}_{q}^{+}}^{21}, \bar{\Omega}_{25} = I - L_{\mathbf{h}_{q}^{+}}^{22}, \\ \bar{\Omega}_{26} &= -L_{\mathbf{h}_{q}^{+}}^{23}, \bar{\Omega}_{33} = -I + He(L_{\mathbf{h}_{q}^{+}}^{31}C_{1h}^{T}), \\ \bar{\Omega}_{34} &= -L_{\mathbf{h}_{q}^{+}}^{31} + C_{1h}G_{\mathbf{h}_{q}^{+}}^{11}, \bar{\Omega}_{35} = -L_{\mathbf{h}_{q}^{+}}^{32} + C_{1h}G_{\mathbf{h}_{q}^{+}}^{12}, \\ \bar{\Omega}_{36} &= I - L_{\mathbf{h}_{q}^{+}}^{33} + C_{1h}G_{\mathbf{h}_{q}^{+}}^{13}, \bar{\Omega}_{44} = He(-G_{\mathbf{h}_{q}^{+}}^{11}), \\ \bar{\Omega}_{45} &= -G_{\mathbf{h}_{q}^{+}}^{12} - G_{\mathbf{h}_{q}^{+}}^{21^{T}}, \bar{\Omega}_{46} = -G_{\mathbf{h}_{q}^{-}}^{13} - G_{\mathbf{h}_{q}^{-}}^{31^{T}}, \\ \bar{\Omega}_{55} &= He(-G_{\mathbf{h}_{q}^{-}}^{22}), \bar{\Omega}_{56} = -G_{\mathbf{h}_{q}^{-}}^{23} - G_{\mathbf{h}_{q}^{-}}^{32^{T}}, \bar{\Omega}_{66} = He(-G_{\mathbf{h}_{q}^{-}}^{33}), \end{split}$$

According to **Property 2**, (16) is warranted by (7) with  $|\dot{h}_i| \le \mu_i$ . On the other hand, we discuss  $|\dot{h}_j| \le \mu_j$ . Let

$$|\dot{h}_j| = |(\frac{\partial h_j}{\partial x})^T \varepsilon_x (A_h x + B_{2h} F_{\mathbf{h}_q^+} P_{\mathbf{h}_q}^{-1} x + B_{1h} \omega)| \le \mu_j.$$
(17)

Notation: If *h* contains all state *x*, then  $\varepsilon_x$  is the unit matrix. If *h* only contains a part of state, for instance,  $x = [x_1, x_2, x_3, x_4]^T$ , *h* depends on  $x_1, x_3$ , then  $\varepsilon_x = diag\{1, 0, 1, 0\}$ .

. .

For (17), one has

$$\eta^T \Delta^T \frac{\partial h_j}{\partial x} (\frac{\partial h_j}{\partial x})^T \Delta \eta < \mu_j^2, \tag{18}$$

where  $\Delta = \varepsilon_x [A_h + B_{2h} F_{\mathbf{h}_q^+} P_{\mathbf{h}_q}^{-1}, B_{1h}], \eta = [x; \omega].$ It is known that  $(\frac{\partial h_j}{\partial x})^T \frac{\partial h_j}{\partial x} \leq \sigma_j^2 \Leftrightarrow \frac{\partial h_j}{\partial x} (\frac{\partial h_j}{\partial x})^T \leq \sigma_j^2 I$  and  $\eta^T diag\{P_{\mathbf{h}_q}^{-1}, I\}\eta \leq \rho + \delta^2.$ Equation (18) is guaranteed by

$$\frac{(\rho+\delta^2)\sigma_j^2}{\alpha^2\mu_j^2}\Delta^T\Delta - diag\{P_{\mathbf{h}_q}^{-1}, I\} < 0,$$
(19)

Utilizing Schur complement to (19) and multiplying left and right by  $diag\{P_{\mathbf{h}_a}, I\}$ , one has г D. \* \* ٦

$$\begin{bmatrix} & F_{\mathbf{h}_{q}} & * & * \\ 0 & I & * \\ \varepsilon_{x}(A_{h}P_{\mathbf{h}_{q}} + B_{2h}F_{\mathbf{h}_{q}^{+}}) & B_{1h} & \frac{\alpha^{2}\mu_{j}^{2}}{\rho\sigma_{j}^{2}} \end{bmatrix} > 0.$$
(20)

Therefore, if (7) and (8) hold, we have  $\Gamma < 0$ , which means  $\int_0^\infty z^T z dt \le \gamma^2 \int_0^\infty \omega^T \omega dt$ . The proof is completed.

**Remark 1.** TDMFs in [20,23] were bounded without disturbance and some assumed conditions such as matrix  $P(P_i > \frac{\lambda_x^2 + \lambda_k^2}{2\beta_k}I)$  and the free variable  $S(S_h = P_h)$  in [28,33] were limited. This paper eliminates these restrictions or assumptions.

**Remark 2.** Due to  $(\frac{\partial h_j}{\partial x})^T \frac{\partial h_j}{\partial x} \leq \sigma_j^2$  being restricted, Theorem 1 means a local result. The Lyapunov level  $\Omega_x(x^T P_{\mathbf{h}_q}^{-1} x \leq \rho)$  is an estimated region for  $H_\infty$  performance, which must be contained in  $\Theta_x$ . Applying the Lagrange multiplier method, we get

$$\begin{bmatrix} \frac{\lambda_{xi}^2}{\rho} & e_i P_{\mathbf{h}_q} \\ P_{\mathbf{h}_q} e_i^T & P_{\mathbf{h}_q} \end{bmatrix} \ge 0$$
(21)

Thus, according to Property 3, conditions (22) guarantee (21).

$$\begin{bmatrix} g_{\mathbf{i}_{q}} \frac{\lambda_{x_{i}}^{2}}{\rho} & e_{i} P_{\mathcal{P}(\mathbf{i}_{q})} \\ * & P_{\mathcal{P}(\mathbf{i}_{q})} \end{bmatrix} \geq 0, i = 1, 2, \dots, n$$

$$(22)$$

**Remark 3.** There are two parameters  $\mu_j$  and  $\rho$  in Theorem 1 to be searched. By solving the following optimization problem, one can get  $H\infty$  performance with given  $\mu_j$  and  $\rho$ .

*Compared with the methods in* [28,33], *if*  $|\dot{h}_i| \le \mu_i$  *is decomposed, that is,* 

$$\left|\left(\frac{\partial h_j}{\partial x}\right)^T \varepsilon_x (A_h + B_{2h} F_{\mathbf{h}_q^+} P_{\mathbf{h}_q}^{-1}) x\right| \leq \alpha \mu_j$$
(24)

$$\left|\left(\frac{\partial n_j}{\partial x}\right)^T \varepsilon_x B_{1h}\omega\right| \leq (1-\alpha)\mu_j \tag{25}$$

then (8) will be substituted with (26), (27).

**Corollary 1.** Given  $\mu_j > 0$ ,  $\|\frac{\partial h_j}{\partial x}\| \le \sigma_j$ ,  $j \in \mathbb{I}_1$ ,  $\|\omega\| \le \delta$ ,  $\alpha \in (0, 1)$ , *if* (7), (26), (27), (22) hold,

$$\Omega_{\mathbf{i}_{q+1}}^{(2)} = \begin{bmatrix} \sum_{i=1}^{r} P_{\mathcal{P}(\mathbf{i}_{q+1}/i)} & (*) \\ \Omega_{21}^{(2)} & g_{\mathbf{i}_{q+1}} \frac{\alpha^2 \mu_j^2}{\rho \sigma_j^2} \end{bmatrix} > 0,$$
(26)

$$\frac{(1-\alpha)^2 \mu_j^2}{\varepsilon_x B_{1i_k}} \quad \begin{pmatrix} * \\ \sigma_j^2 \end{pmatrix} > 0, \tag{27}$$

Then, the closed-loop system (1) in  $\Omega_x$  (:= { $x : x^T P_{\mathbf{h}_q}^{-1} x \leq \rho$ }) is locally asymptotically stable with a disturbance attenuation level  $\gamma$  under the controller (9).

**Proof.** The proof is similar to that of Theorem 1, and is thus omitted here.  $\Box$ 

#### 4. Simulation Example

All the experiments were performed on a computer with an Intel(R) Core(TM)i5-7200U CPU @ 2.50 GHz 2.70 GHz, 12 GB(RAM), using Matlab2017a.

Example 1. Considering the following nonlinear system form [33].

$$\begin{cases} \dot{x}_1 = -6x_1 - 4.33x_2 + 7.59x_1sin^2(x_1) - 2.96x_2sin^2(x_1) \\ + (sin^2(x_1) - 2.06cos^2(x_1))u + (0.1 + 0.01cos^2(x_1))\omega \\ \dot{x}_2 = x_1sin^2(x_1) + 5x_2 - 5sin^2(x_1)x_2 - cos^2(x_1)u \\ + (0.01sin^2(x_1) + 0.03)\omega \end{cases}$$
(28)

The above system is expressed as a T-S fuzzy system with two rules in the  $\mathbb{C}_x = \{x : |x_i| \le \frac{\pi}{2}, i = 1, 2\}$ . One gets the system matrices:

$$A_{1} = \begin{bmatrix} 1.59 & -7.29 \\ 1 & 0 \end{bmatrix}, A_{2} = \begin{bmatrix} -6 & -4.33 \\ 0 & 5 \end{bmatrix}, B_{11} = \begin{bmatrix} 0.01 \\ 0.04 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.011 \\ 0.03 \end{bmatrix}, B_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{22} = \begin{bmatrix} -2.06 \\ -1 \end{bmatrix}, C_{11} = C_{12} = \begin{bmatrix} 1 & 2 \end{bmatrix}, D_{11} = D_{12} = D_{21} = D_{22} = 0, h_{1} = sin^{2}(x_{1}), h_{2} = 1 - h_{1}.$$
  
Since  $\frac{\partial h_{1}}{\partial x} = 2sin(x_{1})cos(x_{1})$  and  $\frac{\partial h_{2}}{\partial x} = -2sin(x_{1})cos(x_{1})$ , thus

$$(\frac{\partial h_1}{\partial x})^T \frac{\partial h_1}{\partial x} \leq 1, (\frac{\partial h_2}{\partial x})^T \frac{\partial h_2}{\partial x} \leq 1,$$

one has  $\sigma_i^2 = 1$ .

The external disturbance signal  $\omega(t) = \begin{cases} \sin(t), 5 \le t \le 15 \text{ s} \\ 0, \text{ else} \end{cases}$ . The parameters required by Theorem 1 in the region  $\Omega_x(x^T P_{\mathbf{h}_q}^{-1}x \le \rho = 1)$  are as follows:  $\sigma_i^2 = 1, \lambda_{x_i}^2 = \frac{\pi^2}{4}, \delta^2 = 1, \mu_i = 1000$ .

All results are solved by function *minx* in Matlab Toolbox. The minimal  $\gamma$  obtained are shown in Table 1 under different methods. Notice, a line indicates that SeDuMi solver is unable to converge to a solution.

**Table 1.** Minimum  $H\infty$  performance  $\gamma$ .

Methods	γ	<b>Computational Time</b>
[23]	_	
[34] (Theorem 1)	0.1896	0.2015 s
[33] (Theorem 1)	0.0483	1.4590 s
[27] (Theorem 1)	0.0434	0.3305 s
Corollary 1 ( $q = 1$ )	0.0511	0.4590 s
Corollary 1 ( $q = 2$ )	0.0409	0.9586 s
Corollary 1 ( $q = 3$ )	0.0345	1.6205 s
Theorem 1 ( $q = 1$ )	0.0506	0.5160 s
Theorem 1 ( $q = 2$ )	0.0407	1.0272 s
Theorem 1 ( $q = 3$ )	0.0338	2.2055 s

This clearly shows that the Theorem 1 proposed in this paper is less conservative than other methods. Moreover, as *q* increases, better results are obtained.

On the other hand, when systems include unknown parameters, such as  $B_{22} = \begin{bmatrix} -2.06 + \lambda \\ -1 \end{bmatrix}$ , we compare the minimal  $\gamma$  with the different  $\lambda$  shown in Table 2. Although the minimal  $\gamma$  increases with the parameter  $\lambda$ , Theorem 1's results are better than other ones under the same parameter  $\lambda$ .

Methods	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.7$	$\lambda = 1$
Corollary 1 ( $q = 1$ )	0.0538	0.0567	0.0598	0.0630	0.0742	0.0878
Corollary 1 ( $q = 2$ )	0.0391	0.0374	0.0359	0.344	0.0307	0.0281
Corollary 1 ( $q = 3$ )	0.0334	0.0323	0.0313	0.0304	0.0280	0.0260
Theorem 1 $(q = 1)$	0.0506	0.0480	0.0457	0.0414	0.0361	0.0332
Theorem 1 $(q = 2)$	0.0387	0.0368	0.0350	0.0334	0.0294	0.0271
Theorem 1 $(q = 3)$	0.0327	0.0317	0.0307	0.0298	0.0275	0.0259

**Table 2.** Minimum  $H\infty$  performance  $\gamma$  with different  $\lambda$ .

When  $\lambda = 1$  and q = 3, one gets  $\gamma = 0.0259$  and the gain matrices as follow.

$P_{111} = [$	$1.4172 \\ -1.3062$	$\begin{bmatrix} -1.3062\\ 1.2039 \end{bmatrix}, B$	$F_{111} = \begin{bmatrix} -13\\ 10.4 \end{bmatrix}$	$\begin{bmatrix} .5716 \\ .026 \end{bmatrix}^T_{-}$
$P_{112} = [$	$0.0001 \\ -0.0044$	$\begin{bmatrix} -0.0044 \\ 1.4004 \end{bmatrix}$ , $\frac{1}{2}$	$F_{112} = \begin{bmatrix} -48\\ 22.6 \end{bmatrix}$	$\begin{bmatrix} .4813 \\ .5372 \end{bmatrix}^T$
$P_{122} = [$	7.0579 3 3.9604 2	$\left[ \begin{array}{c} .9604 \\ .2224 \end{array} \right], F_{122}$	$= \left[ \begin{array}{c} 51.1398\\ 55.7751 \end{array} \right]$	]
$P_{222} = [$	2.4673 1 1.0802 0	$\left[\begin{array}{c} .0802\\ .4736\end{array}\right], F_{222}$	$= \begin{bmatrix} -3.3849\\5.5163 \end{bmatrix}$	$\left]^{T}$ .

Choosing the initial four points

$$\begin{array}{c} x_{0}^{(1)} = \begin{bmatrix} 0.1 & 0.01 \end{bmatrix}^{T}, x_{0}^{(2)} = \begin{bmatrix} -0.1 & -0.01 \end{bmatrix}^{T}, \\ x_{0}^{(3)} = \begin{bmatrix} 0 & -0.011 \end{bmatrix}^{T}, x_{0}^{(4)} = \begin{bmatrix} 0 & 0.011 \end{bmatrix}^{T}, \end{array}$$

four trajectory curves in the domain of attraction  $\Omega_x$  are shown in Figure 1, which are asymptotically driven to the origin under the controller. Therefore, this shows that the designed controller is effective.



**Figure 1.** Trajectory curves start from  $\Omega_x$ .

**Example 2.** The state equation of motion for the inverted pendulum controlled by a separately excited direct current (DC) motor from [35].

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \frac{g}{l} sinx_1(t) + \frac{NK_1}{ml^2} x_3(t) + \omega, \\ 0 = -NK_2 x_2(t) - \bar{R} x_3(t) + u, \end{cases}$$
(29)

where  $x_1 \in [-\pi \pi]$  denotes the angle,  $x_2$  denotes the angular velocity,  $x_3$  represents the current of the DC motor, u is the control input voltage. The parameter values of the system are m = 1 kg, l = 1 m, g = 9.8 m/s<sup>2</sup>, N = 10,  $K_2 = 0.1$  Vs/rad,  $K_1 = 0.1$  Nm/A and  $0.6 \Omega \le \overline{R} \le 3.5 \Omega$ .

Equation (29) can be converted to

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \frac{g}{l} sinx_1(t) - \frac{N^2 K_1 K_2}{m l^2 \bar{R}} x_2(t) + \frac{N K_1}{m l^2 \bar{R}} u + \omega. \end{cases}$$
(30)

We consider  $z = x_1 + x_2$ . Applying local approximation method, the T-S fuzzy model (1) is given with  $A_1 = \begin{bmatrix} 0 & 1 \\ 9.8 & -\frac{5}{3} \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{5}{3} \end{bmatrix}$ ,  $B_{21} = \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix}$ ,  $B_{22} = \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix}$ ,  $B_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $B_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $C_{11} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ ,  $C_{12} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ ,  $D_{11} = D_{12} = D_{21} = D_{22} = 0$ .  $h_1 = \frac{\sin(x_1)+1}{2}$ ,  $h_2 = 1 - h_1$  defined in the compact set  $\mathbb{C} = \{x : |x_1| \le \pi, |x_2| \le \pi\}$ . Since  $\frac{\partial h_1}{\partial x} = \begin{bmatrix} \frac{\cos(x_1)}{2} & 0 \end{bmatrix}^T$  and  $\frac{\partial h_2}{\partial x} = \begin{bmatrix} -\frac{\cos(x_1)}{2} & 0 \end{bmatrix}^T$ , thus  $(\frac{\partial h_1}{\partial x})^T \frac{\partial h_1}{\partial x} \le \frac{1}{4}$ ,  $(\frac{\partial h_2}{\partial x})^T \frac{\partial h_2}{\partial x} \le \frac{1}{4}$ , one has  $\mu_i^2 = \frac{1}{4}$ . Selecting  $\sigma_i^2 = \frac{1}{4}$ ,  $\lambda_x^2 = 2\pi^2$ ,  $\rho = 1$ . Here, we consider that the derivative of MFs ( $\mu_1 = \mu_2$ ) in Theorem 1 affects the

Here, we consider that the derivative of MFs ( $\mu_1 = \mu_2$ ) in Theorem 1 affects the conservativeness of the different method. The results are shown in Table 3. It can be seen that [23,27,33] cannot converge to a solution when  $\mu = 1$ , but Theorem 1 can. Moreover, as  $\mu_i$  increases,  $\gamma$  decreases, and Theorem 1's results with q = 2 are better than other ones under the same parameter  $\mu$ .

**Table 3.** Minimum  $H\infty$  performance  $\gamma$  with different  $\mu_i$ .

Methods	$\mu = 1$	$\mu = 10$	$\mu = 100$	$\mu = 1000$
[23] (Theorem 1)	_	$1.6751\times 10^{-5}$	$7.2367\times10^{-6}$	$3.1836\times 10^{-6}$
[33] (Theorem 1)	-	$1.7001  imes 10^{-5}$	$8.7385  imes 10^{-6}$	$4.0402 imes10^{-6}$
[27] (Theorem 1)	-	$2.866 imes10^{-5}$	$2.7622  imes 10^{-5}$	$2.7572  imes 10^{-5}$
Theorem 1 ( $q = 1$ )	$1.5039 imes10^{-5}$	$2.1596  imes 10^{-5}$	$2.1523 imes10^{-5}$	$2.1224 imes10^{-5}$
Theorem 1 ( $q = 2$ )	$2.5539\times10^{-5}$	$1.5306\times10^{-5}$	$6.3759\times10^{-6}$	$2.6786\times10^{-6}$

Choosing q = 2,  $\mu = 100$ , we get  $\gamma = 6.3759 \times 10^{-6}$  and control gain matrixes as follow:

$P_{11} = 10^6 \times $	6.8474	-6.7747	$\Gamma = 10^5 \times$	[ -5.6759 ]	T
	-6.7747	6.8257	$, F_{11} = 10^{-1} \times$	6.5064	'
$P_{12} = 10^6 \times \left[ \right]$	7.1174	-6.9996	$], F_{12} = 10^6 \times [$	[ 0.5910 ]	Т
	-6.9996	7.0022			'
$P_{22} = 10^6 \times$	6.9380	-6.8626	$\Gamma = 106 \times$	[ -0.0242 ]	Т
	-6.8626	6.9128	$, F_{22} = 10^{\circ} \times$	-7.6379	•

From the starting point  $x_0 = [3, -3]^T$ , the response trajectories of the states, Lyapunov function and control input are shown in Figures 2–4, respectively. Therefore, the closed-loop system (30) is locally asymptotically stable with disturbance attenuation level  $\gamma$  under the controller (9).



Figure 2. States response for closed-loop systems.



Figure 3. Lyapunov curve for closed-loop systems.



Figure 4. Control input curve for closed-loop systems.

#### 5. Conclusions

This paper has presented the local  $H\infty$  control for continuous-time T-S fuzzy systems via generalized nonquadratic Lyapunov functions. Using polynomial technology, relaxed LMIs conditions satisfying  $H\infty$  performance are obtained, which are easily solved by the optimization problem. The simulation examples provided show the validity of the proposed method.

Furthermore, the method proposed in this paper could be generalized to handle problems regarding output feedback controllers, filters or observers. Now, our group is considering the output feedback controller design for nonlinear systems, finite-time boundedness, finite-time annular domain stability and mean-square strong stability for stochastic systems using our method.

**Author Contributions:** Formal analysis, Z.Y. and G.H.; methodology, G.H.; funding acquisition, Z.Y.; investigation, software, and writing—original draft preparation and editing, J.Z.; review and editing, Z.Y. and G.H. All authors have read and agreed to the published version of the manuscript.

**Funding:** This paper is supported in part by the National Natural Science Foundation of China under Grants (61877062, 61977043).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

## References

- Takagi, T.; Sugeno, M. Fuzzy identification of systems and its applications to modeling and control. *IEEE Trans. Syst. Man Cybern.* 1985, 15, 387–403. [CrossRef]
- Su, X.J.; Wu, L.G.; Shi, P.; Song, Y.D. A novel approach to output feedback control of fuzzy stochastic systems. *Automatica* 2014, 50, 3268–3275. [CrossRef]
- 3. Vafamand, N.; Asemani, M.H.; Khayatiyan, A.; Khooban, M.H.; Dragicevic, T. TS Fuzzy Model-Based Controller Design for a Class of Nonlinear Systems Including Nonsmooth Functions. *IEEE Trans. Syst. Man Cybern. Syst.* 2020, *50*, 233–244. [CrossRef]
- 4. Achour, H.; Boukhetala, D.; Labdelaoui, H. An observer-based robust H∞ controller design for uncertain Takagi-Sugeno fuzzy systems with unknown premise variables using particle swarm optimisation. *Int. J. Syst. Sci.* **2020**, *51*, 2563–2581. [CrossRef]
- Yan, Z.; Zhang, J.; Hu, G. A new approach to fuzzy output feedback controller design of continuous-time Takagi-Sugeno fuzzy systems. *Int. J. Fuzzy Syst.* 2020, 22, 2223–2235. [CrossRef]
- 6. Chang, X.H.; Yang, G.H.; Wang, H. Observer-based H∞ control for discrete-time T-S fuzzy systems. *Int. J. Syst. Sci.* 2011, 42, 1801–1809. [CrossRef]
- 7. Wang, L.K.; Zhang, H.G.; Liu, X.D. H∞ Observer Design for Continuous-Time Takagi-Sugeno Fuzzy Model With Unknown Premise Variables via Nonquadratic Lyapunov Function. *IEEE Trans. Cybern.* **2016**, *46*, 1986–1996. [CrossRef]
- 8. Zhang, Z.; Lin, C.; Chen, B. New decentralized H∞ filter design for nonlinear interconnected systems based on Takagi-Sugeno fuzzy models. *IEEE Trans. Cybern.* 2015, 45, 2914–2924. [CrossRef] [PubMed]
- 9. Shi, P.; Su, X.; Li, F. Dissipativity-based filtering for fuzzy switched systems with stochastic perturbation. *IEEE Trans. Autom. Control* **2016**, *61*, 1694–1699. [CrossRef]
- Dong, J.; Yang, G. H<sub>∞</sub> Filtering for Continuous-Time T-S Fuzzy Systems with Partly Immeasurable Premise Variables. *IEEE Trans.* Syst. Man, Cybern. Syst. 2017, 47, 1931–1940. [CrossRef]
- 11. Tanaka, K.; Hori, T.; Wang, H.O. A multiple Lyapunov function approach to stabilization of fuzzy control systems. *IEEE Trans. Fuzzy Syst.* **2003**, *11*, 582–589. [CrossRef]
- 12. Guerra, T.M.; Vermeiren, L. LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno's form. *Automatica* 2004, 40, 823–829. [CrossRef]
- Rhee, B.J.; Won, S. A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design. *Fuzzy Sets Syst.* 2006, 157, 1211–1228. [CrossRef]
- Bernal, M.; Guerra, T.M. Generalized nonquadratic stability of continuous-time Takagi-Sugeno models. *IEEE Trans. Fuzzy Syst.* 2010, 18, 815–822. [CrossRef]
- 15. Ding, B. Homogeneous polynomially nonquadratic stabilization of discrete-time Takagi-Sugeno systems via nonparallel distributed compensation law. *IEEE Trans. Fuzzy Syst.* **2010**, *18*, 994–1000. [CrossRef]
- 16. Zhang, H.; Xie, X. Relaxed stability conditions for continuous-time T-S fuzzy-control systems via augmented multi-indexed matrix approach. *IEEE Trans. Fuzzy Syst.* **2011**, *19*, 478–492. [CrossRef]
- 17. Hu, G.; Liu, X.; Wang, L.; Li, H. Relaxed stability and stabilisation conditions for continuous-time Takagi-Sugeno fuzzy systems using multiple-parameterised approach. *IET Control Theory Appl.* **2017**, *11*, 774–780. [CrossRef]
- Delmotte, F.; Guerra, T.M.; Ksantini, M. Continuous Takagi-Sugeno's models: Reduction of the number of LMI conditions in various fuzzy control design technics. *IEEE Trans. Fuzzy Syst.* 2007, 15, 426–438. [CrossRef]
- 19. Sala, A.; Arino, C. Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets Syst.* **2007**, *158*, 2671–2686. [CrossRef]
- Vafamand, N.; Khooban, M.H.; Khayatian, A.; Blaabjerg, F. Design of Robust Double-Fuzzy-Summation Non-PDC Controller for Chaotic Power Systems. J. Dyn. Syst. Meas. Control 2018, 140, 031004. [CrossRef]
- 21. Wang, L.; Lam, H. H∞ control for continuous-time Takagi-Sugeno fuzzy model by applying generalized Lyapunov function and introducing outer variables. *Automatica* 2021, 125, 109409. [CrossRef]

- Faria, F.A.; Silva, G.N.; Oliveira, V.A. Reducing the conservatism of LMI-based stabilisation conditions for T-S fuzzy systems using fuzzy Lyapunov functions. Int. J. Syst. Sci. 2013, 44, 1956–1969. [CrossRef]
- 23. Pan, J.; Guerra, T.M.; Fei, S.; Jaadari, A. Nonquadratic stabilization of continuous T-S fuzzy models: LMI solution for a local approach. *IEEE Trans. Fuzzy Syst.* 2012, 20, 594–602. [CrossRef]
- Marquez, R.; Guerra, T.M.; Bernal, M.; Kruszewski, A. Asymptotically necessary and sufficient conditions for Takagi-Sugeno models using generalized non-quadratic parameter-dependent controller design. *Fuzzy Sets Syst.* 2017, 306, 48–62. [CrossRef]
- 25. Vafamand, N.; Asemani, M.H.; Khayatian, A. Robust *L*<sub>1</sub> Observer-based Non-PDC Controller Design for Persistent Bounded Disturbed TS Fuzzy Systems. *IEEE Trans. Fuzzy Syst.* **2017**, *26*, 1401–1413. [CrossRef]
- 26. Wang, L.; Liu, J.; Lam, H.K. Further study on stabilization for continuous-time Takagi-Sugeno fuzzy systems with time delay. *IEEE Trans. Cybern.* **2021**, *51*, 5637–5643. [CrossRef]
- Hu, G.; Wang, L.; Liu, X. A new approach to local H∞ control for continuous-time T-S fuzzy models. J. Intell. Fuzzy Syst. 2020, 39, 213–220. [CrossRef]
- 28. Wang, L.; Peng, J.; Liu, X. An approach to observer design of continuous-time Takagi-Sugeno fuzzy model with bounded disturbances. *Inf. Sci.* 2015, 324, 108–125. [CrossRef]
- Yan, Z.; Zhang, M.; Chang, G.; Lv, H.; Park, J. Finite-time annular domain stability and stabilization of Ito stochastic systems with Wiener noise and Poisson jumps-differential Gronwall inequality approach. *Appl. Math. Comput.* 2022, 412, 1–12. [CrossRef]
- 30. Yan, Z.; Su, F.; Gao, Z. Mean-square strong stability and stabilization of discrete-time stochastic systems with multiplicative noises. *Int. J. Robust Nonlinear Control* **2022**, *32*, 6767–6784. [CrossRef]
- 31. Hu, G.; Liu, X.; Wang, L.; Li, H. An extended approach to controller design of continuous-time Takagi-Sugeno fuzzy model. *J. Intell. Fuzzy Syst.* **2018**, *34*, 2235–2246. [CrossRef]
- Xie, X.P.; Liu, Z.W.; Zhu, X.L. An efficient approach for reducing the conservatism of LMI-based stability conditions for continuous-time TiCS fuzzy systems. *Fuzzy Sets Syst.* 2015, 263, 71–81. [CrossRef]
- 33. Wang, L.; Liu, X.; Zhang, H. Further studies on H∞ observer design for continuous-time T-S fuzzy model. *Inf. Sci.* 2018, 422, 396–407. [CrossRef]
- Kim, S.H. Relaxation technique for a T-S fuzzy control design based on a continuous-time fuzzy weighting-dependent Lyapunov function. *IEEE Trans. Fuzzy Syst.* 2013, 21, 761–766. [CrossRef]
- 35. Chen, S.H.; Ho, W.H.; Tsai, J.T.; Chou, J.H. Regularity and controllability robustness of TS fuzzy descriptor systems with structured parametric uncertainties. *Inf. Sci.* **2014**, 277, 36–55. [CrossRef]