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An Algebraic Characterization of Prefix-Strict Languages

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Abstract: Let Σ^+ be the set of all finite words over a finite alphabet Σ. A word u is called a strict prefix of a word v, if u is a prefix of v and there is no other way to show that u is a subword of v. A language $L \subseteq \Sigma^+$ is said to be prefix-strict, if for any $u, v \in L$, u is a subword of v always implies that u is a strict prefix of v. Denote the class of all prefix-strict languages in Σ^+ by $\mathcal{P}(\Sigma^+)$. This paper characterizes $\mathcal{P}(\Sigma^+)$ as a universe of a model of the free object for the ai-semiring variety satisfying the additional identities $x + yx \approx x$ and $x + yxz \approx x$. Furthermore, the analogous results for so-called suffix-strict languages and infix-strict languages are introduced.

Keywords: free algebra; formal languages; embedding order; ai-semirings variety

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1. Introduction

In algebraic theory of formal languages, there are two common methods to cluster languages. One is constructing algebra structure for a given class of languages, the other is collecting languages that satisfy a property with respect to a binary relation over the free monoid Σ^* (generated by a finite alphabet Σ). This paper aims at constructing an algebra structure for a class of languages that is defined by a partial order. Furthermore, we characterize the algebra structure as a model of a free object for a variety.

It is noted that algebraic and combinatorial properties of languages and words play a role in both clustering methods mentioned above. This is the case for the regular languages which can be defined by regular expressions. Let $\mathrm{Reg}(\Sigma^*)$ denote the class of regular languages over Σ and let $\cup$, $\circ$ and * denote the well-known regular operations, i.e., set union, catenation and Kleene closure, respectively. It can be obtained from the combinatorial properties of regular languages that $\mathrm{Reg}(\Sigma^*)$ is closed under all these operations. Hence, $(\mathrm{Reg}(\Sigma^*), \cup, \circ, ^*)$ forms an algebra structure [1], which contains all the regular expressions as its elements. This algebra has been widely applied in theoretical computer and information science. Another example is the semiring of finite languages. Let $\mathcal{F}(\Sigma^+)$ denote the class of all finite languages over Σ. It easy to see that $\mathcal{F}(\Sigma^+)$ is closed under the operations $\cup$ and $\circ$, and so $(\mathcal{F}(\Sigma^+), \cup, \circ)$ forms an algebra structure. However, language classes are not always closed under regular operations.

On the other hand, a partial order seems a more convenient tool for defining a language. Generally, for a given partial order $\leq$ over Σ^*, three types of language might be proposed. A language $L \subseteq \Sigma^*$ is said to be *convex* with respect to $\leq$, if for any $u, w \in L$ with $u \leq w$, the inequalities $u \leq v$ and $v \leq w$ always imply that $v \in L$, where $v \in \Sigma^*$. Further, L is said to be *closed* with respect to $\leq$, if $u \in L$ and $v \in u$ imply that $v \in L$. Furthermore, L is said to be *free* with respect to $\leq$, if it is an independent set with respect to $\leq$.

Many convex and free languages with respect to various binary relations were introduced by G. Thierrin, M. Ito and their co-researchers, and further studied by T. Ang and J. Brzozowski. They established the algebraic properties, combinatorial structures and decision algorithms of these three types of languages with respect to prefix relation, suffix relation, outfix relation, infix relation, factor relation, subword relation and so on. We refer the reader to [2–5] for details. In particular, a *hypercode* is a free language with respect to the embedding order (also known as a subword relation) also studied by L. Haines in [6]. Here, an *embedding order*, denoted by $\leq_{\mathcal{H}}$, is a partial order over Σ^* defined by: for any $u, v \in \Sigma^*$, $u \leq_{\mathcal{H}} v$ if and only if $u = x_1x_2 \cdots x_n$ and $v = y_0x_1y_1 \cdots x_ny_n$, where n is a positive integer and $x_i, y_i \in \Sigma^*$, $i = 0, 1, \cdots, n$. A language L is a hypercode means that for any $u, v \in L$, $(u, v) \notin \leq_{\mathcal{H}}$.

Since the definition of embedding order explicates directly the combination characterization of words involved in it, the combinatorial properties of hypercodes are almost certain to relate to them. L. Haines proved that every hypercode was finite. H. Shyr and G. Thierrin [3] proved that the class of all hypercodes over Σ, denoted by $\mathcal{H}(\Sigma^+)$, was closed under the regular operation $\circ$. Further, Z. Wang et al. defined in [7] a binary operation $+_{\mathcal{H}}$ in $\mathcal{H}(\Sigma^+)$ by picking out the minimal elements (in the sense of $\leq_{\mathcal{H}}$) from the union $A \cup B$ of $A, B \in \mathcal{H}(\Sigma^+)$. It was shown that $\mathcal{H}(\Sigma^+)$ was closed under $+_{\mathcal{H}}$ and hence $(\mathcal{H}(\Sigma^+), +_{\mathcal{H}}, \circ)$ formed an algebra structure.

Moreover, to construct an algebra structure for a language class also makes some sense in the effort to find a model of a free object for a variety. The well-known examples are structures of free semigroups and free commutative (noncommutative) algebras. The operation rules in these structures reflect the combinatorial properties among words and commutative (noncommutative) polynomials, which represent, respectively, the combinatorial properties of elements in a semigroup and commutative (noncommutative) algebra. When it comes to an algebra structure of a class of languages, if its operation rules reflect (or are defined by) the combinatorial characterizations of languages, then this structure has a probability to be a model of a free object for a variety, just as a free semigroup does.

By an *additively idempotent semiring* (ai-semiring for short) we mean a semiring whose additive reduct is a semilattice, i.e., a commutative idempotent semigroup. The variety of all ai-semirings is denoted by **AI**. M. Kuřil and L. Polák initiated the studies in the field of constructing a model of a free object for an ai-semiring variety by an algebra structure of a class of languages. In [8], they proved that the structure $(\mathcal{F}(\Sigma^+), \cup, \circ)$ was freely generated by Σ in the variety **AI**. We refer the reader to [9] for more detail on subvarieties of semilattice ordered algebras.

In addition, the algebra $(\mathcal{H}(\Sigma^+), +_{\mathcal{H}}, \circ)$ was also characterized as a model of a free object for an ai-semiring variety satisfying the additional identities $x + xy \approx x$ and $x + yx \approx x$ (see [7] for details). Undoubtedly, these two identities (named absorption laws) reflect some special combinatorial properties of hypercodes, which are derived from the embedding order.

Further, more ai-semiring varieties with absorption laws as additional identities were studied in [10]. The authors established combinatorial properties of three classes of languages containing hypercodes and constructed algebra structures for these classes, respectively. All these three structures were proved to be models of free objects for ai-semiring varieties satisfying $x + xy \approx x$, $x + yx \approx x$ and $xz + xyz \approx xz$, respectively. For literature on studying ai-semiring varieties by establishing combinatorial properties of words, we refer the reader to [11–17].

Following the study in [10], this paper focuses on ai-semiring varieties with absorption laws as additional identities. We define a class of so-called prefix-strict languages, denoted by $\mathcal{P}(\Sigma^+)$, and recall some notions in Section 2 as preliminary. In Section 3, we study a subset of the embedding order, which might be proved a partial order, say $\leq_{\mathcal{P}}$. It is shown that the class of free languages with respect to $\leq_{\mathcal{P}}$ coincides with the class $\mathcal{P}(\Sigma^+)$. Further, we establish in Section 4 some combinatorial properties for languages in $\mathcal{P}(\Sigma^+)$, which are used to verify the operation properties of the algebra structure construct for $\mathcal{P}(\Sigma^+)$. In Section 5, the class $\mathcal{P}(\Sigma^+)$ is characterized as a universe of an ai-semiring, which is freely generated by Σ in the variety with additional identities $x + yx \approx x$ and $x + yxz \approx x$. Moreover, some parallel concepts and results are introduced in this paper, simultaneously.

2. Preliminaries

Let Σ be a nonempty finite alphabet and Σ^* the set of all finite words. Denote the *empty word* by ε and let $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. Given two words $u, v \in \Sigma^*$, we say u is a *prefix* (*suffix*) of v, if there exists $x \in \Sigma^*$ such that $v = ux$ ($v = xu$). Furthermore, u is an *infix* of v, if $v = xuy$ for some $x, y \in \Sigma^*$. Clearly, a prefix or a suffix is also an infix.

Let $u, v \in \Sigma^+$. u is called a *subword* of v, if $u \leq_{\mathcal{H}} v$. Further, if $u \leq_{\mathcal{H}} v$ and $u \neq v$, then u is a *proper subword* of v. Furthermore, u is said to be a *factor* of v, if there exist $x, y \in \Sigma^*$ such that $v = xuy$. Thus, a prefix (suffix) or an infix of a word v must be its subword and factor as well.

Suppose that u is a subword of v. The following example shows that u, as a string of letters, may be embedded letter by letter into a word for obtaining v in different ways.

Example 1. *Let $u = ab$ and $v = ababab$ be two words in $\{a, b\}^*$. It is easy to see that ab is a prefix (suffix) and an infix of v. Now, if we consider $v = ababab$ as ay_1by_2, where $y_1 = ba$ and $y_2 = ab$, then we get another case to show that u is a subword of v.*

We concern ourselves with the case that being a prefix (infix) is the unique way to show that u is a subword v. Formally, we have the following definition.

Definition 1. *Let $u, v \in \Sigma^+$. For any $x_1, x_2, \cdots, x_n \in \Sigma^+$ and any $y_0, y_1, \cdots, y_n \in \Sigma^*$, if $u = x_1x_2\cdots x_n$, together with $v = y_0x_1y_1x_2y_2\cdots x_ny_n$, always implies that $y_0y_1\cdots y_{n-1} = \varepsilon$ ($y_1y_2\cdots y_n = \varepsilon$, $y_1\cdots y_{n-1} = \varepsilon$, respectively), then u is called a strict prefix (strict suffix, strict infix, respectively) of v. In particular, if $u \in \Sigma$ and there is only one occurrence of u in the word v, then u is also a strict infix of v.*

By this definition, we know that the word $u = ab$ in Example 1 is neither a strict prefix (strict suffix) nor a strict infix of v. Furthermore, it is easy to see that a strict prefix (strict suffix) must be a strict infix. The following two propositions give necessary and sufficient conditions for a prefix and infix to be strict, respectively.

Proposition 1. *Let $u = x\sigma$ and $v = x\sigma z$ be two words in Σ^+, where $\sigma \in \Sigma$ and $x, z \in \Sigma^*$. Then, u is a strict prefix of v if and only if there is no occurrence of σ in the word z.*

Proof. From the assumptions $u = x\sigma$ and $v = x\sigma z$, we know that u is a prefix of v.

Suppose that u is a strict prefix of v. Assume that σ is a factor of z. Then, $z = a\sigma b$ for some $a, b \in \Sigma^*$. Thus, the equality $v = x\sigma z = x\sigma a\sigma b$ holds. If we write $y_0xy_1\sigma b$ as $x\sigma a\sigma b$, where $y_0 = \varepsilon$ and $y_1 = \sigma a$, then $y_0y_1 \neq \varepsilon$, a contradiction. Therefore, σ is not a factor of z, as required.

Conversely, suppose that there is no occurrence of σ in the word z. Note that u is a subword of v, since u is a prefix of v. Given $x_1, x_2, \cdots, x_n \in \Sigma^*$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$, if both $u = x\sigma = x_1x_2\cdots x_n$ and $v = x\sigma z = y_0x_1y_1\cdots x_ny_n$ hold, then there exists $x'_n \in \Sigma^*$ such that

$$u = x\sigma = x_1x_2\cdots x'_n\sigma \text{ and } v = x\sigma z = y_0x_1y_1\cdots x'_n\sigma y_n,$$

since σ is the right-most letter of u. This means that $x = x_1x_2\cdots x'_n$. In the following, we show that σ is not a factor of y_n. In fact, if assume that if σ is a factor of y_n, then there exists $y'_n \in \Sigma^*$ such that $y_n = y'_n\sigma z$. Hence, we have that $v = x\sigma z = y_0x_1y_1\cdots x'_n\sigma y'_n\sigma z$. It follows that $x = y_0x_1y_1\cdots x'_n\sigma y'_n$. Therefore, $x_1x_2\cdots x'_n = y_0x_1y_1\cdots x'_n\sigma y'_n$ and so $|x_1x_2\cdots x'_n| = |y_0x_1y_1\cdots x'_n\sigma y'_n|$. Since $\sigma \neq \varepsilon$, we also have $|x_1x_2\cdots x'_n| < |y_0x_1y_1\cdots x'_n\sigma y'_n|$, a contradiction.

Now, we know that there is no occurrence of σ in the y_n. Then, we have that $y_n = z$ and so $x_1x_2\cdots x'_n = x = y_0x_1y_1\cdots x'_n$. This implies that $y_0y_1\cdots y_{n-1} = \varepsilon$. Therefore, u is a strict prefix of v, as required. □

In an analogue fashion, we can verify the following proposition and we omit the proof.

Proposition 2. *Let $u = \theta x \sigma$ and $v = y\theta x\sigma z$ be two words in Σ^+, where $\theta, \sigma \in \Sigma$, $x, y, z \in \Sigma^*$. Then, u is a strict infix of v if and only if there is no occurrence of θ in y and there is no occurrence of σ in z.*

Definition 2. *A language $L \subseteq \Sigma^+$ is said to be prefix-strict (suffix-strict, infix-strict, respectively) if and only if for any $u, v \in L$, $u \leq_{\mathcal{H}} v$ implies that u is a strict prefix (strict suffix, strict infix, respectively) of v. The class of all prefix-strict, suffix-strict and infix-strict languages in Σ^+ are denoted by $\mathcal{P}(\Sigma^+)$, $\mathcal{S}(\Sigma^+)$ and $\mathcal{I}(\Sigma^+)$, respectively.*

Let $w = \sigma_1\sigma_2 \cdots \sigma_n$ be a word with $\sigma_i \in \Sigma$, $i = 1, 2, \cdots, n$. Then, n is called the *length* of w and is denoted by $|w|$. Suppose that L is prefix-strict and that $u, v \in L$ with $|u| \leq |v|$. Then, we know from Definition 2 that either u is a strict prefix of v or $(u, v) \notin \leq_{\mathcal{H}}$. Hence, we have that a free language with respect to $\leq_{\mathcal{H}}$ must be a strict prefix. This means that $\mathcal{H}(\Sigma^+) \subseteq \mathcal{P}(\Sigma^+)$. Similarly, we can get that $\mathcal{H}(\Sigma^+) \subseteq \mathcal{S}(\Sigma^+)$ and $\mathcal{H}(\Sigma^+) \subseteq \mathcal{I}(\Sigma^+)$.

Example 2. *Let $\Sigma = \{a, b, c\}$. Suppose that $A = \{a, ab\}$, $B = \{b, ac\}$. Then, $C = \{ac, abc\}$. It is easy to see that A is prefix-strict and infix-strict but is not suffix-strict. B is prefix-strict (suffix-strict) and infix-strict, since $B \in \mathcal{H}(\Sigma^+)$. C is neither prefix-strict (suffix-stric) nor infix-strict, since ab is neither a strict prefix (strict suffix) nor a strict infix of abc, even if it is a subword.*

Now, recall the three subsets of the embedding order introduced in [10]. For any $u, v \in \Sigma^+$, $u \leq_{\mathcal{R}}$ ($u \leq_{\mathcal{L}}$) if and only if there exist some $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_1, y_2, \cdots, y_n \in \Sigma^*$ such that $u = x_1x_2 \cdots x_n$ and $v = y_1x_1y_2x_2 \cdots y_nx_n$ ($v = x_1y_1x_2y_2 \cdots x_ny_n$). In addition, $u \leq_{\mathcal{O}} v$ if and only if there exist some $x_0, x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_1, y_2, \cdots, y_n \in \Sigma^*$ such that $u = x_0x_1 \cdots x_n$ and $v = x_0y_1x_1y_2 \cdots x_{n-1}y_nx_n$.

All these binary relations prove to be partial orders over Σ^+. They show different manners to embed some string of words $y_1, y_2, \cdots, y_n$ into the word u for obtaining v. Denote the class of all free languages with respect to $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{O}}$ by $\mathcal{L}(\Sigma^+)$, $\mathcal{R}(\Sigma^+)$ and $\mathcal{O}(\Sigma^+)$, respectively. It is true that $\mathcal{H}(\Sigma^+) \subseteq \mathcal{X}(\Sigma^+)$, for all $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{O}\}$.

In the following, we recall the formal definitions of notions mentioned in the introduction section.

By a semiring, we mean an algebra $(S, +, \cdot)$ such that:

- The additive reduct $(S, +)$ is a commutative semigroup;
- The multiplicative reduct $(S, \cdot)$ is a semigroup;
- $(S, +, \cdot)$ satisfies the identities $x(y + z) \approx xy + xz$ and $(y + z)x \approx yx + zx$.

A semiring $(S, +, \cdot)$ is called an *ai-semiring*, if it satisfies the identity $x + x \approx x$. An algebra $(S, +, \cdot)$ is called a $(2, 2)$-*type* algebra if there are two binary operations involved in this algebra. In this manner, the additive reduct $(S, +)$ of $(S, +, \cdot)$ is a (2)-type algebra. For the formal definition of a type of an algebra and more examples, we refer the reader to Definition 1.2 in [18].

By a *variety*, we mean a class of algebras of the same type that is closed under subalgebras, homomorphic images and direct products. It is well known (Birkhoff's theorem) that a class of algebras of the same type is a variety if and only if it is an *equational class*, i.e., a class of algebras that satisfies all the members in a given set of identities.

An ai-semiring identity over Σ is an expression of the form $u \approx v$, where $u, v \in \mathcal{F}(\Sigma^+)$. For the free object $(\mathcal{F}(\Sigma^+), \cup, \circ)$ in **AI**, we write $+$ as $\cup$ and write

$$u_1 + u_2 + \cdots + u_k \approx v_1 + v_2 + \cdots + v_l$$

as the ai-semiring identity $\{u_i | 1 \leq k\} \approx \{v_i | 1 \leq l\}$, for convenience. We give an example in the following to show that a variety is an equational class.

Example 3. *Given a set of identities*

$$E = \{(x+y)+x \approx x+(y+z), x+y \approx y+x, (xy)z \approx x(yz), x(y+z) \approx xy+xz,$$
$$(y+z)x \approx yx+zx, x+x \approx x\}.$$

Since all ai-semirings satisfy the identities in E, the variety **AI** *is an equational class.*

Furthermore, we denote an ai-semiring variety satisfying the additional identities $u_i \approx v_i$, by $[u_1 \approx v_1, u_2 \approx v_2, \cdots, u_n \approx v_n]$, where $i = 1, 2, \cdots, n$ and n is a positive integer. Then, $[u_1 \approx v_1, u_2 \approx v_2, \cdots, u_n \approx v_n]$ is an equational class defined by the set $E \cup \{u_1 \approx v_1, u_2 \approx v_2, \cdots, u_n \approx v_n\}$, where E is the set of identities given by Example 2.

Let **V** be an algebra variety of type $\mathcal{F}$ and let $U(\Sigma)$ be an algebra of type $\mathcal{F}$ which is generated by Σ. If for every $K \in \mathbf{V}$ and for every map

$$\alpha : \Sigma \to K,$$

there is a unique homomorphism

$$\beta : U(\Sigma) \to K,$$

which extends α (i.e., $\alpha(\Sigma) = \beta(\Sigma)$ for $\sigma \in \Sigma$), then $U(\Sigma)$ is said to be *a free object* in **V** generated by Σ (or $U(\Sigma)$ is freely generated by Σ in **V**). For more details on free algebra, we refer the reader to [18].

In this paper, we take the following steps to show an algebra structure to be a model of a free object for an ai-semiring variety. Firstly, we verify that this algebra structure is a member of the given variety. Secondly, we prove that it is a free object in the variety.

In the sequel, u, v and w are words in Σ^+, unless otherwise specified.

3. Partial Orders

In this section, we shall characterize the class $\mathcal{P}(\Sigma^+)$ as a independent set of a certain partial order, namely, to show that languages in $\mathcal{P}(\Sigma^+)$ are free with respect to a partial order. Similar results for $\mathcal{S}(\Sigma^+)$ and $\mathcal{I}(\Sigma^+)$ are obtained. Furthermore, we study the inclusion relations among those classes of languages we mentioned in the last section.

Definition 3. *Let $\leq_{\mathcal{P}}$ be a binary relation over Σ^+. For any $u, v \in \Sigma^+$, $u \leq_{\mathcal{P}} v$ if and only if there exist $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$ such that $u = x_1x_2\cdots x_n$ and $v = y_0x_1y_1x_2\cdots y_{n-1}x_ny_n$ and such that the implication $y_0y_1\cdots y_{n-1} = \varepsilon \Rightarrow y_n = \varepsilon$ holds as well.*

It is easy to see that $\leq_{\mathcal{P}} \subseteq \leq_{\mathcal{H}}$. The following example shows that the implication in the definition of $\leq_{\mathcal{P}}$ may not always hold for every finite sequence $y_0, y_1, \cdots, y_n$ to state $u \leq_{\mathcal{H}} v$, even if $u \leq_{\mathcal{P}} v$.

Example 4. *Let $\Sigma = \{x_1, x_2, y\}$. Suppose that $u = x_1x_2$, $v = x_1yx_2$ and $w = x_1x_2y$. By the definition of $\leq_{\mathcal{P}}$, we have that $u \leq_{\mathcal{P}} v$ but $u \not\leq_{\mathcal{P}} w$. Further, assume that $w' = x_1x_2x_2$. Then, $u \leq_{\mathcal{P}} w'$, since we can set $w' = y_0x_1y_1x_2y_2$, where $y_0 = y_2 = \varepsilon$ and $y_1 = x_2$. From this, we deduce that the implication holds. However, if we write $y_0x_1y_1x_2y_2$ as w', where $y_0 = y_1 = \varepsilon$ and $y_2 = x_2$, then the implication $y_0y_1 = \varepsilon \Rightarrow y_2 = \varepsilon$ is not true.*

Similar to the definition of $\leq_{\mathcal{P}}$, we have:

Definition 4. *Let $\leq_{\mathcal{S}}$ be a binary relation over Σ^+. For any $u, v \in \Sigma^+$, $u \leq_{\mathcal{S}} v$ if and only if there exist $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$ such that $u = x_1x_2\cdots x_n$ and $v = y_0x_1y_1x_2\cdots y_{n-1}x_ny_n$ and such that the implication $y_1y_2\cdots y_n = \varepsilon \Rightarrow y_0 = \varepsilon$ holds as well.*

Let u be a proper subword of v. In the following, we present a necessary and sufficient condition for $u \leq_{\mathcal{P}} v$ to hold.

Assume that $u \leq_{\mathcal{P}} v$ and $u \neq v$. Then, there must exist $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$ such that $u = x_1 x_2 \cdots x_n$ and $v = y_0 x_1 y_1 x_2 \cdots y_{n-1} x_n y_n$ and scuh that $y_0 y_1 \cdots, y_{n-1} \neq \varepsilon$. In fact, if $y_0 y_1 \cdots, y_{n-1} = \varepsilon$, we know from the implication in Definition 1 that $y_n = \varepsilon$. This yields $u = v$, a contradiction.

Conversely, assume that $u = x_1 x_2 \cdots x_n$ and $v = y_0 x_1 y_1 x_2 \cdots y_{n-1} x_n y_n$ for some $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$, where $y_0 y_1 \cdots, y_{n-1} \neq \varepsilon$. In this case, the implication in Definition 1 holds. We thus have that u is a proper subword of v, since $y_0 y_1 \cdots, y_{n-1} \neq \varepsilon$.

Summarizing the above discussion with a similar condition for $u \leq_{\mathcal{S}} v$ to hold, we give the following remark to interpret the relations $\leq_{\mathcal{P}}$ and $\leq_{\mathcal{S}}$ in detail.

Remark 1. *Let u be a proper subword of v. Then, $u \leq_{\mathcal{P}} v$ ($u \leq_{\mathcal{S}} v$) if and only if $u = x_1 x_2 \cdots x_n$ and $v = y_0 x_1 y_1 x_2 \cdots x_n y_n$ for some $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1 \cdots, y_n \in \Sigma^*$, where $y_0 y_1 \cdots, y_{n-1} \neq \varepsilon$ ($y_1 y_2 \cdots, y_n \neq \varepsilon$).*

Let u be a proper subword of v. Suppose now that u is a strict prefix of v. Then, for any $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1 \cdots, y_n \in \Sigma^*$, $u = x_1 x_2 \cdots x_n$ together with $v = y_0 x_1 y_1 x_2 \cdots x_n y_n$ implies that $y_0 y_1 \cdots, y_{n-1} = \varepsilon$. This means that $(u, v) \notin \leq_{\mathcal{P}}$.

For any two words u and v, the following proposition gives the necessary and sufficient condition for $u \leq_{\mathcal{P}} v$ to hold, which may be used handily.

Proposition 3. *Let $u = x\sigma$ and $v = y\sigma z$ be two words in Σ^*, where $\sigma \in \Sigma$, $x, y, z \in \Sigma^*$ and there is no occurrence of σ in the word z. Then, $u \leq_{\mathcal{P}} v$ if and only if either x is a proper subword of y or $u = v$.*

Proof. Suppose that $u \leq_{\mathcal{P}} v$ and so $x\sigma \leq_{\mathcal{H}} y\sigma z$. Since there is no occurrence of σ in the word z, we get that $x \leq_{\mathcal{H}} y$. If x is not a proper subword of y, then $x = y$. This deduce that $v = x\sigma z$ and that $x\sigma \leq_{\mathcal{H}} x\sigma z$. By Proposition 1, we get that $x\sigma$ is a strict prefix of $x\sigma z$. Thus, we know from the discussion in Remark 1 that $x\sigma = x\sigma z$ and hence $u = v$.

Conversely, is it a trivial that $u = v$ implies $u \leq_{\mathcal{P}} v$. Suppose now that x is a proper subword of y. Then, there exist some $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1 \cdots, y_n \in \Sigma^*$ such that $u = x_1 x_2 \cdots x_n \sigma$ and $v = y_0 x_1 y_1 x_2 \cdots x_n y_n \sigma z$, where $y_0 y_1 \cdots y_n \neq \varepsilon$. By Remark 1, $u \leq_{\mathcal{P}} v$, as required. □

In the following, we shall prove that $\leq_{\mathcal{P}}$ and $\leq_{\mathcal{S}}$ are partial orders. By Definition 4, $\leq_{\mathcal{P}}$ and $\leq_{\mathcal{S}}$ are reflexive. Furthermore, it is a routine matter to verify that they are antisymmetric. Then, we have proved part of the following theorem.

Theorem 1. *Both $\leq_{\mathcal{P}}$ and $\leq_{\mathcal{S}}$ are partial orders.*

Proof. We only need to prove that $\leq_{\mathcal{P}}$ and $\leq_{\mathcal{S}}$ are transitive.

Assume that $u \leq_{\mathcal{P}} v$ and $v \leq_{\mathcal{P}} w$. If $u = v$ or $v = w$, it is easy to see that $u \leq_{\mathcal{P}} w$. Otherwise, u is a proper subword of v and v is a proper subword of w. Thus, u is a proper subword of w. Suppose now that $u = x\sigma$ and $v = y\sigma z$, where $\sigma \in \Sigma$, $x, y, z \in \Sigma^*$ and there is no occurrence of σ in the word z. By Proposition 3, x is a proper subword of y. Furthermore, there must exist $b, c \in \Sigma^*$ such that $w = b\sigma c$ (since σ is a factor of v an so a factor of w), where there is no occurrence of σ in c. Associate this fact with the truth $y\sigma z \leq_{\mathcal{H}} b\sigma c$, we can verify $y \leq_{\mathcal{H}} b$ as a matter of routine. Hence, we get that x is a proper subword of b. Therefore, we know from Proposition 3 that $u \leq_{\mathcal{P}} w$ and so $\leq_{\mathcal{P}}$ is transitive, as required.

A similar result is also true for $\leq_{\mathcal{S}}$ and we omit the proof. □

Definition 5. *Let $\leq_{\mathcal{I}}$ be a binary relation over Σ^+. For any $u, v \in \Sigma^+$, $u \leq_{\mathcal{I}} v$ if and only if there exist $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$ such that $u = x_1x_2\cdots x_n$ and $v = y_0x_1y_1x_2\cdots y_{n-1}x_ny_n$ and that the implication $y_1y_2\cdots y_{n-1} = \varepsilon \Rightarrow y_0y_n = \varepsilon$ holds as well. In particular, if $|u| = 1$, then $u \leq_{\mathcal{I}} v$ if only if $u = v$.*

The following example shows that the implication in Definition 5 may not always hold for every finite sequence $y_0, y_1, \cdots, y_n$ to state $u \leq_{\mathcal{H}} v$, even if $u \leq_{\mathcal{I}} v$.

Example 5. *Let $\Sigma = \{x_1, x_2, a, b, c\}$. Suppose that $u = x_1x_2$, $v = ax_1bx_2c$ and $w = ax_1x_2b$. By the definition of $\leq_{\mathcal{I}}$, we have that $u \leq_{\mathcal{I}} v$ but $u \not\leq_{\mathcal{I}} w$. Further, assume that $w' = x_1x_1x_2x_2$. Then, $u \leq_{\mathcal{I}} w'$, since we can set $w' = y_0x_1y_1x_2y_2$, where $y_0 = \varepsilon$, $y_1 = x_1$ and $y_2 = x_2$. Thus, we deduce that the implication holds. However, if we write $y_0x_1y_1x_2y_2$ as w', where $y_0 = x_1$, $y_1 = \varepsilon$ and $y_2 = x_2$, then the implication $y_1 = \varepsilon \Rightarrow y_0y_2 = \varepsilon$ is not true.*

Similar to Remark 1, the following remark give a further specification for Definition 5.

Remark 2. *Let u (with $|u| \geq 2$) be a proper subword of v. Then, $u \leq_{\mathcal{I}} v$ if and only if $u = x_1x_2\cdots x_n$ and $v = y_0x_1y_1x_2\cdots x_ny_n$ for some $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1 \cdots, y_n \in \Sigma^*$, where $y_1y_2\cdots y_{n-1} \neq \varepsilon$.*

Let u be a proper subword of v. Suppose now that u is a strict infix of v. Then, for any $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1 \cdots, y_n \in \Sigma^*$, $u = x_1x_2\cdots x_n$ together with $v = y_0x_1y_1x_2\cdots x_ny_n$ implies that $y_1y_2\cdots, y_{n-1} = \varepsilon$. This means that $(u, v) \notin \leq_{\mathcal{L}}$.

From Remarks 1 and 2, we have $\leq_{\mathcal{I}} \subseteq \leq_{\mathcal{P}}$ and $\leq_{\mathcal{I}} \subseteq \leq_{\mathcal{S}}$.

We give a necessary and sufficient condition for $u \leq_{\mathcal{I}} v$ to hold. Since the proof process is similar to Proposition 3, we omit the proof.

Proposition 4. *Let $u = \theta a\sigma$ and $v = x\theta y\sigma z$, where $\theta, \sigma \in \Sigma$, $a, x, y, z \in \Sigma^*$ and there is no occurrence of θ and σ in the words x and z, respectively. Then, $u \leq_{\mathcal{I}} v$ if and only if either a is a proper subword of y or $u = v$.*

Using this proposition, we can prove the following theorem.

Theorem 2. *$\leq_{\mathcal{I}}$ is a partial order.*

Proof. It is a routine matter to verify that $\leq_{\mathcal{I}}$ is reflexive and antisymmetric. Now, we show that it is also transitive.

Assume that $u \leq_{\mathcal{I}} v$ and $v \leq_{\mathcal{I}} w$. If $u = v$ or $v = w$, it is easy to see that $u \leq_{\mathcal{I}} w$. Otherwise, u is a proper subword of v and v is a proper subword of w. Thus, u is a proper subword of w.

Suppose now that $u = \theta a\sigma$ and $v = x\theta y\sigma z$, where $\theta, \sigma \in \Sigma$, $a, x, y, z \in \Sigma^*$ and there are no occurrences of θ and σ in the words x and z, respectively. By Proposition 4, a is a proper subword of y. Furthermore, there must exist $b, c, d \in \Sigma^*$ such that $w = b\theta c\sigma d$ (since both θ and σ are factors of v and so are factors of w), where there are no occurrences of θ and σ in the words b and d, respectively. Associate this fact with the truth $x\theta y\sigma z \leq_{\mathcal{H}} b\theta c\sigma d$, we deduce that $y \leq_{\mathcal{H}} c$. Hence, we get that a is a proper subword of c. Therefore, we know from Proposition 4 that $u \leq_{\mathcal{P}} w$ and so $\leq_{\mathcal{I}}$ is transitive. Then, we obtain that $\leq_{\mathcal{I}}$ is a partial order, as required. □

Recall that the classes $\mathcal{P}(\Sigma^+)$, $\mathcal{S}(\Sigma^+)$ and $\mathcal{I}(\Sigma^+)$ are collections of all prefix-strict, suffix-strict and infix-strict languages in Σ^+, respectively. In the following, we show that for any $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$, $\mathcal{X}(\Sigma^+)$ is exactly the collection of all free languages with respect to the partial order $\leq_{\mathcal{X}}$.

Proposition 5. *$\mathcal{P}(\Sigma^+)$ and $\mathcal{S}(\Sigma^+)$ are the classes of all free languages with respect to $\leq_\mathcal{P}$ and $\leq_\mathcal{S}$, respectively.*

Proof. Let $L \subseteq \Sigma^+$ be a prefix-strict language and u, v be distinct words in L. On one hand, if u and v are incomparable under the relation $\leq_\mathcal{H}$, they are also incomparable under $\leq_\mathcal{P}$. On the other hand, we suppose that u is a proper subword of v. Then, u is a strict prefix of v. By Remark 1, $(u,v) \notin \leq_\mathcal{P}$ and hence L is a free language with respect to $\leq_\mathcal{P}$.

Conversely, let L be a free language with respect to $\leq_\mathcal{P}$ and $u, v \in L$. Suppose that $u \leq_\mathcal{H} v$. For any $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$, if $u = x_1 x_2 \cdots x_n$ and $v = y_0 x_1 y_1 x_2 y_2 \cdots x_n y_n$, then it is true that $y_0 y_1 \cdots y_{n-1} = \varepsilon$ (in fact, if $y_0 y_1 \cdots y_{n-1} \neq \varepsilon$, we get that $u \leq_\mathcal{P} v$, contradict to the assumption) and it follows that u is a strict prefix of v. Therefore, L is prefix-strict.

We can prove that $\mathcal{S}(\Sigma^+)$ is the class of all free languages with respect to $\leq_\mathcal{S}$ in a similar way. Therefore, the proof is omitted. □

Proposition 6. *$\mathcal{I}(\Sigma^+)$ is the class of all free languages with respect to $\leq_\mathcal{I}$,*

Proof. Let $L \subseteq \Sigma^+$ be an infix-strict language and u, v be distinct words in L. If $(u,v) \notin \leq_\mathcal{H}$, then $(u,v) \notin \leq_\mathcal{I}$. Otherwise, assume that u is a proper subword of v. Then, u is a strict infix of v. By Remark 2, $(u,v) \notin \leq_\mathcal{I}$. Therefore, L is a free language with respect to $\leq_\mathcal{I}$.

Conversely, let L be a free language with respect to $\leq_\mathcal{I}$ and u, v be two distinct words in L. For any $x_1, x_2, \cdots, x_n \in \Sigma^+$ and any $y_0, y_1, \cdots, y_n \in \Sigma^*$, if $u = x_1 x_2 \cdots x_n$ and $v = y_0 x_1 y_1 x_2 y_2 \cdots x_n y_n$, then it is true that $y_1 y_2 \cdots y_{n-1} = \varepsilon$ and it follows that u is a strict infix of v. Therefore, L is infix-strict. □

Recall that a binary relation ρ on Σ^+ is said to be *strict* ([4]) if for all $u, v \in \Sigma^+$,

(1) $u \rho u$;
(2) $u \rho v \Rightarrow |u| \leq |v|$;
(3) $u \rho v, |u| = |v| \Rightarrow u = v$.

It is can be easily verified that for any $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{O}, \mathcal{H}, \mathcal{P}, \mathcal{S}, \mathcal{I}\}$, $\leq_\mathcal{X}$ is strict. Based on the following lemma, we can figure out the inclusion relation about all these strict relations.

Lemma 1 ([4]). *Let ρ_1, ρ_2 be two strict binary relations on Σ^+ and I_{ρ_1} and I_{ρ_2} be the classes of all independent sets with respect to ρ_1 and ρ_2, respectively. Then, $\rho_1 \subseteq \rho_2$ if and only if $I_{\rho_1} \supseteq I_{\rho_2}$.*

Since $\leq_\mathcal{I} \subseteq \leq_\mathcal{X} \subseteq \leq_\mathcal{H}$ for any $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}\}$, we know form Lemma 1 that

$$\mathcal{H}(\Sigma^+) \subseteq \mathcal{X}(\Sigma^+) \subseteq \mathcal{I}(\Sigma^+).$$

Furthermore, it is routine to verify that $\leq_\mathcal{R} \subseteq \leq_\mathcal{P}$, $\leq_\mathcal{L} \subseteq \leq_\mathcal{S}$ and $\leq_\mathcal{O} \subseteq \leq_\mathcal{I}$. We then have that

$$\mathcal{P}(\Sigma^+) \subseteq \mathcal{R}(\Sigma^+),\ \mathcal{S}(\Sigma^+) \subseteq \mathcal{L}(\Sigma^+) \text{ and } \mathcal{I}(\Sigma^+) \subseteq \mathcal{O}(\Sigma^+).$$

Inaddition, it is shown in [10] that

$$\mathcal{H}(\Sigma^+) \subseteq \mathcal{Y}(\Sigma^+) \subseteq \mathcal{O}(\Sigma^+),$$

where $\mathcal{Y} \in \{\mathcal{L}, \mathcal{R}\}$. We illustrate all above inclusion relations by Figure 1.

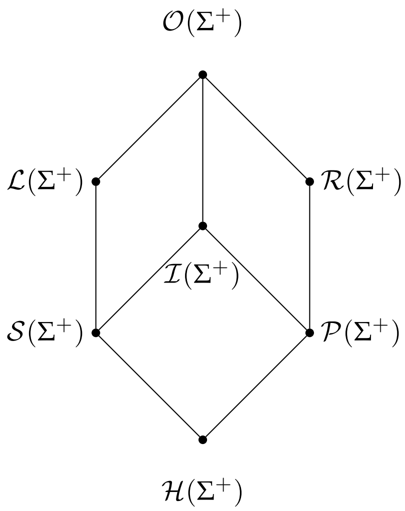

Figure 1. Inclusion relations among subsets of $\mathcal{O}(\Sigma^+)$.

Since it was proved in [10] that every language in $\mathcal{O}(\Sigma^+)$ is finite, we get that any language in $\mathcal{P}(\Sigma^+) \cup \mathcal{S}(\Sigma^+) \cup \mathcal{B}(\Sigma^+)$ is also finite.

4. Combinatorial Properties

In this section, we study the combinatorial properties of languages we defined in the last section. Let A, B be languages of Σ^*. We write $A \circ B$ to mean

$$\{ab \mid a \in A, b \in B\}.$$

Shyr and Thierrin [3] proved that the class of $\mathcal{H}(\Sigma^+)$ was closed under the operation $\circ$ and Ito et al. [2] showed a similar result for the class of outfix-free languages. However, for two prefix-strict (suffix-strict, infix-strict, respectively) languages A and B, $A \circ B$ does not need to be prefix-strict (suffix-strict, infix-strict, respectively), as the following example shows.

Example 6. *Let $\Sigma = \{a, b, c\}$. Suppose that $A = \{a, ab\}$, $B = \{c\}$. Then, $A \circ B = \{ac, abc\}$. It is easy to see that A and B are prefix-strict, but $A \circ B$ is not prefix-strict, since $ac \leq_{\mathcal{P}} abc$.*

Next, we give a necessary and sufficient condition for $\mathcal{X}(\Sigma^+)$ to be closed under the operation $\circ$, where $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$.

Proposition 7. *Let $A, B \in \mathcal{P}(\Sigma^+)$. Then, $A \circ B \in \mathcal{P}(\Sigma^+)$ if and only if $A \in \mathcal{H}(\Sigma^+)$.*

Proof. Let $A, B \in \mathcal{P}(\Sigma^+)$. Then, for any $a, a' \in A$ with $|a| < |a'|$, we have that either $(a, a') \notin \leq_{\mathcal{H}}$ or a is a strict prefix of a'.

Assume that $A \circ B \in \mathcal{P}(\Sigma^+)$. Then, it is a truth that a is not a strict prefix of a'. In fact, if $a' = ac$ for some $c \in \Sigma^+$, then for any $b \in B$, we have that $ab, acb \in A \circ B$. Since $ab \leq_{\mathcal{P}} acb$, we get $A \circ B \notin \mathcal{P}(\Sigma^+)$, a contradiction. This shows that $(a, a') \notin \leq_{\mathcal{H}}$ and hence $A \in \mathcal{H}(\Sigma^+)$.

Conversely, assume that $A \in \mathcal{H}(\Sigma^+)$ and $B \in \mathcal{P}(\Sigma^+)$. Given $u, v \in A \circ B$. Suppose that u is a subword of v. Then, $u = x_1 x_2 \cdots x_n$ and $v = y_0 x_1 y_1 x_2 y_2 \cdots y_{n-1} x_n y_n$ for some $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, y_2, \cdots, y_n \in \Sigma^*$. Now, we prove that u is a strict prefix of v.

Let i and j be two integers such that $x_1 x_2 \cdots x_i \in A,\ x_{i+1} x_{i+2} \cdots x_n \in B$ and

$$y_0 x_1 y_1 \cdots y_{j-1} x_j \in A,\ y_j x_{j+1} \cdots x_n y_n \in B$$

$$(\text{ or } y_0 x_1 y_1 \cdots y_{j-1} x_j y_j \in A,\ x_{j+1} y_{j+1} \cdots x_n y_n \in B).$$

It istrue that $i \leq j$. In fact, if $i > j$, then we have from the fact $x_{j+1} \neq \varepsilon$ that

$$x_{i+1}x_{i+2}\cdots x_n \leq_{\mathcal{P}} y_j x_{j+1}\cdots y_i x_{i+1}\cdots x_n y_n \text{ or}$$

$$x_{i+1}x_{i+2}\cdots x_n \leq_{\mathcal{P}} x_{j+1}\cdots y_i x_{i+1}\cdots x_n y_n.$$

Bothof these two cases contradict to $B \in \mathcal{P}(\Sigma^+)$. Furthermore, we have $j \leq i$. In fact, if $j > i$, then we have that

$$x_1x_2\cdots x_i \leq_{\mathcal{H}} y_0x_1y_1\cdots x_iy_{i+1}\cdots y_{j-1}x_j$$

$$\text{or } x_1x_2\cdots x_i \leq_{\mathcal{H}} y_0x_1y_1\cdots x_iy_{i+1}\cdots y_{j-1}x_jy_j,$$

which contradicts $A \in \mathcal{H}(\Sigma^+)$. Hence, we have $i = j$. It follows that $y_0y_1\cdots y_{i-1} = \varepsilon$ (or $y_0y_1\cdots y_i = \varepsilon$) and that $y_iy_{i+1}\cdots y_{n-1} = \varepsilon$ (or $y_{i+1}\cdots y_{n-1} = \varepsilon$). This implies that u is a strict prefix of v. Therefore, $A \circ B \in \mathcal{P}(\Sigma^+)$. $\square$

By a similar method, one can verify the following proposition.

Proposition 8. *Let $A, B \in \mathcal{S}(\Sigma^+)$. Then, $A \circ B \in \mathcal{S}(\Sigma^+)$ if and only if $B \in \mathcal{H}(\Sigma^+)$.*

Proposition 9. *Let $A, B \in \mathcal{I}(\Sigma^+)$. Then, $A \circ B \in \mathcal{I}(\Sigma^+)$ if and only if $A \in \mathcal{S}(\Sigma^+)$ and $B \in \mathcal{P}(\Sigma^+)$.*

Proof. Let $A, B \in \mathcal{I}(\Sigma^+)$. Suppose that $A \circ B \in \mathcal{I}(\Sigma^+)$. If we assume that there exist $u, v \in A$ such that $u \leq_{\mathcal{S}} v$, then there exist $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, \cdots, y_n \in \Sigma^*$ such that $u = x_1x_2\cdots x_n$ and $v = y_0x_1y_1x_2\cdots x_ny_n$ with $y_1y_2\cdots y_n \neq \varepsilon$. It follows that

$$x_1x_2\cdots x_nb \leq_{\mathcal{I}} y_0x_1y_1x_2\cdots y_{n-1}x_ny_nb$$

for any $b \in B$, which is a contradiction with $A \circ B \in \mathcal{I}(\Sigma^+)$. Hence, we deduce that $A \in \mathcal{S}(\Sigma^+)$. In a similar way, we can prove that $A \in \mathcal{P}(\Sigma^+)$.

Conversely, assume that $A \in \mathcal{S}(\Sigma^+)$ and $B \in \mathcal{P}(\Sigma^+)$. Given $u, v \in A \circ B$. Suppose that u is a subword of v. Then, $u = x_1x_2\cdots x_n$ and $v = y_0x_1y_1x_2y_2\cdots y_{n-1}x_ny_n$ for some $x_1, x_2, \cdots, x_n \in \Sigma^+$ and $y_0, y_1, y_2, \cdots, y_n \in \Sigma^*$. Now, we prove that u is a strict infix of v.

Let i and j be two integers such that $x_1x_2\cdots x_i \in A,\ x_{i+1}x_{i+2}\cdots x_n \in B$ and

$$y_0x_1y_1\cdots y_{j-1}x_j \in A,\ y_jx_{j+1}\cdots x_ny_n \in B$$

$$(\text{ or } y_0x_1y_1\cdots y_{j-1}x_jy_j \in A,\ x_{j+1}y_{j+1}\cdots x_ny_n \in B).$$

It istrue that $i \leq j$. In fact, if $i > j$, then we have from the fact $x_{j+1} \neq \varepsilon$ that

$$x_{i+1}x_{i+2}\cdots x_n \leq_{\mathcal{P}} y_jx_{j+1}\cdots y_ix_{i+1}\cdots x_ny_n \text{ or}$$

$$x_{i+1}x_{i+2}\cdots x_n \leq_{\mathcal{P}} x_{j+1}\cdots y_ix_{i+1}\cdots x_ny_n.$$

Bothof these two cases contradict $B \in \mathcal{P}(\Sigma^+)$. Furthermore, we have $j \leq i$. In fact, if $j > i$, then we have that

$$x_1x_2\cdots x_i \leq_{\mathcal{S}} y_0x_1y_1\cdots x_iy_{i+1}\cdots y_{j-1}x_j$$

$$\text{or } x_1x_2\cdots x_i \leq_{\mathcal{S}} y_0x_1y_1\cdots x_iy_{i+1}\cdots y_{j-1}x_jy_j,$$

which contradicts to $A \in \mathcal{S}(\Sigma^+)$. Hence, we have $i = j$. If follows that $x_1x_2\cdots x_i$ and $y_0x_1y_1\cdots y_{i-1}x_i$ (or $y_0x_1y_1\cdots y_{i-1}x_iy_i$) are elements in A. This implies that $y_0y_1\cdots y_{i-1} = \varepsilon$ (or $y_0y_1\cdots y_i = \varepsilon$). Furthermore, from the fact that $x_{i+1}x_{i+2}\cdots x_n$ and $y_ix_{i+1}y_{i+1}\cdots x_ny_n$ (or $x_{i+1}y_{i+1}\cdots x_ny_n$) are elements in B, we have that $y_iy_{i+1}\cdots y_{n-1} = \varepsilon$ (or $y_{i+1}\cdots y_{n-1} = \varepsilon$). Hence, u is a strict infix of v. Therefore, $A \circ B \in \mathcal{I}(\Sigma^+)$. $\square$

Let $A \in \mathcal{F}(\Sigma^+)$. For any $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$. We denote the set

$$\{a \in A \mid (\forall b \in A)\ b \leq_{\mathcal{X}} a \Rightarrow b = a\}$$

by $A^{\mathcal{X}}$, which is a free language with respect to $\leq_{\mathcal{X}}$. Thus, $A^{\mathcal{X}} \in \mathcal{X}(\Sigma^+)$. It is easy to see that $A \in \mathcal{X}(\Sigma^+)$ if and only if $A^{\mathcal{X}} = A$. Then, we have $(A^{\mathcal{X}})^{\mathcal{X}} = A^{\mathcal{X}}$. Further, we have:

Lemma 2. *Both* $(A^{\mathcal{I}})^{\mathcal{X}} = A^{\mathcal{X}}$ *and* $(A^{\mathcal{X}})^{\mathcal{H}} = A^{\mathcal{H}}$ *hold for any* $A \in \mathcal{F}(\Sigma^+)$ *and* $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}, \mathcal{H}\}$.

Proof. We only prove the equality $(A^{\mathcal{I}})^{\mathcal{P}} = A^{\mathcal{P}}$. The other one can be proved in analogous fashion.

Suppose that $a \in A^{\mathcal{P}}$. Then, a is a minimal element in A with respect to $\leq_{\mathcal{P}}$. Since $A^{\mathcal{P}} \subseteq A^{\mathcal{I}} \subseteq A$, a is also a minimal element in $A^{\mathcal{I}}$ with respect to $\leq_{\mathcal{P}}$. That is to say, $a \in (A^{\mathcal{I}})^{\mathcal{P}}$ and so $A^{\mathcal{P}} \subseteq (A^{\mathcal{I}})^{\mathcal{P}}$.

On the other hand, suppose that $a \in (A^{\mathcal{I}})^{\mathcal{P}}$. We now show that a is a minimal element in A with respect to $\leq_{\mathcal{P}}$. Let $b \leq_{\mathcal{P}} a$ for some $b \in A$. If $b \in A^{\mathcal{I}}$, then $b = a$, since a a minimal element in $A^{\mathcal{I}}$ with respect to $\leq_{\mathcal{P}}$; otherwise, $b \in A \setminus A^{\mathcal{I}}$. Then, there exists $c \in A^{\mathcal{I}}$ such that $c \leq_{\mathcal{I}} b$ and so $c \leq_{\mathcal{P}} b$, since $\leq_{\mathcal{I}} \subseteq \leq_{\mathcal{P}}$. We thus have $c \leq_{\mathcal{P}} b \leq_{\mathcal{P}} a$. This implies that $c = a$ and so $b = a$. Therefore, $a \in A^{\mathcal{P}}$ and hence $(A^{\mathcal{I}})^{\mathcal{P}} \subseteq A^{\mathcal{P}}$, as required. □

We conclude this section with the following results.

Proposition 10. *Let* $A, B \in \mathcal{F}(\Sigma^+)$. *Then*

(1) $(A \circ B)^{\mathcal{I}} = A^{\mathcal{S}} \circ B^{\mathcal{P}}$;
(2) $(A \circ B)^{\mathcal{P}} = A^{\mathcal{H}} \circ B^{\mathcal{P}}$;
(3) $(A \circ B)^{\mathcal{S}} = A^{\mathcal{S}} \circ B^{\mathcal{H}}$.

Proof. (1) Since $A^{\mathcal{S}} \in \mathcal{S}(\Sigma^+)$ and $B^{\mathcal{P}} \in \mathcal{P}(\Sigma^+)$, we know from Proposition 9 that $A^{\mathcal{S}} \circ B^{\mathcal{P}} \in \mathcal{I}(\Sigma^+)$. It follows that $(A^{\mathcal{S}} \circ B^{\mathcal{P}})^{\mathcal{I}} = A^{\mathcal{S}} \circ B^{\mathcal{P}}$. Notice that $A^{\mathcal{S}} \subseteq A$ and $B^{\mathcal{P}} \subseteq B$. Then, $A^{\mathcal{S}} \circ B^{\mathcal{P}} \subseteq A \circ B$. We thus have $(A^{\mathcal{S}} \circ B^{\mathcal{P}})^{\mathcal{I}} \subseteq (A \circ B)^{\mathcal{I}}$ and so $A^{\mathcal{S}} \circ B^{\mathcal{P}} \subseteq (A \circ B)^{\mathcal{I}}$

On the other hand, let $a \in A \setminus A^{\mathcal{S}}$. Then, $a' \leq_{\mathcal{S}} a$ for some $a' \in A^{\mathcal{S}}$. Thus, $a'b \leq_{\mathcal{I}} ab$ for any $b \in B$. It follows that $ab \in (A \circ B) \setminus (A \circ B)^{\mathcal{I}}$. This shows that

$$ab \in (A \circ B) \setminus (A^{\mathcal{S}} \circ B) \Rightarrow ab \in (A \circ B) \setminus (A \circ B)^{\mathcal{I}}.$$

Hence,we have that $(A \circ B)^{\mathcal{I}} \subseteq A^{\mathcal{S}} \circ B$. Further, if $b \in B \setminus B^{\mathcal{P}}$ then $b' \leq_{\mathcal{P}} b$ for some $b \in B^{\mathcal{P}}$. We thus have $ab' \leq_{\mathcal{I}} ab$ for any $a \in A^{\mathcal{S}}$ and so $ab \notin (A \circ B)^{\mathcal{I}}$. This shows that

$$ab \in (A^{\mathcal{S}} \circ B) \setminus (A^{\mathcal{S}} \circ B^{\mathcal{P}}) \Rightarrow ab \in (A^{\mathcal{S}} \circ B) \setminus (A \circ B)^{\mathcal{I}},$$

which means that $(A \circ B)^{\mathcal{I}} \subseteq A^{\mathcal{S}} \circ B^{\mathcal{P}}$. Then, we obtain that $(A \circ B)^{\mathcal{I}} = A^{\mathcal{S}} \circ B^{\mathcal{P}}$, as required.

In an analogous fashion, we can prove (2) and (3) by using Propositions 7 and 8, respectively. So we omit the proof. □

5. Algebraic Characterizations

In this section, for any $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$, we construct an algebra structure for the class $\mathcal{X}(\Sigma^+)$, by defining two binary operations. The operation properties of these algebra structures are dominated by the combinatorial properties of languages discussed in the last section. Further, we prove that these algebra structures are ai-semirings by showing each of them is isomorphic to a quotient algebra of $\mathcal{F}(\Sigma^+)$ over an ai-semiring congruence. Furthermore, we show that the algebra $\mathcal{X}(\Sigma^+)$ is free generated by Σ in a subvariety of **AI**. This gives an algebraic characterization for the class $\mathcal{X}(\Sigma^+)$.

Let $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$. Define two operations on $\mathcal{X}(\Sigma^+)$ as follows:

$$(\forall A, B \in \mathcal{X}(\Sigma^+))\ A +_{\mathcal{X}} B = (A \cup B)^{\mathcal{X}},\ A \times_{\mathcal{X}} B = (A \circ B)^{\mathcal{X}}.$$

In this section, we show that $(\mathcal{X}(\Sigma^+), +_{\mathcal{X}}, \times_{\mathcal{X}})$ is free generated by Σ in some ai-semiring variety.

5.1. Congruences

For every $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$, we define a binary relation $\sim_{\mathcal{X}}$ on $\mathcal{F}(\Sigma^+)$ by

$$A \sim_{\mathcal{X}} B \Leftrightarrow A^{\mathcal{X}} = B^{\mathcal{X}}.$$

Lemma 3. *For any $A, B \in \mathcal{F}(\Sigma^+)$ and any $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$, we have that*

$$\begin{aligned}(A^{\mathcal{X}} \cup B)^{\mathcal{X}} &= (A \cup B)^{\mathcal{X}} = (A \cup B^{\mathcal{X}})^{\mathcal{X}},\\ (A^{\mathcal{X}} \circ B)^{\mathcal{X}} &= (A \circ B)^{\mathcal{X}} = (A \circ B^{\mathcal{X}})^{\mathcal{X}}.\end{aligned}$$

Proof. For any $A, B \in \mathcal{F}(\Sigma^+)$, in order to pick out all the minimal elements with respect to $\leq_{\mathcal{X}}$ from $A \cup B$, we firstly pick out the minimal elements from A and B, respectively. Thus, it is true that $(A \cup B)^{\mathcal{X}} \subseteq A^{\mathcal{X}} \cup B^{\mathcal{X}}$. Next, we pick out the minimal elements from $A^{\mathcal{X}} \cup B^{\mathcal{X}}$, then we get all the minimal elements in $A \cup B$. Hence, $(A \cup B)^{\mathcal{X}} = (A^{\mathcal{X}} \cup B^{\mathcal{X}})^{\mathcal{X}}$. Therefore, we have that

$$\begin{aligned}(A^{\mathcal{X}} \cup B)^{\mathcal{X}} &= ((A^{\mathcal{X}})^{\mathcal{X}} \cup B^{\mathcal{X}})^{\mathcal{X}} = (A^{\mathcal{X}} \cup B^{\mathcal{X}})^{\mathcal{X}},\\ (A \cup B^{\mathcal{X}})^{\mathcal{X}} &= (A^{\mathcal{X}} \cup (B^{\mathcal{X}})^{\mathcal{X}})^{\mathcal{X}} = (A^{\mathcal{X}} \cup B^{\mathcal{X}})^{\mathcal{X}}.\end{aligned}$$

These show that $(A^{\mathcal{X}} \cup B)^{\mathcal{X}} = (A \cup B)^{\mathcal{X}} = (A \cup B^{\mathcal{X}})^{\mathcal{X}}$.

To prove the remaining equalities, there are three cases to consider:

(1) $\mathcal{X} = \mathcal{I}$. By Proposition 10(1) and Lemma 2, we have that

$$\begin{aligned}(A^{\mathcal{I}} \circ B)^{\mathcal{I}} &= (A^{\mathcal{I}})^{\mathcal{S}} \circ B^{\mathcal{P}} = A^{\mathcal{S}} \circ B^{\mathcal{P}} = (A \circ B)^{\mathcal{I}},\\ (A \circ B^{\mathcal{I}})^{\mathcal{I}} &= A^{\mathcal{S}} \circ (B^{\mathcal{I}})^{\mathcal{P}} = A^{\mathcal{S}} \circ B^{\mathcal{P}} = (A \circ B)^{\mathcal{I}}.\end{aligned}$$

(2) $\mathcal{X} = \mathcal{P}$. By Proposition 10(2) and Lemma 2, we have that

$$\begin{aligned}(A^{\mathcal{P}} \circ B)^{\mathcal{P}} &= (A^{\mathcal{P}})^{\mathcal{H}} \circ B^{\mathcal{P}} = A^{\mathcal{H}} \circ B^{\mathcal{P}} = (A \circ B)^{\mathcal{P}},\\ (A \circ B^{\mathcal{P}})^{\mathcal{P}} &= A^{\mathcal{H}} \circ (B^{\mathcal{P}})^{\mathcal{P}} = A^{\mathcal{H}} \circ B^{\mathcal{P}} = (A \circ B)^{\mathcal{P}}.\end{aligned}$$

(3) $\mathcal{X} = \mathcal{S}$. By Proposition 10(3) and Lemma 2, we have that

$$\begin{aligned}(A^{\mathcal{S}} \circ B)^{\mathcal{S}} &= (A^{\mathcal{S}})^{\mathcal{S}} \circ B^{\mathcal{H}} = A^{\mathcal{S}} \circ B^{\mathcal{H}} = (A \circ B)^{\mathcal{S}},\\ (A \circ B^{\mathcal{S}})^{\mathcal{S}} &= A^{\mathcal{S}} \circ (B^{\mathcal{S}})^{\mathcal{H}} = A^{\mathcal{S}} \circ B^{\mathcal{H}} = (A \circ B)^{\mathcal{S}}.\end{aligned}$$

Therefore, $(A^{\mathcal{X}} \circ B)^{\mathcal{X}} = (A \circ B)^{\mathcal{X}} = (A \circ B^{\mathcal{X}})^{\mathcal{X}}$ for every $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$, as required. □

Proposition 11. *For any $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$, $\sim_{\mathcal{X}}$ is a congruence on the free ai-semirng $(\mathcal{F}(\Sigma^+), \cup, \circ)$.*

Proof. Let $A, B, C \in \mathcal{F}(\Sigma^+)$ and $A \sim_{\mathcal{X}} B$. Then, $A^{\mathcal{X}} = B^{\mathcal{X}}$ and consequently by Lemma 3,

$$\begin{aligned}(A \cup C)^{\mathcal{X}} &= (A^{\mathcal{X}} \cup C)^{\mathcal{X}} = (B^{\mathcal{X}} \cup C)^{\mathcal{X}} = (B \cup C)^{\mathcal{X}},\\ (A \circ C)^{\mathcal{X}} &= (A^{\mathcal{X}} \circ C)^{\mathcal{X}} = (B^{\mathcal{X}} \circ C)^{\mathcal{X}} = (B \circ C)^{\mathcal{X}}.\end{aligned}$$

Thus, $A \cup C \sim_{\mathcal{X}} B \cup C$ and $A \circ C \sim_{\mathcal{X}} B \circ C$. From this and its dual it follows that $\sim_{\mathcal{X}}$ is a congruence on the ai-semiring $\mathcal{F}(\Sigma^+)$. □

Let $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$. We know from Proposition 11 that the quotient algebra $\mathcal{F}(\Sigma^+)/\sim_{\mathcal{X}}$ is an ai-semiring. Furthermore, it is easy to see that $\mathcal{F}(\Sigma^+)/\sim_{\mathcal{X}}$ is isomorphic to the algebra $\mathcal{X}(\Sigma^+)$ and hence $\mathcal{I}(\Sigma^+)$, $\mathcal{S}(\Sigma^+)$ and $\mathcal{P}(\Sigma^+)$ are ai-semirings.

5.2. Models for Free Objects in Three Subvarieties of AI

In the sequel, we denote the following ai-semiring varieties

$$[xz + xyz \approx xz, xz + xyzw \approx xz, xz + wxyz \approx xz, xz + w_1xyzw_2 \approx xz],$$
$$[x + yx \approx x, x + yxz \approx x], \text{ and}$$
$$[x + xy \approx x, x + yxz \approx x]$$

by $\mathbf{AI}_{\mathcal{I}}$, $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$, respectively. We show that $\mathcal{X}(\Sigma^+)$ is a member of $\mathbf{AI}_{\mathcal{X}}$, for all $\mathcal{X} \in \{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$, by verifying the algebra $\mathcal{X}(\Sigma^+)$ satisfies the corresponding identities.

It is a routine matter to verify that for any $A_1, A_2, B_1, B_2, B_3 \in \mathcal{I}(\Sigma^+)$, the following equalities are true.

$$A_1 \times_{\mathcal{I}} A_2 +_{\mathcal{I}} A_1 \times_{\mathcal{I}} B_1 \times_{\mathcal{I}} A_2 = A_1 \times_{\mathcal{I}} A_2;$$
$$A_1 \times_{\mathcal{I}} A_2 +_{\mathcal{I}} A_1 \times_{\mathcal{I}} B_1 \times_{\mathcal{I}} A_2 \times_{\mathcal{I}} B_2 = A_1 \times_{\mathcal{I}} A_2;$$
$$A_1 \times_{\mathcal{I}} A_2 +_{\mathcal{I}} B_1 \times_{\mathcal{I}} A_1 \times_{\mathcal{I}} B_2 \times_{\mathcal{I}} A_2 = A_1 \times_{\mathcal{I}} A_2;$$
$$A_1 \times_{\mathcal{I}} A_2 +_{\mathcal{I}} B_1 \times_{\mathcal{I}} A_1 \times_{\mathcal{I}} B_2 \times_{\mathcal{I}} A_2 \times_{\mathcal{I}} B_3 = A_1 \times_{\mathcal{I}} A_2.$$

This means that the ai-semiring $\mathcal{I}(\Sigma^+) \in \mathbf{AI}_{\mathcal{I}}$. Furthermore, it can be verified that for any $A, B, C \in \mathcal{P}(\Sigma^+)$,

$$A +_{\mathcal{P}} B \times_{\mathcal{P}} A = A;$$
$$A +_{\mathcal{P}} B \times_{\mathcal{P}} A \times_{\mathcal{P}} C = A,$$

and for any $A, B, C \in \mathcal{S}(\Sigma^+)$,

$$A +_{\mathcal{S}} A \times_{\mathcal{P}} B = A;$$
$$A +_{\mathcal{S}} B \times_{\mathcal{S}} A \times_{\mathcal{S}} C = A.$$

Then, $\mathcal{P}(\Sigma^+)$ and $\mathcal{S}(\Sigma^+)$ belong to the subvarieties $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$, respectively.

It easy to see that $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$ are subvarieties of $[x + yx \approx x]$ and $[x + xy \approx x]$, respectively. By Lemma 16 in [10], we immediately have the following lemma.

Lemma 4. *For any integer $n \geq 1$, $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$ satisfy*

$$x_1x_2 \cdots x_n + y_1x_1y_2x_2 \cdots y_nx_n \approx x_1x_2 \cdots x_n \text{ and}$$
$$x_1x_2 \cdots x_n + x_1y_1x_2y_2 \cdots x_ny_n \approx x_1x_2 \cdots x_n,$$

respectively.

Further, if we premultiply (postmultiply) both sides by x (z) to the identity $z + yz \approx z$ $(x + xy \approx x)$ and apply the distribution law, we know that both $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$ satisfy the identity $xz + xyz \approx xz$. Hence, by Lemma 15 in [10], we have:

Lemma 5. *For any $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$ and any integer $n \geq 2$, $\mathbf{AI}_{\mathcal{X}}$ satisfies the identity*

$$x_1x_2 \cdots x_n + x_1y_1x_2 \cdots y_{n-1}x_n \approx x_1x_2 \cdots x_n.$$

The following two lemmas show more identities hold in $\mathbf{AI}_{\mathcal{P}}$, $\mathbf{AI}_{\mathcal{S}}$ and $\mathbf{AI}_{\mathcal{I}}$.

Lemma 6. *For any integer $n \geq 1$, $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$ satisfy the identity*

$$x_1x_2\cdots x_n + y_0x_1y_1\cdots x_ny_n \approx x_1x_2\cdots x_n.$$

Proof. It is clear that both $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$ satisfy the identity $x_1 + y_0x_1y_1 \approx x_1$. From Lemma 4, we know that these two subvarieties satisfy

$$x_1x_2\cdots x_n + x_1y_1x_2\cdots y_{n-1}x_n \approx x_1x_2\cdots x_n$$

for any $n \geq 2$. If we premultiply both sides by y_0 and postmultiply both sides by y_n and apply the distribution law as well, we obtain

$$y_0x_1x_2\cdots x_ny_n + y_0x_1y_1x_2\cdots y_{n-1}x_ny_n \approx y_0x_1x_2\cdots x_ny_n.$$

Now, adding $x_1x_2\cdots x_n$ to the both side of this identity, we have that

$$x_1x_2\cdots x_n + y_0x_1x_2\cdots x_ny_n + y_0x_1y_1x_2\cdots y_{n-1}x_ny_n \approx x_1x_2\cdots x_n + y_0x_1x_2\cdots x_ny_n.$$

Notice that $x_1x_2\cdots x_n + y_0x_1x_2\cdots x_ny_n \approx x_1x_2\cdots x_n$, since $\mathbf{AI}_{\mathcal{P}}$ and $\mathbf{AI}_{\mathcal{S}}$ satisfy the identity $x + yxz \approx x$. We hence have that

$$x_1x_2\cdots x_n + y_0x_1y_1\cdots x_ny_n \approx x_1x_2\cdots x_n,$$

as required. □

Lemma 7. *For any integer $n \geq 2$, $\mathbf{AI}_{\mathcal{I}}$ satisfies the following identities.*

$$x_1x_2\cdots x_n + y_0x_1y_1x_2\cdots y_{n-1}x_n \approx x_1x_2\cdots x_n, \tag{1}$$
$$x_1x_2\cdots x_n + x_1y_1x_2y_2\cdots x_ny_n \approx x_1x_2\cdots x_n, \tag{2}$$
$$x_1x_2\cdots x_n + y_0x_1y_1x_2y_2\cdots x_ny_n \approx x_1x_2\cdots x_n. \tag{3}$$

Proof. Firstly, we prove identity (1) is true for any $n \geq 2$. Since $\mathbf{AI}_{\mathcal{I}}$ satisfies the identity $xz + wxyz \approx xz$, the identity (1) is true when $n = 2$. Assume that $\mathbf{AI}_{\mathcal{I}}$ satisfies

$$x_1x_2\cdots x_k + y_0x_1y_1x_2\cdots y_{k-1}x_k \approx x_1x_2\cdots x_k,$$

where $k \geq 2$ is an integer. We postmultiply both sides by y_kx_{k+1} and apply the distribution law. A routine calculation gives

$$x_1x_2\cdots x_ky_kx_{k+1} + y_0x_1y_2x_2\cdots y_{k-1}x_ky_kx_{k+1} \approx x_1x_2\cdots x_ky_kx_{k+1}.$$

If we add $x_1x_2\cdots x_kx_{k+1}$ to both sides of above equality, we get

$$\begin{aligned} &x_1x_2\cdots x_kx_{k+1} + x_1x_2\cdots x_ky_kx_{k+1} + y_0x_1y_1x_2\cdots y_{k-1}x_ky_kx_{k+1} \\ &\approx x_1x_2\cdots x_kx_{k+1} + x_1x_2\cdots x_ky_kx_{k+1}. \end{aligned}$$

Since $\mathbf{AI}_{\mathcal{I}}$ satisfies $xz + xyz \approx xz$, we obtain

$$x_1x_2\cdots x_kx_{k+1} + y_0x_1y_1x_2\cdots y_{k-1}x_ky_kx_{k+1} \approx x_1x_2\cdots x_kx_{k+1}.$$

This means that $\mathbf{AI}_{\mathcal{I}}$ satisfies (1) when $n = k + 1$, and hence it satisfies (1) for all integer $n \geq 2$, as required.

In an analogous fashion, we can prove that $\mathbf{AI}_{\mathcal{I}}$ satisfies (2) for all $n \geq 2$, so we omit the details.

Lastly, we prove that identity (3) is true for any $n \geq 2$. Since $\mathbf{AI}_{\mathcal{I}}$ satisfies the identity $xz + w_1xyzw_2 \approx xz$, (3) is true when $n = 2$. Assume that $\mathbf{AI}_{\mathcal{I}}$ satisfies

$$x_1x_2\cdots x_k + y_0x_1y_1x_2\cdots y_{k-1}x_k \approx x_1x_2\cdots x_k,$$

where $k \geq 2$ is an integer. We postmultiply both sides by $y_kx_{k+1}y_{k+1}$ and apply the distribution law. Then, we get

$$x_1x_2\cdots x_ky_kx_{k+1}y_{k+1} + y_0x_1y_2x_2\cdots y_{k-1}x_ky_kx_{k+1}y_{k+1} \approx x_1x_2\cdots x_ky_kx_{k+1}y_{k+1}.$$

If we add $x_1x_2\cdots x_kx_{k+1}$ to both sides of above equality, we get

$$\begin{aligned} x_1x_2\cdots x_kx_{k+1} + x_1x_2\cdots x_ky_kx_{k+1}y_{k+1} + y_0x_1y_1x_2\cdots y_kx_{k+1}y_{k+1} \\ \approx x_1x_2\cdots x_kx_{k+1} + x_1x_2\cdots x_ky_kx_{k+1}y_{k+1}. \end{aligned}$$

Since $\mathbf{AI}_{\mathcal{I}}$ satisfies $xz + xyzw \approx xz$, we obtain

$$x_1x_2\cdots x_kx_{k+1} + y_0x_1y_1x_2\cdots y_kx_{k+1}y_{k+1} \approx x_1x_2\cdots x_kx_{k+1}.$$

This means that $\mathbf{AI}_{\mathcal{I}}$ satisfies (3) when $n = k + 1$, and hence it satisfies (3) for all integer $n \geq 2$, as required. □

For a nonempty finite set Σ and $\sigma \in \Sigma$, we have $\{\sigma\} \in \mathcal{X}(\Sigma^+)$. Then the mapping $\iota_{\mathcal{X}} : \Sigma \to \mathcal{X}(\Sigma^+)$, $\sigma \mapsto \{\sigma\}$ is one-to-one. We set out to prove that the algebra $\mathcal{X}(\Sigma^+)$ is freely generated by Σ in $\mathbf{AI}_{\mathcal{X}}$ for every $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$.

Let $K \in \mathbf{AI}_{\mathcal{X}}$ and $\psi_{\mathcal{X}} : \Sigma \to K$ a mapping. Suppose that $\theta_{\mathcal{X}} : \Sigma^+ \to K$ is the multiplicative homomorphism which extends $\psi_{\mathcal{X}}$. From Lemmas 3–6, we immediately have the following lemma.

Lemma 8. *For any $A \in \mathcal{F}(\Sigma^+)$ and any $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$,*

$$\sum_{w \in A} \theta_{\mathcal{X}}(w) = \sum_{w \in A^{\mathcal{X}}} \theta_{\mathcal{X}}(w).$$

Now we can formulate and prove the main result of this paper.

Theorem 3. *Let Σ be a nonempty set. Then, for any $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$, the algebra $\mathcal{X}(\Sigma^+)$ is freely generated by Σ in $\mathbf{AI}_{\mathcal{X}}$.*

Proof. Let $(K, +, \cdot) \in \mathbf{AI}_{\mathcal{X}}$ and $\psi_{\mathcal{X}} : \Sigma \to S$ a mapping. Suppose that $\theta_{\mathcal{X}} : \Sigma^+ \to K$ is the multiplicative homomorphism which extends $\psi_{\mathcal{X}}$. Define the mapping

$$\varphi_{\mathcal{X}} : \mathcal{X}(\Sigma^+) \to K,\ A \mapsto \sum_{w \in A} \theta_{\mathcal{X}}(w).$$

For every $\sigma \in \Sigma$, we then have

$$\varphi_{\mathcal{X}}(\iota_{\mathcal{X}}(\sigma)) = \varphi(\{\sigma\}) = \theta_{\mathcal{X}}(\sigma) = \psi_{\mathcal{X}}(\sigma).$$

Therefore,

$$\begin{tikzcd} \Sigma \arrow[r, "\iota_{\mathcal{X}}"] \arrow[d, "\psi_{\mathcal{X}}"'] & \mathcal{X}(\Sigma^+) \arrow[dl, "\varphi_{\mathcal{X}}"] \\ K & \end{tikzcd}$$

is a commutative diagram. We need to prove that $\varphi_{\mathcal{X}}$ is an ai-semigring homomorphism.

Let $A, B \in \mathcal{X}(\Sigma^+)$. Then, by Lemma 8,

$$\begin{aligned} \varphi_{\mathcal{X}}(A) + \varphi_{\mathcal{X}}(B) &= \sum_{w \in A} \theta_{\mathcal{X}}(w) + \sum_{w \in B} \theta_{\mathcal{X}}(w) \\ &= \sum_{w \in A \cup B} \theta_{\mathcal{X}}(w) = \sum_{w \in (A \cup B)^{\mathcal{X}}} \theta_{\mathcal{X}}(w) \\ &= \sum_{w \in A +_{\mathcal{X}} B} \theta_{\mathcal{X}}(w) = \varphi_{\mathcal{X}}(A +_{\mathcal{X}} B), \end{aligned}$$

$$\begin{aligned} \varphi_{\mathcal{X}}(A)\varphi_{\mathcal{X}}(B) &= \Big(\sum_{w \in A} \theta_{\mathcal{X}}(w)\Big)\Big(\sum_{w \in B} \theta_{\mathcal{X}}(w)\Big) \\ &= \sum_{u \in A, v \in B} \theta_{\mathcal{X}}(u)\theta_{\mathcal{X}}(v) = \sum_{u \in A, v \in B} \theta_{\mathcal{X}}(uv) \\ &= \sum_{w \in A \circ B} \theta_{\mathcal{X}}(w) = \sum_{w \in (A \circ B)^{\mathcal{X}}} \theta_{\mathcal{X}}(w) \\ &= \varphi_{\mathcal{X}}(A \circ B)^{\mathcal{X}} = \varphi_{\mathcal{X}}(A \times_{\mathcal{X}} B). \end{aligned}$$

Therefore, $\varphi_{\mathcal{X}}$ is a homomorphism for any $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}, \mathcal{I}\}$. $\square$

6. Discussion

In this paper, we introduced three classes of formal languages over a finite alphabet, and we described them as independent sets with respect to partial orders contained in the embedding order. Then, we discussed the combinatorial properties of words involved in these partial orders. Furthermore, we established combinatorial properties of languages of interest, in the sense of set catenation and partial order. In addition, we constructed algebra structures for these three classes of languages, by defining two binary operations on each class. At last, we characterized these algebra structures as free objects of ai-semiring varieties.

We developed in this paper a method to decompose (or compose) free languages with respect to a particular partial order, which is useful for clustering languages in the sense of algebra structure. However, we are not sure whether this method can be extended to a more general case. Furthermore, it is still unknown how to cluster languages with an algebra structure, which is a free object in the variety $[x + yxz \approx x]$.

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