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Modeling and Analysis of the Influence of Fear on a Harvested Food Web System

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Abstract: The food web is a crucial conceptual tool for understanding the dynamics of energy transfer in an ecosystem, as well as the feeding relationships among species within a community. It also reveals species interactions and community structure. As a result, an ecological food web system with two predators competing for prey while experiencing fear was developed and studied. The properties of the solution of the system were determined, and all potential equilibrium points were identified. The dynamic behavior in their immediate surroundings was examined both locally and globally. The system's persistence demands were calculated, and all conceivable forms of local bifurcations were investigated. With the aid of MATLAB, a numerical simulation was used to clarify the control set of parameters and comprehend the overall dynamics. For the system to continue, it was determined that extremely high levels of either fear or harvesting lead to the extinction of one of the predator species. Moreover, in contrast to the ecological assumption that if two species are vying for the same resources, population values cannot be constant, this study showed that it is possible for two competing species to subsist on the same resources.

Keywords: food web; fear; harvesting; stability; bifurcation; persistence

MSC: 92D25; 34D20; 37G10

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1. Introduction

A community of creatures and their physical surroundings can interact to form a structure known as an ecosystem. The food web system is essential to ecology, as one of the essential components for maintaining life and developing a species is food. A "food web" describes the flow of energy through several species in a given area; it can generally be thought of as being crucial to the survival of organisms in nature [1]. In this paper, the food web depicts a single prey being consumed by two predators. Predation may be regarded as a direct link between nutrition and the existence of species in nature, because it can result in the extinction of one species without the occurrence of another or in the dominance of one species over the other. If the type of predation induces fear in the prey, the number of predators of one prey can have an effect on the survival and diversity of the species. Although predators with a single predation process can enhance their share by congregating on a single prey, the focus of this study is on the sensation of dread experienced by the prey during predation. Notably, a fear response in prey can cause it to die and become unavailable to predators. As a result, this behavior may have an impact on the predator's ability to exist.

For the food web model, functional response is crucial. It describes how each predator hunts prey based on its density [2]. The three functional responses that Holling hypothesized for the predator are known as Holling types I, II, and III [3]. Despite including the inhibitory impact at high concentrations, Andrews proposed the Monod-Haldane function for low concentrations [4]. Predation decreases when prey populations are high, because the prey can better protect themselves or blend in under such conditions [5]. This phenomenon is referred to as group defense.

On the other hand, predation fear plays a significant role in the development of predator–prey relationships, mainly by making it more challenging for predators to hunt [6–10]. Fear of predation (perceived predation risk) caused by the mere presence of a predator within an ecosystem is increasingly regarded as an ecological force that rivals or exceeds that of direct killing [11]. Therefore, in recent years, many scientists have started investigating the predator–prey model based on the fear component [12–21]. By taking into account both fear and group defense of prey in the presence of a predator, Sasmal and Takeuchi [18] developed an ODE model on predator–prey interactions. They considered the Monod-Haldane type response function, which can capture the group defense of prey. The combined impacts of fear, prey refuge, and extra food for the predator in a predator–prey system with Beddington-type functional response were examined by SK et al. [22]. A modified Leslie-Gower predator–prey model incorporating the fear effect and nonlinear harvesting was created and studied by Al-Momen and Naji [23]. Those authors employed the Holling type-II functional response to model the predator's eating process. A predator–prey system with a Holling type II functional response that combines predation fear with predator-dependent prey refuge was conceived and explored by Ibrahim et al. in [24]. The effects of anti-predator behavior resulting from fear of predator species were investigated by Xie and Zhang [25] in a predator–prey system with Holling III type functional response and prey shelters. Kumar and Kumari [26] devised and investigated a fractional-order delayed predator–prey system which took into account the fear effect. Since they considered the time delay in terms of the effects of fear, the system did not undergo any dynamic changes as a result of an absence of fear.

Later, several investigations were conducted on the impact of predation fear on the dynamic of the food chains of three species and food webs systems [27–34]. In a three-species food chain model, Kumar and Kumari [27] investigated the effects of fear on the dynamics in cases whereby the top predator's fear inhibits the growth rate of the intermediate predator, while the intermediate predator's fear suppresses the growth rate of the prey. For the purpose of examining the effects of fear, Cong et al. [30] developed a three-species food chain model to determine the cost and benefit of anti-predator behaviors. They did this by applying the traditional Holling's time budget argument to calculate the predator's functional response. Rahi et al. [31] proposed a predator–prey interaction model in which a predator's population is divided into two stages. To include the additional supply of food for the predator, they changed Holling's disk equation to describe how the prey is consumed. To better understand how predation fear affects the dynamics of a food chain system comprising three species, Maghool and Naji [32] developed and investigated a novel model. As each prey in the system has an anti-predator trait, they assumed that food is moved from the lower to the upper level by a Sokol-Howell type functional response. A tritrophic food-chain model with the inclusion of prey refuge terms was developed by Saha and Samanta [33], in which consumers hunt for prey using Holling type-III functional responses. Maghool and Naji [34] mathematically constructed a three-species food web model with two competing prey species and one predator experiencing fear. They took advantage of modified Holling type II functional responses as well as intraspecific competition within the predator population.

Harvesting can cause severe damage to the ecosystem of a given region. As such, if the activity is inevitable, then the governing authority of that area should implement a regulating policy that would minimize such damage. Harvesting has a substantial impact on the dynamic evolution of a population subjected to it [35]. Therefore, it is important to take into account the harvesting of species in predator–prey models from the standpoint of financial income. Numerous harvesting techniques have been applied in the literature. While some of them employed age selection, proportional, and constant harvesting, others thought about nonlinear harvesting. Investigations of how harvesting affects ecological system dynamics have attracted a lot of attention [23,36–43].

Keeping these notions in mind, the issue of hunting when two predators are competing for the same prey is addressed in the current work. It is assumed that the prey species have a capacity for collective defense. Therefore, the dynamics and bifurcations of a two-predator model feeding on a single prey with a functional response of the Monod-Haldane type are investigated in the presence of harvesting.

2. The Model Formulation

In this section, a real-world food web system under the influence of fear and harvesting is formulated using a mathematical model. The following hypotheses are used to create the model, with prey, first predator density, and second predator density at time t represented by $X(T)$, $Y(T)$, and $Z(T)$, respectively.

1. Fear imposed by predators has several effects on prey populations, including foraging frequency, habitat utilization, reproductive speed, and physiological changes [33]. As a result, it is hypothesized that predators use the fear function $\left(\frac{1}{1+k_1Y+k_2Z}\right)$ to influence the prey's growth rate.
2. The prey population grows logistically in the absence of predation.
3. The predators consume the prey depending on the using Monod–Haldane-type response function; their numbers are reduced due to intraspecific competition. Moreover, it is assumed that interspecies competition exists.
4. Finally, the system is imposed under the effect of fixed-offer harvesting.

Depending on the above hypotheses, the dynamics of a food-web system can be simulated using the following set of differential equations.

$$\begin{aligned}\frac{dX}{dT} &= \left(\frac{rX}{1+k_1Y+k_2Z}\right) - b_0X - c_0X^2 - \frac{a_1XY}{b_1+c_1X+X^2} - \frac{a_2XZ}{b_2+c_2X+X^2} - h_1X,\\ \frac{dY}{dT} &= \frac{e_1a_1XY}{b_1+c_1X+X^2} - u_1Y^2 - v_1YZ - d_1Y - h_2Y,\\ \frac{dZ}{dT} &= \frac{e_2a_2XZ}{b_2+c_2X+X^2} - u_2Z^2 - v_2YZ - d_2Z - h_3Z,\end{aligned} \tag{1}$$

where the system domain is given by $\mathbb{R}_+{}^3 = \left\{(X,Y,Z) \in \mathbb{R}^3 : X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0\right\}$. The system parameters, all assumed to be positive, are described in Table 1.

The existence of a large number of parameters in any dynamic system presents difficulties in terms of performing analyses. Therefore, in the interests of simplification, the following dimensionless variables and parameters are utilized in the present system (1).

$$t = rT, x = \frac{c_0}{r}X, y = \frac{a_1c_0^2}{r^3}Y, z = \frac{a_2\ c_0^2}{r^3}Z.$$

$$p_1 = \frac{k_1r^3}{a_1c_0^2}, p_2 = \frac{k_2r^3}{a_2c_0^2}, p_3 = \frac{b_0}{r}, p_4 = \frac{b_1c_0^2}{r^2}, p_5 = \frac{c_1c_0}{r}, p_6 = \frac{b_2\ c_0^2}{r^2},$$

$$p_7 = \frac{c_2\ c_0}{r}, p_8 = \frac{h_1}{r}, p_9 = \frac{e_1a_1\ c_0}{r^2}, p_{10} = \frac{u_1r^2}{a_1c_0^2}, p_{11} = \frac{v_1r^2}{a_2c_0^2}, p_{12} = \frac{d_1}{r}, p_{13} = \frac{h_2}{r},$$

$$p_{14} = \frac{e_2a_2\ c_0}{r^2}, p_{15} = \frac{u_2r^2}{a_2c_0^2}, p_{16} = \frac{v_2\ r^2}{a_1c_0^2}, p_{17} = \frac{d_2}{r}, p_{18} = \frac{h_3}{r}.$$

Then, the resulting dimensionless system can be written as

$$\begin{aligned}\frac{dx}{dt} &= \frac{x}{1+p_1y+p_2z} - p_3x - x^2 - \frac{xy}{p_4+p_5x+x^2} - \frac{xz}{p_6+p_7x+x^2} - p_8x,\\ \frac{dy}{dt} &= \frac{p_9xy}{p_4+p_5x+x^2} - p_{10}y^2 - p_{11}\ yz - p_{12}y - p_{13}y,\\ \frac{dz}{dt} &= \frac{p_{14}xz}{p_6+p_7x+x^2} - p_{15}z^2 - p_{16}yz - p_{17}\ z - p_{18}z,\end{aligned} \tag{2}$$

with the initial condition $x(0) \geq 0$, $y(0) \geq 0$, and $z(0) \geq 0$.

Table 1. Description of parameters.

Parameter	Descriptions	Unite
r	The birth rate of prey.	T^{-1}
b_0	The natural death rate of prey.	T^{-1}
c_0	The intraspecies competition of the prey.	$D^{-1}.T^{-1}$
k_1	The fear rate of prey of predator Y.	D^{-1}
k_2	The fear rate of prey of predator Z.	D^{-1}
a_1	The attack rate of the first predator of the prey.	$D.T^{-1}$
a_2	The attack rate of the second predator of the prey.	$D.T^{-1}$
b_1	The half-saturation constant of first predator Y in the absence of a direct measure of the inhibitory effect.	D^2
b_2	The half-saturation constant of second predator Z in the absence of a direct measure of the inhibitory effect.	D^2
c_1	The inhibitory effect at high concentrations for predator Y.	D
c_2	The inhibitory effect at high concentrations for predator Z.	D
$h_i;\ i = 1, 2, 3$	The harvest rate of species i.	T^{-1}
e_1 *and* e_2	The conversion rates of the first and second predators, respectively.	non
u_1 *and* u_2	The intraspecific competition of the first and second predators, respectively.	$D^{-1}.T^{-1}$
v_1 *and* v_2	The interspecific competition between the first and second predators, respectively.	$D^{-1}.T^{-1}$
d_1 *and* d_2	The natural death rate of the first and second predators, respectively.	T^{-1}

Note: D represents the population density symbol.

3. Properties of the Solution

According to the right-hand side of the system (2), all the interaction functions are continuous and have continuous partial derivatives with respect to variables x, y, and z. Therefore, they are Lipschitz functions, and as such, the system with an initial condition has a unique solution [44]. Moreover, the following theorem can determine the bounds of all existing solutions.

Theorem 1. *In the region*$\Lambda \subseteq {\mathbb{R}_+}^3$*, all existing system (2) solutions are uniformly bounded, where*

$$\Lambda = \left\{ (x, y, z) \in {\mathbb{R}_+}^3 : 0 \leq x \leq (1 - p_3),\ x + \frac{\mathrm{y}}{\mathrm{p}_9} + \frac{\mathrm{z}}{\mathrm{p}_{14}} \leq \frac{2(1 - p_3 - p_8)(1 - p_3)}{\mu} \right\}$$

Proof. From the first equation, it is observed that

$$\frac{dx}{dt} \leq (1 - p_3)x - x^2 = (1 - p_3)x\left(1 - \frac{x}{(1 - P_3)}\right)$$

Direct computation shows that $\lim_{t\to\infty} Sup x(t) \leq (1 - p_3)$. Now, let us define the function $W = x + \frac{\mathrm{y}}{\mathrm{p}_9} + \frac{\mathrm{z}}{\mathrm{p}_{14}}$. Differentiating the function W with respect to time t, it is obtained, such that:

$$\begin{aligned} \tfrac{dw}{dt} &= \tfrac{x}{B_1} - p_3 x - x^2 - \tfrac{xy}{B_2} - \tfrac{xz}{B_3} - p_8 x + \tfrac{xy}{B_2} - \tfrac{p_{10}}{p_9} y^2 - \tfrac{p_{11}}{p_9} yz \\ &- \tfrac{(p_{12}+p_{13})}{p_9} y + \tfrac{xz}{B_3} - \tfrac{p_{15}}{p_{14}} z^2 - \tfrac{p_{16}}{p_{14}} yz - \tfrac{(p_{17}+p_{18})}{p_{14}} z, \end{aligned}$$

where $B_1 = 1 + p_1 y + p_2 z$, $B_2 = p_4 + p_5 x + x^2$, and $B_3 = p_6 + p_7 x + x^2$. Then

$$\frac{dw}{dt} \leq (1 - p_3 - p_8)x - \frac{(p_{12} + p_{13})}{p_9} y - \frac{(p_{17} + p_{18})}{p_{14}} z$$

Using some simple calculations yields:

$$\frac{dw}{dt} \leq 2(1 - p_3 - p_8)x - \mu\left(x + \frac{y}{p_9} + \frac{z}{p_{14}}\right)$$

where $\mu = \min\{1 - p_3 - p_8, p_{12} + p_{13}, p_{17} + p_{18}\}$. Note that $1 - p_3 - p_8 > 0$, due to the biological meaning of these parameters, which is known as a survival condition of x.

Accordingly, it is determined that

$$\frac{dw}{dt} + \mu W \leq 2(1 - p_3 - p_8)(1 - p_3)$$

Hence, direct computation indicates that for $t \to \infty$, $W \leq \frac{2(1-p_3-p_8)(1-p_3)}{\mu}$, thereby completing the proof. □

4. Stability Analysis

In this section, the possible equilibrium points (EQPs) are determined, and then a stability analysis for them is undertaken. Different EQPs may exist for the system (2); these are summarized as:

The evanescence equilibrium point (EEQP), denoted by $E_0 = (0, 0, 0)$, always exists.

The predation-free equilibrium point (PDFEQP), denoted by $E_1 = (\hat{x}, 0, 0)$, with $\hat{x} = 1 - p_3 - p_8$, which exists under the survival condition $1 - p_3 - p_8 > 0$.

The second predator-free equilibrium point (SPFEQP) is denoted by $E_2 = (\check{x}, \check{y}, 0)$, where $\check{y}$ is given by:

$$\check{y} = \frac{1}{p_{10}}\left[\frac{p_9 \check{x}}{p_4 + p_5 \check{x} + \check{x}^2} - (p_{12} + p_{13})\right] \tag{3}$$

When $\check{x}$ is the positive root of the equation:

$$A_1 x^7 + A_2 x^6 + A_3 x^5 + A_4 x^4 + A_5 x^3 + A_6 x^2 + A_7 x + A_8 = 0 \tag{4}$$

where

$$A_1 = -p_{10}[p_{10} - p_1(p_{12} + p_{13})]$$

$$\begin{aligned} A_2 = {} & -p_1 p_{10}[p_9 - (p_3 + p_8)(p_{12} + p_{13})] + {p_{10}}^2[1 - (p_3 + p_8)] \\ & -3p_5 p_{10}[p_{10} - p_1(p_{12} + p_{13})], \end{aligned}$$

$$\begin{aligned} A_3 = {} & p_{10}[-p_1 p_9(p_3 + 2p_5 + p_8) - 3p_4 p_{10} + 3p_5 p_{10}(1 - p_3 - p_8) \\ & -3p_5^2[p_{10} - p_1(p_{12} + p_{13})] + 3p_1(p_{12} + p_{13})(p_4 + p_3 p_5 + p_5 p_8)], \end{aligned}$$

$$\begin{aligned} A_4 = {} & -2p_1 p_4 p_9 p_{10} - 2p_1 p_3 p_5 p_9 p_{10} - p_1 p_5^2 p_9 p_{10} - 2p_1 p_5 p_8 p_9 p_{10} + 3p_4 p_{10}^2 \\ & -3p_3 p_4 p_{10}^2 - 6p_4 p_5 p_{10}^2 + 3p_5^2 p_{10}^2 - 3p_3 p_5^2 p_{10}^2 - p_5^3 p_{10}^2 - 3p_4 p_8 p_{10}^2 \\ & -3p_5^2 p_8 p_{10}^2 + p_{10} p_{12} + 3p_1 p_3 p_4 p_{10} p_{12} + 6p_1 p_4 p_5 p_{10} p_{12} + 3p_1 p_3 p_5^2 p_{10} p_{12} \\ & +p_1 p_5^3 p_{10} p_{12} + 3p_1 p_4 p_8 p_{10} p_{12} + 3p_1 p_5^2 p_8 p_{10} p_{12} - p_1 p_{12}^2 + p_{10} p_{13} \\ & +3p_1 p_3 p_4 p_{10} p_{13} + 6p_1 p_4 p_5 p_{10} p_{13} + 3p_1 p_3 p_5^2 p_{10} p_{13} + p_1 p_5^3 p_{10} p_{13} \\ & +3p_1 p_4 p_8 p_{10} p_{13} + 3p_1 p_5^2 p_8 p_{10} p_{13} - 2p_1 p_{12} p_{13} - p_1 p_{13}^2, \end{aligned}$$

$$A_5 = -p_9p_{10} - 2p_1p_3p_4p_9p_{10} - 2p_1p_4p_5p_9p_{10} - p_1p_3p_5^2p_9p_{10} - 2p_1p_4p_8p_9p_{10} \\ -p_1p_5^2p_8p_9p_{10} - 3p_4^2p_{10}^2 + 6p_4p_5p_{10}^2 - 6p_3p_4p_5p_{10}^2 - 3p_4p_5^2p_{10}^2 + p_5^3p_{10}^2 \\ -p_3p_5^3p_{10}^2 - 6p_4p_5p_8p_{10}^2 - p_5^3p_8p_{10}^2 + 2p_1p_9p_{12} + 3p_1p_4^2p_{10}p_{12} \\ +2p_5p_{10}p_{12} + 6p_1p_3p_4p_5p_{10}p_{12} + 3p_1p_4p_5^2p_{10}p_{12} + p_1p_3p_5^3p_{10}p_{12} \\ +6p_1p_4p_5p_8p_{10}p_{12} + p_1p_5^3p_8p_{10}p_{12} - 2p_1p_5p_{12}^2 + 2p_1p_9p_{13} + 3p_1p_4^2p_{10}p_{13} \\ +2p_5p_{10}p_{13} + 6p_1p_3p_4p_5p_{10}p_{13} + 3p_1p_4p_5^2p_{10}p_{13} + p_1p_3p_5^3p_{10}p_{13} \\ +6p_1p_4p_5p_8p_{10}p_{13} + p_1p_5^3p_8p_{10}p_{13} - 4p_1p_5p_{12}p_{13} - 2p_1p_5p_{13}^2,$$

$$A_6 = -p_1p_9^2 - p_1p_4^2p_9p_{10} - p_5p_9p_{10} - 2p_1p_3p_4p_5p_9p_{10} - 2p_1p_4p_5p_8p_9p_{10} \\ +3p_4^2p_{10}^2 - 3p_3p_4^2p_{10}^2 - 3p_4^2p_5p_{10}^2 + 3p_4p_5^2p_{10}^2 - 3p_3p_4p_5^2p_{10}^2 - 3p_4^2p_8p_{10}^2 \\ -3p_4p_5^2p_8p_{10}^2 + 2p_1p_5p_9p_{12} + 2p_4p_{10}p_{12} + 3p_1p_3p_4^2p_{10}p_{12} \\ +3p_1p_4^2p_5p_{10}p_{12} + p_5^2p_{10}p_{12} + 3p_1p_3p_4p_5^2p_{10}p_{12} + 3p_1p_4^2p_8p_{10}p_{12} \\ +3p_1p_4p_5^2p_8p_{10}p_{12} - 2p_1p_4p_{12}^2 - p_1p_5^2p_{12}^2 + 2p_1p_5p_9p_{13} \\ +2p_4p_{10}p_{13} + 3p_1p_3p_4^2p_{10}p_{13} + 3p_1p_4^2p_5p_{10}p_{13} + p_5^2p_{10}p_{13} \\ +3p_1p_3p_4p_5^2p_{10}p_{13} + 3p_1p_4^2p_8p_{10}p_{13} + 3p_1p_4p_5^2p_8p_{10}p_{13} \\ -4p_1p_4p_{12}p_{13} - 2p_1p_5^2p_{12}p_{13} - 2p_1p_4p_{13}^2 - p_1p_5^2p_{13}^2,$$

$$A_7 = -p_4p_9p_{10} - p_1p_3p_4^2p_9p_{10} - p_1p_4^2p_8p_9p_{10} - p_4^3p_{10}^2 + 3p_4^2p_5p_{10}^2 \\ -3p_3p_4^2p_5p_{10}^2 - 3p_4^2p_5p_8p_{10}^2 + 2p_1p_4p_9p_{12} + p_1p_4^3p_{10}p_{12} \\ +2p_4p_5p_{10}p_{12} + 3p_1p_3p_4^2p_5p_{10}p_{12} + 3p_1p_4^2p_5p_8p_{10}p_{12} - 2p_1p_4p_5p_{12}^2 \\ +2p_1p_4p_9p_{13} + p_1p_4^3p_{10}p_{13} + 2p_4p_5p_{10}p_{13} + 3p_1p_3p_4^2p_5p_{10}p_{13} \\ +3p_1p_4^2p_5p_8p_{10}p_{13} - 4p_1p_4p_5p_{12}p_{13} - 2p_1p_4p_5p_{13}^2,$$

$$A_8 = p_4^3p_{10}^2(1 - p_3 - p_8) + p_4^2p_{10}(p_{12} + p_{13})[1 + p_1p_4(p_3 + p_8)] - p_1p_4^2(p_{12} + p_{13})^2$$

As a result, there is at least one positive SPFEQP, provided that the following requirements are met.

$$(p_{12} + p_{13}) < \frac{p_9\check{x}}{p_4 + p_5\check{x} + \check{x}^2} \tag{5}$$

$$\left.\begin{array}{c} A_1 > 0, \quad A_8 < 0 \\ \text{or} \\ A_1 < 0, \quad A_8 > 0 \end{array}\right\} \tag{6}$$

The first predator-free equilibrium point (FPFEQP) is denoted by $E_3 = (\overline{x}, 0, \overline{z})$, where $\overline{z}$ is given by:

$$\overline{z} = \frac{1}{p_{15}}\left[\frac{p_{14}\overline{x}}{p_6 + p_7\overline{x} + \overline{x}^2} - (p_{17} + p_{18})\right] \tag{7}$$

where $\overline{x}$ is a positive root of the following equation.

$$D_1x^7 + D_2x^6 + D_3x^5 + D_4x^4 + D_5x^3 + D_6x^2 + D_7x + D_8 = 0 \tag{8}$$

where

$$D_1 = -p_{15}[p_{15} - p_2(p_{17} + p_{18})]$$

$$D_2 = -p_2p_{15}[p_{14} - (p_3 + p_8)(p_{17} + p_{18})] + p_{15}^2[1 - (p_3 + p_8)] \\ -3p_7p_{15}[p_{15} - p_2(p_{17} + p_{18})],$$

$$D_3 = p_{15}[-p_2p_{14}(p_3 + 2p_7 + p_8) - 3p_6p_{15} + 3p_7p_{15}(1 - p_3 - p_8) \\ -3p_7^2[p_{15} - p_2(p_{17} + p_{18})] + 3p_2(p_{17} + p_{18})(p_6 + p_7p_3 + p_7p_8)],$$

$$D_4 = -2p_2p_6p_{14}p_{15} - 2p_2p_3p_7p_{14}p_{15} - p_2p_7^2p_{14}p_{15} - 2p_2p_7p_8p_{14}p_{15} + 3p_6p_{15}^2 \\ -3p_3p_6p_{15}^2 - 6p_6p_7p_{15}^2 + 3p_7^2p_{15}^2 - 3p_3p_7^2p_{15}^2 - p_7^3p_{15}^2 - 3p_6p_8p_{15}^2 \\ -3p_7^2p_8p_{15}^2 + p_{15}p_{17} + 3p_2p_3p_6p_{15}p_{17} + 6p_2p_6p_7p_{15}p_{17} \\ +3p_2p_3p_7^2p_{15}p_{17} + p_2p_7^3p_{15}p_{17} + 3p_2p_6p_8p_{15}p_{17} + 3p_2p_7^2p_8p_{15}p_{17} \\ -p_2p_{17}^2 + p_{15}p_{18} + 3p_2p_3p_6p_{15}p_{18} + 6p_2p_6p_7p_{15}p_{18} + 3p_2p_3p_7^2p_{15}p_{18} \\ +p_2p_7^3p_{15}p_{18} + 3p_2p_6p_8p_{15}p_{18} + 3p_2p_7^2p_8p_{15}p_{18} - 2p_2p_{17}p_{18} - p_2p_{18}^2,$$

$$\begin{aligned}
D_5 = & -p_{14}p_{15} - 2p_2p_3p_6p_{14}p_{15} - 2p_2p_6p_7p_{14}p_{15} - p_2p_3p_7^2p_{14}p_{15} \\
& -2p_2p_6p_8p_{14}p_{15} - p_2p_7^2p_8p_{14}p_{15} - 3p_6^2p_{15}^2 + 6p_6p_7p_{15}^2 - 6p_3p_6p_7p_{15}^2 \\
& -3p_6p_7^2p_{15}^2 + p_7^3p_{15}^2 - p_3p_7^3p_{15}^2 - 6p_6p_7p_8p_{15}^2 - p_7^3p_8p_{15}^2 + 2p_2p_{14}p_{17} \\
& +3p_2p_6^2p_{15}p_{17} + 2p_7p_{15}p_{17} + 6p_2p_3p_6p_7p_{15}p_{17} + 3p_2p_6p_7^2p_{15}p_{17} \\
& +p_2p_3p_7^3p_{15}p_{17} + 6p_2p_6p_7p_8p_{15}p_{17} + p_2p_7^3p_8p_{15}p_{17} - 2p_2p_7p_{17}^2 \\
& +2p_2p_{14}p_{18} + 3p_2p_6^2p_{15}p_{18} + 2p_7p_{15}p_{18} + 6p_2p_3p_6p_7p_{15}p_{18} \\
& +3p_2p_6p_7^2p_{15}p_{18} + p_2p_3p_7^3p_{15}p_{18} + 6p_2p_6p_7p_8p_{15}p_{18} \\
& +p_2p_7^3p_8p_{15}p_{18} - 4p_2p_7p_{17}p_{18} - 2p_2p_7p_{18}^2,
\end{aligned}$$

$$\begin{aligned}
D_6 = & -p_2p_{14}^2 - p_2p_6^2p_{14}p_{15} - p_7p_{14}p_{15} - 2p_2p_3p_6p_7p_{14}p_{15} - 2p_2p_6p_7p_8p_{14}p_{15} \\
& +3p_6^2p_{15}^2 - 3p_3p_6^2p_{15}^2 - 3p_6^2p_7p_{15}^2 + 3p_6p_7^2p_{15}^2 - 3p_3p_6p_7^2p_{15}^2 \\
& -3p_6^2p_8p_{15}^2 - 3p_6p_7^2p_8p_{15}^2 + 2p_2p_7p_{14}p_{17} + 2p_6p_{15}p_{17} + 3p_2p_3p_6^2p_{15}p_{17} \\
& +3p_2p_6^2p_7p_{15}p_{17} + p_7^2p_{15}p_{17} + 3p_2p_3p_6p_7^2p_{15}p_{17} + 3p_2p_6^2p_8p_{15}p_{17} \\
& +3p_2p_6p_7^2p_8p_{15}p_{17} - 2p_2p_6p_{17}^2 - p_2p_7^2p_{17}^2 + 2p_2p_7p_{14}p_{18} + 2p_6p_{15}p_{18} \\
& +3p_2p_3p_6^2p_{15}p_{18} + 3p_2p_6^2p_7p_{15}p_{18} + p_7^2p_{15}p_{18} + 3p_2p_3p_6p_7^2p_{15}p_{18} \\
& +3p_2p_6^2p_8p_{15}p_{18} + 3p_2p_6p_7^2p_8p_{15}p_{18} - 4p_2p_6p_{17}p_{18} \\
& -2p_2p_7^2p_{17}p_{18} - 2p_2p_6p_{18}^2 - p_2p_7^2p_{18}^2,
\end{aligned}$$

$$\begin{aligned}
D_7 = & -p_6p_{14}p_{15} - p_2p_3p_6^2p_{14}p_{15} - p_2p_6^2p_8p_{14}p_{15} - p_6^3p_{15}^2 + 3p_6^2p_7p_{15}^2 \\
& -3p_3p_6^2p_7p_{15}^2 - 3p_6^2p_7p_8p_{15}^2 + 2p_2p_6p_{14}p_{17} + p_2p_6^3p_{15}p_{17} \\
& +2p_6p_7p_{15}p_{17} + 3p_2p_3p_6^2p_7p_{15}p_{17} + 3p_2p_6^2p_7p_8p_{15}p_{17} - 2p_2p_6p_7p_{17}^2 \\
& +2p_2p_6p_{14}p_{18} + p_2p_6^3p_{15}p_{18} + 2p_6p_7p_{15}p_{18} + 3p_2p_3p_6^2p_7p_{15}p_{18} \\
& +3p_2p_6^2p_7p_8p_{15}p_{18} - 4p_2p_6p_7p_{17}p_{18} - 2p_2p_6p_7p_{18}^2,
\end{aligned}$$

$$D_8 = p_6^3p_{15}^2(1 - p_3 - p_8) + p_6^2p_{15}(p_{17} + p_{18})[1 + p_2p_6(p_3 + p_8)] - p_2p_6^2(p_{17} + p_{18})^2$$

Similarly, there is at least one positive FPFEQP, provided that the following requirements are met.

$$(p_{17} + p_{18}) < \frac{p_{14}\overline{x}}{p_6 + p_7\overline{x} + \overline{x}^2} \tag{9}$$

$$\left.\begin{array}{c} D_1 > 0, \quad D_8 < 0 \\ \text{or} \\ D_1 < 0, \quad D_8 > 0 \end{array}\right\} \tag{10}$$

The interior equilibrium point (IEQP), denoted by $E_4 = (x^*, y^*, z^*)$, can be determined by solving the following system.

$$\left.\begin{array}{l} f_1 = \frac{1}{1+p_1y+p_2z} - p_3 - x - \frac{y}{p_4+p_5x+x^2} - \frac{z}{p_6+p_7x+x^2} - p_8 = 0, \\ f_2 = \frac{p_9x}{p_4+p_5x+x^2} - p_{10}y - p_{11}z - p_{12} - p_{13} = 0, \\ f_3 = \frac{p_{14}x}{p_6+p_7x+x^2} - p_{15}z - p_{16}y - p_{17} - p_{18} = 0. \end{array}\right\} \tag{11}$$

Consequently, solving the last equation of (11) with respect to z gives:

$$z = \frac{p_{14}x - \left(p_6 + p_7x + x^2\right)(p_{16}y + p_{17} + p_{18})}{p_{15}(p_6 + p_7x + x^2)} \tag{12}$$

Substituting, Equation (12) in the first two equations of (11) gives the following two isoclines:

$$\left.\begin{array}{l} h_1(x,y) = \frac{1}{1+p_1y+p_2\frac{p_{14}x-\left(p_6+p_7x+x^2\right)(p_{16}y+p_{17}+p_{18})}{p_{15}(p_6+p_7x+x^2)}} - p_3 - x - \frac{y}{p_4+p_5x+x^2} \\ \quad - \frac{p_{14}x-\left(p_6+p_7x+x^2\right)(p_{16}y+p_{17}+p_{18})}{p_{15}(p_6+p_7x+x^2)^2} - p_8 = 0. \\ h_2(x,y) = \frac{p_9x}{p_4+p_5x+x^2} - p_{10}y - (p_{12} + p_{13}) \\ \quad - p_{11}\frac{p_{14}x-(p_6+p_7x+x^2)(p_{16}y+p_{17}+p_{18})}{p_{15}(p_6+p_7x+x^2)} = 0. \end{array}\right\} \tag{13}$$

Remember that if the two isoclines given in (13) meet at a single point, say (x^*, y^*) in the first quadrant of the xy plane, which satisfies the positivity of $z(x^*, y^*)$, the IEQP will exist uniquely in the interior of the state space of system (2). After some algebraic work, the isoclines in (13) are as follows when $x \to 0$:

$$\left.\begin{array}{l} h_1(0, y) = N_1 y^2 + N_2 y + N_3 = 0 \\ h_2(0, y) = C_1 y + C_2 = 0. \end{array}\right\} \tag{14}$$

where

$$N_1 = p_6^2(p_6 p_{15} - p_4 p_{16})(p_1 p_{15} - p_2 p_{16})$$

$$\begin{array}{c} N_2 = p_6^3 p_{15}^2(1 + p_1 p_4 p_3 + p_1 p_4 p_8) - p_4 p_6^2 p_{15}(p_{16} + p_1 p_{17} + p_1 p_{18}) \\ -p_2 p_4 p_6^3 p_{15} p_{16}(p_3 + p_8) - p_2 p_6^3 p_{15}(p_{17} + p_{18}) + 2 p_2 p_4 p_6^2 p_{16}(p_{17} + p_{18}), \end{array}$$

$$\begin{array}{c} N_3 = -p_4 p_6^3 p_{15}^2(1 - p_3 - p_8) - p_4 p_6^2 p_{15}(p_{17} + p_{18})[1 + p_2 p_6(p_3 + p_8)] \\ +p_2 p_4 p_6^2 (p_{17} + p_{18})^2, \end{array}$$

$$C_1 = p_4 p_6(p_{10} p_{15} - p_{11} p_{16})$$

$$C_2 = p_4 p_6[p_{15}(p_{12} + p_{13}) - p_{11}(p_{17} + p_{18})]$$

It is clear that the two isoclines in Equation (14) cross the y axis in their respective locations at those places.

$$y_1 = \frac{-N_2 - \sqrt{N_2{}^2 - 4N_1 N_3}}{2N_1} > 0,\ y_2 = \frac{-N_2 + \sqrt{N_2{}^2 - 4N_1 N_3}}{2N_1} < 0 \tag{15a}$$

$$y_3 = -\frac{C_2}{C_1} > 0 \tag{15b}$$

provided that the following conditions are satisfied:

$$\left.\begin{array}{c} N_1 < 0; N_3 > 0, \\ C_1 > 0; C_2 < 0\ OR\ C_1 < 0; C_2 > 0 \end{array}\right\} \tag{16a}$$

It should be noted that by using the condition $N_1 < 0$, the isocline $h_1(0, y) = 0$ is guaranteed to have its maximum on the right side of the xy plane. Consequently, if the following sufficient set of requirements is satisfied in addition to condition (16a), it is simple to verify that the two isoclines produced by Equation (13) intersect at a singular positive point, i.e., (x^*, y^*), in the first quadrant of the xy plane.

$$\left.\begin{array}{c} y_1 > y_3 \\ \frac{dx}{dy} = -\frac{\partial h_2/\partial y}{\partial h_2/\partial x} > 0 \\ p_{14}x^* > \left(p_6 + p_7 x^* + x^{*2}\right)(p_{16}y^* + p_{17} + p_{18}) \end{array}\right\} \tag{16b}$$

By finding the Jacobian matrix (JM) and then their eigenvalues, the local stability near the above EQPs can be investigated. The JM of System (2) at (x, y, z) is as follows:

$$J(x, y, z) = \begin{bmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{bmatrix} \tag{17}$$

where

$$\frac{\partial f_1}{\partial x} = -1 + \frac{y(p_5 + 2x)}{B_2{}^2} + \frac{z(p_7 + 2x)}{B_3{}^2},\ \frac{\partial f_1}{\partial y} = -\frac{p_1}{B_1{}^2} - \frac{1}{B_2},\ \frac{\partial f_1}{\partial z} = -\frac{p_2}{B_1{}^2} - \frac{1}{B_3},$$

$$\frac{\partial f_2}{\partial x} = \frac{p_9\left(p_4 - x^2\right)}{B_2{}^2},\ \frac{\partial f_2}{\partial y} = -p_{10},\ \frac{\partial f_2}{\partial z} = -p_{11}\,,$$

$$\frac{\partial f_3}{\partial x} = \frac{p_{14}\left(p_6 - x^2\right)}{{B_3}^2},\ \frac{\partial f_3}{\partial y} = -p_{16},\ \frac{\partial f_3}{\partial z} = -p_{15}.$$

Therefore, the JM at EEQP can be written as:

$$J(E_0) = \begin{bmatrix} 1-(p_3+p_8) & 0 & 0 \\ 0 & -(p_{12}+p_{13}) & 0 \\ 0 & 0 & -(p_{17}+p_{18}) \end{bmatrix} \tag{18}$$

Accordingly, the eigenvalues of $J(E_0)$ can be written as:

$$\lambda_{01} = 1-(p_3+p_8),\ \lambda_{02} = -(p_{12}+p_{13}) < 0 \text{ and } \lambda_{03} = -(p_{17}+p_{18}) < 0 \tag{19}$$

Therefore, EEQP is a locally asymptotically stable (LAS) if and only if the following condition is met:

$$1 < (p_3+p_8). \tag{20}$$

The JM at PDFEQP is determined as:

$$J(E_1) = \begin{bmatrix} -\hat{x} & -p_1\hat{x} - \frac{\hat{x}}{\hat{B}_2} & -p_2\hat{x} - \frac{\hat{x}}{\hat{B}_3} \\ 0 & \frac{p_9\hat{x}}{\hat{B}_2} - (p_{12}+p_{13}) & 0 \\ 0 & 0 & \frac{p_{14}\hat{x}}{\hat{B}_3} - (p_{17}\ + p_{18}) \end{bmatrix} \tag{21}$$

where $\hat{B}_2 = p_4 + p_5\hat{x} + \hat{x}^2$, and $\hat{B}_3 = p_6 + p_7\hat{x} + \hat{x}^2$. Consequently, the eigenvalues of $J(E_1)$ are given by

$$\lambda_{11} = -\hat{x},\ \lambda_{12} = \frac{p_9\hat{x}}{\hat{B}_2} - (p_{12}+p_{13}),\lambda_{13} = \frac{p_{14}\hat{x}}{\hat{B}_3} - (p_{17}\ + p_{18}). \tag{22}$$

Clearly, all the above eigenvalues are negative. Then, PDFEQP is LAS if the following requirements are satisfied:

$$\frac{p_9\hat{x}}{\hat{B}_2} < (p_{12}+p_{13}) \tag{23}$$

$$\frac{p_{14}\hat{x}}{\hat{B}_3} < (p_{17}\ + p_{18}) \tag{24}$$

At SPFEQP, the JM can be written as:

$$J(E_2) = \left[\breve{a}_{ij}\right] \tag{25}$$

where

$$\breve{a}_{11} = \breve{x}\left[-1 + \frac{\breve{y}(p_5+2\breve{x})}{{\breve{B}_2}^2}\right],\ \breve{a}_{12} = -\frac{p_1\breve{x}}{{\breve{B}_1}^2} - \frac{\breve{x}}{\breve{B}_2},\ \breve{a}_{13} = -\frac{p_2\breve{x}}{{\breve{B}_1}^2} - \frac{\breve{x}}{\breve{B}_3},$$

$$\breve{a}_{21} = \frac{p_9\left(p_4 - \breve{x}^2\right)\breve{y}}{{\breve{B}_2}^2},\ \breve{a}_{22} = -p_{10}\breve{y},\ \breve{a}_{23} = -p_{11}\breve{y},$$

$$\breve{a}_{31} = \breve{a}_{32} = 0,\ \breve{a}_{33} = \frac{p_{14}\breve{x}}{\breve{B}_3} - p_{16}\breve{y} - (p_{17}\ + p_{18})$$

with $\breve{B}_1 = 1 + p_1\breve{y}$, $\breve{B}_2 = p_4 + p_5\breve{x} + \breve{x}^2$, and $\breve{B}_3 = p_6 + p_7\breve{x} + \breve{x}^2$. Therefore, the characteristic equation of $J(E_2)$ can be written as:

$$\left(\lambda^2 - \mathrm{Tr}_1\lambda + \mathrm{Det}_1\right)(\breve{a}_{33} - \lambda) = 0, \tag{26}$$

where $\text{Tr}_1 = \check{a}_{11} + \check{a}_{22}$, and $\text{Det}_1 = \check{a}_{11}\check{a}_{22} - \check{a}_{12}\check{a}_{21}$. Direct computation shows that all the eigenvalues of $J(E_2)$ and roots of the Equation (26) have negative real parts, provided that the following conditions are met.

$$\frac{p_{14}\check{x}}{\check{B}_3} < p_{16}\check{y} + p_{17} + p_{18} \tag{27}$$

$$\frac{\check{y}(p_5 + 2\check{x})}{\check{B}_2{}^2} < 1 \tag{28}$$

$$\check{x}^2 < p_4 \tag{29}$$

Similarly, at FPFEQP, JM can be written as:

$$J(E_3) = [\overline{a}_{ij}] \tag{30}$$

where

$$\overline{a}_{11} = \overline{x}\left[-1 + \frac{\overline{z}(p_7 + 2\overline{x})}{\overline{B}_3{}^2}\right],\ \overline{a}_{12} = -\frac{p_1\overline{x}}{\overline{B}_1{}^2} - \frac{\overline{x}}{\overline{B}_2},\ \overline{a}_{13} = -\frac{p_2\overline{x}}{\overline{B}_1{}^2} - \frac{\overline{x}}{\overline{B}_3},$$

$$\overline{a}_{21} = \overline{a}_{23} = 0,\ \overline{a}_{22} = \frac{p_9\overline{x}}{\overline{B}_2} - p_{11}\,\overline{z} - (p_{12} + p_{13}),$$

$$\overline{a}_{31} = \frac{p_{14}(p_6 - \overline{x}^2)\overline{z}}{\overline{B}_3{}^2},\ \overline{a}_{32} = -p_{16}\overline{z},\ \overline{a}_{33} = -p_{15}\overline{z},$$

with $\overline{B}_1 = 1 + p_2\overline{z}$, $\overline{B}_2 = p_4 + p_5\overline{x} + \overline{x}^2$, and $\overline{B}_3 = p_6 + p_7\overline{x} + \overline{x}^2$. Therefore, the equation of $J(E_3)$ can be written as:

$$\left(\lambda^2 - \text{Tr}_2\lambda + \text{Det}_2\right)(\overline{a}_{22} - \lambda) = 0, \tag{31}$$

where $\text{Tr}_2 = \overline{a}_{11} + \overline{a}_{33}$ and $\text{Det}_1 = \overline{a}_{11}\overline{a}_{33} - \overline{a}_{13}\overline{a}_{31}$. Direct computation shows that all the eigenvalues of $J(E_3)$ and roots of Equation (31) have negative real parts, provided that the following conditions are met.

$$\frac{p_9\overline{x}}{\overline{B}_2} < p_{11}\,\overline{z} + p_{12} + p_{13} \tag{32}$$

$$\frac{\overline{z}(p_7 + 2\overline{x})}{\overline{B}_3{}^2} < 1 \tag{33}$$

$$\overline{x}^2 < p_6 \tag{34}$$

Now the JM at the IEQP can be represented as $E_4 = (x^*, y^*, z^*)$

$$J(E_4) = [a_{ij}] \tag{35}$$

where

$$a_{11} = x^*\left(-1 + \frac{(p_5 + 2x^*)y^*}{B_2^{*2}} + \frac{(p_7 + 2x^*)z^*}{B_3^{*2}}\right),\ a_{12} = -x^*\left(\frac{p_1}{B_1^{*2}} + \frac{1}{B_2^*},\right),$$

$$a_{13} = -x^*\left(\frac{p_2}{B_1^{*2}} + \frac{1}{B_3^*}\right),\ a_{21} = y^*\frac{p_9(p_4 - x^{*2})}{B_2^{*2}},\ a_{22} = -p_{10}y^*,$$

$$a_{23} = -p_{11}y^*,\ a_{31} = z^*\frac{p_{14}(p_6 - x^{*2})}{B_3^{*2}}, a_{32} = -p_{16}z^*,\ a_{33} = -p_{15}z^*,$$

with $B_1^* = 1 + p_1y^* + p_2z^*$, $B_2^* = p_4 + p_5x^* + x^{*2}$, and $B_3^* = p_6 + p_7x^* + x^{*2}$. As a result, the following theorem establishes the local stability criteria of IEQP.

Theorem 2. *Assuming that System (2) has an IEQP, it is LAS if the following conditions are met.*

$$\frac{(p_5 + 2x^*)y^*}{B_2^{*2}} + \frac{(p_7 + 2x^*)z^*}{B_3^{*2}} < 1 \tag{36}$$

$$x^{*2} < \min\{p_4,\ p_6\} \tag{37}$$

$$p_{11}p_{16} < p_{10}p_{15} \tag{38}$$

$$\frac{p_9 p_{16}(p_4 - x^{*2})}{p_{10}p_{14}\ B_2^{*2}} < \frac{(p_6 - x^{*2})}{B_3^{*2}} < \frac{p_9 p_{15}(p_4 - x^{*2})}{p_{11}p_{14}\ B_2^{*2}} \tag{39}$$

$$p_{11}y^* \frac{p_{14}(p_6 - x^{*2})}{B_3^{*2}} < p_{16}x^*\left(\frac{p_2}{B_1^{*2}} + \frac{1}{B_3^*}\right) \tag{40}$$

Proof. The characteristic equation of $J(E_4)$ is determined as

$$\lambda^3 + \Gamma_1\lambda^2 + \Gamma_2\lambda + \Gamma_3 = 0 \tag{41}$$

where

$$\Gamma_1 = -(a_{11} + a_{22} + a_{33})$$

$$\Gamma_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}$$

$$\Gamma_3 = -[a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{22}a_{31} - a_{21}a_{32})]$$

with

$$\begin{aligned}\Delta = \Gamma_1\Gamma_2 - \Gamma_3 = &-(a_{11} + a_{22})[a_{11}a_{22} - a_{12}a_{21}] - (a_{11} + a_{33})[a_{11}a_{33} - a_{13}a_{31}] \\ &-(a_{22} + a_{33})[a_{22}a_{33} - a_{23}a_{32}] - 2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}.\end{aligned}$$

If $\Gamma_1 > 0$, $\Gamma_3 > 0$, and $\Delta > 0$, then using the Routh-Hurwitz criterion, all the roots of Equation (41) have negative real parts, and thus, IEQP is LAS. All Routh-Hurwitz constraints are satisfied under the stated sufficient conditions according to direct computation. □

5. Global Stability Analysis

In this part, the basin of attraction of each asymptotically stable EQP is determined using Lyapunov functions, as proven in the theorems below.

Theorem 3. *The EEQP is globally stable locally (GAS) whenever it is LAS.*

Proof. Now we define V_0 as a real-valued function that is given by

$$V_0 = c_1x + c_2y + c_3z,$$

where c_1, c_2, and c_3 are the positive constants to be determined. V_0 is a positive definite function that is defined on ${\mathbb{R}_+}^3$. Furthermore, $\frac{dV_0}{dt}$ can be written as follows:

$$\begin{aligned}\tfrac{dV_0}{dt} = c_1x(1 - p_3 - p_8) - c_1x^2 - (c_1 - c_2p_9)\tfrac{xy}{p_4 + p_5x + x^2} - (c_1 - c_3p_{14})\tfrac{xz}{p_6 + p_7x + x^2} \\ -c_2p_{10}y^2 - (c_2p_{11} + c_3p_{16})yz - c_2y(p_{12} + p_{13}) \\ -c_3p_{15}z^2 - c_3z(p_{17} + p_{18})\end{aligned}$$

Now, by choosing $c_1 = 1$, $c_2 = \frac{1}{p_9}$, and $c_3 = \frac{1}{p_{14}}$ as the positive constants, it is determined that:

$$\frac{dV_0}{dt} \leq x(1 - p_3 - p_8) - \frac{(p_{12} + p_{13})}{\mathrm{p}_9}y - \frac{(p_{17} + p_{18})}{\mathrm{p}_{14}}z$$

Then, using the LAS condition (20), the derivative $\frac{dV_0}{dt}$ becomes a negative definite function. Hence, EEQP is GAS. □

Theorem 4: *The PDFEQP is GAS if the following conditions are met.*

$$p_1\check{x} + \frac{\check{x}}{p_4} < \frac{(p_{12} + p_{13})}{p_9} \tag{42}$$

$$p_2\check{x} + \frac{\check{x}}{p_6} < \frac{(p_{17} + p_{18})}{p_{14}} \tag{43}$$

Proof: Now we define V_1 as a real-valued function that is given by

$$V_1 = q_1\left[x - \check{x} - \check{x}ln\frac{x}{\check{x}}\right] + q_2y + q_3z$$

where q_1, q_2, and q_3 are the positive constants to be determined. V_1 is a positive definite function that is defined on $\left\{(x, y, z) \in \mathbb{R}^3 : x > 0,\ y \geq 0,\ z \geq 0\right\}$. Furthermore, after doing some simplifications steps, $\frac{dV_1}{dt}$ can be written as follows:

$$\begin{aligned}\tfrac{dV_1}{dt} \leq\ & q_1p_1\check{x}y + q_1p_2\check{x}z - q_1(x-\check{x})^2 - (q_1 - q_2p_9)\tfrac{xy}{p_4+p_5x+x^2} + \tfrac{q_1}{p_4}\check{x}y \\ & -(q_1 - q_3p_{14})\tfrac{xz}{p_6+p_7x+x^2} + \tfrac{q_1}{p_6}\check{x}z - q_2p_{10}y^2 - q_2(p_{12}+p_{13})y \\ & -(q_2p_{11} + q_3p_{16})yz - q_3p_{15}z^2 - q_3(p_{17}+p_{18})z\end{aligned}$$

Now, by choosing the positive constants as $q_1 = 1$, $q_2 = \frac{1}{p_9}$, and $q_3 = \frac{1}{p_{14}}$, it is determined that

$$\frac{dV_1}{dt} \leq -(x - \check{x})^2 - \left[\frac{(p_{12} + p_{13})}{p_9} - p_1\check{x} - \frac{\check{x}}{p_4}\right]y - \left[\frac{(p_{17} + p_{18})}{p_{14}} - p_2\check{x} - \frac{\check{x}}{p_6}\right]z$$

Then, by using the above sufficient conditions, the derivative $\frac{dV_1}{dt}$ becomes a negative definite function, and hence, the PDFEQP is GAS. □

Theorem 5: *The SPFEQP is GAS if the following sufficient conditions are met.*

$$\frac{(p_5 + x + \hat{x})\hat{y}}{B_2\hat{B}_2} < 1 \tag{44}$$

$$\frac{p_2\hat{x}}{B_1\hat{B}_1} + \frac{\hat{x}}{B_3} + \frac{p_{11}\hat{B}_2}{p_9p_4}\hat{y} < \frac{(p_{17} + p_{18})}{p_{14}} \tag{45}$$

$$\left[\frac{p_1}{B_1\hat{B}_1} + \frac{\hat{B}_2\hat{x}x}{p_4B_2\hat{B}_2}\right]^2 < 4\frac{p_{10}\hat{B}_2}{p_9p_4}\left[1 - \frac{(p_5 + x + \hat{x})\hat{y}}{B_2\hat{B}_2}\right] \tag{46}$$

where all the symbols are defined in the proof.

Proof: Next, we define V_2 as a real-valued function that is given by

$$V_2 = \mu_1\left[x - \hat{x} - \hat{x}ln\frac{x}{\hat{x}}\right] + \mu_2\left[y - \hat{y} - \hat{y}ln\frac{y}{\hat{y}}\right] + \mu_3z$$

where μ_1, μ_2, and μ_3 are the positive constants to be determined. V_2 is a positive definite function that is defined on $\{(x, y, z) \in \mathbb{R}^3 : x > 0,\ y > 0,\ z \geq 0\}$. Furthermore, after doing some simplifications steps, $\frac{dV_2}{dt}$ can be written as follows:

$$\begin{aligned}\frac{dV_2}{dt} = &-\mu_1\left[1 - \frac{(p_5+x+\hat{x})\hat{y}}{B_2\hat{B}_2}\right](x-\hat{x})^2 - \mu_2 p_{10}\,(y-\hat{y})^2 - \mu_3 p_{15} z^2 \\ &-\left[\frac{\mu_1 p_1}{B_1\hat{B}_1} + \frac{\mu_1\hat{B}_2}{B_2\hat{B}_2} - \frac{\mu_2 p_9 p_4}{B_2\hat{B}_2} + \frac{\mu_2 p_9\hat{x}x}{B_2\hat{B}_2}\right](x-\hat{x})(y-\hat{y}) \\ &-\left[\frac{\mu_1 p_2}{B_1\hat{B}_1} + \frac{\mu_1}{B_3} - \frac{\mu_3 p_{14}}{B_3}\right]xz - [\mu_2 p_{11} + \mu_3 p_{16}]yz \\ &-\left[\mu_3(p_{17}+p_{18}) - \frac{\mu_1 p_2\hat{x}}{B_1\hat{B}_1} - \frac{\mu_1\hat{x}}{B_3} - \mu_2 p_{11}\hat{y}\right]z,\end{aligned}$$

where B_1, B_2, and B_3 are given in Theorem (1), while $\hat{B}_2$, and $\hat{B}_3$ are given in Equation (21) with $\hat{B}_1 = 1 + p_1\hat{y}$. Now, by choosing $\mu_1 = 1$, $\mu_2 = \frac{\hat{B}_2}{p_9 p_4}$, and $\mu_3 = \frac{1}{p_{14}}$ as the positive constants, it is determined after simple calculation that:

$$\begin{aligned}\frac{dV_2}{dt} \leq &-\left[1 - \frac{(p_5+x+\hat{x})\hat{y}}{B_2\hat{B}_2}\right](x-\hat{x})^2 - \frac{p_{10}\hat{B}_2}{p_9 p_4}\,(y-\hat{y})^2 \\ &-\left[\frac{p_1}{B_1\hat{B}_1} + \frac{\hat{B}_2\hat{x}x}{p_4 B_2\hat{B}_2}\right](x-\hat{x})(y-\hat{y}) \\ &-\left[\frac{(p_{17}+p_{18})}{p_{14}} - \frac{p_2\hat{x}}{B_1\hat{B}_1} - \frac{\hat{x}}{B_3} - \frac{p_{11}\hat{B}_2}{p_9 p_4}\hat{y}\right]z.\end{aligned}$$

Then, using Condition (46) yields:

$$\begin{aligned}\frac{dV_2}{dt} \leq &-\left[\sqrt{1 - \frac{(p_5+x+\hat{x})\hat{y}}{B_2\hat{B}_2}}(x-\hat{x}) + \sqrt{\frac{p_{10}\hat{B}_2}{p_9 p_4}}(y-\hat{y})\right]^2 \\ &-\left[\frac{(p_{17}+p_{18})}{p_{14}} - \frac{p_2\hat{x}}{B_1\hat{B}_1} - \frac{\hat{x}}{B_3} - \frac{p_{11}\hat{B}_2}{p_9 p_4}\hat{y}\right]z.\end{aligned}$$

According to the given sufficient conditions, derivative $\frac{dV_2}{dt}$ becomes a negative definite function, and hence, SPFEQP is GAS. □

Theorem 6: *The FPFEQP is GAS if the following sufficient conditions are met.*

$$\frac{(p_7 + x + \overline{x})\overline{z}}{B_3\overline{B}_3} < 1 \tag{47}$$

$$\frac{p_1\overline{x}}{B_1\overline{B}_1} + \frac{\overline{x}}{B_2} + \frac{p_{16}\overline{B}_3}{p_6 p_{14}}\overline{z} < \frac{(p_{12}+p_{13})}{p_9} \tag{48}$$

$$\left[\frac{p_2}{B_1\overline{B}_1} + \frac{\overline{B}_3\overline{x}x}{p_6 B_3\overline{B}_3}\right]^2 < 4\frac{p_{15}\overline{B}_3}{p_6 p_{14}}\left[1 - \frac{(p_7 + x + \overline{x})\overline{z}}{B_3\overline{B}_3}\right] \tag{49}$$

where all the symbols are given in the proof.

Proof: Next, we define V_3 as a real-valued function that is given by

$$V_3 = \Omega_1\left[x - \overline{x} - \overline{x}ln\frac{x}{\overline{x}}\right] + \Omega_2[y] + \Omega_3\left[z - \overline{z} - \overline{z}ln\frac{z}{\overline{z}}\right]$$

where Ω_1, Ω_2, and Ω_3 are the positive constants to be determined. V_3 is a positive definite function that is defined on $\{(x, y, z) \in \mathbb{R}^3 : x > 0,\ y \geq 0,\ z > 0\}$. Furthermore, after doing some simplifications steps, $\frac{dV_3}{dt}$ can be written as follows:

$$\begin{aligned}\tfrac{dV_3}{dt} = &-\Omega_1\left[1-\tfrac{(p_7+x+\overline{x})\overline{z}}{B_3\overline{B}_3}\right](x-\overline{x})^2-\Omega_2 p_{10}\,y^2-\Omega_3 p_{15}(z-\overline{z})^2\\ &-\left[\tfrac{\Omega_1 p_2}{B_1\overline{B}_1}+\tfrac{\Omega_1\overline{B}_3}{B_3\overline{B}_3}-\tfrac{\Omega_3 p_6 p_{14}}{B_3\overline{B}_3}+\tfrac{\Omega_3 p_{14}\overline{x}x}{B_3\overline{B}_3}\right](x-\overline{x})(z-\overline{z})\\ &-\left[\tfrac{\Omega_1 p_1}{B_1\overline{B}_1}+\tfrac{\Omega_1}{B_2}-\tfrac{\Omega_2 p_9}{B_2}\right]xy-[\Omega_2 p_{11}+\Omega_3 p_{16}]yz\\ &-\left[\Omega_2(p_{12}+p_{13})-\tfrac{\Omega_1 p_1\overline{x}}{B_1\overline{B}_1}-\tfrac{\Omega_1\overline{x}}{B_2}-\Omega_3 p_{16}\overline{z}\right]y,\end{aligned}$$

where B_1, B_2, and B_3 are given in Theorem (1), while $\overline{B}_1$, $\overline{B}_2$, and $\overline{B}_3$ are given in Equation (30). Now, by choosing $\Omega_1 = 1$, $\Omega_2 = \frac{1}{p_9}$, and $\Omega_3 = \frac{\overline{B}_3}{p_6 p_{14}}$ as the positive constants, it is determined after a simple calculation that:

$$\begin{aligned}\tfrac{dV_3}{dt} \leq &-\left[1-\tfrac{(p_7+x+\overline{x})\overline{z}}{B_3\overline{B}_3}\right](x-\overline{x})^2-\tfrac{p_{15}\overline{B}_3}{p_6 p_{14}}(z-\overline{z})^2\\ &-\left[\tfrac{p_2}{B_1\overline{B}_1}+\tfrac{\overline{B}_3\overline{x}x}{p_6 B_3\overline{B}_3}\right](x-\overline{x})(z-\overline{z})\\ &-\left[\tfrac{(p_{12}+p_{13})}{p_9}-\tfrac{p_1\overline{x}}{B_1\overline{B}_1}-\tfrac{\overline{x}}{B_2}-\tfrac{p_{16}\overline{B}_3}{p_6 p_{14}}\overline{z}\right]y.\end{aligned}$$

Then, using Condition (49) yields:

$$\begin{aligned}\tfrac{dV_3}{dt} \leq &-\left[\sqrt{1-\tfrac{(p_7+x+\overline{x})\overline{z}}{B_3\overline{B}_3}}(x-\overline{x})+\sqrt{\tfrac{p_{15}\overline{B}_3}{p_6 p_{14}}}(z-\overline{z})\right]^2\\ &-\left[\tfrac{(p_{12}+p_{13})}{p_9}-\tfrac{p_1\overline{x}}{B_1\overline{B}_1}-\tfrac{\overline{x}}{B_2}-\tfrac{p_{16}\overline{B}_3}{p_6 p_{14}}\overline{z}\right]y.\end{aligned}$$

Then, according to the given sufficient conditions, the derivative $\frac{dV_3}{dt}$ becomes a negative definite function, and hence, FPFEQP is GAS. □

Theorem 7: *The IEQP is GAS if the following sufficient conditions are met.*

$$\frac{(p_5+x+x^*)y^*}{B_2 B_2^*}+\frac{(p_7+x+x^*)z^*}{B_3 B_3^*}<1, \tag{50}$$

$${\sigma_{12}}^2<\sigma_{11}\sigma_{22}, \tag{51}$$

$${\sigma_{13}}^2<\sigma_{11}\sigma_{33}, \tag{52}$$

$${\sigma_{23}}^2<\sigma_{22}\sigma_{33}, \tag{53}$$

where all the symbols are given in the proof.

Proof: Next, we define V_4 as a real-valued function that is given by

$$V_4=\omega_1\left[x-x^*-x^* ln\frac{x}{x^*}\right]+\omega_2\left[y-y^*-y^* ln\frac{y}{y^*}\right]+\omega_3\left[z-z^*-z^* ln\frac{z}{z^*}\right]$$

where ω_1, ω_2, and ω_3 are the positive constants to be determined. V_4 is a positive definite function that is defined on $\{(x,y,z) \in \mathbb{R}^3 : x > 0,\ y > 0,\ z > 0\}$. Furthermore, after doing some simplifications steps, $\frac{dV_4}{dt}$ can be written as follows:

$$
\begin{aligned}
\frac{dV_4}{dt} = -\omega_1 \left[1 - \frac{(p_5+x+x^*)y^*}{B_2 B_2^*} - \frac{(p_7+x+x^*)z^*}{B_3 B_3^*}\right](x-x^*)^2 - \omega_2 p_{10}(y-y^*)^2 \\
-\left[\frac{\omega_1 p_1}{B_1 B_1^*} + \frac{\omega_1 B_2^*}{B_2 B_2^*} - \frac{\omega_2 p_9 (p_4 - x^* x)}{B_2 B_2^*}\right](x-x^*)(y-y^*) \\
-\left[\frac{\omega_1 p_2}{B_1 B_1^*} + \frac{\omega_1 B_3^*}{B_3 B_3^*} - \frac{\omega_3 p_{14} (p_6 - x^* x)}{B_3 B_3^*}\right](x-x^*)(z-z^*) \\
-[\omega_2 p_{11} + \omega_3 p_{16}](y-y^*)(z-z^*) - \omega_3 p_{15}(z-z^*)^2,
\end{aligned}
$$

where B_1, B_2, and B_3 are given in Theorem (1), while B_1^*, B_2^*, and B_3^* are given in Equation (35). Now, by choosing the positive constants as $\omega_1 = 1$, $\omega_2 = \frac{B_2^*}{p_4 p_9}$, and $\omega_3 = \frac{B_3^*}{p_6 p_{14}}$, then it is determined after simple calculation that:

$$
\begin{aligned}
\frac{dV_4}{dt} = -\left[1 - \frac{(p_5+x+x^*)y^*}{B_2 B_2^*} - \frac{(p_7+x+x^*)z^*}{B_3 B_3^*}\right](x-x^*)^2 - \frac{p_{10} B_2^*}{p_4 p_9}(y-y^*)^2 \\
-\frac{p_{15} B_3^*}{p_6 p_{14}}(z-z^*)^2 - \left[\frac{p_1}{B_1 B_1^*} + \frac{x^* x}{B_2 B_2^*}\right](x-x^*)(y-y^*) \\
-\left[\frac{p_2}{B_1 B_1^*} + \frac{x^* x}{B_3 B_3^*}\right](x-x^*)(z-z^*) \\
-\left[\frac{p_{11} B_2^*}{p_4 p_9} + \frac{p_{16} B_3^*}{p_6 p_{14}}\right](y-y^*)(z-z^*).
\end{aligned}
$$

Using the above sufficient conditions yields:

$$
\begin{aligned}
\frac{dV_4}{dt} \le -\tfrac{1}{2}\left[\sqrt{\sigma_{11}}(x-x^*) + \sqrt{\sigma_{22}}(y-y^*)\right]^2 \\
-\tfrac{1}{2}\left[\sqrt{\sigma_{11}}(x-x^*) + \sqrt{\sigma_{33}}(z-z^*)\right]^2 \\
-\tfrac{1}{2}\left[\sqrt{\sigma_{22}}(y-y^*) + \sqrt{\sigma_{33}}(z-z^*)\right]^2,
\end{aligned}
$$

where $\sigma_{11} = 1 - \frac{(p_5+x+x^*)y^*}{B_2 B_2^*} - \frac{(p_7+x+x^*)z^*}{B_3 B_3^*}$, $\sigma_{22} = \frac{p_{10} B_2^*}{p_4 p_9}$, $\sigma_{33} = \frac{p_{15} B_3^*}{p_6 p_{14}}$, $\sigma_{12} = \frac{p_1}{B_1 B_1^*} + \frac{x^* x}{B_2 B_2^*}$, $\sigma_{13} = \frac{p_2}{B_1 B_1^*} + \frac{x^* x}{B_3 B_3^*}$, and $\sigma_{23} = \frac{p_{11} B_2^*}{p_4 p_9} + \frac{p_{16} B_3^*}{p_6 p_{14}}$.

Then, the derivative $\frac{dV_4}{dt}$ becomes a negative definite function, and hence, IEQP is GAS. □

6. Persistence

Persistence is commonly understood as a global attribute of a dynamic system. Rather than the interior solution space, it is dependent on the solution behavior around the extinction boundaries. Biologically, the persistence of a system entails the survival of all of the system's populations in the future. In mathematical terms, this indicates that strictly positive solutions do not have an omega limit on the boundary of the non-negative cone. As a result, if the dynamic system does not continue, it is doomed to extinction.

We must first analyze the global dynamics in boundary planes xy and xz, as illustrated below, before examining the persistence of the food web model using the average Lyapunov function approach [45]. It is simple to verify that System (2) includes two subsystems when the first subsystem is:

$$
\begin{aligned}
\frac{dx}{dt} &= x\left(\frac{1}{1+p_1 y} - p_3 - x - \frac{y}{p_4 + p_5 x + x^2} - p_8\right) = k_1(x,y), \\
\frac{dy}{dt} &= y\left(\frac{p_9 x}{p_4 + p_5 x + x^2} - p_{10} y - p_{12} - p_{13}\right) = k_2(x,y).
\end{aligned}
\tag{54}
$$

However, the second subsystem can be written as:

$$\begin{aligned} \frac{dx}{dt} &= x\left(\frac{1}{1+p_2z} - p_3 - x - \frac{z}{p_6+p_7x+x^2} - p_8\right) = k_3(x,z), \\ \frac{dz}{dt} &= z\left(\frac{p_{14}x}{p_6+p_7x+x^2} - p_{15}z - p_{17} - p_{18}\right) = k_4(x,z). \end{aligned} \tag{55}$$

It is observed that these subsystems have unique interior equilibrium points in the positive quadrants of their xy and xz planes, which coincide with the projection of the boundary corresponding to the planar equilibrium points of System (2).

Here, we define $\theta_1(x,y) = \frac{1}{xy}$, a continuously differential function in the interior of the positive quadrant of the xy plane, which represents a Dulac function which computes the following expression:

$$\Delta(x,y) = \frac{\partial}{\partial x}(k_1\theta_1) + \frac{\partial}{\partial y}(k_2\theta_1) = -\frac{1}{y} + \frac{p_5+2x}{B_2{}^2} - \frac{p_{10}}{x}$$

Clearly, the expression $\Delta(x,y)$ is not identically zero, and their sign does not change if the following sufficient condition is met:

$$\frac{p_5+2x}{B_2{}^2} < \frac{1}{y} + \frac{p_{10}}{x} \tag{56}$$

As a result, using the Bendixson-Dulac criterion, Subsystem (54) has no periodic dynamic in the interior of the positive quadrant of the xy plane. Consequently, using the Poincare-Bendixson theorem, the interior EQP of Subsystem (54), as given by $(\check{x},\check{y})$ is, in fact, a GAS in the interior of a positive quadrant of the xy plane whenever it exists under Condition (56).

Similarly, for Subsystem (55), there are no periodic dynamics in the interior of the positive quadrant of the xz plane, and the interior EQP, given by $(\overline{x},\overline{z})$, will be a GAS provided that the following condition is met.

$$\frac{p_7+2x}{B_3{}^2} < \frac{1}{z} + \frac{p_{15}}{x} \tag{57}$$

Theorem 8. *Assuming that Conditions (56) and (57) are met, then System (2) is uniformly persistent, provided that the following conditions are satisfied.*

$$(p_3+p_8) < 1 \tag{58}$$

$$\left.\begin{aligned} p_{12}+p_{13} &< \frac{p_9\tilde{x}}{\hat{B}_2} \\ p_{17}+p_{18} &< \frac{p_{14}\tilde{x}}{\hat{B}_3} \end{aligned}\right\} \tag{59}$$

$$p_{16}\check{y} + p_{17} + p_{18} < \frac{p_{14}\check{x}}{\check{B}_3} \tag{60}$$

$$p_{11}\,\overline{z} + p_{12} + p_{13} < \frac{p_9\overline{x}}{\overline{B}_2} \tag{61}$$

Proof. Consider the following function: $\delta(x,y,z) = x^{\alpha 1}y^{\alpha 2}z^{\alpha 3}$, where α_i, i $= 1,2,3$ are positive constants. $\delta(x,y,z)$ is a non-negative, continuously differentiable function, and as such, $\delta(x,y,z) \to 0$ if any of the variables x, y, or z approach zero.

Therefore, the following is determined:

$$\varphi(x,y,z) = \frac{\delta'(x,y,z)}{\delta(x,y,z)} = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$$

where the functions $f_i,\ i = 1,2,3$, are given in Equation (11). Accordingly, substituting the functions leads to:

$$\varphi(x,y,z) = \alpha_1\left[\tfrac{1}{B_1} - p_3 - x - \tfrac{y}{B_2} - \tfrac{z}{B_3} - p_8\right] + \alpha_2\left[\tfrac{p_9 x}{B_2} - p_{10}y - p_{11}\,z - p_{12} - p_{13}\right]$$
$$+\alpha_3\left[\tfrac{p_{14}x}{B_3} - p_{15}z - p_{16}y - p_{17} - p_{18}\right]$$

The proof is now complete [45] and System (2) is uniformly persistent if $\varphi(\mathrm{Y}) > 0$, where Y belongs to the omega limit set of the boundary planes of the system. Because only the EQPs are found in the omega limit set of System (2) in the boundary planes, it is deduced that:

$$\varphi(E_0) = \alpha_1[1 - p_3 - p_8] - \alpha_2[p_{12} + p_{13}] - \alpha_3[p_{17} + p_{18}]$$

Clearly, by choosing a constant $\alpha_1 > 0$ which is sufficiently large with respect to the positive constants α_2 and α_3, it is determined that $\varphi(E_0) > 0$, provided that Condition (58) remains to be met.

$$\varphi(E_1) = \alpha_2\left[\frac{p_9\hat{x}}{\hat{B}_2} - p_{12} - p_{13}\right] + \alpha_3\left[\frac{p_{14}\hat{x}}{\hat{B}_3} - p_{17} - p_{18}\right]$$

$\varphi(E_1) > 0$ under the Conditions (59), or when at least one of them is met with a suitable choice of the arbitrary positive constants α_2 and α_3.

$$\varphi(E_2) = \alpha_3\left[\frac{p_{14}\check{x}}{\check{B}_3} - p_{16}\check{y} - p_{17} - p_{18}\right]$$

Similarly, $\varphi(E_2) > 0$ under Condition (60).

$$\varphi(E_3) = \alpha_2\left[\frac{p_9\overline{x}}{\overline{B}_2} - p_{11}\,\overline{z} - p_{12} - p_{13}\right]$$

Again, $\varphi(E_3) > 0$, provided that Condition (61) is satisfied. As a result, no omega limit set is placed in the boundary planes of System (2) under the provided set of conditions, and hence, System (2) is uniformly persistent. □

7. Bifurcation Analysis

This section looks at how the equilibrium configurations of System (2) are affected by the parameters that define the model. Indeed, as one of the parameters approaches a certain value, the solution may trend toward a different equilibrium position. The goal of this section is to investigate the bifurcation of System (2) that can occur as the parameter values change.

Here, we rewrite System (2) in the form:

$$\frac{dX}{dt} = F(X), \tag{62}$$

where $X = (x,y,z)^T$ and $F(X) = (xf_1, yf_2, zf_3)^T$. Therefore, the second derivative of $F(X)$, with respect to the vector X, can be written as:

$$D^2F(X)(V,V) = [c_{i1}]_{3\times 1}, \tag{63}$$

where $V = (v_1, v_2, v_3)$ is a non-zero real vector, with

$$c_{11} = 2\left[-1 + \tfrac{3p_4x + p_4p_5 - x^3}{B_2{}^3}y + \tfrac{3p_6x + p_6p_7 - x^3}{B_3{}^3}z\right]v_1^2 + 2\tfrac{x(p_1v_2 + p_2v_3)^2}{B_1{}^3}$$
$$-2\tfrac{p_1v_1v_2}{B_1{}^2} + 2\tfrac{(x^2 - p_4)v_1v_2}{B_2{}^2} + 2\left[-\tfrac{p_2}{B_1{}^2} + \tfrac{x^2 - p_6}{B_3{}^2}\right]v_1v_3.$$

$$c_{21} = \frac{2\mathrm{p}_9 y\left(x^3 - 3p_4 x - p_4 p_5\right)v_1^2}{{B_2}^3} + \frac{2\mathrm{p}_9\left(p_4 - x^2\right)v_1 v_2}{{B_2}^2} - 2v_2(p_{10}v_2 + p_{11}v_3)$$

$$c_{31} = \frac{2\mathrm{p}_{14} z\left(x^3 - 3p_6 x - p_6 p_7\right)v_1^2}{{B_3}^3} + \frac{2\mathrm{p}_{14}\left(p_6 - x^2\right)v_1 v_3}{{B_3}^2} - 2v_3(p_{16}v_2 + p_{15}v_3)$$

Meanwhile, the third derivative of $F(X)$ with respect to the vector X can be written as:

$$D^3F(X)(V, V,\ V) = [d_{i1}]_{3\times 1}, \tag{64}$$

where:

$$\begin{gathered} d_{11} = 6\left[\frac{x^4 y}{{B_2}^4} + \frac{y p_4^2}{{B_2}^4} - \frac{y p_4\left(6x^2 + 4x p_5 + p_5^2\right)}{{B_2}^4} + \frac{z\left[x^4 + p_6(p_6 - p_7(4x + p_7)) - 6x^2\right]}{{B_3}^4}\right]v_1^3 \\ +6\frac{v_1(p_1 v_2 + p_2 v_3)^2}{{B_1}^3} - 6\frac{x(p_1 v_2 + p_2 v_3)^3}{{B_1}^4} \\ +6\left[\frac{\left(p_4(3x + p_5) - x^3\right)v_2}{{B_2}^3} + \frac{\left(p_6(3x + p_7) - x^3\right)v_3}{{B_3}^3}\right]v_1^2. \end{gathered}$$

$$d_{21} = \frac{6p_9 v_1^2}{{B_2}^4}\left(-y\left[x^4 + p_4\left(p_4 - p_5(4x + p_5) - 6x^2\right)\right]v_1 + B_2\left[x^3 - p_4(3x + p_5)\right]v_2\right)$$

$$d_{31} = \frac{6p_{14} v_1^2}{{B_3}^4}\left(-z\left[x^4 + p_6\left(p_6 - p_7(4x + p_7) - 6x^2\right)\right]v_1 + B_3\left[x^3 - p_6(3x + p_7)\right]v_3\right)$$

Theorem 9. *System (2) undergoes transcritical bifurcation (TBF) near the EEQP when p_3 passes through the value $p_3^* = 1 - p_8$.*

Proof. The JM of System (2) at the EEQP with $p_3 = p_3^*$ can be written in the form:

$$J_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(p_{12} + p_{13}) & 0 \\ 0 & 0 & -(p_{17} + p_{18}) \end{bmatrix}$$

The eigenvalues of J_0 are given by $\lambda_{01}^* = 0$, $\lambda_{02}^* = -(p_{12} + p_{13}) < 0$, and $\lambda_{03}^* = -(p_{17} + p_{18}) < 0$. Hence, the EEQP is a non-hyperbolic point.

Let vectors $V_0 = (v_{01}, v_{02}, v_{03})^T$ and $\Psi_0 = (\psi_{01}, \psi_{02}, \psi_{03})^T$ represent the eigenvectors corresponding to $\lambda_{01}^* = 0$ of J_0 and ${J_0}^T$ respectively. Hence, direct computation shows that:

$$V_0 = (1, 0, 0)^T, \text{ and } \Psi_0 = (1, 0, 0)^T$$

Moreover, direct computation shows the following:

$$\frac{\partial}{\partial p_3}F(X, p_3) = (-x, 0, 0)^T \Rrightarrow F_{p_3}(E_0,\ p_3^*) = (0, 0, 0)^T \Rrightarrow \Psi_0^T F_{p_3}(E_0,\ p_3^*) = 0.$$

$$DF_{p_3}(X, p_3)V_0 = (-1, 0, 0)^T \Rrightarrow \Psi_0^T\left[DF_{p_3}(E_0,\ p_3^*)V_0\right] = -1 \neq 0,$$

where DF_{p_3} represents the directional derivatives of F_{p_3} with respect to X. Furthermore, using Equation (63) yields:

$$D^2F(E_0,\ p_3^*)(V_0, V_0) = (-2, 0, 0)^T \Rrightarrow \Psi_0^T\left[D^2F(E_0,\ p_3^*)(V_0, V_0)\right] = -2 \neq 0$$

Therefore, in the sense of Sotomayor's theorem [44], System (2) possesses a TBF and the proof is complete. □

Theorem 10. *Assuming that Condition (24) holds, then System (2) undergoes a TBF near the PDFEQP when* p_9 *passes through the value* $\hat{p}_9 = (p_{12} + p_{13})\frac{\hat{B}_2}{\hat{x}}$*, provided that the following condition is met.*

$$-2\hat{p}_9\left[\frac{p_4 - \hat{x}^2}{\hat{B}_2{}^2}\right]\left(p_1 + \frac{1}{\hat{B}_2}\right) - 2p_{10} \neq 0 \tag{65}$$

Otherwise, System (2) undergoes a pitchfork bifurcation (PBF) near the PDFEQP, provided that the following condition is met.

$$\hat{x}^3 - p_4(3\hat{x} + p_5) \neq 0 \tag{66}$$

Proof. The JM of System (2) at the PDFEQP with $p_9 = \hat{p}_9$ can be written in the form:

$$J_1 = J(E_1, \hat{p}_9) = \begin{bmatrix} -\hat{x} & -p_1\hat{x} - \frac{\hat{x}}{\hat{B}_2} & -p_2\hat{x} - \frac{\hat{x}}{\hat{B}_3} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{p_{14}\hat{x}}{\hat{B}_3} - (p_{17} + p_{18}) \end{bmatrix}$$

The eigenvalues of J_1 are given by $\hat{\lambda}_{11} = -\hat{x}$, $\hat{\lambda}_{12} = 0$, and $\hat{\lambda}_{13} = \frac{p_{14}\hat{x}}{\hat{B}_3} - (p_{17} + p_{18})$, which are negative under Condition (24). Hence, the PDFEQP is a non-hyperbolic point.

Let vectors $V_1 = (v_{11}, v_{12}, v_{13})^T$ and $\Psi_1 = (\psi_{11}, \psi_{12}, \psi_{13})^T$ represent the eigenvectors corresponding the $\hat{\lambda}_{12} = 0$ of J_1 and $J_1{}^T$, respectively. Hence, direct computation reveals that:

$$V_1 = \left(-p_1 - \frac{1}{\hat{B}_2}, 1, 0\right)^T, \text{ and } \Psi_1 = (0, 1, 0)^T$$

Moreover, direct computation shows the following:

$$\frac{\partial}{\partial p_9}F(X, p_9) = \left(0, \frac{xy}{B_2}, 0\right)^T \Rightarrow F_{p_9}(E_1, \ \hat{p}_9) = (0, 0, 0)^T \Rightarrow \Psi_1^T F_{p_9}(E_1, \ \hat{p}_9) = 0.$$

$$DF_{p_9}(E_1, \ \hat{p}_9)V_1 = \left(0, \frac{\hat{x}}{\hat{B}_2}, 0\right)^T \Rightarrow \Psi_1^T\left[DF_{p_9}(E_1, \ \hat{p}_9)V_1\right] = \frac{\hat{x}}{\hat{B}_2} \neq 0.$$

Furthermore, using Equation (63) yields:

$$D^2F(E_1, \ \hat{p}_9)(V_1, V_1) = [\hat{c}_{i1}]_{3\times 1},$$

where:

$$\hat{c}_{11} = 2\hat{x}p_1^2 + 2p_1\left(p_1 + \frac{1}{\hat{B}_2}\right) + \frac{2(\hat{x}^2 - p_4)v_1}{\hat{B}_2{}^2} - 2\left(p_1 + \frac{1}{\hat{B}_2}\right)^2$$
$$\hat{c}_{21} = -2p_{10} - \frac{2\hat{p}_9(p_4 - \hat{x}^2)v_1}{\hat{B}_2{}^2}\left(p_1 + \frac{1}{\hat{B}_2}\right)$$
$$\hat{c}_{31} = 0$$

Therefore, it is determined that:

$$\Psi_1^T\left[D^2F(E_1, \ \hat{p}_9)(V_1, V_1)\right] = -2\hat{p}_9\left[\frac{p_4 - \hat{x}^2}{\hat{B}_2{}^2}\right]\left(p_1 + \frac{1}{\hat{B}_2}\right) - 2p_{10}$$

Therefore, in the sense of Sotomayor's theorem, System (2) possesses a TBF, provided that Condition (65) holds. However, if Condition (65) is not satisfied, then, using Equation (64), it is obtained that:

$$D^3F(E_1,\ \hat{p}_9)(V_1, V_1, V_1) = \left[\hat{d}_{i1}\right]_{3\times 1},$$

where:

$$\hat{d}_{11} = -6\left(p_1 + \frac{1}{\hat{B}_2}\right)p_1^2 - 6\hat{x}p_1^3 - \frac{6}{\hat{B}_2{}^3}\left[\hat{x}^3 - 3p_4\hat{x} - p_4p_5\right]\left(p_1 + \frac{1}{\hat{B}_2}\right)^2$$

$$\hat{d}_{21} = \frac{6p_9}{\hat{B}_2{}^3}\left(p_1 + \frac{1}{\hat{B}_2}\right)^2\left[\hat{x}^3 - p_4(3\hat{x} + p_5)\right]$$

$$\hat{d}_{31} = 0$$

Therefore, by using Condition (66), it is shown that:

$$\Psi_1^T\left[D^3F(E_1,\ \hat{p}_9)(V_1, V_1, V_1)\right] = \frac{6p_9}{\hat{B}_2{}^3}\left(p_1 + \frac{1}{\hat{B}_2}\right)^2\left[\hat{x}^3 - p_4(3\hat{x} + p_5)\right] \neq 0,$$

which means a PBF has taken place and the proof is complete. □

Theorem 11. *Assuming that Conditions (28) and (29) hold, then System (2) undergoes a TBF near the SPFEQP when* p_{14} *passes through the value* $\check{p}_{14} = \frac{\check{B}_3}{\check{x}}[p_{16}\check{y} + p_{17} + p_{18}]$, *provided that the following condition is met.*

$$2\check{p}_{14}\left[\frac{p_6 - \check{x}^2}{\check{B}_3{}^2}\right]h_1 - 2p_{16}h_2 - 2p_{15} \neq 0. \quad (67)$$

Otherwise, System (2) undergoes a PBF near the SPFEQP, provided that the following condition is met.

$$\check{x}^3 - p_6(3\check{x} + p_7) \neq 0. \quad (68)$$

Proof. The JM of System (2) at the SPFEQP with $p_{14} = \check{p}_{14}$ can be written in the form:

$$J_2 = J(E_2, \check{p}_{14}) = \begin{bmatrix} \check{x}\left[-1 + \frac{\check{y}(p_5 + 2\check{x})}{\check{B}_2{}^2}\right] & -\frac{p_1\check{x}}{\check{B}_1{}^2} - \frac{\check{x}}{\check{B}_2} & -\frac{p_2\check{x}}{\check{B}_1{}^2} - \frac{\check{x}}{\check{B}_3} \\ \frac{p_9\left(p_4 - \check{x}^2\right)\check{y}}{\check{B}_2{}^2} & -p_{10}\check{y} & -p_{11}\check{y} \\ 0 & 0 & 0 \end{bmatrix} = (\check{a}_{ij})$$

The eigenvalues of J_2 are given by $\check{\lambda}_{21},\ \check{\lambda}_{22} = \frac{\mathrm{Tr}_1}{2} \pm \frac{\sqrt{\mathrm{Tr}_1{}^2 - 4\mathrm{Det}_1}}{2}$ and $\check{\lambda}_{23} = 0$ where Tr_1, and Det_1 are given by Equation (26) and the two eigenvalues $\check{\lambda}_{21},\ \check{\lambda}_{22}$ have negative real parts under Conditions (28) and(29). Hence, the SPFEQP is a non-hyperbolic point.

If vectors $V_2 = (v_{21}, v_{22}, v_{23})^T$ and $\Psi_2 = (\psi_{21}, \psi_{22}, \psi_{23})^T$ represent the eigenvectors corresponding the $\check{\lambda}_{23} = 0$ of J_2, and $J_2{}^T$, respectively, then direct computation reveals that

$$V_2 = (h_1, h_2, 1)^T, \text{ and } \Psi_2 = (0, 0, 1)^T,$$

where $h_1 = \frac{\check{a}_{12}\check{a}_{23} - \check{a}_{13}\check{a}_{22}}{\check{a}_{11}\check{a}_{22} - \check{a}_{12}\check{a}_{21}}$, $h_2 = \frac{\check{a}_{21}\check{a}_{13} - \check{a}_{11}\check{a}_{23}}{\check{a}_{11}\check{a}_{22} - \check{a}_{12}\check{a}_{21}} < 0$.

Moreover, direct computation shows the following:

$$\frac{\partial}{\partial p_{14}}F(X, p_{14}) = \left(0, 0, \frac{xz}{B_3}\right)^T \Rrightarrow F_{p_{14}}(E_2, \check{p}_{14}) = (0, 0, 0)^T \Rrightarrow \Psi_2^T F_{p_{14}}(E_2, \check{p}_{14}) = 0.$$

$$DF_{p_{14}}(E_2, \check{p}_{14})V_2 = \left(0, 0, \frac{\check{x}}{\check{B}_3}\right)^T \Rrightarrow \Psi_2^T\left[DF_{p_{14}}(E_2, \check{p}_{14})V_2\right] = \frac{\check{x}}{\check{B}_3} \neq 0.$$

Furthermore, using Equation (63) yields:

$$D^2F(E_2, \check{p}_{14})(V_2, V_2) = [\check{c}_{i1}]_{3\times 1},$$

where

$$\begin{aligned}\check{c}_{11} = -2h_1^2 &+ \tfrac{2\check{x}h_2^2p_1^2}{\check{B}_1{}^3} - \tfrac{2h_1h_2p_1}{\check{B}_1{}^2} + \tfrac{4\check{x}h_2p_1p_2}{\check{B}_1{}^3} - \tfrac{2h_1p_2}{\check{B}_1{}^2} + \tfrac{2\check{x}p_2^2}{\check{B}_1{}^3} - \tfrac{2\check{x}^3\check{y}h_1^2}{\check{B}_2{}^3} + \tfrac{6\check{x}\check{y}h_1^2p_4}{\check{B}_2{}^3} \\ &+ \tfrac{2\check{y}h_1^2p_4p_5}{\check{B}_2{}^3} + \tfrac{2\check{x}^2h_1h_2}{\check{B}_2{}^2} - \tfrac{2h_1h_2p_4}{\check{B}_2{}^2} + \tfrac{2\check{x}^2h_1}{\check{B}_3{}^2} - \tfrac{2h_1p_6}{\check{B}_3{}^2},\end{aligned}$$

$$\check{c}_{21} = \frac{2p_9\check{x}^3\check{y}h_1^2}{\check{B}_2{}^3} - \frac{6p_9\check{x}\check{y}h_1^2p_4}{\check{B}_2{}^3} - \frac{2p_9\check{y}h_1^2p_4p_5}{\check{B}_2{}^3} - \frac{2p_9\check{x}^2h_1h_2}{\check{B}_2{}^2} + \frac{2p_9h_1h_2p_4}{\check{B}_2{}^2} - 2h_2^2p_{10} - 2h_2p_{11}$$

$$\check{c}_{31} = -\frac{2\check{p}_{14}\check{x}^2h_1}{\check{B}_3{}^2} + \frac{2\check{p}_{14}h_1p_6}{\check{B}_3{}^2} - 2p_{15} - 2h_2p_{16}$$

Therefore, it is determined that

$$\Psi_2^T\left[D^2F(E_2, \check{p}_{14})(V_2, V_2)\right] = 2\check{p}_{14}\left[\frac{p_6 - \check{x}^2}{\check{B}_3{}^2}\right]h_1 - 2p_{16}h_2 - 2p_{15}$$

Therefore, in the sense of Sotomayor's theorem, System (2) possesses a TBF, provided that Condition (67) holds. However, if that condition is not met, then, using Equation (64), it may be obtained that:

$$D^3F(E_2, \check{p}_{14})(V_2, V_2, V_2) = \left[\check{d}_{i1}\right]_{3\times 1},$$

where

$$\begin{aligned}\check{d}_{11} \quad &= -\tfrac{6\check{x}h_2^3p_1^3}{\check{B}_1{}^4} + \tfrac{6h_1h_2^2p_1^2}{\check{B}_1{}^3} - \tfrac{18\check{x}h_2^2p_1^2p_2}{\check{B}_1{}^4} + \tfrac{12h_1h_2p_1p_2}{\check{B}_1{}^3} - \tfrac{18\check{x}h_2p_1p_2^2}{\check{B}_1{}^4} + \tfrac{6h_1p_2^2}{\check{B}_1{}^3} - \tfrac{6\check{x}p_2^3}{\check{B}_1{}^4} \\ &+ \tfrac{6\check{x}^4\check{y}h_1^3}{\check{B}_2{}^4} - \tfrac{36\check{x}^2\check{y}h_1^3p_4}{\check{B}_2{}^4} + \tfrac{6\check{y}h_1^3p_4^2}{\check{B}_2{}^4} - \tfrac{24\check{x}\check{y}h_1^3p_4p_5}{\check{B}_2{}^4} - \tfrac{6\check{y}h_1^3p_4p_5^2}{\check{B}_2{}^4} - \tfrac{6\check{x}^3h_1^2h_2}{\check{B}_2{}^3} + \tfrac{18\check{x}h_1^2h_2p_4}{\check{B}_2{}^3} \\ &+ \tfrac{6h_1^2h_2p_4p_5}{\check{B}_2{}^3} - \tfrac{6\check{x}^3h_1^2}{\check{B}_3{}^3} + \tfrac{18\check{x}h_1^2p_6}{\check{B}_3{}^3} + \tfrac{6h_1^2p_6p_7}{\check{B}_3{}^3}\end{aligned}$$

$$\begin{aligned}\check{d}_{21} \quad &= -\tfrac{6p_9\check{x}^4\check{y}h_1^3}{\check{B}_2{}^4} + \tfrac{6p_9\check{x}^5h_1^2h_2}{\check{B}_2{}^4} + \tfrac{36p_9\check{x}^2\check{y}h_1^3p_4}{\check{B}_2{}^4} - \tfrac{12p_9\check{x}^3h_1^2h_2p_4}{\check{B}_2{}^4} - \tfrac{6p_9\check{y}h_1^3p_4^2}{\check{B}_2{}^4} \\ &- \tfrac{18p_9\check{x}h_1^2h_2p_4^2}{\check{B}_2{}^4} + \tfrac{6p_9\check{x}^4h_1^2h_2p_5}{\check{B}_2{}^4} + \tfrac{24p_9\check{x}\check{y}h_1^3p_4p_5}{\check{B}_2{}^4} - \tfrac{24p_9\check{x}^2h_1^2h_2p_4p_5}{\check{B}_2{}^4} \\ &- \tfrac{6p_9h_1^2h_2p_4^2p_5}{\check{B}_2{}^4} + \tfrac{6p_9\check{y}h_1^3p_4p_5^2}{\check{B}_2{}^4} - \tfrac{6p_9\check{x}h_1^2h_2p_4p_5^2}{\check{B}_2{}^4}.\end{aligned}$$

$$\check{d}_{31} = \frac{6\check{p}_{14}h_1^2}{\check{B}_2{}^3}\left[\check{x}^3 - p_6(3\check{x} + p_7)\right]$$

Therefore, by using Condition (68), it is determined that

$$\Psi_2^T\left[D^3F(E_2, \check{p}_{14})(V_2, V_2, V_2)\right] = \frac{6\check{p}_{14}h_1^2}{\check{B}_2{}^3}\left[\check{x}^3 - p_6(3\check{x} + p_7)\right] \neq 0,$$

which means that a PBF has taken place and the proof is complete. □

Theorem 12. *Assuming that Conditions (33) and (34) hold, then System (2) undergoes a TBF near the FPFEQP when p_9 passes through the value $\overline{p}_9 = \frac{\overline{B}_2}{\overline{x}}[p_{11}\,\overline{z} + p_{12} + p_{13}]$, provided that the following condition is met.*

$$2\overline{p}_9\left[\frac{p_4-\overline{x}^2}{\overline{B}_2{}^2}\right]h_3-2p_{10}-2p_{11}h_4\neq 0 \tag{69}$$

Otherwise, System (2) undergoes a PBF near the FPFEQP, provided that the following condition is met.

$$\overline{x}^3-p_4(3\overline{x}+p_5)\neq 0 \tag{70}$$

Proof. The JM of System (2) at the FPFEQP with $p_{14}=\check{p}_{14}$ can be written in the form:

$$J_3=J(E_3,\overline{p}_9)=\begin{bmatrix}\overline{x}\left[-1+\frac{\overline{z}(p_7+2\overline{x})}{\overline{B}_3{}^2}\right] & -\frac{p_1\overline{x}}{\overline{B}_1{}^2}-\frac{\overline{x}}{\overline{B}_2} & -\frac{p_2\overline{x}}{\overline{B}_1{}^2}-\frac{\overline{x}}{\overline{B}_3}\\ 0 & 0 & 0\\ \frac{p_{14}\left(p_6-\overline{x}^2\right)\overline{z}}{\overline{B}_3{}^2} & -p_{16}\overline{z} & -p_{15}\overline{z}\end{bmatrix}=\left(\overline{a}_{ij}\right)$$

The eigenvalues of J_3 are given by $\overline{\lambda}_{31},\ \overline{\lambda}_{33}=\frac{\mathrm{Tr}_2}{2}\pm\frac{\sqrt{\mathrm{Tr}_2{}^2-4\mathrm{Det}_2}}{2}$ and $\overline{\lambda}_{32}=0$, where Tr_2 and Det_2 are given by Equation (31) and the two eigenvalues $\overline{\lambda}_{31},\ \overline{\lambda}_{33}$ have negative real parts under Condition (33) and (34). Hence, the FPFEQP is a non-hyperbolic point.

If vectors $V_3=(v_{31},v_{32},v_{33})^T$ and $\Psi_3=(\psi_{31},\psi_{32},\psi_{33})^T$ represent the eigenvectors corresponding the $\overline{\lambda}_{32}=0$ of J_3, and $J_3{}^T$ respectively, direct computation reveals that:

$$V_3=(h_3,1,h_4)^T,\ \text{and}\ \Psi_3=(0,1,0)^T,$$

where $h_3=\frac{\overline{a}_{13}\overline{a}_{32}-\overline{a}_{12}\overline{a}_{33}}{\overline{a}_{11}\overline{a}_{33}-\overline{a}_{13}\overline{a}_{31}}$, and $h_4=\frac{\overline{a}_{12}\overline{a}_{31}-\overline{a}_{11}\overline{a}_{32}}{\overline{a}_{11}\overline{a}_{33}-\overline{a}_{13}\overline{a}_{31}}<0$.

Moreover, direct computation shows the following:

$$\frac{\partial}{\partial p_9}F(X,p_9)=\left(0,\frac{xy}{B_2},0\right)^T\Rightarrow F_{p_9}(E_3,\overline{p}_9)=(0,0,0)^T\Rightarrow\Psi_3^TF_{p_9}(E_3,\overline{p}_9)=0.$$

$$DF_{p_9}(E_3,\overline{p}_9)V_3=\left(0,\frac{\overline{x}}{\overline{B}_2},0\right)^T\Rightarrow\Psi_3^T\left[DF_{p_9}(E_3,\overline{p}_9)V_3\right]=\frac{\overline{x}}{\overline{B}_2}\neq 0.$$

Furthermore, using Equation (63), we determine that:

$$D^2F(E_3,\overline{p}_9)(V_3,V_3)=[\overline{c}_{i1}]_{3\times1},$$

where:

$$\begin{aligned}\overline{c}_{11}=&-2h_3^2+\frac{2\overline{x}p_1^2}{\overline{B}_1{}^3}+\frac{4\overline{x}h_4p_1p_2}{\overline{B}_1{}^3}+\frac{2\overline{x}h_4^2p_2^2}{\overline{B}_1{}^3}-\frac{2h_3p_1}{\overline{B}_1{}^2}-\frac{2h_3h_4p_2}{\overline{B}_1{}^2}\\&+\frac{2\overline{x}^2h_3}{\overline{B}_2{}^2}-\frac{2h_3p_4}{\overline{B}_2{}^2}-\frac{2\overline{x}^3\overline{z}h_3^2}{\overline{B}_3{}^3}+\frac{6\overline{x}\overline{z}h_3^2p_6}{\overline{B}_3{}^3}\\&+\frac{2\overline{z}h_3^2p_6p_7}{\overline{B}_3{}^3}+\frac{2\overline{x}^2h_3h_4}{\overline{B}_3{}^2}-\frac{2h_3h_4p_6}{\overline{B}_3{}^2}\end{aligned}$$

$$\overline{c}_{21}=2p_9\left[\frac{p_4-\overline{x}^2}{\overline{B}_2{}^2}\right]h_3-2p_{10}-2p_{11}h_4,$$

$$\overline{c}_{31}=\frac{2p_{14}\overline{x}^3\overline{z}h_3^2}{\overline{B}_3{}^3}-\frac{6p_{14}\overline{x}\overline{z}h_3^2p_6}{\overline{B}_3{}^3}-\frac{2p_{14}\overline{z}h_3^2p_6p_7}{\overline{B}_3{}^3}-\frac{2p_{14}\overline{x}^2h_3h_4}{\overline{B}_3{}^2}+\frac{2p_{14}h_3h_4p_6}{\overline{B}_3{}^2}-2h_4^2p_{15}-2h_4p_{16}$$

Therefore, it is shown that

$$\Psi_3^T\left[D^2F(E_3,\overline{p}_9)(V_3,V_3)\right]=2p_9\left[\frac{p_4-\overline{x}^2}{\overline{B}_2{}^2}\right]h_3-2p_{10}-2p_{11}h_4$$

In the sense of Sotomayor's theorem, System (2) possesses a TBF provided that Condition (69) holds. However, if that condition is not met, then, by using Equation (64), it is determined that:

$$D^3F(E_3,\overline{p}_9)(V_3,V_3,V_3) = \left[\overline{d}_{i1}\right]_{3\times 1}$$

where:

$$\begin{aligned}\overline{d}_{11} = &-\frac{6\overline{x}p_1^3}{\overline{B}_1{}^4} - \frac{18\overline{x}h_4p_1^2p_2}{\overline{B}_1{}^4} - \frac{18\overline{x}h_4^2p_1p_2^2}{\overline{B}_1{}^4} - \frac{6\overline{x}h_4^3p_2^3}{\overline{B}_1{}^4} + \frac{6h_3p_1^2}{\overline{B}_1{}^3} + \frac{12h_3h_4p_1p_2}{\overline{B}_1{}^3} + \frac{6h_3h_4^2p_2^2}{\overline{B}_1{}^3} \\ &- \frac{6\overline{x}^3h_3^2}{\overline{B}_2{}^3} + \frac{18\overline{x}h_3^2p_4}{\overline{B}_2{}^3} + \frac{6h_3^2p_4p_5}{\overline{B}_2{}^3} + \frac{6\overline{x}^4\overline{z}h_3^3}{\overline{B}_3{}^4} - \frac{36\overline{x}^2\overline{z}h_3^3p_6}{\overline{B}_3{}^4} + \frac{6\overline{z}h_3^3p_6^2}{\overline{B}_3{}^4} \\ &- \frac{24\overline{x}\overline{z}h_3^3p_6p_7}{\overline{B}_3{}^4} - \frac{6\overline{z}h_3^3p_6p_7^2}{\overline{B}_3{}^4} - \frac{6\overline{x}^3h_3^2h_4}{\overline{B}_3{}^3} + \frac{18\overline{x}h_3^2h_4p_6}{\overline{B}_3{}^3} + \frac{6h_3^2h_4p_6p_7}{\overline{B}_3{}^3}\end{aligned}$$

$$\overline{d}_{21} = \frac{6\overline{p}_9h_3^2}{\overline{B}_2{}^3}\left[\overline{x}^3 - p_4(3\overline{x}+p_5)\right]$$

$$\begin{aligned}\overline{d}_{31} = &-\frac{6p_{14}\overline{x}^4\overline{z}h_3^3}{\overline{B}_3{}^4} + \frac{6p_{14}\overline{x}^5h_3^2h_4}{\overline{B}_3{}^4} + \frac{36p_{14}\overline{x}^2\overline{z}h_3^3p_6}{\overline{B}_3{}^4} - \frac{12p_{14}\overline{x}^3h_3^2h_4p_6}{\overline{B}_3{}^4} - \frac{6p_{14}\overline{z}h_3^3p_6^2}{\overline{B}_3{}^4} \\ &- \frac{18p_{14}\overline{x}h_3^2h_4p_6^2}{\overline{B}_3{}^4} + \frac{6p_{14}\overline{x}^4h_3^2h_4p_7}{\overline{B}_3{}^4} + \frac{24p_{14}\overline{x}\overline{z}h_3^3p_6p_7}{\overline{B}_3{}^4} - \frac{24p_{14}\overline{x}^2h_3^2h_4p_6p_7}{\overline{B}_3{}^4} \\ &- \frac{6p_{14}h_3^2h_4p_6^2p_7}{\overline{B}_3{}^4} + \frac{6p_{14}\overline{z}h_3^3p_6p_7^2}{\overline{B}_3{}^4} - \frac{6p_{14}\overline{x}h_3^2h_4p_6p_7^2}{\overline{B}_3{}^4}.\end{aligned}$$

By using Condition (70), it is determined that

$$\Psi_3^T\left[D^3F(E_3,\overline{p}_9)(V_3,V_3,V_3)\right] = \frac{6\overline{p}_9h_3^2}{\overline{B}_2{}^3}\left[\overline{x}^3 - p_4(3\overline{x}+p_5)\right] \neq 0,$$

which means that a PBF has taken place and the proof is complete. □

Theorem 13. *Assuming that Conditions (36) and (37) hold, then System (2) undergoes a saddle-node bifurcation (SNB) near the IEQP when* p_{15} *passes through the value* $p_{15}^* = \frac{[-a_{11}a_{23}a_{32}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}]}{z^*(a_{11}a_{22}-a_{12}a_{21})}$*, provided that the following conditions are met.*

$$\frac{(p_6 - x^{*2})}{B_3^{*2}} < \frac{p_9p_{16}(p_4 - x^{*2})}{p_{10}p_{14}\,B_2^{*2}}, \tag{71}$$

$$c_{11}^*h_7 + c_{21}^*h_8 + c_{31}^* \neq 0 \tag{72}$$

where all the new symbols are given in the proof.

Proof. The JM of System (2) at the IEQP with $p_{15} = p_{15}^*$ can be written in the form:

$$J_4 = J(E_4,p_{15}^*) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}^* \end{bmatrix},$$

where $a_{33}^* = -p_{15}^*z^*$, while the other $a_{ij},\ i,j = 1,2,3$ are given in Equation (35). Conditions (36), (37), and (71) guarantee that $p_{15}^* > 0$. Moreover, it is observed that the determinant of J_4 that is given by Γ_3 in Equation (41) at $p_{15} = p_{15}^*$ is zero. Consequently, J_4 has an eigenvalue $\lambda_4^* = 0$, and as such, the IEQP becomes a non-hyperbolic point.

If vectors $V_4 = (v_{41},v_{42},v_{43})^T$ and $\Psi_4 = (\psi_{41},\psi_{42},\psi_{43})^T$ represent the eigenvectors corresponding to the $\lambda_4^* = 0$ of J_4 and $J_4{}^T$, respectively, then direct computation reveals that:

$$V_4 = (h_5,h_6,1)^T, \text{ and } \Psi_4 = (h_7,h_8,1)^T,$$

where $h_5 = \frac{a_{12}a_{23} - a_{22}a_{13}}{a_{11}a_{22} - a_{12}a_{21}}$, $h_6 = \frac{a_{21}a_{13} - a_{11}a_{23}}{a_{11}a_{22} - a_{12}a_{21}} < 0$, $h_7 = \frac{a_{21}a_{32} - a_{22}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}$, and $h_8 = \frac{a_{12}a_{31} - a_{11}a_{32}}{a_{11}a_{22} - a_{12}a_{21}} < 0$. Moreover, direct computation shows the following:

$$\frac{\partial}{\partial p_{15}} F(X, p_{15}) = \left(0, 0, -z^2\right)^T \Rightarrow F_{p_{15}}(E_4, p_{15}^*) = \left(0, 0, -z^{*2}\right)^T \Rightarrow \Psi_4^T F_{p_9}(E_4, p_{15}^*) = -z^{*2}.$$

Accordingly, the first requirement of the SNB is satisfied. Furthermore, Equation (63) shows that:

$$D^2F(E_4, p_{15}^*)(V_4, V_4) = [c_{i1}^*]_{3\times 1},$$

where:

$$\begin{aligned} c_{11}^* = -2h_5^2 - \tfrac{2h_5h_6p_4}{B_2^{*2}} + \tfrac{2h_5h_6(x^*)^2}{B_2^{*2}} - \tfrac{2h_5p_6}{B_3^{*2}} + \tfrac{2h_5(x^*)^2}{B_3^{*2}} + \tfrac{2h_5^2p_4p_5y^*}{B_2^{*3}} + \tfrac{6h_5^2p_4x^*y^*}{B_2^{*3}} \\ - \tfrac{2h_5^2(x^*)^3y^*}{B_2^{*3}} + \tfrac{2h_5^2p_6p_7z^*}{B_3^{*3}} + \tfrac{6h_5^2p_6x^*z^*}{B_3^{*3}} - \tfrac{2h_5^2(x^*)^3z^*}{B_3^{*3}} + \tfrac{2h_6^2p_1^2x^*}{B_1^{*3}} \\ + \tfrac{4h_6p_1p_2x^*}{B_1^{*3}} + \tfrac{2p_2^2x^*}{B_1^{*3}} - \tfrac{2h_5h_6p_1}{B_1^{*2}} - \tfrac{2h_5p_2}{B_1^{*2}}. \end{aligned}$$

$$c_{21}^* = -p_9\left[\frac{2p_4p_5y^* + 6p_4x^*y^* - 2x^{*3}y^*}{B_2^{*3}}\right]h_5^2 + 2p_9\left[\frac{p_4 - x^{*2}}{B_2^{*2}}\right]h_5h_6 - 2p_{10}h_6^2 - 2p_{11}h_6$$

$$c_{31}^* = -p_{14}\left[\frac{2p_6p_7z^* + 6p_6x^*z^* - 2x^{*3}z^*}{B_3^{*3}}\right]h_5^2 + 2p_{14}\left[\frac{p_6 - x^{*2}}{B_3^{*2}}\right]h_5 - 2p_{16}h_6 - 2p_{15}^*$$

Therefore, it is determined that

$$\Psi_4^T\left[D^2F(E_4, p_{15}^*)(V_4, V_4)\right] = c_{11}^*h_7 + c_{21}^*h_8 + c_{31}^*.$$

In the sense of Sotomayor's theorem, System (2) possesses an SNB near the IEQP provided that condition (72) holds, and the proof is complete. □

8. Numerical Simulation

This section comprises a numerical analysis of the dynamics of the system. MATLAB R2013a (8.1.0) was used to examine the behavior of System (2) with various parameter values. The following set of hypothetical parameters was used.

$$\left.\begin{array}{l} p_1 = 0.2,\ p_2 = 0.2,\ p_3 = 0.01,\ p_4 = 0.3,\ p_5 = 0.2,\ p_6 = 0.3, \\ p_7 = 0.2,\ p_8 = 0.1, p_9 = 0.7,\ p_{10} = 0.2,\ p_{11} = 0.1,\ p_{12} = 0.1, \\ p_{13} = 0.1,\ p_{14} = 0.7, p_{15} = 0.2,\ p_{16} = 0.1,\ p_{17} = 0.1,\ p_{18} = 0.1 \end{array}\right\} \tag{73}$$

The main objective of this study is to confirm our theoretical findings and specify a parameter set that controls the dynamic system. It was observed that, for Dataset (73), System (2) has a unique IEQP that is a GAS; see Figure 1.

Throughout the following figures, magenta represents the solution of System (2), while blue, green, and red represent the trajectories of x, y, and z, respectively.

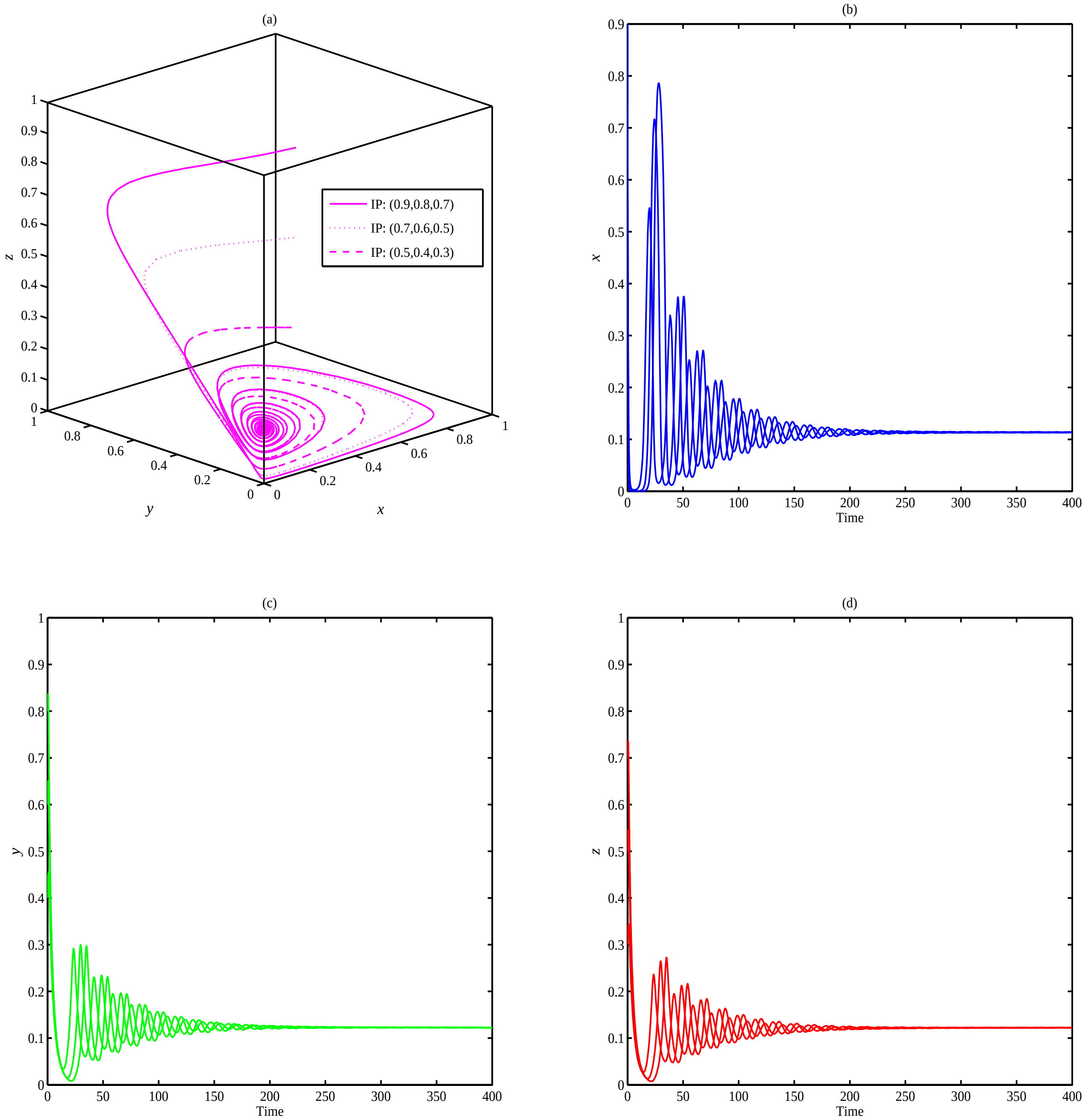

Figure 1. The IEQP of System (2) is a GAS using Dataset (73) and various initial points. (**a**) Phase portrait. (**b**) Trajectories of x versus time. (**c**) Trajectories of y versus time. (**d**) Trajectories of z versus time.

Figure 1 shows the ownership of System (2) using Dataset (73) with a unique IEQP that is GAS. The influence of varying p_1 and p_2 on the dynamics of System (2) is shown in Figure 2.

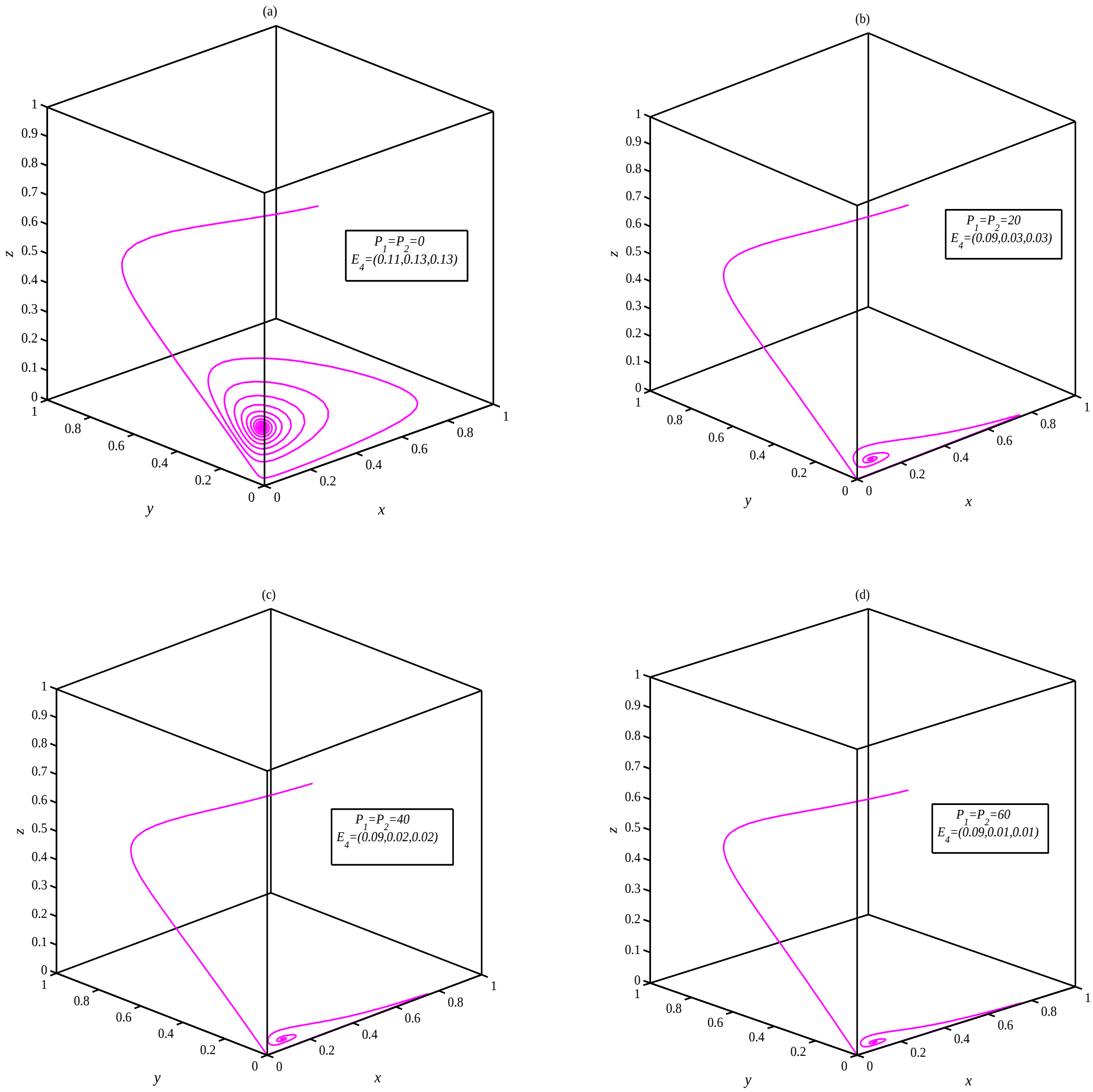

Figure 2. The trajectory of System (2) using Dataset (73) with different values of fear factors approaching IEQP. (**a**) Phase portrait when $p_1 = p_2 = 0$. (**b**) Phase portrait when $p_1 = p_2 = 20$. (**c**) Phase portrait when $p_1 = p_2 = 40$. (**d**) Phase portrait when $p_1 = p_2 = 60$.

As shown in Figure 2, increasing of p_1, and p_2 forces the IEQP to gradually converge to PDFEQP. The role of varying p_3 on the behavior of System (2) is illustrated in Figure 3.

Increasing the value of p_3 reduces the stability of IEQP, and the system approaches PDFEQP first and then approaches EEQP. The impact of the functional response parameters—which are responsible for the transfer of food x to predator y—on the dynamics of System (2) is summarized in Figures 4 and 5.

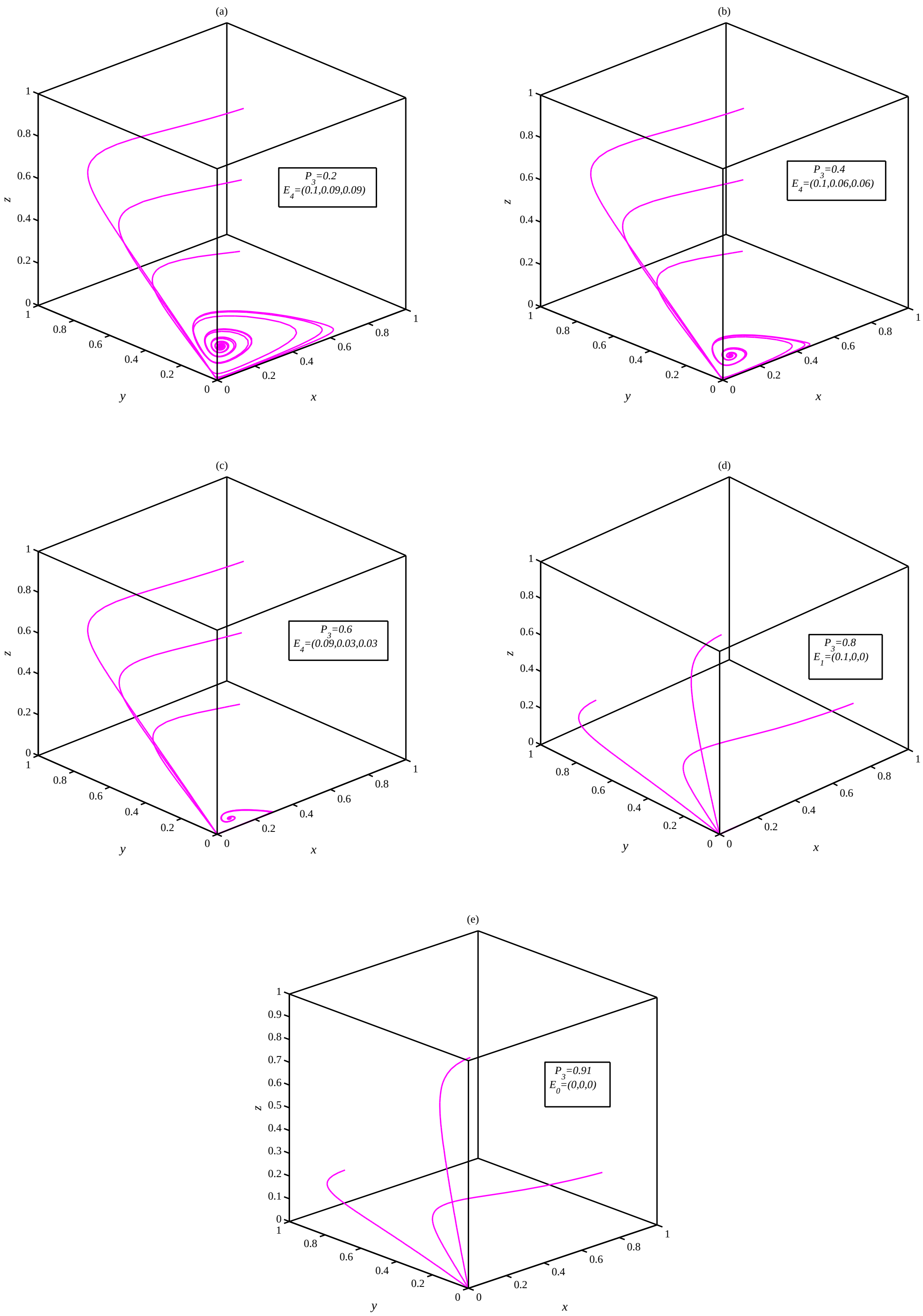

Figure 3. The trajectories of System (2) using Dataset 73 with different values of p_3 and various initial points. (**a**) The phase portrait approaches IEQP when $p_3 = 0.2$. (**b**) The phase portrait approaches IEQP when $p_3 = 0.4$. (**c**) The phase portrait approaches IEQP when $p_3 = 0.6$. (**d**) The phase portrait approaches PDFEQP when $p_3 = 0.8$. (**e**) The phase portrait approaches EEQP when $p_3 = 0.91$.

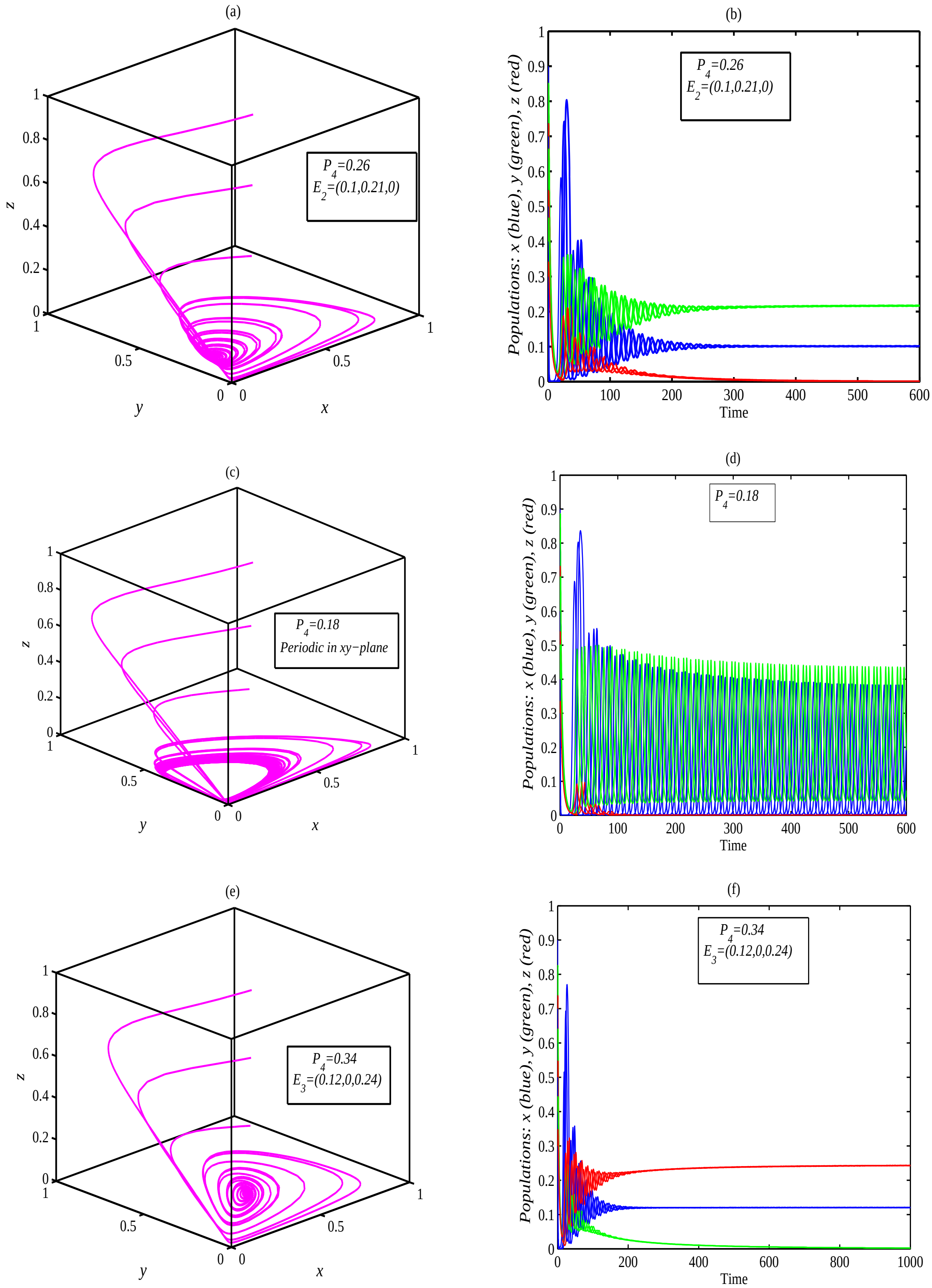

Figure 4. The trajectories of System (2) using Dataset 73 and various initial points. (**a**) The phase portrait approaches SPFEQP when $p_4 = 0.26$. (**b**) Time series for the phase portrait at $p_4 = 0.26$. (**c**) The phase portrait approaches a periodic attractor when $p_4 = 0.18$. (**d**) Time series for the phase portrait at $p_4 = 0.18$. (**e**) The phase portrait approaches FPFEQP when $p_4 = 0.34$. (**f**) Time series for the phase portrait at $p_4 = 0.34$.

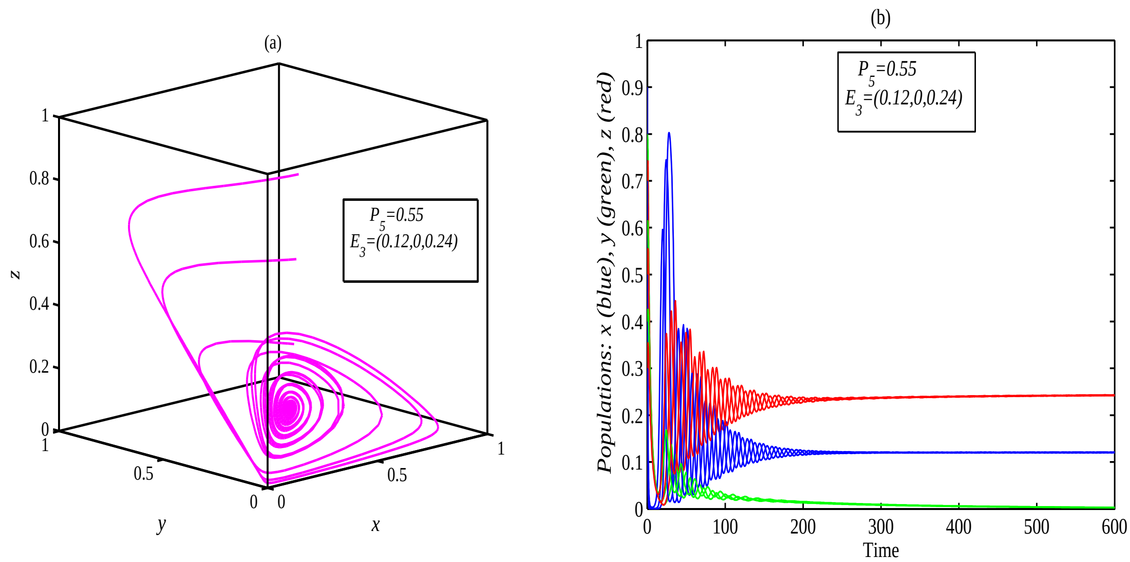

Figure 5. The trajectories of System (2) using Dataset 73 and various initial points. (**a**) The phase portrait approaches FPFEQP when $p_5 = 0.55$. (**b**) Time series for the phase portrait at $p_5 = 0.55$.

According to Figures 4 and 5, the dynamics of System (2) are sensitive to variations of p_4, switching from a periodic dynamic on the xy plane to SPFEQP, and then to IEQP, and finally, to FPFEQP. However, it has less sensitivity to varyiations in p_5, whereby the dynamic transfer is from IEQP to FPFEQP. The impact of varying the parameters of the functional response that are responsible for transferring the food from x to z is shown in Figure 6.

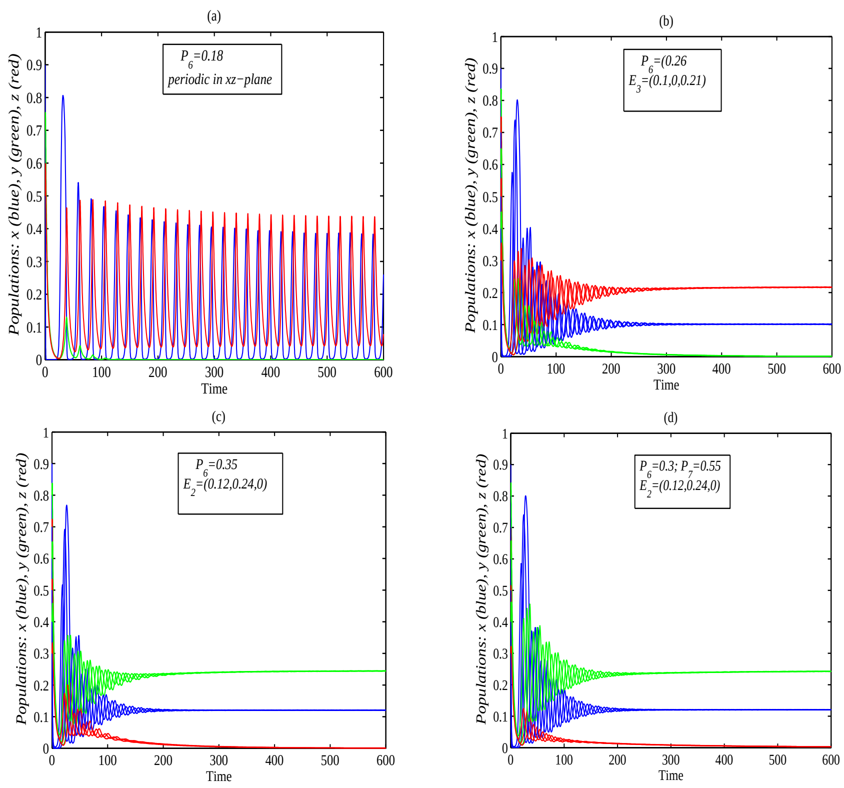

Figure 6. Time series of the trajectories of System (2) using Dataset 73 and various initial points. (**a**) Periodic dynamics when $p_6 = 0.18$. (**b**) Approaching FPFEQP when $p_6 = 0.26$. (**c**) Approaching SPFEQP when $p_6 = 0.35$. (**d**) Approaching SPFEQP when $p_6 = 0.3$ and $p_7 = 0.55$.

Figure 6 shows that p_6 has the opposite effect to p_4 (and p_7 has the opposite effect to p_5) on the dynamics of System (2). Figure 7 illustrates the influence of varying p_8 on the dynamics of System (2).

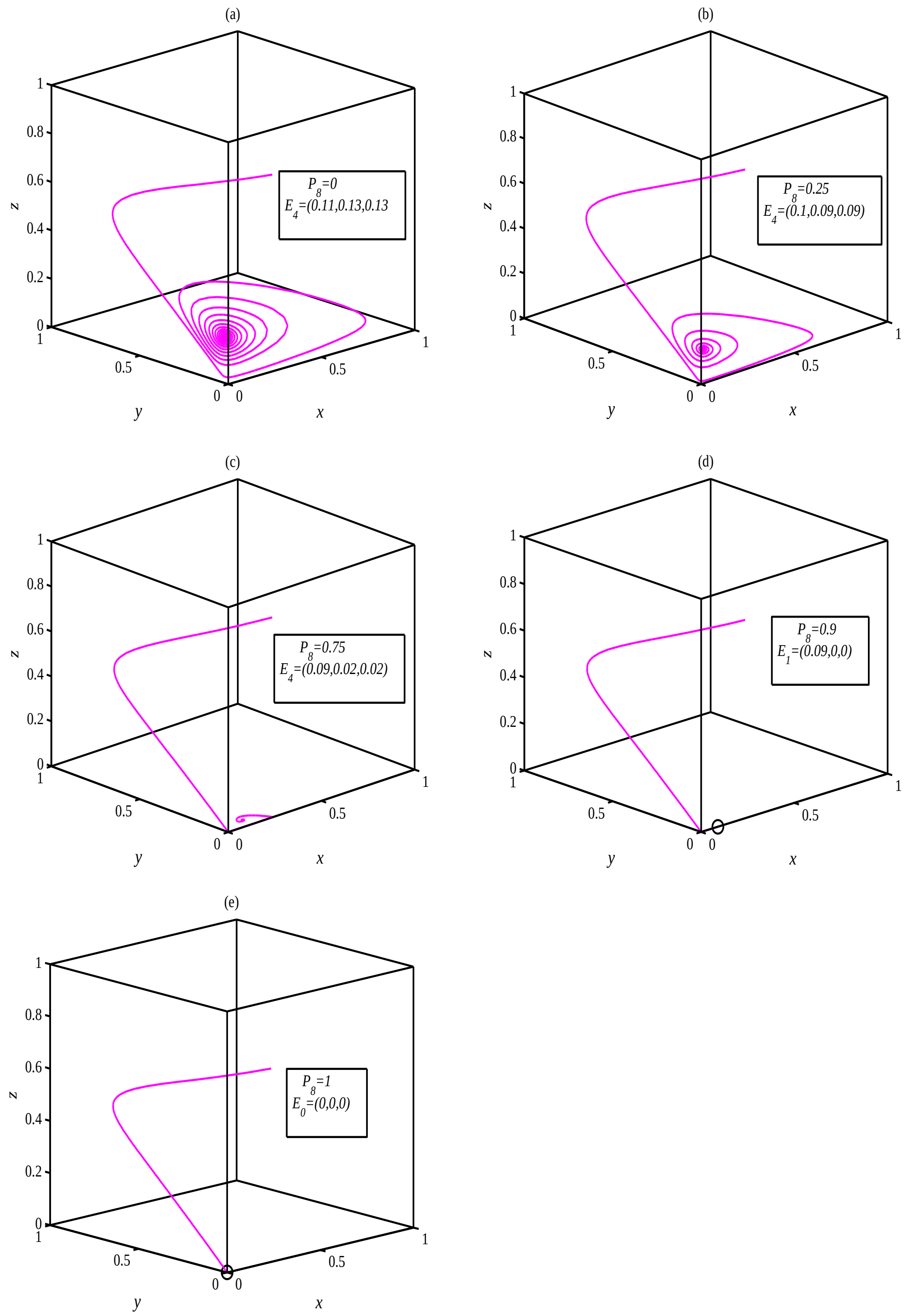

Figure 7. The trajectory of System (2) using Dataset 73 with different values of p_8. (**a**) Approaching IEQP when $p_8 = 0$. (**b**) Approaching IEQP when $p_8 = 0.25$. (**c**) Approaching IEQP when $p_8 = 0.75$. (**d**) Approaching PDFEQP when $p_8 = 0.9$. (**e**) Approaching EEQP when $p_8 = 1$.

The parameter p_8 significantly influences the dynamics of the system, similar to p_3. The impact of the other harvesting rates p_{13} and p_{18} is illustrated in Figure 8.

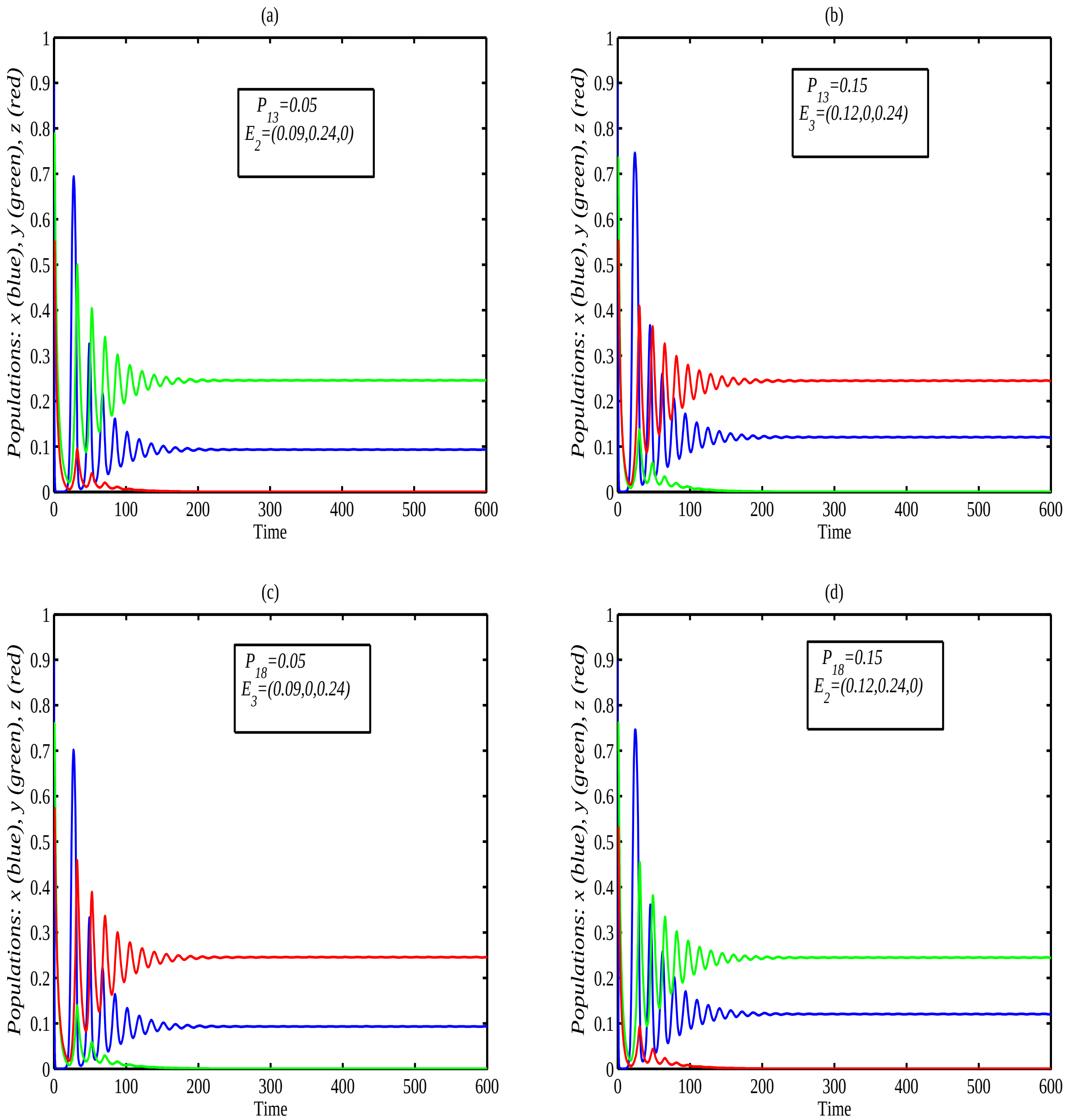

Figure 8. Time series of the trajectories of System (2) using Dataset 73 with different values of p_{13} or p_{18}. (**a**) Approaching SPFEQP when $p_{13} = 0.05$. (**b**) Approaching FPFEQP when $p_{13} = 0.15$. (**c**) Approaching FPFEQP when $p_{18} = 0.05$. (**d**) Approaching FPFEQP when $p_{18} = 0.15$.

Parameters p_{13} and p_{18} have opposite influences on the dynamics of the system. We further investigated the impact of varying the parameters p_9 and p_{14}. It was observed that p_9 has an effect on the dynamics of System (2) similar to that of p_{18}, while the impact of p_{14} is similar to that of p_{13}. Finally, the effects of varying the intra-n and inter- specific competition on the dynamics of System (2) are illustrated in Figures 9–11, respectively.

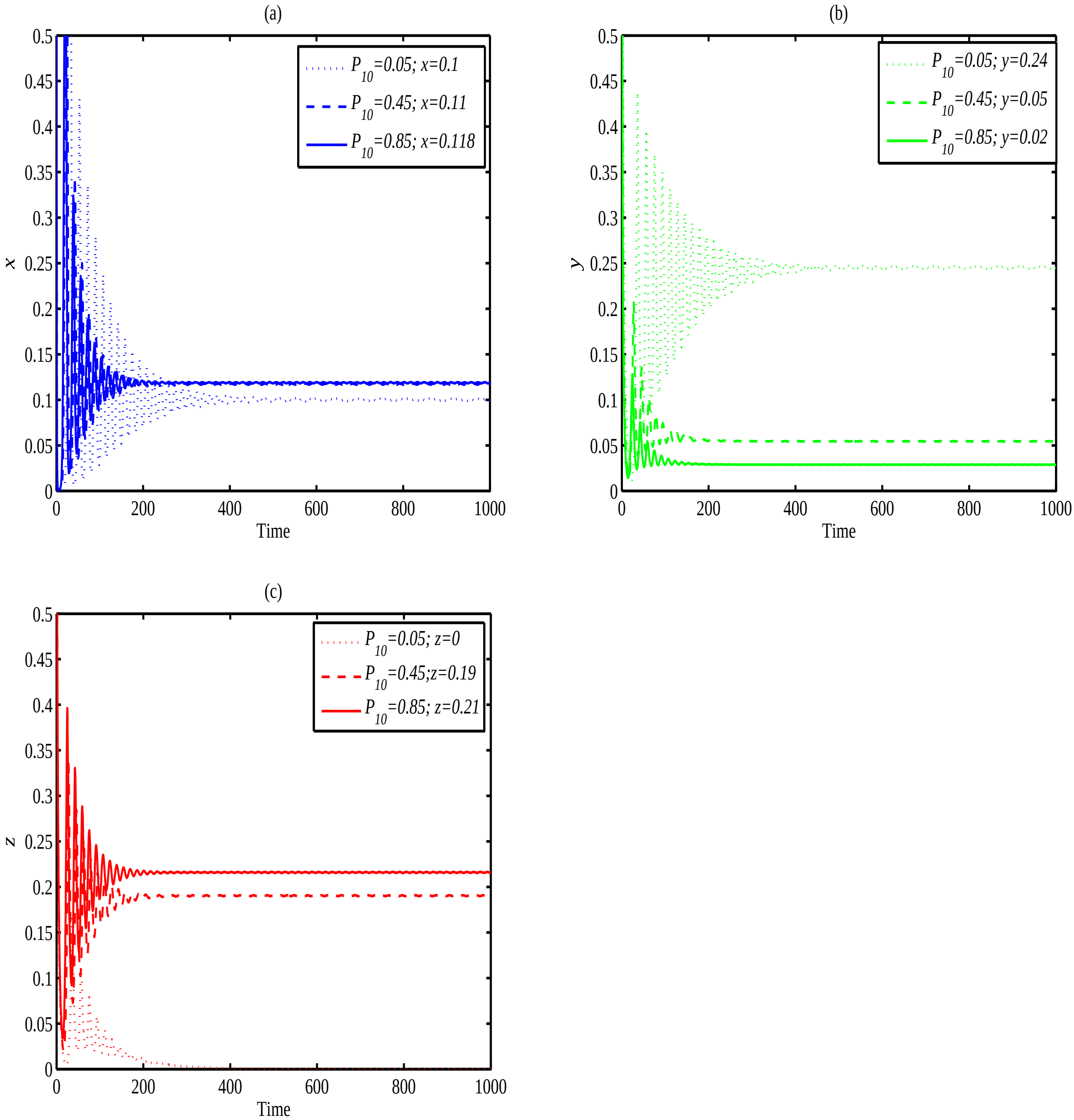

Figure 9. Time series of the trajectories of System (2) using Dataset 73 with different values of p_{10}. (**a**) Trajectories of x for $p_{10} = 0.05, 0.45, 0.85$. (**b**) Trajectories of y for $p_{10} = 0.05, 0.45, 0.85$. (**c**) Trajectories of z for $p_{10} = 0.05, 0.45, 0.85$.

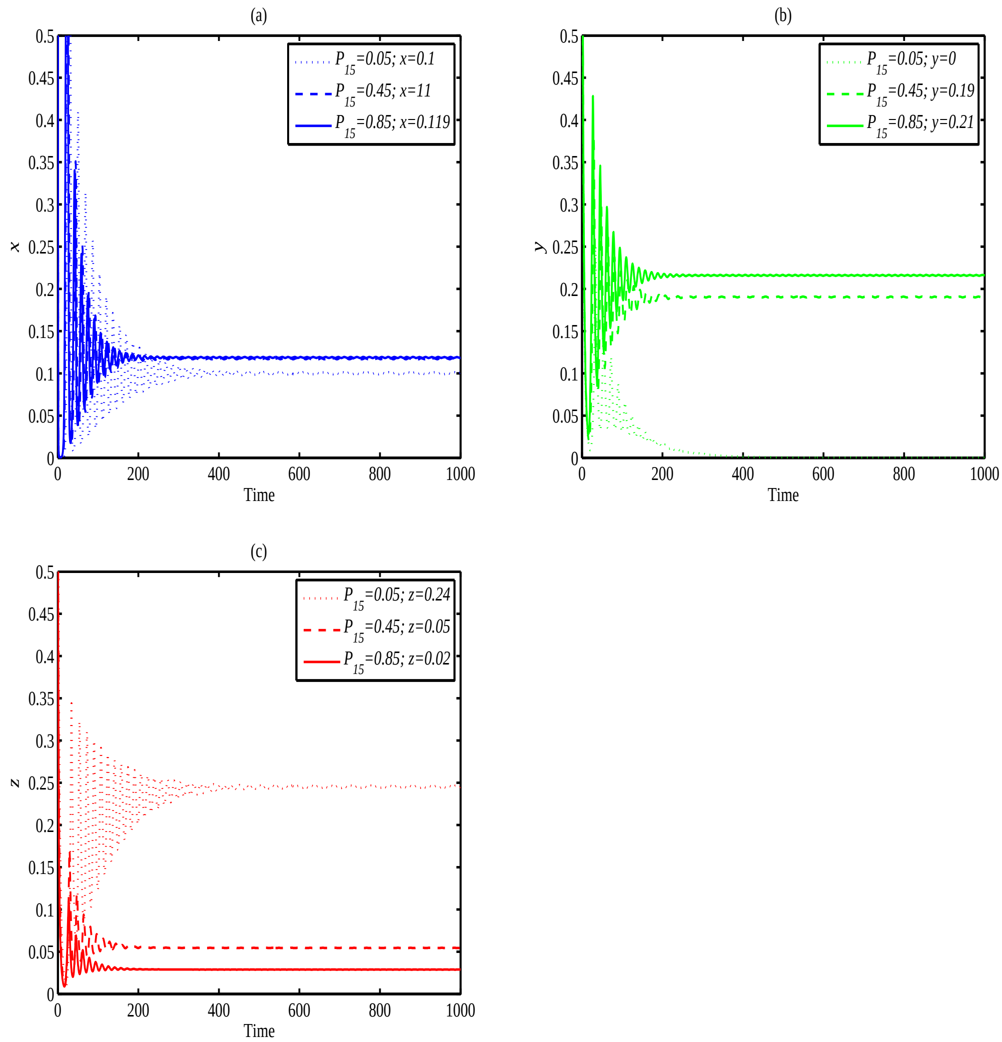

Figure 10. Time series of the trajectories of System (2) using Dataset 73 with different values of p_{15}. (**a**) Trajectories of x for $p_{15} = 0.05, 0.45, 0.85$. (**b**) Trajectories of y for $p_{15} = 0.05, 0.45, 0.85$. (**c**) Trajectories of z for $p_{15} = 0.05, 0.45, 0.85$.

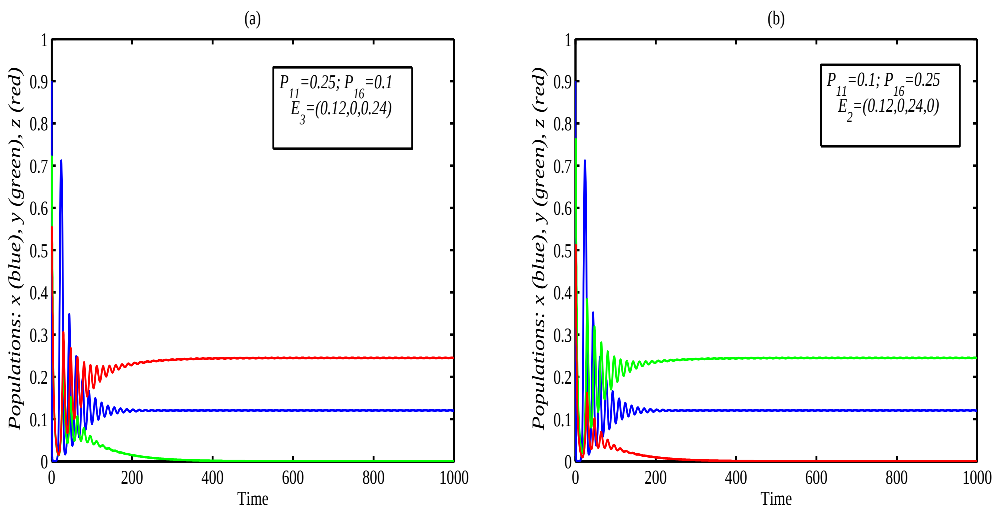

Figure 11. Time series of the trajectories of System (2) using Dataset 73 and various values of p_{11} and p_{16}. (**a**) Approaching FPFEQP when $p_{11} = 0.25$ and $p_{16} = 0.1$. (**b**) Approaching SPFEQP when $p_{11} = 0.1$ and $p_{16} = 0.25$.

As shown in Figures 9 and 10, parameters p_{10} and p_{15} have opposite influences on the dynamics of the system.

As shown in Figure 11, p_{11} and p_{16} also have opposite influences on the dynamics of the system.

9. Discussion

This work proposes and investigates a mathematical model that simulates the dynamics of a food-web system with two competing predators and prey. It was our goal to examining how fear and harvesting affect the dynamics of such a system. Food transition at the food-web level was described using a Monod-Haldane-type response function. The characteristics of the solution were investigated. It was established that various equilibrium points exist. All species' needs for survival were identified. When feasible, the topics of local and global stability were covered. The Sotomayor theorem for local bifurcation was used to examine the impact of changing the parameter values.

It is noted that System (2) has a variety of conditionally asymptotic stable equilibrium locations. Additionally, the system experiences different forms of local bifurcation as the parameter values change. Finally, System (2) was solved, the theoretical conclusion was verified, and the control set of parameters was specified using numerical simulation. Using a hypothetical set of parameters given by Equation (73), the following results were obtained.

System (2) asymptotically approaches IEQP from several sets of initial points. Due to declining predator numbers, the IEQP gradually converges to a PDFEQP as the fear rates rise, demonstrating the influence of fear on the dynamics of the system. System (2) shifts the stability of the predators from IEQP to PDFEQP when the mortality rate of the prey increases. As the death rate increases further, the system shifts its limit point from PDFEQP to EEQP. By varying the predator death rates, it was demonstrated that the predator death rates play a similar role to the prey death rates regarding the survival of the predator species. For example, lowering the value of the first predator death rate resulted in the extinction of the second predator, and the system approached SPFEQP (FPFEQP). However, increasing this value caused the first predator to go extinct, and System (2) approached FPFEQP (SPFEQP).

The influence of the functional response parameters was explored. It was found that lowering the value of the half-saturation constant of the first predator resulted in the extinction of the second predator, and the system approached SPFEQP (FPFEQP). Further decreasing the value of the half-saturation constant of the first predator made the SPFEQP (FPFEQP) unstable, and System (2) had periodic dynamics in the xy plane. However, increasing its value caused the first predator to go extinct, and System (2) approached FPFEQP (SPFEQP). Moreover, increasing the value of the inhibitory effect at high concentrations for the first predator led to the first predator going extinct; then, the system approached FPFEQP (SPFEQP).

For the harvesting parameters of System (2), it was noted that with an increase in prey harvesting, the system dynamics transferred from IEQP to the PDFEQP, and then to the EEQP. However, lowering the harvesting rate of the first predator caused the second predator to go extinct, and the trajectories of System (2) approached SPFEQP (FPFEQP). Meanwhile, increasing its value above a specific point causes the extinction of the first predator, and the trajectories moved asymptotically to the FPFEQP (SPFEQP). Otherwise, the trajectories of System (2) stalled at IEQP. Similar observations were made when the conversion rates of the first and second predators were lowered; however, the trajectories still approached IEQP.

Finally, decreasing the intraspecific competition rate of the first predator led to the extinction of the second predator, and the trajectories of System (2) approached SPFEQP. However, decreasing the intraspecific competition rate of the second predator led to the extinction of the first predator, and the trajectories of System (2) approached FPFEQP. Otherwise, the trajectories of System (2) stalled at IEQP.

10. Conclusions

Gause's law is another name for the competitive exclusion concept [46]. It is a term used in ecology to describe a situation when two species are vying for the same resources. Gause noted that under such conditions, population values cannot remain constant. However, in this work, our System (2) solution approaches IEQP from distinct initial positions, indicating that the presence of prey allows two competing species to persist on the same resources. There is a threshold value of fear over which System (2) loses persistence due to the extinction of predators, even though the system approaches the IEQP asymptotically with different fear values. It has been noted that if the value of a prey species reaches a vital level, harvesting it may cause System (2) to collapse. The harvesting of the predator, however, causes the competitive predator to survive or become extinct, depending on how much its value increases or decreases; after that, the solution for System (2) approaches one of the boundary planes.

The system persists for a small range of the predator equation parameters, since any modification may provide the competitive predator with a chance to become stronger and survive. It is thus determined that System (2) is particularly sensitive to changing predator equation-related values.

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Abbreviation

The following abbreviation symbols are used in this manuscript.

(EQPs)	represents the equilibrium points.
(EEQP)	represents the evanescence equilibrium point.
(PDFEQP)	represents the predation-free equilibrium point.
(SPFEQP)	represents the second predator-free equilibrium point.
(FPFEQP)	represents the first predator-free equilibrium point.
(IEQP)	represents the interior equilibrium point.
(JM)	represents the Jacobian matrix.
(LAS)	represents the locally asymptotically stable.
(GAS)	represents the globally stable locally.
(TBF)	represents a transcritical bifurcation.
(PBF)	represents a pitchfork bifurcation.
(SNB)	represents a saddle-node bifurcation.

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