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Characterizations of the Exponential Distribution by Some Random Hazard Rate Sequences

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Abstract: In this paper, several characterizations for exponential distribution are derived from a new relative hazard rate measure. This measure is closely related to the concept of remaining lifetime at a random time. It considers the random times specified by the order statistics of a sample, the convolution of random variables, and the record values of a sequence of random variables. The concept of completeness in functional analysis provides the technical background to obtain the main results.

Keywords: hazard rate; lifetime; characterization; completeness; residual life at random time

MSC: 60E05; 46N30; 62N05

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1. Introduction and Preliminaries

The characterization of distributions has long been an important problem in connection with probability and statistics. The property of characterizing a distribution arises from certain relationships between statistical and probabilistic quantities such as the moment of a particular statistic derived from samples or the probability to have it in certain intervals (see, e.g., Shanbhag [1], Kotz and Shanbhag [2], Rao and Shanbhag [3], and Galambos and Kotz [4]). In reliability and survival analysis, lifetime distributions play an essential role in detecting various trends in lifetime data (see, for instance, Lai and Xie [5] and Marshall and Olkin [6]). The aging process of a lifetime unit is considered informative. There are many aspects of aging of a lifetime unit. Exponential distribution as a standard distribution plays a central role among lifetime distributions with different patterns of aging. Namely, the exponential distribution has a lack of memory property in aging, which means that a unit with a lifetime that follows the exponential distribution will never show signs of wear. Therefore, necessary and sufficient conditions for a life unit to detect a random life with the exponential distribution can be useful in various situations (see Azlarov et al. [7]).

In the context of lifetime science, the hazard rate (HR) function is a measure that has attracted considerable interest because it characterizes a lifetime distribution. In the literature, the HR function has been used as a useful quantity for constructing various models in survival analysis, including the well-known proportional hazard rate model. The behavior of the graph of an HR function of a unit's lifespan with respect to time, when it ranges from zero to infinity, can uniquely describe the aging process of that unit. This has been widely used in research papers in recent decades, as researchers usually intend to find novel lifetime distributions with an HR function that have different shapes to provide more flexibility. The HR function is defined for unrepairable populations as the (instantaneous) failure rate of survivors at time t during the next time instant. The detection of HR refers to the entire time interval. However, it may also be of interest to define the failure probability of a device directly according to specific times at which certain events occur, for example, the probability of failure of a coherent system that fails with the successive failures of its components (which turn out to be order statistics of the lifetime of the components in the system) when its components fail may be an informative quantity. In addition, events that represent records, such as the magnitude of earthquakes or new records set by an athlete in a sport, may also be of interest. Thus, the probability that future values represent a new record may be of interest as well.

The goal of the current study is to find some new characterizations of the exponential distribution using a new relative hazard rate measure. This measure adjusts the instantaneous risks of failure of a device in a random time. The random time is considered as an order statistic of a sample or a record statistic of a sequence of random variables. The results of the characterization are regularly proven as an application of the concept of completeness in functional analysis, which gives more tractability to the problem from a mathematical point of view.

Here, a number of preliminary concepts and also technical notions used throughout the paper are given. We also introduce some useful notions that are utilized in subsequent sections. The concept of completeness in mathematical analysis and functional analysis has been frequently applied in different contexts, including probability and statistics.

Definition 1. *The sequence $\phi_1, \phi_2, \cdots$ in a given Hilbert space H is complete if the only member in H that is orthogonal to every ϕ_n is the null member, that is*

$$\langle f, \phi_n \rangle = 0, \ \forall\, n \geq 1 \Rightarrow f = 0,$$

where 0 signifies the zero member in H.

The notation $\langle \cdot, \cdot \rangle$ is used to represent the inner product of H. In what follows, the Hilbert space $L^2[a,b]$ is considered to have an inner product as

$$\langle m_1, m_2 \rangle = \int_a^b m_1(x) m_2(x) dx,$$

in which m_1 and m_2 are two real-valued square integrable functions in $[a,b]$. It is noticeable that if $\phi_1, \phi_2, \cdots$ is a complete sequence in the Hilbert space H, then $\sum a_n \phi_n$ with $a_n = \langle m, \phi_n \rangle$ converges in H, provided that $\sum |a_n|^2 < \infty$, and the limit corresponds with m. For example, Higgins [8] discussed further details in this context.

Lemma 1 (Hwang [9])**.** *Suppose that ψ is an absolutely continuous function defined on $[a,b]$ with $\psi(a)\psi(b) \geq 0$, and let its derivative satisfy $\psi'(x) \neq 0$ a.e. on (a,b). Then, under the assumption*

$$\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty, \qquad \text{where } 1 \leq \lambda_1 < \lambda_2 < \cdots,$$

the sequence $\psi^{\lambda_1}, \psi^{\lambda_2}, \cdots$ is complete on (a,b) if and only if the function ψ is monotone on (a,b).

In the rest of the paper, we assume that X is a lifetime random variable with cumulative distribution function (CDF) F and survival function (SF) $\bar{F} \equiv 1 - F$. If F is absolutely continuous, the probability density function (PDF) of X exists and we denote it by f. One of the important reliability measures is the hazard rate (HR) function, which is closely related to the probability of instantaneous failure of an aging item. The age of lifetime units is a quantity of time. To recognize an aging item at different ages, the residual lifetime of the item after different ages is a useful quantity. The RV $X_t := (X - t | X > t)$ for all $t : \bar{F}(t) > 0$ is called the residual life of X after time t, where t is the current age of an item. The hazard rate of X is then defined as

$$h_X(t) := \lim_{\delta \to 0^+} \frac{1}{\delta} P(X_t \leq \delta) = \frac{f(t)}{\bar{F}(t)}.$$

Note that X_t has PDF $f_t(x) = \frac{f(t+x)}{\bar{F}(t)}$, $x \geq 0$. It can be observed that $h_X(t) = f_t(0)$ for all $t > 0$, which shows that the HR is connected to the instantaneous failure of an item with age t. This observation will become an aid in this paper to construct random hazard rates as a more flexible measure. The HR uniquely determines the underlying distribution as the following recursive relation confirms:

$$\bar{F}(t) = P(X > t) = \exp\left(-\int_0^t h_X(x)dx\right), \ \ t > 0, \tag{1}$$

from which it follows that, if $h_X(x) = h_Y(x)$ for all $x > 0$, then $P(X > t) = P(Y > t)$ for all $t \geq 0$, which means that the nonnegative RVs X and Y with HR functions h_X and h_Y, respectively, are equal in distribution. From the lack of memory property of the exponential distribution, it follows that X is exponential if and only if X_t for all $t > 0$ is equal in distribution with X. The HR function is a unique characteristic of a distribution. From (1), one may thus realize that X is exponential with SF function $\bar{F}(t) = \exp(-\lambda t)$ if and only if it has a constant HR that is not tied up with t, that is $h_X(t) = \frac{1}{\lambda}$, for all $t > 0$.

Let X be a non-negative random variable representing the lifetime of a unit or a device. The random variable $X_T := [X - T|X > T]$, which is called residual life at random time (RLRT), plays an important role in the study of the lifetime of the unit with random lifetime X relative to a unit with random life T. Suppose X and T have CDF's F and G and PDF's f and g, respectively. Let us also assume that T is the lifetime of another lifetime unit and assume for simplicity that X and T are statistically independent. Then, X_T has CDF

$$F_T(x) = \frac{\int_0^{+\infty}[F(x+t) - F(t)]dG(t)}{P(X > T)}, \tag{2}$$

and PDF

$$f_T(x) = \frac{\int_0^{+\infty} f(x+t)dG(t)}{P(X > T)}. \tag{3}$$

The concept of RLRT has been considered by many researchers in recent decades (see, e.g., Dequan and Jinhua [10], Li and Zuo [11], Misra et al. [12], Cai and Zheng [13], Kundu and Patra [14], Misra and Naqvi [15], Patra and Kundu [16], Patra and Kundu [17], and Patra and Kundu [18]). This measure has been applied in different contexts such as reliability theory, actuarial studies and queueing theory. Some authors used the RLRT for stochastic comparisons of lifetime units such as coherent systems (see, for instance, Eryilmaz and Tutuncu [19] and Amini-Seresht et al. [20]).

The remainder of the paper is organized as follows. In Section 2, we introduce a new reliability measure and state a characterization property based on it for the exponential distribution. In Section 3, we apply the characterization result in Section 2 to characterize the exponential distribution based on the relative hazard rate for order statistics. In Section 4, we develop the characterization based on the hazard rate with respect to the convolution of random variables. In Section 5, the exponential distribution is characterized based on the hazard rate relative to the record values. In Section 6, we conclude the paper with additional remarks on the current investigation and give some explanations of future materials that can be studied in this direction.

2. Random Hazard Rate Measure

In this section, we introduce a new reliability measure that is closely related to the hazard rate function. As seen in Section 1, the HR function h_X at the specific time t is related to the residual lifetime of X after time point t, i.e., the conditional RV X_t. The time point t is considered the current age of an item with lifetime X, which is constant. However, to entertain random ages at which one can evaluate the probabilities of instantaneous risks for failure of the item, a new measure can be introduced.

The probability of failure of an item with lifetime X with the random age T to be occur promptly after T can be given a rate. In this regard, the following quantity may be useful

$$\hbar(X,T) := \lim_{\delta \to 0^+} \frac{1}{\delta} P(X_T \le \delta). \tag{4}$$

By appealing Equations (2) and (4), it is found that $\hbar(X,T)$ is the derivative of $F_T(x)$ with respect to x, at $x = 0$; that is, $\hbar(X,T) := f_T(0)$ since

$$F'_T(0) = \lim_{\delta \to 0^+} \frac{F_T(\delta) - F_T(0)}{\delta} = \lim_{\delta \to 0^+} \frac{1}{\delta} P(X_T \le \delta).$$

Therefore, we can consider the following relative measure as the random hazard rate of X relative to T,

$$\hbar(X,T) := \frac{\int_0^{+\infty} f(t) dG(t)}{P(X > T)}. \tag{5}$$

It is acknowledged that when T is equal with $t > 0$ with probability one, i.e., when T is degenerate at t, then $\hbar(X,T)$ corresponds with the ordinary hazard rate function of X at time t. However, otherwise, there is a link between the random hazard rate of X with the usual hazard rate of X. In fact, the random hazard rate of X is the average amount of the ordinary hazard rate measured at a random time. Let us observe that

$$\begin{aligned} \hbar(X,T) &= \frac{\int_0^{+\infty} f(t) dG(t)}{P(X > T)} \\ &= \int_0^{+\infty} \frac{f(t)}{\bar{F}(t)} \frac{\bar{F}(t)}{\int_0^{\infty} \bar{F}(t) dG(t)} dG(t) \\ &= E[h_X(T^*)], \end{aligned} \tag{6}$$

where h_X is the HR function of X and T^* is a non-negative RV with PDF

$$g^*(t) = \frac{\bar{F}(t) g(t)}{\int_0^{\infty} \bar{F}(t) dG(t)}. \tag{7}$$

Let us consider a random sequence of times instead of T at which the random hazard rate is measured and a representation for the random hazard rate is obtained as (6). g^* in (7) is also modified.

We suppose that, for $k = 1, 2, \ldots$, the random time $T(k)$ has PDF

$$g_{[k]}(t) = \frac{w(t)\phi^{\lambda_k}(t)}{\int_0^{+\infty} w(t)\phi^{\lambda_k}(t)dt}, \tag{8}$$

in which w and ϕ are two positive functions such that $\int_0^{+\infty} w(t)\phi^{\lambda_k}(t)dt < +\infty$. The special case where $\lambda_k = k$ for $k = 1, 2, \ldots$ fulfills the result of Lemma 1, as it is known that $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$. Therefore, when ϕ is absolutely continuous and monotone, as a result $\phi(x), \phi^2(x), \cdots$ is a complete sequence of functions.

We prove that the only probability distribution for which the random hazard rate (5) when it is measured at $\{T(k), k = 1, 2, \ldots\}$ in place of T is constant, in the sense that it does not depend on k, is the exponential distribution.

Formally, we present a useful lemma here to characterize the exponential distribution.

Lemma 2. *Let $T(k), k = 1, 2, \ldots$ be a sequence of RVs that are independent of X. Then, X has exponential distribution if and only if $\hbar(X, T(k)) = c$, for all $k = 1, 2, \ldots$ in which $T(k)$ is an RV with PDF (8) with a monotone function ϕ, so that $\sum_{k=1}^{+\infty} \lambda_k^{-1} = +\infty$ where $1 \le \lambda_1 < \lambda_2 < \cdots$.*

Proof. Note that $T(k)$ has PDF (8) and that X has PDF f, SF $\bar{F}$, and HR function h_X. By (5), we have

$$\begin{aligned}\hbar(X,T(k)) &= \int_0^{+\infty} \frac{f(t)g_{[k]}(t)}{P(X>T(k))}dt \\ &= \int_0^{+\infty} h_X(t)\frac{\bar{F}(t)g_{[k]}(t)}{P(X>T(k))}dt.\end{aligned}$$

Therefore, $\hbar(X,T(k)) = c$ for all $k = 1,2,\cdots$ if and only if $E[h_X(T^*(k))] = c$ for all $k = 1,2,\cdots$, where $T^*(k)$ is a nonnegative RV with PDF $g^*_{[k]}(t) = \frac{\bar{F}(t)g_{[k]}(t)}{P(X>T(k))}$, which equivalently holds if

$$\int_0^{+\infty} (h_X(t)-c)\frac{\bar{F}(t)g_{[k]}(t)}{P(X>T(k))}\,dt = 0, \ \text{ for all } k = 1,2,\cdots.$$

Since $P(X > T(k)) > 0$ for all $k = 1,2,\cdots$, thus, the above identity holds if and only if

$$\int_0^{+\infty} (h_X(t)-c)\bar{F}(t)g_{[k]}(t)\,dt = 0, \ \text{ for all } k = 1,2,\cdots.$$

By substituting $g_{[k]}$ from (8) in the above integral, it follows that $\hbar(X,T(k)) = c$ for all $k = 1,2,\cdots$ if and only if

$$\int_0^{+\infty} (h_X(t)-c)\bar{F}(t)w(t)\phi^{\lambda_k}(t)\,dt = 0, \ \text{ for all } k = 1,2,\cdots,$$

which holds if, and only if

$$\langle\psi,\psi_k\rangle = \int_0^{+\infty} \psi(t)\psi_k(t)\,dt = 0, \ \text{ for all } k = 1,2,\cdots, \tag{9}$$

where $\psi(t) = (h_X(t)-c)\bar{F}(t)w(t)$ and $\psi_k(t) = \phi^k(t)$. From Lemma 1, we deduce that the sequence ψ_k fulfills the completeness property on the domain $(0,\infty)$. It thus follows from (9) that $\psi(t) = 0$ for all $t > 0$, from which we infer that $h_X(t) = c$ for all $t \geq 0$. This means that X has exponential distribution. □

In the subsequent sections, the residual life is developed in some typical random sequence of random variables. Suppose that $T_1, T_2, \ldots$ are a sequence of independent random variables that are also independent of X. We want to measure the residual life length of X after a time that is function of T_i, $\mu(T_{i_1}, T_{i_2}, \ldots, T_{i_n})$ where $(i_1, i_2, \ldots, i_n)$ is an increasing arrangement of natural numbers. Let $\mu : \mathbb{R}^n_+ \mapsto \mathbb{R}_+$ be an n-variate function. The random times $T_{i_1}, T_{i_2}, \ldots, T_{i_n}$ may be the time of occurrence of some consecutive events, for example, times of arriving shocks to a system in reliability or a sequence of drought occurrence in aerology. The residual life of X after the random time $\mu(T_{i_1}, T_{i_2}, \ldots, T_{i_n})$, i.e.,

$$X_{\mu(T_{i_1},T_{i_2},\ldots,T_{i_n})} := (X - \mu(T_{i_1}, T_{i_2}, \ldots, T_{i_n})|X > \mu(T_{i_1}, T_{i_2}, \ldots, T_{i_n}))$$

may be an interesting variable for measurement. For several typical functions μ, including different order statistics of $T_1, T_2, \ldots$ and upper and lower record values arising out of this sequence of random variables, we will adopt the sequence $\{X_{\mu(T_{i_1},T_{i_2},\ldots,T_{i_n})} : n \in \mathbb{N}\}$ to present some characterizations for the exponential distribution. In the following sections, we make use of Lemma 2 to present characterizations of exponential distribution.

3. Characterizations Based on Random Hazard Rate Relative to Order Statistics

We assume that $T_1, T_2, \cdots$ is a sequence of independent and identically distributed (IID) nonnegative RVs with PDF g, CDF G, and SF $\bar{G}$. In addition, assume throughout that the lifetime RV X is independent of this sequence. Let $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(i)}$ be the order statistics from the first i elements of the sequence of T_is. For $i = 1, 2, \cdots n$, the PDF of the ith order statistic $T_{(i)}$ is given by

$$g_{(i)}(t) = \frac{n!}{(i-1)!(n-i)!} G^{i-1}(t)\, \bar{G}^{n-i}(t)\, g(t), \quad \text{for all } t > 0. \tag{10}$$

In the sequel, for any $i \in \mathbb{N}$ and for any $n = i, i+1, \ldots$, we consider $X_{T_{(i)}} = (X - T_{(i)} \mid X > T_{(i)})$, which is the additional lifetime of X after $T_{(i)}$, known as the residual lifetime of a lifespan with original lifetime X relative to the lifetime of an $(n-i+1)$-out-of-n system, provided that the lifespan survives the system's lifetime. This is because the RV $T_{(i)}$ represents the lifetime of an $(n-i+1)$-out-of-n system in reliability engineering. A coherent system fails with the consecutive failure of its components. Eryilmaz [19] used the concept of residual life at random time to study the relative behavior of a coherent system with respect to a system. For example, in the assembly of a system of components, it is useful to know what the relative residual lifetime of a coherent system with lifetime T_S with respect to a series system is, given that the coherent system lifetime is greater than the lifetime of the series system. The RV $(T_S - T_{(1)} \mid T_S > T_{(1)})$ studies the relative behavior. For $j > i$, the RV $(T_{(j)} - T_{(i)} \mid T_{(j)} > T_{(i)})$ is identical with $D_{i,j:n} = T_{(j)} - T_{(i)}$ in distribution and therefore the spacings are at the disposal of the residual life at random time. The recent difference provides the spacings between order statistics of a sample, and has found many applications in statistics and life testing (see, e.g., Pledger and Proschan [21]).

Kayid and Izadkhah [22], in Theorem 3.1 of their paper, proved that X has exponential distribution if and only if $E[X - T_{(i)} \mid X > T_{(i)}] = E[X]$ for all $n = i, i+1, \ldots$ when i is a positive fixed integer. Note that $T_{(i)}$ is the ith order statistic among a random sample of size n, which can increase from i to infinity. That is, $T_{(i)}$ for any fixed $i \in \mathbb{N}$ is a sequence in terms of n. Here, a new characterization of the exponential distribution based on random hazard rate (5) is derived. Note that $T_{i:j}$ denotes the ith order statistic among j IID random variables $T_1, T_2, \cdots, T_j$. However, when $j = n$, then $T_{i:n}$ is denoted by $T_{(i)}$, as before.

Theorem 1. *Let $T_1, T_2, \cdots$ constitute a sequence of IID nonnegative RVs with PDF g. Then, the RV X that is independent of T_i's has exponential distribution if and only if $\hbar(X, T_{i:i+k-1}) = c$ for all $k = 1, 2, \cdots$ for a fixed integer $i \in \mathbb{N}$ where c does not depend on k.*

Proof. To make use of Lemma 2, we take $T(k) := T_{i:i+k-1}$ in that lemma. From the PDF given in (10), observe that $T(k)$ follows the PDF

$$\begin{aligned} g_{[k]}(t) &= \frac{(i+k-1)!}{(i-1)!(k-1)!} G^{i-1}(t) \bar{G}^{k-1}(t) g(t) \\ &= \frac{(i+k-1)!}{(i-1)!(k-1)!} \frac{g(t) G^{i-1}(t)}{\bar{G}(t)} \bar{G}^{k}(t), \end{aligned}$$

which coincides with PDF (8) with $w(t) = \frac{g(t)G^{i-1}(t)}{\bar{G}(t)}$ and $\phi(t) = \bar{G}(t)$. It is trivial that ϕ is a decreasing function. Thus, Lemma (2) is applied and provides the proof. □

It is to be mentioned that c in the previous theorem, as well as in the residual theorem, is the reciprocal of the mean of distribution X, i.e., when we write $h_X(t) = c$, it turns out that $h_X(t) = \frac{1}{E(X)}$. In Theorem 1, the random time is considered to be the ith order statistic with i fixed, among partial sets of the sequence of T_i. However, the maximum order statistic cannot be entertained since the size of the partial sets of the sequence is going up, and thus it cannot be fixed. Let us assume that $T_{(k)} = \max\{T_1, T_2, \cdots, T_k\}$ is the order statistic that has the largest amount among $T_1, T_2, \cdots, T_k$. Next, we provide another characterization property of the exponential distribution.

Theorem 2. *Let $T_1, T_2, \cdots$ be a set of IID nonnegative RVs with PDF g. Then, the RV X which is independent of T_i's, follows the exponential distribution if, and only if, $\hbar(X, T_{k:k}) = c$ for all $k = 1, 2, \cdots$, where c is free of k.*

Proof. To make Lemma 2 applicable, one takes $T(k) := T_{k:k}$ in the lemma. From the PDF given in (10), one can see that $T(k)$ follows the PDF

$$
\begin{aligned}
g_{[k]}(t) &= kG^{k-1}(t)g(t) \\
&= k\frac{g(t)}{G(t)}G^k(t),
\end{aligned}
$$

which coincides with PDF (8) with $w(t) = \frac{g(t)}{G(t)}$ and $\phi(t) = G(t)$. It is obvious that ϕ is an increasing function. Thus, Lemma 2 applies and gives the proof. $\square$

Suppose T_0 be a nonnegative RV with CDF G and PDF g. Then, T with CDF $G^\eta, \eta > 0$ is said to have a proportional reversed hazard rate (PRHR) model, since $\widetilde{h}_T(t) = \eta\widetilde{h}_{T_0}(t)$ where $\widetilde{h}_T$ is the reversed hazard rate (RHR) function of T and $\widetilde{h}_{T_0}$ is the RHR function of T_0 given by $\widetilde{h}_{T_0}(t) = \frac{g(t)}{G(t)}$, which is valid for all $t \geq 0$ for which $G(t) > 0$. For further discussion on the PRHR model, we refer the reader to Gupta et al. [23]. In parallel, if the RV T has SF $\bar{G}^\theta, \theta > 0$, then it is said that T satisfies the proportional hazard rate (PHR) model where $h_T(t) = \theta h_{T_0}(t)$, in which h_T and h_{T_0} are the HR function of T and T_0, respectively. For further details on the PHR model in our context, we refer the reader to Kochar and Xu [24].

The results of Theorems 1 and 2 are based on IID random times. In the context of the PRHR model and the PHR model, we characterize the exponential distribution in terms of non-identical independent random times as follows:

Theorem 3. *Let $T_1, T_2, \ldots$ be nonnegative independent RVs that are independent of X, such that*

(a) T_i follows the SF $\bar{G}^{\theta_i}, i = 1, 2, \ldots, k$ where $\theta_1 \geq 1$ and $\theta_i > 0, i = 2, 3, \ldots$. If we assume further $\sum_{k=1}^{+\infty}\left(\sum_{i=1}^k \theta_i\right)^{-1} = +\infty$, then X has exponential distribution if and only if $\hbar(X, T_{1:k}) = c$ for all $k = 1, 2, \cdots$, where c does not depend on k.

(b) T_i follows the CDF $G^{\eta_i}, i = 1, 2, \ldots, k$ so that $\eta_1 \geq 1$ and $\eta_i > 0, i = 2, 3, \ldots$. If we suppose that $\sum_{k=1}^{+\infty}\left(\sum_{i=1}^k \eta_i\right)^{-1} = +\infty$ then X has exponential distribution, if and only if, $\hbar(X, T_{k:k}) = c$ for all $k = 1, 2, \cdots$ where c does not depend on k.

Proof. To establish the result in assertion (a), we make an application of Lemma 2 by taking $T(k) := T_{1:k}$. Note that $T(k)$ follows the PDF

$$
\begin{aligned}
g_{[k]}(t) &= (\sum_{i=1}^k \theta_i)g(t)\bar{G}^{\sum_{i=1}^k \theta_i - 1}(t). \\
&= (\sum_{i=1}^k \theta_i)\frac{g(t)}{\bar{G}(t)}\bar{G}^{\sum_{i=1}^k \theta_i}(t),
\end{aligned}
$$

which matches with PDF (8) with $w(t) = \frac{g(t)}{\bar{G}(t)}$ and $\phi(t) = \bar{G}(t)$, so that $\lambda_k = \sum_{i=1}^k \theta_i$. It is evidently seen that ϕ is a decreasing function. Therefore, Lemma (2) completes the proof of (a). We now prove the assertion (b). In this case, we make another application of Lemma 2 by taking $T(k) := T_{k:k}$. We observe that $T(k)$ follows the PDF

$$\begin{aligned} g_{[k]}(t) &= (\sum_{i=1}^{k} \eta_i) g(t) G^{\sum_{i=1}^{k} \eta_i - 1}(t). \\ &= (\sum_{i=1}^{k} \eta_i) \frac{g(t)}{G(t)} G^{\sum_{i=1}^{k} \eta_i}(t), \end{aligned}$$

which matches with PDF (8) with $w(t) = \frac{g(t)}{G(t)}$ and $\phi(t) = G(t)$ where $\lambda_k = \sum_{i=1}^{k} \eta_i$. Since ϕ is increasing, Lemma (2) is applicable and completes the proof of (b). $\square$

4. Characterizations Based on Random Hazard Rate Relative to Convolution of Random Variables

Here, convolution of random variables is regarded as the random times. The consideration of convolution of random lifetimes has been frequently used in the context of reliability analysis and stochastic comparisons (see Bon, J. L. and Păltănea [25] and Shaked and Suarez-Llorens [26]). The convolution of independent RVs is indeed the lifetime of a standby system, which has been found to be useful to allocate a spare in a system in order to stochastically optimize the lifetime of the resulting system (cf. Boland et al. [27]). Stochastic orderings of a residual lifetime of convolutions have been considered in Amiripour et al. [28], where some applications in reliability and queuing systems were also presented.

In this context, convolutions of heterogenous gamma RVs with different shape parameters but a common scale parameter have been considered in some research work for multiple purposes, including stochastic comparisons. The tail behavior has also been studied (see, e.g., Kochar and Xu [29], Zhao [30], Amiri et al. [31] and Roosta-Khorasani [32]). Let us consider $T_1, T_2, \cdots$ to be a sequence of independent RVs having gamma distribution with shape parameters $\alpha_1, \alpha_2, \cdots$, respectively, and a common scale parameter β. The PDF of T_i, $i = 1, 2, \cdots$ is given by

$$g(t, \alpha_i, \beta) = \frac{t^{\alpha_i - 1} \beta^{\alpha_i} e^{-\beta t}}{\Gamma(\alpha_i)}, \ \ t > 0,$$

in which $\alpha_i > 0$ and $\beta > 0$. Let X be the length of a device. Denote by $S_k = \sum_{i=1}^{k} T_i$ the partial sum of the gamma sequence. In probability theory, it is known that $S_k = T_1 + T_2 + \cdots + T_k$ follows a gamma distribution with PDF

$$g_k(t) = \frac{\beta^{\lambda_k} t^{\lambda_k - 1} e^{-\beta t}}{\Gamma(\lambda_k)}, \quad \text{for all } t > 0, \tag{11}$$

in which $\lambda_k = \sum_{i=1}^{k} \alpha_i$ whenever $k \in \mathbb{N}$. We will utilize the following measure

$$\hbar(X, S_k) = \lim_{\delta \to 0^+} \frac{1}{\delta} P(X_{S_k} \leq \delta), \ \ k \in \mathbb{N}.$$

The following result presents another characterization result for the exponential distribution.

Theorem 4. *Let $T_1, T_2, \ldots$ be independent heterogenous gamma RVs that are independent of X, such that T_i follows a gamma distribution with shape parameter $\alpha_i > 0$ and $\beta > 0$ for $i = 1, 2, \ldots$ where $\alpha \geq 1$. If $\sum_{k=1}^{+\infty} \left(\sum_{i=1}^{k} \alpha_i \right)^{-1} = +\infty$, then X is exponential if and only if $\hbar(X, S_k) = c$ for all $k = 1, 2, \cdots$, where c is independent of k.*

Proof. We apply Lemma 2 once again. Let us take $T(k) := S_k$ in Lemma 2. From the PDF given in (11), we see that $T(k)$ follows the PDF

$$
\begin{aligned}
g_{[k]}(t) &= \frac{\beta^{\lambda_k} t^{\lambda_k - 1} e^{-\beta t}}{\Gamma(\lambda_k)} \\
&= \frac{\beta^{\lambda_k}}{\Gamma(\lambda_k)} \frac{e^{-\beta t}}{t} t^{\lambda_k},
\end{aligned}
$$

where $\lambda_k = \sum_{i=1}^{k} \alpha_i$, which coincides with PDF (8) with $w(t) = \frac{e^{\beta t}}{t}$ and $\phi(t) = t$. It is obvious that ϕ is an increasing function. Thus, Lemma 2 applies in this case and completes the proof. $\square$

5. Characterizations Based on Random Hazard Rate Relative to Record Statistics

In science, nature, and technology, records are important for their content and as evidence of communications, decisions, actions, and history. As public entities, school boards are accountable to the public and government. Records support openness and transparency by documenting and providing evidence of work activities and making them available to the public. Records support the quality of programs and services, provide information for decision making, and help achieve organizational goals. In statistics, a record value or statistic is the largest or smallest value that results from a sequence of random variables (cf. Arnold et al. [33]).

To be more specific, the RV T_i, when it is observed, is called an upper record if its value is greater than the value taken by previously recorded observations. Therefore, T_j is considered to be an upper record if $T_j > T_i$ for every $i < j$. By the identification of the record statistics, a random sequence of time points is generated at which the records occur. We denote the ith element of this sequence by U_i, considered to be the time at which the ith upper record appears. The initial time U_0 is assumed to be zero with probability 1, and for $j \geq 1$, $U_j = \min\{i : T_i > T_{U_{i-1}}\}$. The upper record values are then identified as $\{T_{U_k} : k = 0, 1, 2, \ldots\}$. Since T_is are lifetime RVs, we set $T_0 = 0$. The RV T_{U_k}, which is the kth upper record, has the following PDF

$$
g_{U_k}(t) = \frac{(-\log(\bar{G}(t)))^k}{k!} g(t). \tag{12}
$$

It is of some interest to hold the excess records for future observations. The RV X used to denote the lifetime of a lifespan is considered to be independent of $T_k, k = 1, 2, \ldots$. Therefore, X is also independent of T_{U_k}. The conditional RV $X_{T_{U_k}} = (X - T_{U_k} \mid X > T_{U_k})$ is a tool for measurement of the amount of X, provided that X is greater than the kth upper record of T_is. The RV $X_{T_{U_k}}$ can be used in different contexts; e.g., to evaluate the additional capacity of a dam, we can measure the water level in record times caused by rainfalls.

Next, another characterization of the exponential distribution is presented.

Theorem 5. *Suppose $T_1, T_2, \ldots$ are IID RVs that are independent of X. Then, X is exponential if and only if $\hbar(X, T_{U_k}) = c$ for all $k = 1, 2, \cdots$, where c is not tied to k.*

Proof. We utilize Lemma 2 to obtain the result. We can take $T(k) := T_{U_k}$ in Lemma 2. From the PDF given in (12), it is seen that $T(k)$ follows the PDF

$$
\begin{aligned}
g_{[k]}(t) &= \frac{(-\log(\bar{G}(t)))^k}{k!} g(t) \\
&= \frac{g(t)}{k!} (-\log(\bar{G}(t)))^k,
\end{aligned}
$$

which matches with PDF (8) when $w(t) = g(t)$ and $\phi(t) = -\log(\bar{G}(t))$. It is clear that ϕ is an increasing function. Thus, Lemma 2 applies in this case as well, and provides the proof. □

6. Conclusions

In the study conducted in this paper, new characterizations of the exponential distribution were developed. We have used a new measure obtained by the hazard rate at random time points. It has been shown that the new measure is the average of the hazard rate of the population at an independent random time. Since the population may be exposed to different environments, different random time points may be considered at which the risks of failure in the population are quantified by the new hazard measure. The random time points were defined as order statistics of a sample, convolutions of random time points, and record statistics. The hazard rate function, when measured at each point in the entire time interval, i.e., $[0, +\infty)$, uniquely determines the distribution, and if the amounts it takes for different time points in $[0, +\infty)$ are all the same, then the underlying distribution is exponential, and vice versa. However, in this work, we have shown that the strong condition that the hazard rate function be constant over the entire time interval can be reduced to a weaker condition. Formally, it has been pointed out in various theorems that X must be exponential if the average (the mathematical expectation) of the hazard rate in a sequential random time is a constant, and vice versa.

In future research, the possibility of entertaining dependencies between X and $T_1, T_2, \ldots$ will be sought. In such a situation, a broader desire for applicability of the characterization results is expressed. By considering this possibility, we can characterize distributions using dependent random variables. For example, the sequence of order statistics, the partial sums (convolution) of random variables, and the sequence of record statistics are each dependent random variables. The new measure, called the random rate, is of interest in several contexts. We can use this measure to study the relative behavior of components of a coherent system or the behavior of two coherent systems with respect to each other. We can also study stochastic ordering properties of distributions by their random hazard rate with respect to a reference population.

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References

1. Shanbhag, D.N. The characterizations for exponential and geometric distributions. *J. Am. Stat. Assoc.* **1970**, *65*, 1256–1259. [CrossRef]
2. Kotz, S.; Shanbhag, D.N. Some new approaches to probability distributions. *Adv. Appl. Probab.* **1980**, *12*, 903–921. [CrossRef]
3. Rao, C.R.; Shanbhag, D.N. Recent results on characterization of probability distributions: A unified approach through extensions of Deny and s theorem. *Adv. Appl. Probab.* **1986**, *18*, 660–678. [CrossRef]
4. Galambos, J.; Kotz, S. *Characterizations of Probability Distributions: A Unified Approach with an Emphasis on Exponential and Related Models*; Springer: Berlin/Heidelberg, Germany, 2006; Volume 675.
5. Lai, C.D.; Xie, M. *Stochastic Ageing and Dependence for Reliability*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2006.

6. Marshall, A.W.; Olkin, I. *Life Distributions*; Springer: New York, NY, USA, 2007; Volume 13.
7. Azlarov, T.A.; Volodin, N.A.; Olkin, I. *Characterization Problems Associated with the Exponential Distribution*; Springer: New York, NY, USA, 1986.
8. Higgins, J.R. *Completeness and Basis Properties of Sets of Special Functions (No. 72)*; Cambridge University Press: Cambridge, UK, 2004.
9. Hwang, J.S.; Lin, G.D. On a generalized moment problem II. *Proc. Am. Math. Soc.* **1984**, *91*, 577–580. [CrossRef]
10. Dequan, Y.; Jinhua, C. Some results on the residual life at random time. *Acta Math. Appl. Sin.* **2000**, *16*, 435–443. [CrossRef]
11. Li, X.; Zuo, M.J. Stochastic comparison of residual life and inactivity time at a random time. *Stoch. Model.* **2004**, *20*, 229–235. [CrossRef]
12. Misra, N.; Gupta, N.; Dhariyal, I.D. Stochastic properties of residual life and inactivity time at a random time. *Stoch. Model.* **2008**, *24*, 89–102. [CrossRef]
13. Cai, N.; Zheng, Y. Preservation of generalized ageing classes on the residual life at random time. *J. Stat. Plan. Inference* **2012**, *142*, 148154. [CrossRef]
14. Kundu, C.; Patra, A. Some results on residual life and inactivity time at random time. *Commun.-Stat.-Theory Methods* **2018**, *47*, 372–384. [CrossRef]
15. Misra, N.; Naqvi, S. Some unified results on stochastic properties of residual lifetimes at random times. *Braz. J. Probab. Stat.* **2018**, *32*, 422–436. [CrossRef]
16. Patra, A.; Kundu, C. On generalized orderings and ageing classes for residual life and inactivity time at random time. *Metrika* **2019**, *82*, 691–704. [CrossRef]
17. Patra, A.; Kundu, C. Further results on residual life and inactivity time at random time. *Commun.-Stat.-Theory Methods* **2020**, *49*, 1261–1271. [CrossRef]
18. Patra, A.; Kundu, C. Stochastic comparisons and ageing properties of residual lifetime mixture models. *Math. Methods Oper. Res.* **2021**, *94*, 123–143. [CrossRef]
19. Eryilmaz, S.; Tutuncu, G.Y. Relative behavior of a coherent system with respect to another coherent system. *Stat. Pap.* **2015**, *56*, 519–529. [CrossRef]
20. Amini-Seresht, E.; Kelkinnama, M.; Zhang, Y. On the residual and past lifetimes of coherent systems under random monitoring. *Probab. Eng. Inf. Sci.* **2021**, *35*, 465–480. [CrossRef]
21. Pledger, G.; Proschan, F. Comparisons of order statistics and of spacings from heterogeneous distributions. In *Optimizing Methods in Statistics*; Academic Press: Cambridge, MA, USA, 1971; pp. 89–113.
22. Kayid, M.; Izadkhah, S. Characterizations of the exponential distribution by the concept of residual life at random time. *Stat. Probab. Lett.* **2015**, *107*, 164–169. [CrossRef]
23. Gupta, R.C.; Gupta, P.L.; Gupta, R.D. Modeling failure time data by Lehman alternatives. *Commun.-Stat.-Theory Methods* **1998**, *27*, 887–904. [CrossRef]
24. Kochar, S.; Xu, M. Stochastic comparisons of parallel systems when components have proportional hazard rates. *Probab. Eng. Inf. Sci.* **2007**, *21*, 597–609. [CrossRef]
25. Bon, J.L.; Pǎltǎnea, E. Ordering properties of convolutions of exponential random variables. *Lifetime Data Anal.* **1999**, *5*, 185–192. [CrossRef]
26. Shaked, M.; Suarez-Llorens, A. On the comparison of reliability experiments based on the convolution order. *J. Am. Stat. Assoc.* **2003**, *98*, 693–702. [CrossRef]
27. Boland, P.J.; El-Neweihi, E.; Proschan, F. Stochastic order for redundancy allocations in series and parallel systems. *Adv. Appl. Probab.* **1992**, *24*, 161–171. [CrossRef]
28. Amiripour, F.; Khaledi, B.E.; Shaked, M. Stochastic orderings of convolution residuals. *Metrika* **2013**, *76*, 559–576. [CrossRef]
29. Kochar, S.; Xu, M. The tail behavior of the convolutions of gamma random variables. *J. Stat. Plan. Inference* **2011**, *141*, 418–428. [CrossRef]
30. Zhao, P. Some new results on convolutions of heterogeneous gamma random variables. *J. Multivar. Anal.* **2011**, *102*, 958–976. [CrossRef]
31. Amiri, L.; Khaledi, B.E.; Samaniego, F.J. On skewness and dispersion among convolutions of independent gamma random variables. *Probab. Eng. Inf. Sci.* **2011**, *25*, 55–69. [CrossRef]
32. Roosta-Khorasani, F.; Székely, G.J. Schur properties of convolutions of gamma random variables. *Metrika* **2015**, *78*, 997–1014. [CrossRef]
33. Arnold, B.C.; Balakrishnan, N.; Nagaraja, H.N. *Records*; John Wiley and Sons: Hoboken, NJ, USA, 2011; Volume 768.