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Oscillation of Second Order Nonlinear Neutral Differential Equations

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Abstract: The study of the oscillatory behavior of solutions to second order nonlinear differential equations is motivated by their numerous applications in the natural sciences and engineering. In the presented research, some new oscillation criteria for a class of damped second order neutral differential equations with noncanonical operators are established. The results extend and improve on those reported in the literature. Moreover, some examples are provided to show the significance of the results.

Keywords: oscillation criteria; Emden–Fowler differential equation; half-linear neutral differential equation

MSC: 34C10; 34K11

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1. Introduction

In this paper, we consider a damped second order neutral functional differential equation with the noncanonical operators

$$\left(r(t)|z'(t)|^{\alpha-1}z'(t)\right)' + p(t)|z'(t)|^{\alpha-1}z'(t) + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0, \quad (1)$$

where $z(t) = x(t) + c(t)x(\tau(t)), t \geq t_0, \alpha > 0$, and $\beta > 0$. Here, we use the following assumptions:

($\mathbf{C}_1$) $r \in C^1([t_0,\infty),(0,\infty)),\ \ r'(t) \geq 0,\ \ c(t) \in C([t_0,\infty),\mathbb{R}),\ \ 0 \leq c(t) < 1$;

($\mathbf{C}_2$) $p, q \in C([t_0,\infty),(0,\infty)),\ \ q(t)$ is not eventually zero on $[t^*,\infty)$ for $t^* \geq t_0$;

($\mathbf{C}_3$) $\tau \in C([t_0,\infty),\mathbb{R}),\ \sigma \in C^1([t_0,\infty),\mathbb{R}),\ \tau(t) \leq t,\ \sigma(t) \leq t,\ \sigma'(t) > 0$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

Let $T_x = \min\{\tau(t), \sigma(t)\}, t \geq t_0$. A function $x(t) \in C^1([T_x,\infty),\mathbb{R}), T_x \geq t_0$ is called a solution of Equation (1) if it has the property $r(t)|z'|^{\alpha-1}z'(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies Equation (1) on $[T_x,\infty)$. We only consider the nontrivial solutions of Equation (1), which ensure $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (1) is said to be oscillatory if it has an arbitrarily large zero point on $[T_x,\infty)$; otherwise, it is called non-oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Recently, the study of the oscillation criteria for neutral and damped second order differential equations has been motivated by their applications in the natural sciences and engineering; for example, see [1–28]. However, most of them are aimed at their spacial cases. For Equation (1), one important spacial case is (when $\alpha = \beta, p(t) = 0$)

$$\left(r(t)|z'(t)|^{\alpha-1}z'(t)\right)' + q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t)) = 0. \tag{2}$$

This equation is called the half-linear neutral differential equation, and it has attracted many studies since the 1970s (see [4]).

Another important spacial case of Equation (1) is (when $\alpha = 1, p(t) = 0$)

$$\left(r(t)(x(t) + c(t)x(\tau(t)))'\right)' + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0, \tag{3}$$

which is called the Emden–Fowler neutral differential equation, and it has been widely applied in mathematics and theoretical physics (see [1,25–27]).

Equation (2) can be understood as the half-linear differential equation

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + q(t)y^{\alpha}(t) = 0, \tag{4}$$

where α is the ratio of odd positive integers. Assume that

$$\pi(t) = \int_t^{\infty} r^{-\frac{1}{\alpha}}(s)ds < \infty.$$

Then, from [6], we have the following Kneser-type oscillation theorem for Equation (4).

Theorem 1. *Assume that*

$$\lim_{t\to\infty} r^{\frac{1}{\alpha}}(t)\pi^{\alpha+1}(t)q(t) > (\frac{\alpha}{\alpha+1})^{\alpha+1}.$$

Then, Equation (4) is oscillatory.

In 2020, Jadlovská [6] studied a general case of (4), such that

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + q(t)y^{\alpha}(\sigma(t)) = 0, \tag{5}$$

where $\sigma(t) \geq t$, and obtained the corresponding oscillation criteria.

Note that the half-linear neutral differential Equation (2) and the Emden–Fowler neutral Equation (3) are not mutually inclusive of each other. However, Equations (2) and (3) are included in Equation (1). Therefore, it will be of great interest to find some oscillation criteria for the neutral differential Equation (1). Our aim in this paper is to use the Riccati transformation technique (rather than comparison principles; e.g., see [26,28] for more details) to establish some new sufficient conditions for the oscillation criteria of (1). To the best of our knowledge, very little is known regarding the oscillation of (1). The relevance of our theorems becomes clear in the carefully selected examples.

The rest of paper is organized as follows. In Section 2, we establish several new oscillation criteria for Equation (1). In Section 3, we present six examples to illustrate our results.

2. Main Results

The following inequalities contain the variable t, in which we assume that the inequalities hold for a sufficiently large t if there is no special note. Without loss of generality, we only deal with the positive solution for Equation (1) in the proofs of our results.

In this paper, we study the noncanonical case of Equation (1). Let $R(t) = E(t)r(t)$, where

$$E(t) = \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)}ds\right).$$

We define the functions

$$\phi(t) := \int_t^\infty R^{-\frac{1}{\alpha}}(s)ds,\ t \geq t_0, \tag{6}$$

$$Q(t) := E(t)q(t)\left(1 - c(\sigma(t))\frac{\phi(\tau(\sigma(t)))}{\phi(\sigma(t))}\right)^\beta, \tag{7}$$

and

$$Q_1 := E(t)q(t)(1 - c(\sigma(t)))^\beta. \tag{8}$$

Then, we have the following lemma.

Lemma 1. *Let $x(t)$ be an eventually positive solution of Equation (1). Assume that $\phi(t) < \infty$ and $z'(t) < 0$. Then,*

$$\left(R(t)(-z'(t))^\alpha\right)' - Q(t)z^\beta(t) \geq 0, \quad t \geq t_1. \tag{9}$$

Proof. Let $x(t)$ be an eventually positive solution of Equation (1), then there exists a $t_1 \geq t_0$, such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Multiplying both sides of (1) by $E(t)$, we have the following equation without a damped term:

$$\left(R(t)|z'(t)|^{\alpha-1}z'(t)\right)' + E(t)q(t)x^\beta(\sigma(t)) = 0, \quad t \geq t_0. \tag{10}$$

Since $z'(t) < 0$, from (10) we get

$$\left(R(t)(-z'(t))^\alpha\right)' = E(t)q(t)x^\beta(\sigma(t)) \geq 0. \tag{11}$$

It follows that

$$z'(s) \leq \left(\frac{R(t)}{R(s)}\right)^{\frac{1}{\alpha}} z'(t), \quad s \geq t \geq t_1.$$

Integrating the above inequality from t to l, we obtain

$$z(l) - z(t) \leq R^{\frac{1}{\alpha}}(t)z'(t)\int_t^l R^{-\frac{1}{\alpha}}(s)ds,$$

which implies that

$$z(t) \geq R^{\frac{1}{\alpha}}(t)(-z'(t))\phi(t), \quad t \geq t_1. \tag{12}$$

Hence,

$$\left(\frac{z(t)}{\phi(s)}\right)' \geq 0.$$

In view of the definition of $z(t)$, we obtain the following for $t \geq t_1$:

$$x(t) = z(t) - c(t)x(\tau(t)) \geq z(t) - c(t)z(\tau(t)) \geq z(t)\left(1 - c(t)\frac{\phi(\tau(t))}{\phi(t)}\right). \tag{13}$$

By combining (11) and (13), with $z'(t) < 0$, we thus deduce that (9) holds. The proof is complete. □

Define a function $v(t)$ by

$$v(t) := \frac{R(t)(-z'(t))^{\alpha}}{z^{\beta}(t)}, \quad t \geq t_1. \tag{14}$$

We then have the following lemma.

Lemma 2. *Let $x(t)$ be an eventually positive solution of Equation (1). Assume that $\phi(t) < \infty$ and $z'(t) < 0$. Then,*
(i) $v(t)\phi^{\mu}(t)$ is bounded;
(ii) $v'(t) \geq Q(t) + m\beta R^{-\frac{1}{\alpha}}(t) v^{\frac{\mu+1}{\mu}}(t)$, $t > T_1$,
where m is a positive constant and $\mu = \max\{\alpha, \beta\}$.

Proof. (i). By Lemma 1, we have $(R(t)(-z'(t))^{\alpha})' \geq 0$, which implies that $R(t)(-z'(t))^{\alpha}$ is non-decreasing. From (12), we get

$$z^{\alpha}(t) \geq R(t)(-z'(t))^{\alpha}\phi^{\alpha}(t) = z^{\beta}(t)v(t)\phi^{\alpha}(t).$$

It follows that

$$z^{\alpha-\beta}(t) \geq v(t)\phi^{\alpha}(t), \quad t \geq t_1. \tag{15}$$

If $\alpha > \beta$, using $z'(t) < 0$ in (15), we then find that the positive function $v(t)\phi^{\alpha}(t)$ is bounded.
Now, if $\beta \geq \alpha$, and once again using (12), we obtain

$$z^{\beta}(t) \geq [R^{\frac{1}{\alpha}}(t)(-z'(t))]^{\beta-\alpha+\alpha}\phi^{\beta}(t), \tag{16}$$

which implies that

$$[R^{\frac{1}{\alpha}}(t)(-z'(t))]^{\alpha-\beta} \geq v(t)\phi^{\beta}(t).$$

Since $[R^{\frac{1}{\alpha}}(t)(-z'(t))]^{\alpha-\beta}$ is decreasing, then $v(t)\phi^{\beta}(t)$ is bounded. Therefore, the function $v(t)\phi^{\mu}(t)$ is bounded, where $\mu = \max\{\alpha, \beta\}$.

(ii). In view of the definitions of $v(t)$ and (9), we have

$$v'(t) = \frac{(R(t)(-z'(t))^{\alpha})'}{z^{\beta}(t)} + \frac{\beta R(t)(-z'(t))^{\alpha+1}}{z^{\beta+1}(t)} \geq Q(t) + \frac{\beta}{R^{\frac{1}{\alpha}}(t)} z^{\frac{\beta-\alpha}{\alpha}}(t) v^{\frac{\alpha+1}{\alpha}}(t).$$

If $\alpha > \beta$, and taking into account that $z'(t) < 0$ for $t \geq T$, then $z^{\frac{\beta-\alpha}{\alpha}}(t)$ is increasing. By letting $m_1 = z^{\frac{\beta-\alpha}{\alpha}}(t)$ (if $\beta = \alpha$, then $m_1 = 1$), the above inequality becomes

$$v'(t) \geq Q(t) + \beta m_1 R^{-\frac{1}{\alpha}}(t) v^{\frac{\alpha+1}{\alpha}}(t), \quad t \geq T. \tag{17}$$

Now, if $\beta \geq \alpha$, we have

$$v'(t) \geq Q(t) + \beta R^{-\frac{1}{\beta}}(t)\left(-z'(t)\right)^{\frac{\beta-\alpha}{\beta}} v^{\frac{\beta+1}{\beta}}(t). \tag{18}$$

Since $(R^{-\frac{1}{\alpha}}(t)(-z'(t)))^{\frac{\beta-\alpha}{\beta}}$ is an increasing function, then from (18) we obtain

$$\begin{aligned} v'(t) &\geq Q(t) + \beta R^{-\frac{1}{\alpha}}(t)\left(R^{\frac{1}{\alpha}}(t)(-z'(t))\right)^{\frac{\beta-\alpha}{\beta}} v^{\frac{\beta+1}{\beta}}(t) \\ &\geq Q(t) + \beta m_2 R^{-\frac{1}{\alpha}}(t) v^{\frac{\beta+1}{\beta}}(t), \ \ t \geq T_1 \geq T, \end{aligned} \tag{19}$$

where $m_2 = \left(R^{\frac{1}{\alpha}}(T_1)(-z'(T_1))\right)^{\frac{\beta-\alpha}{\beta}}$ (if $\alpha = \beta$, then $m_2 = 1$).

Combining (17) and (19) yields

$$v'(t) \geq Q(t) + \frac{\beta m}{R^{\frac{1}{\alpha}}(t)} v^{\frac{\mu+1}{\mu}}(t), \quad t \geq T_1, \tag{20}$$

where $\mu = \max\{\alpha, \beta\}$, and $m = \begin{cases} 1, & \alpha = \beta, \\ const > 0, & \alpha \neq \beta. \end{cases}$

The proof is complete. $\square$

Theorem 2. *Assume that $(C_1) - (C_3)$ hold, $\phi(t) < \infty$ and $c(t) < \frac{\phi(t)}{\phi(\tau(t))}$. If there exists a positive non-decreasing function $\rho \in C^1([t_0, \infty), (0, \infty))$, such that*

$$\limsup_{t\to\infty} \int_T^t \left[\rho(s) Q_1(s) - \frac{R(\theta(s))(\rho'(s))^{\nu+1}}{(\nu+1)^{\nu+1}(K\rho(s)\sigma'(s))^{\nu}} \right] ds = \infty \tag{21}$$

and

$$\limsup_{t\to\infty} \int_T^t \left[\phi^{\mu}(s) Q(s) - \frac{L}{\phi(s) R^{\frac{1}{\alpha}}(s)} \right] ds = \infty \tag{22}$$

hold for all sufficiently large $T \geq t_0$, where $K > 0$, $\mu = \max\{\alpha, \beta\}$, $\nu = \min\{\alpha, \beta\}$, and
$\theta(t) = \begin{cases} t, & \alpha > \beta, \\ \sigma(t), & \alpha \leq \beta, \end{cases}$ $L = \begin{cases} (\frac{\mu}{\mu+1})^{\mu+1}(\frac{\mu}{\beta m})^{\mu}, & \alpha \neq \beta, \\ (\frac{\alpha}{\alpha+1})^{\alpha+1}, & \alpha = \beta, \end{cases}$ $m = \begin{cases} 1, & \alpha = \beta, \\ const > 0, & \alpha \neq \beta, \end{cases}$
then Equation (1) is oscillatory.

Proof. Suppose the contrary where Equation (1) has an eventually positive solution $x(t)$, i.e., there exists a $t_1 \geq t_0$, such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Considering the fact that $z(t) \geq x(t) > 0$ for $t \geq t_1$ and (10), we have

$$\left(R(t) |z'(t)|^{\alpha-1} z'(t) \right)' = -E(t) q(t) x^{\beta}(\sigma(t)) \leq 0,$$

which implies that $R(t)|z'(t)|^{\alpha-1}z'(t)$ is non-increasing. Therefore, there exists a $t_2 \geq t_1$, such that either $z'(t) < 0$ or $z'(t) > 0$ for all $t \geq t_2$.

Case I. $z'(t) < 0$ for $t > t_1$. By Lemma 1, we obtain

$$\left(R(t)(-z'(t))^{\alpha} \right)' - Q(t) z^{\beta}(t) \geq 0, \quad t \geq t_1.$$

Let $v(t)$ be defined by (14) for $t \geq t_2 \geq t_1$. It then follows that $v(t) > 0$ for all $t \geq t_2$. From Lemma 2, we get

$$v'(t) \geq Q(t) + m\beta R^{-\frac{1}{\alpha}}(t) v^{\frac{\mu+1}{\mu}}(t), \quad t \geq t_2. \tag{23}$$

Multiplying (23) by $\phi^{\mu}(t)$ and integrating the resulting inequality from $T \geq t_2$ to t, we have

$$\begin{aligned} &\int_T^t \phi^{\mu}(s) Q(s) ds \\ \leq\ &\int_T^t \phi^{\mu-1}(s) R^{-\frac{1}{\alpha}}(s) \left[\mu v(s) - \beta m \phi(s) v^{\frac{\mu+1}{\mu}}(s) \right] ds + \phi^{\mu}(t) v(t). \end{aligned} \tag{24}$$

Using the following inequality ([2], Lemma 2.1) in (24),

$$-Cv^{\frac{\alpha+1}{\alpha}} + Dv \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{D^{\alpha+1}}{C^{\alpha}},\ C > 0, \tag{25}$$

we get

$$\int_T^t \left[\phi^{\mu}(s)Q(s)ds - \frac{L}{\phi(s)R^{\frac{1}{\alpha}}(s)} \right] ds \le \phi^{\mu}(t)v(t),$$

where $L = \begin{cases} (\frac{\mu}{\mu+1})^{\mu+1}(\frac{\mu}{\beta m})^{\mu}, & \alpha \neq \beta, \\ (\frac{\alpha}{\alpha+1})^{\alpha+1}, & \alpha = \beta. \end{cases}$

From Lemma 2 we see that $\phi^{\mu}(t)v(t)$ is bounded. Letting $t \to \infty$ in the above inequality, we obtain a contradiction with (22).

Case II. $z'(t) > 0$ for $t \ge t_1$. Recall that $x(t) = z(t) - c(t)x(\tau(t))$. Hence, $x(t) \ge (1 - c(t))z(t)$. It then follows from (10) that

$$(R(t)(z'(t))^{\alpha})' \le -Q_1(t)z^{\beta}(\sigma(t)), \tag{26}$$

where $Q_1(t)$ is defined by (8).

Define a function $w(t)$ by

$$w(t) := \rho(t)\frac{R(t)(z'(t))^{\alpha}}{z^{\beta}(\sigma(t))},\ \ t \ge t_1. \tag{27}$$

Then, $w(t) > 0$ and

$$w'(t) \le -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta\sigma'(t)\rho(t)R(t)(z'(t))^{\alpha}z'(\sigma(t))}{z^{\beta+1}(\sigma(t))}. \tag{28}$$

For this inequality, we first treat the case $\alpha < \beta$. Note that $R(t)(z'(t))^{\alpha}$ is a positive non-increasing function, then

$$R^{\frac{1}{\alpha}}(t)z'(t) \le R^{\frac{1}{\alpha}}(\sigma(t))z'(\sigma(t)).$$

In view of (28), we get

$$w'(t) \le -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta\sigma'(t)[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}}}{(\rho(t)R(\sigma(t)))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t).$$

Since $z(\sigma(t))$ is an increasing function, thus there exist the constants $K_1 > 0$ and $t_2 \ge t_1$, such that

$$[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} \ge K_1,\quad t \ge t_2. \tag{29}$$

Hence, we obtain

$$w'(t) \le -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha K_1\sigma'(t)}{(\rho(t)R(\sigma(t)))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t). \tag{30}$$

Note that if $\alpha = \beta$, then $K_1 = 1$; thus, (30) still holds.

Now, if $\alpha > \beta$, and because $r'(t) \geq 0$, we have $R'(t) \geq 0$. Recall that $\left(R(t)(z'(t))^{\alpha}\right)' \leq 0$, hence $z''(t) \leq 0$, which implies that $[z'(t)]^{\frac{\beta-\alpha}{\beta}}$ is non-decreasing. Therefore, there exist constants $K_2 > 0,\ t_3 \geq t_2$, such that

$$\left[z'(t)\right]^{\frac{\beta-\alpha}{\beta}} \geq K_2,\ t \geq t_3. \tag{31}$$

By combining (28) and (31), we then have

$$\begin{aligned} w'(t) &\leq -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta\sigma'(t)[z'(t)]^{\frac{\beta-\alpha}{\beta}}}{(\rho(t)R(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t) \\ &\leq -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta K_2\sigma'(t)}{(\rho(t)R(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t), \end{aligned}$$

which, together with (30), implies that

$$w'(t) \leq -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\nu K\sigma'(t)}{(\rho(t)R(\theta(t)))^{\frac{1}{\nu}}}w^{\frac{\nu+1}{\nu}}(t), \quad t \geq t_3, \tag{32}$$

where $\nu = \min\{\alpha,\beta\}, K = \min\{K_1,K_2\},$ and

$$\theta(t) = \begin{cases} t, & \alpha > \beta, \\ \sigma(t), & \alpha \leq \beta. \end{cases}$$

Using (25) in (32), we find that

$$w'(t) \leq -\rho(t)Q_1(t) + \frac{(\rho'(t))^{\nu+1}R(\theta(t))}{(\nu+1)^{\nu+1}(K\rho(t)\sigma'(t))^{\nu}}, t \geq t_3. \tag{33}$$

Integrating this inequality from $T \geq t_3$ to t, we obtain

$$w(t) \leq w(T) - \int_T^t \left[\rho(s)Q_1(s) - \frac{(\rho'(s))^{\nu+1}R(\theta(s))}{(\nu+1)^{\nu+1}(K\rho(s)\sigma'(s))^{\nu}}\right]ds. \tag{34}$$

Letting $t \to \infty$ in the above inequality, we then get a contradiction with (21). The proof is complete. $\square$

Remark 1. *Theorem 2 improves Theorem 2.2 of [2], Theorem 2.2 of [8], Theorem 2.1 of [9], Theorem 2.1 of [10], Theorem 2.5 of [11], Theorem 2.1 of [12], and Theorem 2.2 of [13]. Those articles only considered the special cases of Equation (1) for $\alpha = \beta, p(t) = 0$, or $\alpha = 1, p(t) = 0$.*

The following theorem is the Kneser-type oscillation theorem for Equation (1).

Theorem 3. *Theorem 2 still holds if conditions (21) and (22) are replaced by*

$$\limsup_{t\to\infty} \int_T^t Q_1(s)ds = \infty \tag{35}$$

and

$$\liminf_{t\to\infty} \phi^{\mu+1}(t)R^{\frac{1}{\alpha}}(t)Q(t) > L, \tag{36}$$

respectively.

Proof. Condition (35) follows by substituting $\rho(t) = 1$ into (21). Now, suppose that (36) holds, then for any $\varepsilon > 0$, there exists a sufficiently large $T \geq t_0$, such that

$$\phi^{\mu}(t)Q(t) > \frac{L-\varepsilon}{\phi(t)R^{\frac{1}{\alpha}}(t)}, \quad t \geq T.$$

Integrating this inequality from T to t, we then obtain

$$\begin{aligned} &\int_T^t \left[\phi^{\mu}(s)Q(s) - \frac{L}{\phi(s)R^{\frac{1}{\alpha}}(s)}\right]ds \\ > \quad & -\int_T^t \frac{\varepsilon}{\phi(s)R^{\frac{1}{\alpha}}(s)}ds = -\varepsilon\int_T^t \frac{d\phi(s)}{\phi(s)} = \varepsilon\left(\ln\frac{1}{\phi(t)} - \ln\frac{1}{\phi(T)}\right). \end{aligned}$$

Letting $t \to \infty$ in the above inequality, we find that (22) holds. The proof is complete. $\square$

Remark 2. *If $p(t) = 0$ and $c(t) = 0$, then Equation (1) degenerates to Equation (4). If we set $E(t) = 1$, $R(t) = r(t)$, $\phi(t) = \pi(t)$, $Q_1(t) = Q(t) = q(t)$, $\mu = \alpha = \beta$, and $L = (\frac{\alpha}{\alpha+1})^{\alpha+1}$, then Theorem 3 simply becomes Theorem 1.*

The following two corollaries are for the half-linear neutral differential Equation (2) and the Emden–Fowler neutral Equation (3), respectively.

Corollary 1. *Assume that $\alpha = \beta$ and $p(t) = 0$. Then, Theorem 3 remains true if condition (36) is replaced by*

$$\liminf_{t\to\infty} \pi^{\alpha+1}(t)r^{\frac{1}{\alpha}}(t)q(t)\left(1 - c(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{37}$$

Corollary 2. *Suppose that $\alpha = 1$ and $p(t) = 0$. Then, Theorem 3 still holds if (36) is replaced by any one of the following conditions:*

(i) $\beta > 1, \quad \liminf_{t\to\infty} \pi^{\beta+1}(t)r(t)q(t)\left(1 - c(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\beta} > \left(\frac{\beta}{\beta+1}\right)^{\beta+1}\left(\frac{\beta}{M}\right)^{\beta}$,

(ii) $\beta < 1, \quad \liminf_{t\to\infty} \pi^{2}(t)r(t)q(t)\left(1 - c(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\beta} > \frac{1}{4M}$,

(iii) $\beta = 1, \quad \liminf_{t\to\infty} \pi^{2}(t)r(t)q(t)\left(1 - c(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right) > \frac{1}{4}$,

where M is a positive constant.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 1 ([7], Example 4.2)**.** *Consider the second order Emden–Flower equation*

$$\left(t^{\frac{3}{2}}y'(t)\right)' + y^{\beta}(t) = 0, \quad t \geq 1, \tag{38}$$

where β is a positive constant.

We shall use Corollary 2 to show that Equation (38) is oscillatory. In fact, Equation (38) is a special case of (3), with $c(t) = 0$. Note that $r(t) = t^{\frac{3}{2}}$, $q(t) = 1$, then (35) holds and

$$\pi(t) = \int_t^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)}ds = \int_t^{\infty} s^{-\frac{3}{2}}ds = 2t^{-\frac{1}{2}}, \quad t \geq 1.$$

By Corollary 2, we can deduce that Equation (38) is oscillatory for $0 < \beta < 2$. However, Ref. [7] Theorem 3.1 shows that the solution $y(t)$ of Equation (38) is oscillatory or satisfies $\lim_{t\to\infty} y(t) = 0$ only if $\beta = 1$. Consequently, Corollary 2 improves [7], Theorem 3.1.

The following example illustrates Corollary 1.

Example 2 ([5], Example 1)**.** *Consider the noncanonical Euler differential equation*

$$\left(t^{\alpha+1}\left(y'(t)\right)^{\alpha}\right)' + q_0 y^{\alpha}(\lambda t) = 0, \ \ t \geq 1, \tag{39}$$

where $\alpha > 0,\ q_0 > 0,\ \lambda \in (0,1]$.

Equation (39) is a special case of (2), with $c(t) = 0, q(t) = q_0$. Observing that $r(t) = t^{\alpha+1}$, thus $\pi(t) = \frac{\alpha}{t^{\frac{1}{\alpha}}}$. It follows that condition (35) holds and

$$\liminf_{t\to\infty} \pi^{\alpha+1}(t) r^{\frac{1}{\alpha}}(t) q(t) \left(1 - c(\sigma(t)) \frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha} = \alpha^{\alpha+1} q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Then, by Corollary 1, we can conclude that Equation (39) is oscillatory if $q_0 > \left(\frac{1}{\alpha+1}\right)^{\alpha+1}$. However, due to [5], Theorem 3, one can conclude that Equation (39) is oscillatory if $q_0 > 1$.

Example 3 ([3], Example 2.11)**.** *Consider the half-linear neutral differential equation*

$$\left(t^{\alpha+1}\left[\left(x(t) + p_0 x(\frac{t}{2})\right)'\right]^{\alpha}\right)' + q_0 x^{\alpha}(\lambda t) = 0, \ \ t \geq 1, \tag{40}$$

where $\alpha > 0$ *is a ratio of an odd positive integer,* $q_0 \in (0,\infty),\ p_0 \in [0, \sqrt[\alpha]{\frac{1}{2}}),\ \ \lambda \in (0,1]$.

We see that Equation (40) is a special case of (2), with $c(t) = p_0,\ \ q(t) = q_0$. In this example, $r(t) = t^{\alpha+1}$; hence, $\pi(t) = \frac{\alpha}{t^{\frac{1}{\alpha}}},\ \frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} = \sqrt[\alpha]{2}$ and

$$\begin{aligned} &\liminf_{t\to\infty} \pi^{\alpha+1}(t) r^{\frac{1}{\alpha}}(t) q(t) \left(1 - c(\sigma(t)) \frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha} \\ = \ &\liminf_{t\to\infty} (\frac{\alpha}{t^{1/\alpha}})^{\alpha+1} t^{\frac{\alpha+1}{\alpha}} q_0 \left(1 - \sqrt[\alpha]{2} p_0\right)^{\alpha} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \end{aligned}$$

which shows that (37) holds. By Corollary 1, we conclude that Equation (40) is oscillatory if

$$q_0 \left(1 - \sqrt[\alpha]{2} p_0\right)^{\alpha} > \left(\frac{1}{\alpha+1}\right)^{\alpha+1}. \tag{41}$$

However, by [3], Theorem 2.2, Equation (40) is oscillatory if

$$\alpha^{\alpha} q_0 \left(1 - \sqrt[\alpha]{2} p_0\right) > 1. \tag{42}$$

This restriction is contained in (41).

In [2], the authors considered a special case of Equation (40), with $\alpha = 1$, i.e.,

$$\left(t^2\left(x(t) + p_0 x(\frac{t}{2})\right)'\right)' + q_0 x(\lambda t) = 0. \tag{43}$$

By [2], Example 3.1, Equation (43) is oscillatory if

$$q_0(1-2p_0) > \frac{1}{4}, \tag{44}$$

which is just a special case of (41) when $\alpha = 1$.

Example 4 ([8], Example 3.2)**.** *Consider the Emden–Fowler neutral differential equation*

$$\left(e^t\left(x(t)+\frac{1}{2}\left(t-\frac{\pi}{4}\right)\right)'\right)' + \lambda e^{(\beta+1)t}x^\beta\left(t-\frac{\pi}{2}\right) = 0, \tag{45}$$

where β is a ratio of an odd positive integer and $\lambda > 0$.

In this example, $r(t) = e^t$, $c(t) = \frac{1}{2}$, $q(t) = \lambda e^{(\beta+1)t}$, $\tau(t) = t-\frac{\pi}{4}$, $\sigma(t) = t-\frac{\pi}{2}$. It is easy to see that $\pi(t) = e^{-t}$, $\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} = e^{\frac{\pi}{4}}$, and conditions (i), (ii), and (iii) of Corollary 2 are satisfied. Thus, Equation (45) is oscillatory if $\beta > 0$. However, by [8] Theorem 2.2, one can deduce that Equation (45) is oscillatory if $\beta > 1$.

Example 5 ([10], Example 2.3)**.** *Consider the Emden–Fowler neutral equation*

$$\left(t^2\left(x(t)+\frac{1}{2}x(t-1)\right)'\right)' + t^4x^\beta\left(\frac{t}{2}\right) = 0, \quad t \geq 1, \tag{46}$$

where β is a ratio of an odd positive integer.

Taking into account that $r(t) = t^2$, $c(t) = \frac{1}{2}$, $q(t) = t^4$, $\tau(t) = t-1$, $\sigma(t) = \frac{t}{2}$, then

$$\pi(t) = \frac{1}{t}, \quad \pi(\sigma(t)) = \frac{2}{t}, \quad \pi(\tau(\sigma(t))) = \frac{2}{t-2}.$$

If $0 < \beta \leq 1$, then we have

$$\begin{aligned}
&\liminf_{t\to\infty} \pi^2(t)r(t)q(t)\left(1-c(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^\beta \\
=\ &\liminf_{t\to\infty}\left(\frac{1}{t}\right)^2 t^2t^4\left(1-\frac{1}{2}\frac{t}{t-2}\right)^\beta = \infty.
\end{aligned}$$

This shows that Corollary 2-(ii), (iii) are satisfied.

Now, for $\beta > 1$, by Corollary 2-(i), we can check that Equation (46) is oscillatory if $1 < \beta < 5$. Therefore, Equation (46) is oscillatory if $0 < \beta < 5$. However, by [10] Theorem 2.2, Equation (46) is oscillatory only if $0 < \beta \leq 1$.

Example 6. *Consider the following damped nonlinear differential equation of a neutral type*

$$\left(|z'(t)|^{\alpha-1}z'(t)\right)' + \frac{2\alpha}{t}|z'(t)|^{\alpha-1}z'(t) + t^{\beta-\alpha-1}|x(t-2)|^{\beta-1}x(t-2) = 0, \tag{47}$$

where $z(t) = x(t) + \frac{1}{2}x(t-1)$.

We claim that this equation satisfies the conditions of Theorem 3. In this equation, $r(t) = 1$, $c(t) = \frac{1}{2}$, $q(t) = t^{\beta-\alpha-1}$, $\tau(t) = t-1$, $\sigma(t) = t-2$. Let $t_0 = 1$, then we have $E(t) = t^{2\alpha}$, $R(t) = E(t)r(t) = t^{2\alpha}$, $\phi(t) = \int_t^\infty R^{-\frac{1}{\alpha}}(s)ds = \frac{1}{t}$, and

$$Q_1(t) = E(t)q(t)(1-c(\sigma(t)))^\beta = \left(\frac{1}{2}\right)^\beta t^{\alpha+\beta-1},$$

which implies that (35) holds. Note that

$$Q(t)=E(t)q(t)\left(1-c(\sigma(t))\frac{\phi(\tau(\sigma(t)))}{\phi(\sigma(t))}\right)^{\beta}=t^{\alpha+\beta-1}\left(1-\frac{t-2}{2(t-3)}\right)^{\beta}.$$

We obtain

$$\begin{aligned}&\liminf_{t\to\infty}\phi^{\mu+1}(t)R^{\frac{1}{\alpha}}(t)Q(t)\\ =\;&\liminf_{t\to\infty}t^{-\mu-1}t^{2}t^{\beta+\alpha-1}\left(1-\frac{t-2}{2(t-3)}\right)^{\beta}\\ =\;&\liminf_{t\to\infty}t^{\beta+\alpha-\mu}\left(\frac{1}{2}\right)^{\beta}=\liminf_{t\to\infty}t^{\nu}\left(\frac{1}{2}\right)^{\beta}=\infty,\end{aligned}$$

where $\mu=\max\{\alpha,\beta\},\ \ \nu=\min\{\alpha,\beta\}$. Hence, (36) is satisfied. Therefore, by Theorem 3, Equation (47) is oscillatory.

4. Conclusions

Theorem 2 (or Theorem 3) gives a new oscillation criterion for Equation (1) and improves those oscillation criteria reported in the literature. It can be applied to deal with the half-linear neutral equations, the noncanonical Euler equations, the damped nonlinear neutral equations, and the Emden–Fowler neutral equations. Moreover, the conditions of the oscillation criteria given by Corollarys 1 and 2 are simpler and only require the identification of limits instead of integrals.

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