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Existence and Asymptotic Behaviors of Ground States for a Fourth-Order Nonlinear Schrödinger Equations with a Potential

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Abstract: In this paper, we study the existence and asymptotic behaviors of ground state solutions to a fourth-order nonlinear Schrödinger equation with mass-critical exponent, where the fourth-order term appears as a perturbation with $\varepsilon > 0$. By considering a constrained variational problem, we first establish the existence of ground state solutions. Then, we prove the asymptotic behaviors of the solutions as $\varepsilon \to 0^+$. The main ingredients of the proofs are some energy estimate arguments. Our results improve somewhat the ones in the existing reference.

Keywords: fourth-order equation; ground states; asymptotic behaviors; energy estimates

MSC: 35J20; 35J35; 35J60

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1. Introduction

We consider the following bi-harmonic nonlinear Schrödinger equation

$$i\Psi_t = \varepsilon\Delta^2\Psi - \Delta\Psi + V(x)\Psi - b|\Psi|^{\frac{4}{N}}\Psi, \tag{1}$$

where $i = \sqrt{-1}$ and $\varepsilon > 0, b > 0$. The unknown $\Psi = \Psi(x,t) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{C}$ is a complex-valued wave function, and $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a given potential satisfying some conditions given below. $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplacian operator in $\mathbb{R}^N$, and Δ^2 is the bi-harmonic operator, hence , Equation (1) is often referred to as the bi-harmonic nonlinear Schrödinger equation (denoted by "BNLS" for short).

The BNLS type Equation (1) was first introduced by Karpman and Shagalov in [1,2], where it took into account the role of a small fourth-order dispersion term in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity.

In this paper, we are concerned with standing waves solutions of (1), namely solutions are of the form $\Psi(x,t) := e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ denotes its frequency. Thus, the function $u(x)$ solves the following elliptic equation

$$\varepsilon\Delta^2 u - \Delta u + V(x)u = \lambda u + b|u|^{\frac{4}{N}}u,\ x \in \mathbb{R}^N. \tag{2}$$

Recall that when $\varepsilon = 0, V(x) = 0$, and the exponent $\frac{4}{N}$ is replaced by $p \in (0, 2^* - 2)$, then (1) becomes the classical nonlinear Schrödinger equation:

$$i\Psi_t = -\Delta\Psi - b|\Psi|^p\Psi,\ (x,t) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{3}$$

It is well known (see [3]) that when $0 < p < \frac{4}{N}$, Equation (3) has orbitally stable standing wave solutions, while when $p \geq \frac{4}{N}$, standing waves are unstable. Namely, the exponent $\frac{4}{N}$ appears as a mass-critical value. However, if the perturbation "$\varepsilon\Delta^2 u$" is involved, then the new mass-critical value is doubled, becoming $\frac{8}{N}$. Thus, the exponent $\frac{4}{N}$ in (1) becomes mass-subcritical, which leads to the existence of stable standing wave solutions, see e.g., [4–8]. Hence, it is an interesting issue to consider the behavior of standing waves solutions as $\varepsilon \to 0^+$. The aim of this paper is twofold: when $V(x) \neq 0$, on one hand, we establish the existence of ground state solutions of (2), then we prove the asymptotic behaviors of the solutions as $\varepsilon \to 0^+$.

To find ground state solutions of (2), we consider the following minimization problem

$$e_b(\varepsilon) := \inf_{u \in B_1} E_b^\varepsilon(u), \tag{4}$$

where

$$E_b^\varepsilon(u) = \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}\Big(|\nabla u|^2 + V(x)|u(x)|^2\Big)dx - \frac{bN}{2N+4}\int_{\mathbb{R}^N}|u|^{2+\frac{4}{N}}dx, \tag{5}$$

and

$$B_1 = \Big\{u \in \mathcal{H} : \int_{\mathbb{R}^N}|u|^2 dx = 1\Big\}.$$

The space $\mathcal{H}$ is defined by

$$\mathcal{H} := \Big\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx < \infty\Big\},$$

with the associated norm $\|u\|_{\mathcal{H}} = \Big(\int_{\mathbb{R}^N}[|\Delta u|^2 + (1+V(x))|u|^2]dx\Big)^{\frac{1}{2}}$.

By standard arguments (see e.g., [4,7,9–11]), one can prove that a minimizer of $e_b(\varepsilon)$ is a ground state solution of (2), where, however, the frequency λ is not fixed anymore, appearing as a Lagrange multiplier. Similar analysis approaches can be referred to in [12].

Before starting to state our result, let us recall some studies in the case where $\varepsilon = 0$, and $V(x)$ satisfies the following conditions:

$$V(x) \in C^\infty_{loc}(\mathbb{R}^N), \ \inf_{x\in\mathbb{R}^N} V(x) = 0, \text{ and } \lim_{|x|\to\infty} V(x) = \infty. \tag{6}$$

Let $Q(x)$ be the unique ground state solution of the Schrödinger Equation (14), whose properties are given in Lemma 2 and Remark 4, and set $b^* = \|Q\|_{L^2}^{\frac{4}{N}}$. It has been proved in [13] that

Lemma 1 ([13], Theorem 1.1)**.** *Assume that $\varepsilon = 0$ and $V(x)$ satisfies (6), then*

(1) if $0 < b < b^*$, then there exists at least one non-negative minimizer of $e_b(0)$;
(2) if $b \geq b^*$, then there is no minimizers of $e_b(0)$.

Moreover, $e_b(0) > 0$ for all $0 < b < b^*$, $e_b(0) = 0$ for $b = b^*$, and $e_b(0) = -\infty$ for $b > b^*$.

Remark 1. *In [13], the authors also give the asymptotic behaviors of minimizers of $e_b(0)$ as $b \nearrow b^*$, precisely, if $b \nearrow b^*$, then minimizers of $e_b(0)$ concentrate at the minimum point of $V(x)$. We remark that part of such important works also have been established by Bao and Cai [9], Maeda [14], and Zhang [15].*

Our first result concerned with the existence reads as the following.

Theorem 1 (Existence)**.** *Assume that $\varepsilon > 0$, and $V(x)$ satisfies (6). Then, for all $b > 0$, $e_b(\varepsilon)$ admits at least one non-negative minimizer.*

Remark 2. *Note that when $\varepsilon > 0$, $V(x) \equiv 0$, sharp conditions for the existence/non-existence of minimizers for $e_b(\varepsilon)$ was established by [7,8].*

We observe that when $b = b^*$ and $\varepsilon = 0$, by Lemma 1 $e_b(0)$ has no minimizers, whereas Theorem 1 shows us that, for all $\varepsilon > 0$, minimizers of $e_b(\varepsilon)$ do exist for $b = b^*$. Hence, in our second theorem, we give the asymptotic behavior of minimizers as $\varepsilon \to 0^+$.

Theorem 2. *Assume that $\varepsilon > 0$, $b = b^*$, and $V(x)$ satisfies (6). Let $u_\varepsilon(x)$ be a non-negative minimizer of $e_b(\varepsilon)$ and x_ε be a maximum point of u_ε. Then,*

$$\lim_{\varepsilon \to 0^+} x_\varepsilon = x_0, \; with \; x_0 \in \mathbb{R}^N \; satisfying \; V(x_0) = 0. \tag{7}$$

Moreover, as $\varepsilon \to 0^+$,

$$w_\varepsilon := t_\varepsilon^{\frac{N}{2}} u_\varepsilon(t_\varepsilon x + x_\varepsilon) \to \frac{Q(x)}{\|Q\|_{L^2}} \; in \; H^1(\mathbb{R}^N), \tag{8}$$

where

$$t_\varepsilon := \left(\frac{Nb^*}{N+2} \int_{\mathbb{R}^N} |u_\varepsilon|^{2+\frac{4}{N}} dx \right)^{-\frac{1}{2}}, \tag{9}$$

and

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx = +\infty, \quad \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} |u_\varepsilon|^{2+\frac{4}{N}} dx = +\infty. \tag{10}$$

Remark 3. *Note that in [5] (Theorem 1.2), the authors also considered the behaviors of solutions of (2) with $V(x) \equiv 0$ as $\varepsilon \to 0^+$, however, our theorem gives more precise asymptotic behaviors. In addition, the blow up property (10) explains the reason for non-existence of solutions in the case $\varepsilon = 0$ and $b = b^*$.*

Finally, let us consider the special potential $V(x) = |x|^2$, namely the so-called harmonic potential, which has a wide usage on the model related to the Bose–Einstein condensates, see, e.g., [9,13,15]. From the technical point of view, $V(x) = |x|^2$ has a unique zero point $x_0 = 0$. In the following theorem, we give the precise energy estimate.

Theorem 3. *Let $V(x) := |x|^2, x \in \mathbb{R}^N$, and $b = b^*$, then as $\varepsilon \to 0^+$,*

$$t_\varepsilon = (1 + o_\varepsilon(1)) \Big(\frac{8\|Q\|_{L^2}^2 \varepsilon \mu_1}{3\mu_2^2} \Big)^{\frac{1}{8}}, \tag{11}$$

and

$$e_{b^*}(\varepsilon) = \frac{3(1 + o_\varepsilon(1))}{2\|Q\|_{L^2}^2} (\varepsilon \mu_1)^{\frac{1}{3}} (\frac{\mu_2}{2})^{\frac{2}{3}}, \tag{12}$$

where

$$\mu_1 := \int_{\mathbb{R}^N} |\Delta Q|^2 dx, \; and \; \mu_2 := \int_{\mathbb{R}^N} |x|^2 |Q(x)|^2 dx. \tag{13}$$

The paper is organized as follows. In Section 2, we show the existence of non-negative minimizers of $e_b(\varepsilon)$, proving Theorem 1. Section 3 is devoted to prove Theorem 2 on the asymptotic behavior of minimizers for $e_b(\varepsilon)$ as $\varepsilon \to 0^+$. Section 4 is to give the proof of Theorem 3 on the energy estimate. Finally, we give a general conclusion of this paper in Section 5.

2. Existence of Minimizers

Lemma 2 ([16,17])**.** *The following nonlinear scalar field equation*

$$-\Delta u + \frac{2}{N} u - |u|^{\frac{4}{N}} u = 0, \; u \in H^1(\mathbb{R}^N), \tag{14}$$

has a unique positive, radially symmetric solution $Q(x)$, which is a non-increasing function of $|x|$. Moreover, the optimal Gagliardo–Nirenberg inequality holds:

$$\int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}} dx \leq \frac{N+2}{N} \left(\int_{\mathbb{R}^N} |Q|^2 dx \right)^{-\frac{2}{N}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{2}{N}}. \tag{15}$$

Remark 4. *By [17], the equality in (15) is achieved by $u(x) = Q(x)$, namely,*

$$\int_{\mathbb{R}^N} |Q|^{2+\frac{4}{N}} dx = \frac{N+2}{N} \int_{\mathbb{R}^N} |\nabla Q|^2 dx. \tag{16}$$

In addition, recall from [18] (Proposition 4.1), we know that

$$Q(x),\ |\nabla Q(x)| = O(|x|^{-\frac{1}{2}} e^{-|x|}),\ as\ |x| \to \infty. \tag{17}$$

In particular, by the classical elliptic regularity theory, $Q(x) \in H^2(\mathbb{R}^N)$, see e.g., [19] (Theorem 8.1.1).

To show the existence result, we need to use the following compactness of embedding.

Lemma 3. *Let $V(x)$ satisfy (6), then the embedding $\mathcal{H} \hookrightarrow L^{q+1}(\mathbb{R}^N)$ is compact, for any $1 \leq q < \frac{2N}{(N-4)^+} - 1$, where $\frac{2N}{(N-4)^+} = \frac{2N}{N-4}$ if $N \geq 5$, and $\frac{2N}{(N-4)^+} = +\infty$ if $1 \leq N \leq 4$.*

Remark 5. *The proof of Lemma 3 basically is the same as the one of [15] (Lemma 5.1) or [9] (Lemma 2.1). Here we omit the details.*

The proof of Theorem 1. For any $u \in B_1$, using the Gagliardo–Nirenberg inequality (15), we have

$$\begin{aligned} E_b^\varepsilon(u) &= \frac{\varepsilon}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)|u(x)|^2 \right) dx - \frac{bN}{2N+4} \int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}} dx \\ &\geq \frac{\varepsilon}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2} \left(1 - \frac{b}{b^*} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx. \end{aligned} \tag{18}$$

Observe that when $0 < b \leq b^*$, (18) shows that $E_b^\varepsilon(u) \geq 0, \forall\, u \in B_1$. When $b > b^*$, then by the inequality $\|\nabla u\|_{L^2}^2 \leq \|\Delta u\|_{L^2} \|u\|_{L^2}$, we use (18) to derive that

$$E_b^\varepsilon(u) \geq \frac{\varepsilon}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2} \left(1 - \frac{b}{b^*} \right) \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{\frac{1}{2}} \geq \min_{t>0} h(t), \tag{19}$$

where $h(t) := \frac{\varepsilon}{2} t^2 + \frac{1}{2}\left(1 - \frac{b}{b^*}\right) t$. Hence, for all $b > 0$, $E_b^\varepsilon(u)$ is bounded from below on B_1. Thus $e_b(\varepsilon) > -\infty, \forall\, b > 0$.

Now we show that $e_b(\varepsilon)$ is reached. Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence of $e_b(\varepsilon)$, satisfying

$$\|u_n\|_{L^2}^2 = 1, \text{ and } \lim_{n \to \infty} E_b^\varepsilon(u_n) = e_b(\varepsilon).$$

From (18) and (19), one may easily observe that $\{u_n\}_{n=1}^\infty$ is bounded in $\mathcal{H}$. Thus by Lemma 3, up to a subsequence, there exists $u \in \mathcal{H}$, such that

$$u_n \rightharpoonup u \text{ weakly in } \mathcal{H},\ u_n \to u \text{ strongly in } L^q(\mathbb{R}^N), \forall q \in [2, \frac{2N}{(N-4)^+}).$$

Then, we deduce that

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \leq \varliminf_{n \to \infty} \int_{\mathbb{R}^N} |\Delta u_n|^2 dx, \tag{20}$$

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx, \tag{21}$$

$$\int_{\mathbb{R}^N} V(x)|u(x)|^2 dx \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n(x)|^2 dx, \tag{22}$$

$$\int_{\mathbb{R}^N} |u|^q dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q dx,\ q = 2, 2 + \frac{4}{N}. \tag{23}$$

It follows from (20)–(23) that

$$e_b(\varepsilon) \leq E_b^\varepsilon(u) \leq \varliminf_{n\to\infty} E_b^\varepsilon(u_n) = e_b(\varepsilon),$$

thus we have

$$E_b^\varepsilon(u) = e_b(\varepsilon),\ u \in B_1.$$

Namely, u is a minimizer of $e_b(\varepsilon)$. Moreover, using [20] (Theorem 6.17), we have,

$$|\nabla|u(x)|| \leq |\nabla u(x)| \text{ and } |\Delta|u(x)|| \leq |\Delta u(x)| \text{ a.e. in } \mathbb{R}^N.$$

Then,

$$e_b(\varepsilon) \leq E_b^\varepsilon(|u|) \leq E_b^\varepsilon(u) = e_b(\varepsilon),\ u \in B_1.$$

Therefore, if u is a minimizer of $e_b(\varepsilon)$, then $|u|$ also is a minimizer of $e_b(\varepsilon)$. This shows that for all $b > 0$, $e_b(\varepsilon)$ admits at least one non-negative minimizer. □

3. Asymptotic Behaviors of Minimizers

In this section, we investigate the asymptotic behaviors of minimizers as $\varepsilon \to 0^+$ and $b = b^*$. To begin with, we first estimate the energy of $e_{b^*}(\varepsilon)$ as $\varepsilon \to 0^+$ in the following lemma, where we shall use some arguments from [21,22], whose basic ideas stem from [13,23].

Lemma 4.

$$\lim_{\varepsilon\to 0^+} e_{b^*}(\varepsilon) = e_{b^*}(0) = 0. \tag{24}$$

Proof. First, by Lemma 1 we have,

$$\varliminf_{\varepsilon\to 0^+} e_{b^*}(\varepsilon) \geq e_{b^*}(0) = 0. \tag{25}$$

On the other hand, choose a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi(x) \equiv 1 \text{ if} |x| < 1, \text{ and } \varphi(x) \equiv 0 \text{ if} |x| > 2. \tag{26}$$

Let $x_0 \in \mathbb{R}^N$ be such that $V(x_0) = 0$. For $\tau > 0, R > 0$, denote

$$u_\tau = \frac{A_{R,\tau}}{\|Q\|_{L^2}} \tau^{\frac{N}{2}} \varphi(\frac{x - x_0}{R}) Q(\tau|x - x_0|), \tag{27}$$

where $A_{R,\tau} > 0$ is chosen so that $u_\tau \in B_1$. By scaling, $A_{R,\tau}$, depends only on the product $R\tau$, and we have

$$1 \leq A_{R,\tau}^2 \leq 1 + Ce^{-R\tau},$$

and also we have that, as $\tau \to +\infty$,

$$\int_{\mathbb{R}^N} |\Delta u_\tau|^2 dx = \frac{\tau^4}{\|Q\|_{L^2}^2} \int_{\mathbb{R}^N} |\Delta Q(x)|^2 dx + O(e^{-R\tau}), \tag{28}$$

$$\int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx = \frac{\tau^2}{\|Q\|_{L^2}^2} \int_{\mathbb{R}^N} |\nabla Q(x)|^2 dx + O(e^{-R\tau}), \tag{29}$$

$$\int_{\mathbb{R}^N} V(x)|u_\tau(x)|^2dx = \frac{A^2_{R,\tau}}{\|Q\|^2_{L^2}} \int_{\mathbb{R}^N} V(\frac{x}{\tau}+x_0)\varphi(\frac{x}{\tau})|Q(x)|^2dx \to 0, \tag{30}$$

$$\int_{\mathbb{R}^N} |u_\tau|^{2+\frac{4}{N}}dx = \frac{N+2}{N}\frac{\tau^2}{\|Q\|^{2+\frac{4}{N}}_{L^2}} \int_{\mathbb{R}^N} |\nabla Q|^2dx + O(e^{-R\tau}). \tag{31}$$

It follows from (28)–(31) that

$$E^\varepsilon_{b^*}(u_\tau) = \frac{\varepsilon\tau^4}{2\|Q\|^2_{L^2}} \int_{\mathbb{R}^N} |\Delta Q(x)|^2dx + O(e^{-R\tau}).$$

Setting $\tau = \varepsilon^{-\frac{1}{8}}$, we have

$$\overline{\lim_{\varepsilon\to0^+}}\, e_{b^*}(\varepsilon) \le \overline{\lim_{\varepsilon\to0^+}}\, E^\varepsilon_{b^*}(u_\tau) = 0.$$

This, together with (25), implies (24). □

The Proof of Theorem 2. Let u_ε be a non-negative minimizer of $e_{b^*}(\varepsilon)$, then standardly, there exists a Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}$, such that $(u_\varepsilon, \lambda_\varepsilon)$ solves weakly

$$\varepsilon\Delta^2 u_\varepsilon - \Delta u_\varepsilon + V(x)u_\varepsilon = \lambda_\varepsilon u_\varepsilon + b^*|u_\varepsilon|^{\frac{4}{N}}u_\varepsilon,\ x \in \mathbb{R}^N. \tag{32}$$

Using (18) and Lemma 4, we have

$$0 \le E^0_{b^*}(u_\varepsilon) \le E^\varepsilon_{b^*}(u_\varepsilon) = e_{b^*}(\varepsilon) \to e_{b^*}(0) = 0,\ \text{as } \varepsilon \to 0^+, \tag{33}$$

and then,

$$\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2dx \to 0,\ \text{as } \varepsilon \to 0^+. \tag{34}$$

Now, we claim that

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2dx = +\infty. \tag{35}$$

In fact, if we argue by contradiction to assume that $\{\|\nabla u_\varepsilon\|_{L^2}\}$ is bounded, then applying the compact embedding in [13] (Lemma 2.1) (similar to Lemma 3), up to a subsequence, there exists $u_0 \in \mathcal{H}$ such that

$$u_\varepsilon \to u_0 \text{ strongly in } L^q(\mathbb{R}^N), \forall q \in [2, \frac{2N}{(N-4)^+}),$$

as $\varepsilon \to 0^+$. Thus, $u_0 \in B_1$, and by (34),

$$0 \le E^0_{b^*}(u_0) \le E^0_{b^*}(u_\varepsilon) \le E^\varepsilon_{b^*}(u_\varepsilon) \to e_{b^*}(0) = 0,$$

this implies that u_0 is a minimizer of $e_{b^*}(0)$, which is contradicts Lemma 1 (2). Then (35) follows.

Using (35) and Lemma 4, we conclude that

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^N} |u_\varepsilon|^{2+\frac{4}{N}}dx = +\infty. \tag{36}$$

Set $t_\varepsilon := \left(\frac{2b^*}{N+2}\int_{\mathbb{R}^N} |u_\varepsilon|^{2+\frac{4}{N}}dx\right)^{-\frac{1}{2}}$, then $\lim_{\varepsilon\to0^+} t_\varepsilon = 0$. Moreover, multiplying (32) by u_ε and integrating by part, we have

$$\lambda_\varepsilon = \varepsilon\int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2dx + \int_{\mathbb{R}^N} \left(|\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon(x)|^2\right)dx - b^*\int_{\mathbb{R}^N} |u_\varepsilon|^{2+\frac{4}{N}}dx, \tag{37}$$

and then,

$$t_\varepsilon^2\lambda_\varepsilon = t_\varepsilon^2\Big(2e_{b^*}(\varepsilon) - \frac{2b^*}{N+2}\int_{\mathbb{R}^N}|u_\varepsilon|^{2+\frac{4}{N}}dx\Big) \to -1, \text{ as } \varepsilon \to 0^+.$$

Now, denote $\overline{w}_\varepsilon(x) := t_\varepsilon^{\frac{N}{2}} u_\varepsilon(t_\varepsilon x)$, or equivalently $u_\varepsilon(x) = t_\varepsilon^{-\frac{N}{2}}\overline{w}_\varepsilon(t_\varepsilon^{-1}x)$, then $\overline{w}_\varepsilon \in B_1$, and

$$t_\varepsilon^2 E_{b^*}^\varepsilon(u_\varepsilon) = \frac{\varepsilon t_\varepsilon^{-2}}{2}\int_{\mathbb{R}^N}|\Delta\overline{w}_\varepsilon|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\Big(|\nabla\overline{w}_\varepsilon|^2 + V(t_\varepsilon x)|\overline{w}_\varepsilon|^2\Big)dx - \frac{b^*N}{2N+4}\int_{\mathbb{R}^N}|\overline{w}_\varepsilon|^{2+\frac{4}{N}}dx.$$

Note that $t_\varepsilon^2 E_{b^*}^\varepsilon(u_\varepsilon) = t_\varepsilon^2 e_{b^*}(\varepsilon) \to 0$, then by (34), we have

$$\lim_{\varepsilon\to 0^+}\Big(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\overline{w}_\varepsilon|^2dx - \frac{b^*N}{2N+4}\int_{\mathbb{R}^N}|\overline{w}_\varepsilon|^{2+\frac{4}{N}}dx\Big) = 0, \tag{38}$$

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon t_\varepsilon^{-2}}{2}\int_{\mathbb{R}^N}|\Delta\overline{w}_\varepsilon|^2dx = 0, \tag{39}$$

and

$$\int_{\mathbb{R}^N}|\overline{w}_\varepsilon|^{2+\frac{4}{N}}dx = \frac{N+2}{2b^*}, \ \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}|\nabla\overline{w}_\varepsilon|^2dx = \frac{N}{2}. \tag{40}$$

Then using the same arguments as the proof of [22] (Lemma 2.5), which is basically the concentration compactness argument, we deduce that there exists $\overline{x}_\varepsilon \in \mathbb{R}^N$, such that

$$\widetilde{w}_\varepsilon(x) := \overline{w}_\varepsilon(x + \overline{x}_\varepsilon) \to \frac{Q(x)}{\|Q\|_{L^2}^2}, \text{ in } H^1(\mathbb{R}^N), \text{ as } \varepsilon \to 0^+, \tag{41}$$

where Q is introduced in Lemma 2. Furthermore, using (34), we have

$$\int_{\mathbb{R}^N} V(t_\varepsilon x + \overline{x}_\varepsilon)|\widetilde{w}_\varepsilon|^2dx \to 0, \text{ as } \varepsilon \to 0^+.$$

Thus, up to a subsequence, there exists $x_0 \in \mathbb{R}^N$, such that

$$V(x_0) = 0, \text{ and } \lim_{\varepsilon\to 0^+}\overline{x}_\varepsilon = x_0. \tag{42}$$

In addition, by (40) and the inequality $\|\nabla\overline{w}_\varepsilon\|_{L^2}^2 \le \|\Delta\overline{w}_\varepsilon\|_{L^2}\|\overline{w}_\varepsilon\|_{L^2}$, we know that $\|\Delta\overline{w}_\varepsilon\|_{L^2} \nrightarrow 0$, which, together with (39), implies that

$$\lim_{\varepsilon\to 0^+}\varepsilon t_\varepsilon^{-2} = 0. \tag{43}$$

From (32), we can check that $\widetilde{w}_\varepsilon$ satisfies

$$\varepsilon t_\varepsilon^{-2}\Delta^2\widetilde{w}_\varepsilon - \Delta\widetilde{w}_\varepsilon + t_\varepsilon^2 V(t_\varepsilon x + \overline{x}_\varepsilon)\widetilde{w}_\varepsilon = \lambda_\varepsilon t_\varepsilon^2\widetilde{w}_\varepsilon + b^*|\widetilde{w}_\varepsilon|^{\frac{4}{N}}\widetilde{w}_\varepsilon. \tag{44}$$

Applying the exponential decay result due to [4] (Theorem 3.10 or Remark 3.11), we have

$$\widetilde{w}_\varepsilon \le Ce^{-\beta|x|}, \text{ for some } \beta > 0, \text{ as } |x| > 0 \text{ large enough.} \tag{45}$$

Let x_ε be a global maximum point of $u_\varepsilon(x)$, then clearly $\widetilde{w}_\varepsilon$ attains its maximum at $x = \frac{x_\varepsilon - \overline{x}_\varepsilon}{t_\varepsilon}$. Thanks to (45), we know that

$$\overline{\lim_{\varepsilon\to 0^+}}\frac{|x_\varepsilon - \overline{x}_\varepsilon|}{t_\varepsilon} < \infty.$$

Set

$$w_\varepsilon(x) := \widetilde{w}_\varepsilon\Big(x + \frac{x_\varepsilon - \overline{x}_\varepsilon}{t_\varepsilon}\Big) = t_\varepsilon^{\frac{N}{2}}u_\varepsilon(t_\varepsilon x + x_\varepsilon),$$

then $w_\varepsilon(x)$ attains its maximum at $x=0$. Therefore, from (41) we conclude that

$$\lim_{\varepsilon\to 0^+} w_\varepsilon(x) \to \frac{Q(x)}{\|Q\|_{L^2}}, \text{ in } H^1(\mathbb{R}^N).$$

Then the proof of Theorem 2 is completed. □

4. Proof of Theorem 3

In this section, we particularly treat the special case $V(x)=|x|^2$.

The Proof of Theorem 3. Note that when $V(x)=|x|^2$, there exists a unique $x_0=0$, such that $V(x_0)=0$. To prove this theorem, we start with the upper bound estimate of the energy $e_{b^*}(\varepsilon)$ as $\varepsilon\to 0^+$. Let u_τ be given by (27), then by (30),

$$\int_{\mathbb{R}^N} V(x)|u_\tau(x)|^2dx = \frac{A_{R,\tau}^2}{\|Q\|_{L^2}^2}\int_{\mathbb{R}^N} V(\frac{x}{\tau})\varphi(\frac{x}{\tau})|Q(x)|^2dx \le \frac{1+Ce^{-R,\tau}}{\tau^2\|Q\|_{L^2}^2}\int_{\mathbb{R}^N}|x|^2|Q(x)|^2dx.$$

Thus, using (28), (29), and (31), we obtain that as $\tau\to+\infty$,

$$E_{b^*}^\varepsilon(u_\tau) \le \frac{\varepsilon\tau^4}{2\|Q\|_{L^2}^2}\int_{\mathbb{R}^N}|\Delta Q|^2dx + \frac{1+Ce^{-R,\tau}}{2\tau^2\|Q\|_{L^2}^2}\int_{\mathbb{R}^N}|x|^2|Q(x)|^2dx + O(e^{-R\tau}). \tag{46}$$

Take $\tau=\left(\frac{\mu_2}{2\varepsilon\mu_1}\right)^{\frac{1}{6}}$ in (46), then we have, as $\varepsilon\to 0^+$,

$$e_{b^*}(\varepsilon) \le \frac{3(1+o_\varepsilon(1))}{2\|Q\|_{L^2}^2}(\varepsilon\mu_1)^{\frac{1}{3}}(\frac{\mu_2}{2})^{\frac{2}{3}}. \tag{47}$$

On the other hand, let u_ε be a non-negative minimizer of $e_{b^*}(\varepsilon)$ and $w_\varepsilon(x)$ be given by (8). Then from (8), we know that $w_\varepsilon(x)\to\frac{1}{\|Q\|_{L^2}}Q(x)$ *a.e.* in $\mathbb{R}^N$. Thus by the weak semi-continuity, we have that

$$\lim_{\varepsilon\to 0^+} t_\varepsilon^4\int_{\mathbb{R}^N}|\Delta u_\varepsilon|^2dx = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}|\Delta w_\varepsilon|^2dx \ge \frac{1}{\|Q\|_{L^2}^2}\int_{\mathbb{R}^N}|\Delta Q|^2dx. \tag{48}$$

Moreover, by direct calculation we have,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{1}{t_\varepsilon^2}\int_{\mathbb{R}^N}V(x)|u_\varepsilon(x)|^2dx &= \lim_{\varepsilon\to 0^+}\frac{1}{t_\varepsilon^2}\int_{\mathbb{R}^N}V(t_\varepsilon x+x_\varepsilon)|w_\varepsilon(x)|^2dx\\ &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}|x+\frac{x_\varepsilon}{t_\varepsilon}|^2|w_\varepsilon(x)|^2dx.\end{aligned} \tag{49}$$

We claim that $\{\frac{x_\varepsilon}{t_\varepsilon}\}$ is bounded. Indeed, if not, then from (49) we obtain that, as any $M>0$ large enough,

$$\lim_{\varepsilon\to 0^+}\frac{1}{t_\varepsilon^2}\int_{\mathbb{R}^N}V(x)|u_\varepsilon(x)|^2dx \ge M.$$

Thus, using (48) and the Young inequality, we derive that, as $\varepsilon\to 0^+$,

$$e_{b^*}(\varepsilon) \ge \frac{\varepsilon\mu_1}{2t_\varepsilon^4\|Q\|_{L^2}^2} + \frac{t_\varepsilon^2 M}{2} \ge \left(\frac{27M^2}{32\|Q\|_{L^2}^2}\right)^{\frac{1}{3}}\left(\varepsilon\mu_1\right)^{\frac{1}{3}},$$

which clearly contradicts (47), provided $M > 0$ large enough. Hence, up to a subsequence if necessary, there exists $y_0 \in \mathbb{R}^N$, such that, $\lim_{\varepsilon\to 0^+} \frac{x_\varepsilon}{t_\varepsilon} = y_0$. Therefore, by (8) and (49), and the Fatou lemma,

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{1}{t_\varepsilon^2}\int_{\mathbb{R}^N} V(x)|u_\varepsilon(x)|^2dx &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N} |x+\frac{x_\varepsilon}{t_\varepsilon}|^2|w_\varepsilon(x)|^2 \\ &\geq \frac{1}{\|Q\|_{L^2}^2}\int_{\mathbb{R}^N}|x+y_0|^2|Q(x)|^2dx \\ &\geq \frac{1}{\|Q\|_{L^2}^2}\int_{\mathbb{R}^N}|x|^2|Q(x)|^2dx. \end{aligned} \tag{50}$$

Thus, it follows from (48), (50) and the Young inequality that, as $\varepsilon \to 0^+$,

$$e_{b^*}(\varepsilon) \geq \frac{(1+o_\varepsilon(1))\varepsilon\mu_1}{2t_\varepsilon^4\|Q\|_{L^2}^2} + \frac{(1+o_\varepsilon(1))\mu_2 t_\varepsilon^2}{2\|Q\|_{L^2}^2} \geq \frac{3(1+o_\varepsilon(1))}{2\|Q\|_{L^2}^2}(\varepsilon\mu_1)^{\frac{1}{3}}(\frac{\mu_2}{2})^{\frac{2}{3}}. \tag{51}$$

Combining (47) and (51), then (12) follows. In addition, it is easy to verify that the equality holds in the second inequality of (51) if and only if (11) holds. Thus we have finished the proof. □

5. Conclusions

In this paper, we consider a global minimization problem on an L^2-norm constrained manifold to obtain the existence of ground state solutions to the stationary equation, which then gives to the existence of the standing wave solutions to the time-dependent equation. Using some energy estimate arguments, we manage to establish the asymptotic behaviors of the solutions we obtained as $\varepsilon \to 0^+$. Precisely, when the perturbation is small enough, solutions concentrate on a zero point of the potential $V(x)$. In particular, we prove the blow-up of the solutions as $\varepsilon \to 0^+$. This information shows us the reason why solutions do not exist when $\varepsilon = 0$ and $b = b^*$. We believe that similar analyses can also be carried out on other equations.

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