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Group Actions and Monotone Quantum Metric Tensors

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Abstract: The interplay between actions of Lie groups and monotone quantum metric tensors on the space of faithful quantum states of a finite-level system observed in recent works is here further developed.

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1. Introduction

In the context of information geometry for finite-dimensional quantum systems, it is well-known that the canonical action $\rho \mapsto U\rho U^{\dagger}$, where $\dagger$ denotes the usual adjoint of an operator, of the unitary group $\mathcal{U}(\mathcal{H})$ on the manifold $\mathscr{S}(\mathcal{H})$ of faithful quantum states provides symmetry transformations for every monotone quantum metric tensor on $\mathscr{S}(\mathcal{H})$ pertaining to Petz's classification [1]. Therefore, the fundamental vector fields generating the canonical action of $\mathcal{U}(\mathcal{H})$ are Killing vector fields for every quantum monotone metric tensor.

It is also known that the canonical action of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ can be seen as the restriction to $\mathcal{U}(\mathcal{H})$ of a nonlinear action of the general linear group $\mathcal{GL}(\mathcal{H})$ given by

$$\rho \mapsto \beta(\mathrm{g}, \rho) = \frac{\mathrm{g}\,\rho\,\mathrm{g}^{\dagger}}{\mathrm{Tr}(\mathrm{g}\,\rho\,\mathrm{g}^{\dagger})}. \tag{1}$$

The action β is transitive on $\mathscr{S}$ and turns it into a homogeneous manifold [2–5]. Therefore, the fundamental vector fields of the canonical action of $\mathcal{U}(\mathcal{H})$ form a Lie-subalgebra of the algebra of fundamental vector fields of the action of $\mathcal{GL}(\mathcal{H})$.

In [6], it is shown that, in order to describe the fundamental vector fields of β, it is sufficient to consider the fundamental vector fields of the canonical action of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ together with the gradient vector fields associated with the expectation-value functions $l_{\mathbf{a}}(\rho) = Tr(\mathbf{a}\rho)$—where $\mathbf{a}$ is any self-adjoint element in the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$—by means of the so-called Bures–Helstrom metric tensor [7–12]. This instance provides an unexpected link between the unitary group $\mathcal{U}(\mathcal{H})$, the $\mathcal{GL}(\mathcal{H})$-homogeneous manifold structure of $\mathscr{S}(\mathcal{H})$, the Bures–Helstrom metric tensor, and the expectation value functions.

However, this is not the only example in which a monotone metric tensor "interacts" with the general linear group $\mathcal{GL}(\mathcal{H})$. Indeed, again in [6], it is also shown that fundamental vector fields of the canonical action of $\mathcal{U}(\mathcal{H})$ together with the gradient vector fields associated with the expectation value functions by means of the Wigner–Yanase metric tensor [13–19] close on a representation of the Lie algebra of $\mathcal{GL}(\mathcal{H})$ that integrates to a group action given by

$$\rho \mapsto \beta^{WY}(\mathrm{g},\rho) = \frac{(\mathrm{g}\sqrt{\rho}\mathrm{g}^{\dagger})^2}{\mathrm{Tr}\Big((\mathrm{g}\sqrt{\rho}\mathrm{g}^{\dagger})^2\Big)} \in \mathscr{S}(\mathcal{H}). \tag{2}$$

Of course, the action β^{WY} is different from the action β, but it is still a transitive action so that $\mathscr{S}(\mathcal{H})$ is a homogeneous manifold also with respect to this action, and the underlying smooth structure coincides with the one related with β. Moreover, a direct inspection shows that β^{WY} can be thought of as a kind of deformation of β by means of the square-root map and its inverse on positive operators. This instance is better described and elaborated upon in the rest of the paper.

Finally, again in [6], it is proved that there is another Lie group "extending" the unitary group $\mathcal{U}(\mathcal{H})$ and for which a construction similar to the one discussed above is possible. This Lie group is the cotangent bundle $T^*\mathcal{U}(\mathcal{H})$ of $\mathcal{U}(\mathcal{H})$ endowed with its canonical Lie group structure [20,21]. In this case, the gradient vector fields of the expectation value functions are built using the Bogoliubov–Kubo–Mori metric tensor [22–26], and the action is given by

$$\rho \mapsto \gamma((\mathbf{U},\mathbf{a}),\rho) = \frac{\mathrm{e}^{U\ln(\rho)U^{\dagger}+\mathbf{a}}}{\mathrm{Tr}\Big(\mathrm{e}^{U\ln(\rho)U^{\dagger}+\mathbf{a}}\Big)}, \tag{3}$$

where $\mathbf{a}$ is a self-adjoint element which is identified with a cotangent vector at U. Once again, we obtain a transitive action on $\mathscr{S}(\mathcal{H})$ associated with a homogeneous manifold structure whose underlying smooth structure coincides with the two other smooth structures previously mentioned.

It is important to note that, when we restrict to the unitary group $\mathcal{U}(\mathcal{H})$, all the group actions we considered reduce to the canonical action

$$(U,\rho) \mapsto \alpha(U,\rho) = U\rho U^{\dagger} \tag{4}$$

of the unitary group whose importance in quantum theories is almost impossible to overestimate.

Once we have these three "isolated" instances, it is only natural to wonder if they are truly isolated cases, or if there are other monotone metric tensors for which a similar construction is possible. In [27], this problem is completely solved in the two-level case in which a direct, coordinate-based solution is possible. The result is that the only two groups for which the aforementioned construction works are precisely the general linear group $\mathcal{GL}(\mathcal{H})$ and the cotangent group $T^*\mathcal{U}(\mathcal{H})$. Moreover, in the case of $T^*\mathcal{U}(\mathcal{H})$, the only compatible action is the action γ already described in [6], while, for $\mathcal{GL}(\mathcal{H})$, there is an entire family of compatible smooth actions parameterized by a real number $\kappa \in (0,1]$ and given by

$$\rho \mapsto \beta^{\kappa}(\mathrm{g},\rho) = \frac{(\mathrm{g}\rho^{\sqrt{\kappa}}\mathrm{g}^{\dagger})^{\frac{1}{\sqrt{\kappa}}}}{\mathrm{Tr}\Big(\big(\mathrm{g}\rho^{\sqrt{\kappa}}\mathrm{g}^{\dagger}\big)^{\frac{1}{\sqrt{\kappa}}}\Big)}. \tag{5}$$

All these actions are connected with a different quantum metric tensor. For instance, when $\kappa = 1$ the Bures–Helstrom metric tensor is recovered, while the Wigner–Yanase metric is recovered when $\kappa = 1/4$. All the other cases correspond to Riemannian metric tensors on $\mathscr{S}(\mathcal{H})$ which are invariant under the standard action of $\mathcal{U}(\mathcal{H})$.

In this work, we further investigate the problem by showing that all the group actions and metric tensors found in [27] for a two-level system actually appear also for a quantum system with an arbitrary, albeit finite, number of levels. Moreover, we characterize all the values of κ for which the Riemannian metric tensor associated with the action β^{κ} in Equation (5) is actually a quantum monotone metric tensor (*cfr.* Proposition 1).

The work is structured as follows. In Section 2, we discuss those differential geometric properties of the manifold of normalized and un-normalized quantum states that are necessary to the proof of our main results. In Section 4, we set up the problem and prove our main results for $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$, namely, Propositions 2 and 3. In Section 5, we discuss our results and some possible future directions of investigation.

2. Geometry of (Un-Normalized) Quantum States

In this section, the construction of the space of quantum states [7,28] is briefly described and some of its geometric features are recalled; this gives the setting for our discussion. Then, we give a hint to the role played by group actions in the context of Quantum Mechanics and introduce some particular group actions that will be needed in order to get to the main result of this work. Finally, the concept of *monotone metric*, which is crucial in the context of Quantum Information Geometry, is introduced.

In standard quantum mechanics [29,30], a quantum system is mathematically described with the aid of a complex Hilbert space $\mathcal{H}$. The bounded observables of the system are identified with the self-adjoint elements in the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$, and the set of all such elements is denoted with $\mathcal{B}_{sa}(\mathcal{H})$. The physical states of the system are identified with the so-called ***density operators*** on $\mathcal{H}$. In order to define what a density operator is, we start recalling that $\omega \in \mathcal{B}_{sa}(\mathcal{H})$ is said to be positive semi-definite if

$$\langle\psi|\omega|\psi\rangle \geq 0 \quad \forall\psi \in \mathcal{H}, \tag{6}$$

and its customary to write $\omega \geq 0$ and $\omega > 0$ when it is also invertible. The space of positive semi-definite operators is denoted by $\overline{\mathcal{P}(\mathcal{H})}$ so that

$$\overline{\mathcal{P}(\mathcal{H})} = \{\omega \in \mathcal{B}(\mathcal{H})|\ \omega \geq 0\}, \tag{7}$$

and its elements may be referred to as ***un-normalized quantum states*** for reasons that are clarified below. From a geometrical point of view, $\overline{\mathcal{P}(\mathcal{H})}$ is a convex cone. A *density operator* ρ is just an element in $\overline{\mathcal{P}(\mathcal{H})}$ satisfying the normalization condition $\mathrm{Tr}\,\rho = 1$. This linear condition defines a hyperplane

$$\mathcal{B}^1_{sa}(\mathcal{H}) = \{a \in \mathcal{B}_{sa}(\mathcal{H})|\ a = a^\dagger,\ \mathrm{Tr}\ a = 1\} \tag{8}$$

in $\mathcal{B}_{sa}(\mathcal{H})$. As anticipated before, physical states are identified with density operators, and thus, the space of quantum states reads

$$\overline{\mathscr{S}(\mathcal{H})} = \{\rho \in \overline{\mathcal{P}(\mathcal{H})},\ \mathrm{Tr}\ \rho = 1\}, \tag{9}$$

and thus, the nomenclature "un-normalized quantum states" for elements in $\overline{\mathcal{P}(\mathcal{H})}$ appears justified. Clearly, $\overline{\mathscr{S}(\mathcal{H})}$ is given by the intersection between the convex cone $\overline{\mathcal{P}(\mathcal{H})}$ and the hyperplane $\mathcal{B}^1_{sa}$ and thus is a convex set.

Remark 1. *It is worth noting that there is a very deep analogy between the space $\overline{\mathcal{P}(\mathcal{H})}$ of positive semi-definite operators and the space $\mathscr{M}_\mu(M)$ of classical measures on the measurable space M which are absolutely continuous with respect to the reference measure μ. This analogy finds the perfect mathematical formalization in the context of the theory of C^*/W^*-algebras [31–34], where it turns out that both $\overline{\mathcal{P}(\mathcal{H})}$ and $\mathscr{M}_\mu(M)$ arise as the space of normal positive linear functionals on suitable C^*/W^*-algebras. This parallel is being exploited to give a unified account of some aspects of classical and quantum information geometry [3,22,24,35,36].*

In the rest of this work, as it is often done in the context of quantum information theory [37], we restrict our attention to the finite-dimensional case in which $\mathcal{H}$ has complex dimension $n < \infty$. Then, it is proved that both $\overline{\mathcal{P}(\mathcal{H})}$ and $\overline{\mathscr{S}(\mathcal{H})}$ may be endowed with the structure of stratified manifold whose underlying topological structure coincides with the topology inherited from $\mathcal{B}(\mathcal{H})$ [38]. It turns out that the *strata* of these stratified manifolds can be described in terms of a particular action of the general linear group $\mathcal{GL}(\mathcal{H})$ [2–5,39]. Specifically, $\mathcal{GL}(\mathcal{H})$ acts on the whole $\mathcal{B}(\mathcal{H})$ according to

$$(\mathrm{g}, a) \mapsto \hat{\beta}(\mathrm{g}, a) = \mathrm{g}\, a\, \mathrm{g}^{\dagger}. \tag{10}$$

It is important to note that, when we restrict the action $\hat{\beta}$ in such a way that it acts only on positive elements and only by means of elements in $\mathcal{U}(\mathcal{H})$, it reduces to the canonical action

$$(U, \omega) \mapsto \hat{\alpha}(U, \omega) = U\omega U^{\dagger} \tag{11}$$

of the unitary group. The action $\hat{\beta}$ is linear, and it is a matter of direct inspection to check that it preserves both $\mathcal{B}_{sa}(\mathcal{H})$ and $\overline{\mathcal{P}(\mathcal{H})}$ and that the orbits through $\overline{\mathcal{P}(\mathcal{H})}$ are made of positive semi-definite operators of the same rank, denoted by $\mathcal{P}^k(\mathcal{H})$ where $k \leq n$ is the rank. These orbits thus become homogeneous manifolds and their underlying smooth structures agree with those associated with the stratification of $\overline{\mathcal{P}(\mathcal{H})}$ [38].

In particular, we are interested in the maximal stratum

$$\mathcal{P}(\mathcal{H}) \equiv \mathcal{P}^n(\mathcal{H}) = \{\omega \in \overline{\mathcal{P}(\mathcal{H})},\ \omega > 0\}, \tag{12}$$

i.e., the space of invertible elements in $\overline{\mathcal{P}(\mathcal{H})}$, which forms the open interior of $\overline{\mathcal{P}(\mathcal{H})}$. The tangent space $T_{\omega}\mathcal{P}(\mathcal{H})$ of $\mathcal{P}(\mathcal{H})$ at $\omega \in \mathcal{P}(\mathcal{H})$ is isomorphic to $\mathcal{B}_{sa}(\mathcal{H})$, since $\mathcal{P}(\mathcal{H})$ is an open set in $\mathcal{B}_{sa}(\mathcal{H})$. Since $\mathcal{P}(\mathcal{H})$ is a homogeneous manifold, the tangent space at each point can be described in terms of the fundamental vector fields of the action $\hat{\beta}$ evaluated at a point $\omega \in \mathcal{P}(\mathcal{H})$ [40]. Recalling that the Lie algebra $\mathfrak{gl}(\mathcal{H})$ of $\mathcal{GL}(\mathcal{H})$ is essentially $\mathcal{B}(\mathcal{H})$ endowed with the standard commutator, a curve in the group $\mathcal{GL}(\mathcal{H})$ can be written as

$$g(t) = e^{\frac{1}{2}t(\mathbf{a} - i\mathbf{b})}, \tag{13}$$

with $\mathbf{a}$ and $\mathbf{b}$ self-adjoint operators. Therefore, the fundamental vector field associated with $\mathbf{a} - i\mathbf{b}$ at the point ω reads

$$\hat{Z}_{\mathbf{a}\,\mathbf{b}}(\omega) = \frac{d}{dt}\beta(g(t), \omega)\big|_{t=0} = [\omega, \mathbf{b}] + \{\omega, \mathbf{a}\} \equiv \hat{X}_{\mathbf{b}}(\omega) + \hat{Y}_{\mathbf{a}}(\omega), \tag{14}$$

where we have used the notation

$$\begin{aligned} [\mathbf{a}, \mathbf{b}] &= \frac{i}{2}(\mathbf{ab} - \mathbf{ba}), \\ \{\mathbf{a}, \mathbf{b}\} &= \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \end{aligned} \tag{15}$$

and we have set

$$\hat{X}_{\mathbf{b}}(\omega) := \hat{Z}_{\mathbf{0}\,\mathbf{b}}(\omega) = [\omega, \mathbf{b}], \tag{16}$$

and

$$\hat{Y}_{\mathbf{a}}(\omega) := \hat{Z}_{\mathbf{a}\,\mathbf{0}}(\omega) = \{\omega, \mathbf{a}\}. \tag{17}$$

As mentioned before, when we restrict $\hat{\beta}$ to $\mathcal{U}(\mathcal{H})$ we obtain the canonical action $\hat{\alpha}$ of $\mathcal{U}(\mathcal{H})$. Therefore, since the Lie algebra $\mathfrak{u}(\mathcal{H})$ is just the space of skew-adjoint elements in $\mathcal{B}(\mathcal{H}) = \mathfrak{gl}(\mathcal{H})$, setting $\mathbf{a} = 0$ in Equation (13), we immediately obtain that the fundamental vector fields of $\hat{\alpha}$ are recovered as the fundamental vector fields $\hat{X}_{\mathbf{b}} = \hat{Z}_{\mathbf{0}\,\mathbf{b}}$ of the action $\hat{\beta}$. Concerning the vector fields of the form $\hat{Y}_{\mathbf{a}}$, taking $\mathbf{a} = \mathbb{I}$, we obtain the vector field

$$\Delta(\omega) := \hat{Y}_{\mathbb{I}}(\omega) = \omega \tag{18}$$

which represents the infinitesimal generator of the Lie group $\mathbb{R}_+$ acting on $\mathcal{P}(\mathcal{H})$ by dilation. However, in general, it turns out that $[\hat{Y}_{\mathbf{a}}, \hat{Y}_{\mathbf{b}}] = \hat{X}_{[\mathbf{b},\mathbf{a}]}$ [2,3] so that they do not form a Lie subalgebra.

Besides $\mathcal{GL}(\mathcal{H})$, also the cotangent Lie group $T^*\mathcal{U}(\mathcal{H})$ acts on $\mathcal{P}(\mathcal{H})$ in such a way that the latter becomes a homogeneous manifold of $T^*\mathcal{U}(\mathcal{H})$. Specifically, the action is given by

$$((U,\mathbf{a}),\omega) \mapsto \hat{\gamma}((U,\mathbf{a}),\rho) = \mathrm{e}^{U \log\rho\, U^\dagger + \mathbf{a}}, \tag{19}$$

where we used the canonical identifications $T^*\mathcal{U}(\mathcal{H}) \cong \mathcal{U}(\mathcal{H}) \times \mathfrak{u}(\mathcal{H})^* \cong \mathcal{U}(\mathcal{H}) \times \mathcal{B}_{sa}(\mathcal{H})$. It is not hard to check that, if we restrict to $\mathcal{U}(\mathcal{H})$ by considering only elements of the type $(U, 0)$, the action $\hat{\gamma}$ reduces to the action $\hat{\alpha}$ of $\mathcal{U}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$. The action $\hat{\gamma}$ is smooth with respect to the previously mentioned smooth structure on $\mathcal{P}(\mathcal{H})$ associated with the action $\hat{\beta}$, and it is a transitive action. Therefore, we conclude that the smooth structure underlying $\mathcal{P}(\mathcal{H})$ when thought of as a homogeneous manifold for $T^*\mathcal{U}(\mathcal{H})$ coincides with the smooth structure underlying $\mathcal{P}(\mathcal{H})$ thought of as a homogeneous manifold for $\mathcal{GL}(\mathcal{H})$. The action $\hat{\gamma}$ is basically related with the isomorphism $\Phi\colon \mathcal{P}(\mathcal{H}) \to \mathcal{B}_{sa}(\mathcal{H})$ given by $\omega \to \Phi(\omega) := \ln(\omega)$ and its inverse $\Phi^{-1}\colon \mathcal{B}_{sa}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ given by $a \to \Phi^{-1}(x) := \mathrm{e}^x$. Indeed, it is clear that $T^*\mathcal{U}(\mathcal{H}) \cong \mathcal{U}(\mathcal{H}) \times \mathfrak{u}(\mathcal{H})^* \cong \mathcal{U}(\mathcal{H}) \times \mathcal{B}_{sa}(\mathcal{H})$ acts on $\mathcal{B}_{sa}(\mathcal{H})$ through

$$x \mapsto \zeta((U,\mathbf{a}),x) = U\mathbf{x}U^\dagger + \mathbf{a}, \tag{20}$$

and it is a matter of direct inspection to show that

$$\hat{\gamma}_{(U,\mathbf{a})} = \Phi^{-1} \circ \zeta_{(U,\mathbf{a})} \circ \Phi. \tag{21}$$

Thinking of $\mathcal{U}(\mathcal{H})$ as a subgroup of the rotation group of the vector space $\mathcal{B}_{sa}(\mathcal{H})$, it follows that the action ζ coincides with the restriction to $T^*\mathcal{U}(\mathcal{H})$ of the standard action of the affine group on $\mathcal{B}_{sa}(\mathcal{H})$. The action $\hat{\gamma}$ cannot be extended to the whole $\overline{\mathcal{P}(\mathcal{H})}$ essentially because Φ and its inverse cannot be extended. Concerning the fundamental vector fields $\hat{W}_{\mathbf{a}\,\mathbf{b}}$ of $\hat{\gamma}$, we have that $\hat{W}_{\mathbf{0}\,\mathbf{b}} = \hat{X}_{\mathbf{b}}$ as in Equation (16), while

$$\hat{W}_{\mathbf{a}\,\mathbf{0}}(\omega) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\ln(\omega)+t\mathbf{a}}\right)_{t=0} \;=\; \int_0^1 \mathrm{d}\lambda\, \left(\omega^\lambda\, \mathbf{a}\, \omega^{1-\lambda}\right), \tag{22}$$

where we used the well-known equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{A(t)} \;=\; \int_0^1 \mathrm{d}\lambda\, \left(\mathrm{e}^{\lambda\, A(t)}\, \frac{\mathrm{d}}{\mathrm{d}t}(A(t))\, \mathrm{e}^{(1-\lambda)A(t)}\right), \tag{23}$$

which is valid for every smooth curve $A(t)$ inside $\mathcal{B}(\mathcal{H})$ (remember that the canonical immersion of $\mathcal{P}(\mathcal{H})$ inside $\mathcal{B}(\mathcal{H})$ is smooth) [41].

We turn now our attention to faithful quantum states. The action in Equation (10) does not preserve the hyperplane $\mathcal{B}^1_{sa}(\mathcal{H})$ in Equation (8), and thus, it also does not preserve $\overline{\mathscr{S}(\mathcal{H})}$. However, as already anticipated in Equation (1), it is possible to suitably renormalize $\hat{\beta}$ to obtain the action

$$(\mathrm{g},\rho) \mapsto \beta(\mathrm{g},\rho) := \frac{\mathrm{g}\,\rho\,\mathrm{g}^\dagger}{\mathrm{Tr}(\mathrm{g}\,\rho\,\mathrm{g}^\dagger)}. \tag{24}$$

The normalization is recovered at the expense of the linearity/convexity of the action. However, when we restrict to the unitary group $\mathcal{U}(\mathcal{H})$, the action β reduces to the canonical action

$$(U,\rho) \mapsto \alpha(U,\rho) = U\rho U^\dagger \tag{25}$$

of the unitary group on the space of states which does preserve convexity. Analogously to what happens for the action $\hat{\beta}$ on $\overline{\mathcal{P}(\mathcal{H})}$, the orbits of β are made up of quantum states with the same fixed rank, and any such orbit is denoted as $\mathscr{S}^k(\mathcal{H})$ where k is the rank.

These orbits thus become homogeneous manifolds and their underlying smooth structures agree with those associated with the stratification of $\overline{\mathscr{S}(\mathcal{H})}$ [38]. Moreover, each manifold $\mathscr{S}^k(\mathcal{H})$ can be seen as a submanifold of $\mathcal{P}^k(\mathcal{H})$ singled out by the intersection with the affine hyperplane $\mathcal{B}^1_{sa}(\mathcal{H})$. It is worth mentioning that the partition of $\overline{\mathscr{S}(\mathcal{H})}$ in terms of manifold of quantum states of fixed rank was also exploited in [42,43]; however, as far as the authors know, the homogeneous manifold structures was firstly understood in [4,5] and the stratified structure in [38].

Remark 2. *Building on Remark 1, for a reader familiar with Classical Information Geometry, it may be useful to think of the space of quantum states of an n-level quantum system as the quantum analogue of the $(n-1)$-simplex, with the* strata *of the space of quantum states taking the place of the* faces *of the simplex. A thorough discussion of this analogy can be found in [44–46].*

In particular, we focus on the stratum of maximal rank, i.e., invertible, or *faithful* states

$$\mathscr{S}^n(\mathcal{H}) \equiv \mathscr{S}(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) \,|\, \operatorname{Tr}\rho = 1\}. \tag{26}$$

The tangent space $T_\rho\mathscr{S}(\mathcal{H})$ of $\mathscr{S}(\mathcal{H})$ at ρ is given by self-adjoint operators with the additional property of being traceless, i.e., we have

$$T_\rho\mathscr{S}(\mathcal{H}) \cong \mathcal{B}^0_{sa}(\mathcal{H}) = \{\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H}) | \ \mathbf{a} = \mathbf{a}^\dagger,\ \operatorname{Tr}\mathbf{a} = 0\}. \tag{27}$$

Since $\mathscr{S}(\mathcal{H})$ is a homogeneous manifold, its tangent space can be described using the fundamental vector fields of the action β following what is done for $\hat{\beta}$. The fundamental vector fields of the action β evaluated at a point $\rho \in \mathscr{S}(\mathcal{H})$ are given by

$$Z_{\mathbf{a}\,\mathbf{b}}(\rho) = \frac{d}{dt}\beta(g(t),\rho)\big|_{t=0} = [\rho,\mathbf{b}] + \{\rho,\mathbf{a}\} - \rho\operatorname{Tr}(\{\rho,\mathbf{a}\}) \equiv X_{\mathbf{b}}(\rho) + Y_{\mathbf{a}}(\rho). \tag{28}$$

where $g(t)$ is defined as in (13) and, now, we have set

$$X_{\mathbf{b}}(\rho) := Z_{\mathbf{0}\,\mathbf{b}}(\rho) = [\rho,\mathbf{b}], \tag{29}$$

and

$$Y_{\mathbf{a}}(\rho) := Z_{\mathbf{a}\,\mathbf{0}}(\rho) = \{\rho,\mathbf{a}\} - \rho\operatorname{Tr}(\{\rho,\mathbf{a}\}). \tag{30}$$

Again in analogy with what happens on $\mathcal{P}(\mathcal{H})$, the fundamental vector fields of the action α of $\mathcal{U}(\mathcal{H})$ are identified with the vector fields $X_{\mathbf{b}}$.

As anticipated in the Introduction, the cotangent Lie group $T^*\mathcal{U}(\mathcal{H})$ also acts on $\mathscr{S}(\mathcal{H})$ through the action γ given in Equation (3). This action is smooth with respect to the previously mentioned smooth structure on $\mathscr{S}(\mathcal{H})$ associated with the action β, and it is a transitive action. Therefore, we conclude that the smooth structure underlying $\mathscr{S}(\mathcal{H})$ when thought of as a homogeneous manifold for $T^*\mathcal{U}(\mathcal{H})$ coincides with the smooth structure underlying $\mathscr{S}(\mathcal{H})$ thought of as a homogeneous manifold for $\mathcal{GL}(\mathcal{H})$. Moreover, when restricting to $\mathcal{U}(\mathcal{H})$, a direct computation shows that the action γ reduces to the standard action α of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$. The fundamental vector fields $W_{\mathbf{a}\,\mathbf{b}}$ of γ are then easily found. In particular, $W_{\mathbf{0}\,\mathbf{b}} = X_{\mathbf{b}}$ as in Equation (29), and

$$W_{\mathbf{a}\,\mathbf{0}}(\rho) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{\ln(\rho)+t\mathbf{a}}}{\operatorname{Tr}\left(\mathrm{e}^{\ln(\rho)+t\mathbf{a}}\right)}\right)_{t=0} = \int_0^1 \mathrm{d}\lambda\,\left(\rho^\lambda\,\mathbf{a}\,\rho^{1-\lambda}\right) - \operatorname{Tr}(\rho\,\mathbf{a})\,\rho\,, \tag{31}$$

where we again exploited Equation (23) (remember that the canonical immersion of $\mathscr{S}(\mathcal{H})$ inside $\mathcal{B}(\mathcal{H})$ is smooth). Concerning the vector fields in Equation (31), it is worth mentioning that they already appeared in [25] in connection with the Bogoliubov–Kubo–Mori metric tensor, and then, in the recent work [47], where the finite transformations they induced are exploited in the definition of a Hilbert space structure on $\mathscr{S}(\mathcal{H})$, which is the quantum counterpart of a classical structure relevant in estimation theory. However, as far as the author know, the group-theoretical aspects relating the vector fields in Equation (31) with the action γ of $T^*\mathcal{U}(\mathcal{H})$ were first investigated in [6].

Despite the lack of a universally recognized physical interpretation for un-normalized quantum states in $\overline{\mathcal{P}(\mathcal{H})}$, it turns out that they provide a more flexible environment in which to perform the mathematics needed to prove the main result of this work. Intuitively speaking, it is already clear from the very definition of the actions $\hat{\beta}$ and β that imposing the linear normalization constraint needed to pass to (normalized) quantum states leads to the emergence of nonlinear aspects which destroy the inherent convexity of the space of quantum states. In fact, following the ideology expressed in [48], it can also be argued that the choice of a normalization has a somewhat arbitrary flavor that does not really encode physical information, because basically nothing really serious happens if we decide to normalize to π^2 rather than to 1. Following this line of thought, we will always work on $\mathcal{P}(\mathcal{H})$ making sure that all the structure and results may be appropriately "projected" to $\mathscr{S}(\mathcal{H})$. For this purpose, it is relevant to introduce a projection map from $\pi\colon \mathcal{P}(\mathcal{H}) \to \mathscr{S}(\mathcal{H})$ as

$$\omega \mapsto \pi(\omega) = \frac{\omega}{\mathrm{Tr}(\omega)}, \tag{32}$$

and an associated section given by the natural immersion map $j\colon \mathscr{S}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ reading

$$\rho \mapsto j(\rho) = \rho. \tag{33}$$

It is not hard to show that j is an embedding, while π is a surjective submersion. Moreover, it is also possible to "extend" these maps to the whole $\overline{\mathcal{P}(\mathcal{H})}$ and $\overline{\mathscr{S}(\mathcal{H})}$ in the obvious way, thus obtaining a continuous projection map and a continuous immersion map that preserve the stratification of $\overline{\mathcal{P}(\mathcal{H})}$ and $\overline{\mathscr{S}(\mathcal{H})}$ and are smooth on each strata.

As mentioned before, bounded physical *observables* are described by means of self-adjoint operators in $\mathcal{B}_{sa}(\mathcal{H})$. Then, to any observable $\mathbf{a}$, it is possible to associate a smooth function $\hat{l}_{\mathbf{a}}(\omega) : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ given by

$$\hat{l}_{\mathbf{a}}(\omega) := \mathrm{Tr}\{\, (\omega\, \mathbf{a})\}, \tag{34}$$

this is referred to as *expectation value function* of the observable $\mathbf{a}$. Of course, expectation value functions can also be defined on the space of quantum states $\mathscr{S}(\mathcal{H})$ setting

$$l_{\mathbf{a}}(\rho) := \mathrm{Tr}\{\, (\rho\, \mathbf{a})\} \in \mathbb{R}, \tag{35}$$

and it turns out that $\hat{l}_a$ is connected to l_a by means of the pull-back with respect to j, i.e., it holds

$$l_{\mathbf{a}} = j^*_{\mathscr{S}} \hat{l}_{\mathbf{a}}. \tag{36}$$

By relaxing smoothness to continuity, it is possible to extend the expectation value functions to the whole $\overline{\mathcal{P}(\mathcal{H})}$ and the whole $\overline{\mathscr{S}(\mathcal{H})}$.

It is a matter of direct calculation using the very definition of fundamental vector fields for both $\hat{\alpha}$ and α (*cfr.* Equations (4) and (11)) to show that

$$\begin{aligned} \mathcal{L}_{\hat{X}_{\mathbf{b}}} \hat{l}_{\mathbf{a}} &= \hat{l}_{[\mathbf{b},\mathbf{a}]} \\ \mathcal{L}_{X_{\mathbf{b}}} l_{\mathbf{a}} &= l_{[\mathbf{b},\mathbf{a}]}, \end{aligned} \tag{37}$$

where $[\cdot,\cdot]$ is as in Equation (15).

By direct computation, it is possible to spot an interesting intertwine between the maps π and i and the actions $\hat{\beta}$ and β given by

$$\beta = \pi \circ \hat{\beta} \circ (\mathrm{Id}_{\mathcal{GL}(\mathcal{H})} \times j) \tag{38}$$

where $\mathrm{Id}_{\mathcal{GL}(\mathcal{H})}$ is the identity map on $\mathcal{GL}(\mathcal{H})$. Analogously, we obtain

$$\gamma = \pi \circ \hat{\gamma} \circ (\mathrm{Id}_{T^*\mathcal{U}(\mathcal{H})} \times j) \tag{39}$$

where $\mathrm{Id}_{T^*\mathcal{U}(\mathcal{H})}$ is the identity map on $T^*\mathcal{U}(\mathcal{H})$. Equations (38) and (39) explain in which sense β and γ are a kind of normalized version of the actions $\hat{\beta}$ and $\hat{\gamma}$, respectively. The immersion map j also allows us to obtain a pointwise relation between the fundamental vector fields $\hat{X}_{\mathbf{b}}$ and $X_{\mathbf{b}}$ and between the fundamental vector fields $\hat{Y}_{\mathbf{a}}$ and $Y_{\mathbf{a}}$ in terms of the tangent map $T_\rho j$ to j at ρ. Indeed, from Equations (16)–(18), (29), and (30), it follows that

$$\begin{aligned} T_\rho j(X_{\mathbf{b}}(\rho)) &= \hat{X}_{\mathbf{b}}(j(\rho)) \\ T_\rho j(Y_{\mathbf{a}}(\rho)) &= \hat{Y}_{\mathbf{a}}(j(\rho)) - \mathrm{Tr}(\rho\, \mathbf{a})\, \Delta(j(\rho)). \end{aligned} \tag{40}$$

Accordingly, we conclude that $X_{\mathbf{b}}$ is j-related with $\hat{X}_{\mathbf{b}}$ while $Y_{\mathbf{a}}$ is j-related with $\hat{Y}_{\mathbf{a}} - \hat{l}_{\mathbf{a}}\Delta$. Analogously, from Equations (22), (18) and (31), it follows that

$$T_\rho j(W_{\mathbf{a}\,\mathbf{0}}(\rho)) = \hat{W}_{\mathbf{a}\,\mathbf{0}}(\rho) - \mathrm{Tr}(\rho\, \mathbf{a})\, \Delta(j(\rho)), \tag{41}$$

which means that $W_{\mathbf{a}\,\mathbf{0}}$ is j-related with $\hat{W}_{\mathbf{a}\,\mathbf{0}} - \hat{l}_{\mathbf{a}}\Delta$.

3. Quantum Monotone Metric Tensors

In the classical case, the Riemannian aspects of most of the manifolds of probability employed in statistics, inference theory, information theory, and information geometry are essentially encoded in a single metric tensor (we are here deliberately "ignoring" all those Wasserstein-type metric tensors simply because their very definition depends on the existence of additional structures on the sample space), namely, the Fisher–Rao metric tensor [49–51]. In the case of finite sample spaces, Cencov's pioneering work [52] investigated the Fisher–Rao metric tensor from a category-theoretic perspective and uncovered the uniqueness of this metric tensor when some invariance conditions are required. Specifically, let $\overline{\mathrm{S}_n}$ denote the n-dimensional simplex in $\mathbb{R}^n$, i.e., the space of probability distributions on a discrete sample space with n elements, and let S_n denote the interior of $\overline{\mathrm{S}_n}$, the space of probability distributions with full support. Note that S_n is a smooth, $(n-1)$-dimensional manifold while $\overline{\mathrm{S}_n}$ is a smooth manifold with corners.

A linear map $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is called a ***Markov morphism*** if $F(\overline{\mathrm{S}_n}) \subset .\overline{\mathrm{S}_m}$, and a Markov morphism F is called a ***congruent embedding*** if $F(\mathrm{S}_n)$ is diffeomorphic to S_n. Congruent embeddings where studied by Cencov who characterized the most general form of these maps (*cfr.* [53] for yet another characterization of congruent embeddings).

According to Cencov, the relevant geometrical structures on S_n must all be left unchanged when suitably acted upon by congruent embeddings. For instance, setting $\mathbb{N}_{>1} = \mathbb{N} \setminus \{\{0\},\{1\}\}$, a family $\{g^n\}_{n\in\mathbb{N}_{>1}}$ with g^n a smooth Riemannian metric tensor on S_n is called ***invariant*** if $F^*g^m = g^n$ for every congruent embedding $F\colon \mathbb{R}^n \to \mathbb{R}^m$. Cencov's incredible result was to show that, up to an overall multiplicative positive constant, there is only one invariant family of Riemannian metric tensor for which g^n coincides with the Fisher–Rao metric tensor. Then, much effort has been devoted to extend Cencov's uniqueness result from the case of finite sample spaces to the case of continuous sample spaces leading, for instance, to a formulation on smooth manifolds [54] and a very general formulation valid for very general parametric models [48].

As already hinted at in Remarks 1 and 2, the manifold $\mathscr{S}(\mathcal{H})$ may be thought of as the quantum analogue of S_n in the case of finite-level quantum systems. Then, the quantum analogue of a Markov morphism is a completely-positive and trace-preserving linear (CPTP) map $F\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ (*cfr.* [55] for the precise definition of CPTP maps and [56,57] for their role in quantum information). Quite trivially, a ***quantum congruent embedding*** could be defined as a CPTP map $F\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ such that $F(\mathscr{S}(\mathcal{H}))$ is diffeomorphic to $\mathscr{S}(\mathcal{H})$. A typical example of quantum congruent embedding is given by $\alpha_U(\mathbf{a}) = \alpha(U, \mathbf{a}) = U\mathbf{a}U^{\dagger}$. As far as the authors know, there seems to be no general characterization of these maps at the moment as there is in the classical case.

Inspired by Cencov's work, Petz investigated the following problem: to characterize the families $\{g^n\}_{n\in\mathbb{N}_{>1}}$ with g^n a smooth Riemannian metric tensor on $\mathscr{S}(\mathbb{C}^n)$ satisfying the monotonicity property

$$(g^n)_\rho(v_\rho, v_\rho) \geq (g^m)_{F(\rho)}(T_\rho(v_\rho), T_\rho(v_\rho)) \tag{42}$$

for every CPTP map $F\colon \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^m)$ and for all $\rho \in \mathscr{S}(\mathbb{C}^n)$. He was able to prove [1] that, up to an overall multiplicative positive constant, these families of ***monotone quantum metric tensors*** are completely characterized by operator monotone functions $f\colon \mathbb{R}^+ \to \mathbb{R}$ [58] satisfying

$$\begin{aligned} f(t) &= t f(t^{-1}), \\ f(1) &= 1. \end{aligned} \tag{43}$$

In particular, if $\{G^n_f\}_{n\in\mathbb{N}_{>1}}$ is a family of monotone metric tensors, then

$$\left(G^n_f\right)_\rho(\mathbf{v}_\rho, \mathbf{w}_\rho) = \kappa\,\mathrm{Tr}_n\left(\mathbf{v}_\rho\left(\mathbf{K}^f_\rho\right)^{-1}(\mathbf{w}_\rho)\right) \tag{44}$$

where $\kappa > 0$ is a constant, $\mathbf{v}_\rho, \mathbf{w}_\rho$ are vectors in $T_\rho\mathscr{S}(\mathbb{C}^n) \cong \mathcal{B}^0_{sa}(\mathbb{C}^n)$, $\mathbf{K}^f_\rho$ is a superoperator on $\mathcal{B}(\mathbb{C}^n)$ given by

$$\mathbf{K}^f_\rho = f(\mathbf{L}_\rho\mathbf{R}^{-1}_\rho)\mathbf{R}_\rho \tag{45}$$

with f the operator monotone function mentioned before, and $\mathbf{L}_\rho$ and $\mathbf{R}_\rho$ are two linear superoperators on $\mathcal{B}(\mathbb{C}^n)$ whose action is given by the left and right multiplication by ρ.

We briefly mention a recent development towards the use of non-monotone metric tensors in quantum information theory [59].

Since every n-dimensional complex Hilbert space $\mathcal{H}$ is isomorphic to $\mathbb{C}^n$, we can almost immediately generalize Equation (44) to define a quantum monotone metric tensor $G^{\mathcal{H}}_f$ on $\mathscr{S}(\mathcal{H})$ setting

$$\left(G^{\mathcal{H}}_f\right)_\rho(\mathbf{v}_\rho, \mathbf{w}_\rho) = \kappa\,\mathrm{Tr}_{\mathcal{H}}\left(\mathbf{v}_\rho\left(\mathbf{K}^f_\rho\right)^{-1}(\mathbf{w}_\rho B)\right). \tag{46}$$

In the following, for the sake of notational simplicity, we often simply write G_f instead of $G^{\mathcal{H}}_f$ because the Hilbert space $\mathcal{H}$ is already clear from the context.

If we introduce the operators $\mathbf{e}^\rho_{lm}$ diagonalizing $\rho \in \mathscr{S}(\mathcal{H})$, that is, such that

$$\rho = \sum_{j=1}^n p^\rho_j\,\mathbf{e}^\rho_{jj}, \tag{47}$$

we can also introduce the superoperators E^ρ_{kj} acting on $\mathcal{B}(\mathcal{H})$ according to

$$E^\rho_{kj}\left(\mathbf{e}^\rho_{lm}\right) \;=\; \delta_{jl}\,\delta_{km}\mathbf{e}^\rho_{jk}, \tag{48}$$

and it is then a matter of straightforward computation to check that

$$K^f_\rho = \sum_{j,k=1}^n p^\rho_k\, f\left(\frac{p^\rho_j}{p^\rho_k}\right) E^\rho_{kj} \tag{49}$$

where $p_1^\rho, \ldots, p_n^\rho$ are the eigenvalues of ρ. Now, whenever $[\mathbf{w}_\rho, \rho] = 0$, from Equations (46) and (49), it follows that

$$(G_f)_\rho(\mathbf{v}_\rho, \mathbf{w}_\rho) = \sum_{j=1}^{n} \frac{v_\rho^{jj} w_\rho^{jj}}{p_j^\rho}, \tag{50}$$

where v_ρ^{jj} and w_ρ^{jj} are the diagonal elements of $\mathbf{v}_\rho$ and $\mathbf{w}_\rho$ with respect to the basis of eigenvectors of ρ. It is relevant to note then that in this case, we have

$$(G_f)_\rho(\mathbf{v}_\rho, \mathbf{w}_\rho) \;=\; (G_{FR})_{\vec{p}}\,(\vec{a}, \vec{b}), \tag{51}$$

where G_{FR} is the classical Fisher–Rao metric tensor on S_n, and we have set $\vec{p} = (p_1^\rho, \ldots, p_n^\rho)$, $\vec{a} = (v_\rho^{11}, \ldots, v_\rho^{nn})$, and $\vec{b} = (w_\rho^{11}, \ldots, w_\rho^{nn})$. Equation (51) holds for every choice of the operator monotone function f.

As mentioned before, the action α of $\mathcal{U}(\mathcal{H})$ in (4) gives rise to CPTP maps from $\mathscr{S}(\mathcal{H})$ into itself. Moreover, these maps are invertible and their inverses are again CPTP maps from $\mathscr{S}(\mathcal{H})$ to itself. Therefore, the monotonicity property in Equation (42) becomes an invariance property, and we conclude that the fundamental vector fields $X_{\mathbf{b}}$ of the action α (*cfr.* Equation (29)) are Killing vector fields for every monotone quantum metric tensor G_f. Consequently, the unitary group $\mathcal{U}(\mathcal{H})$ acts as a sort of universal symmetry group for the metric tensors classified by Petz and thus occupies a prominent role also in the context of Quantum Information Geometry.

To explicitly prove our main results, it is better to work first on $\mathcal{P}(\mathcal{H})$ and then "project" the results down to $\mathscr{S}(\mathcal{H})$. Accordingly, we need a suitable extension of the monotone quantum metric tensors to $\mathcal{P}(\mathcal{H})$, very much in the spirit of Campbell's work on the extension of the Fisher–Rao metric tensor to the non-normalized case of finite measures [53]. Kumagai already investigated this problem and provided a complete solution of Petz's problem when the normalization condition on quantum states is lifted [60]. Quite interestingly, the result very much resembles Campbell's result in the sense that the difference with the normalized case is entirely contained in a function $b\colon \mathbb{R}^+ \to \mathbb{R}$ and a family $\{f_t\}_{t\in\mathbb{R}^+}$ of operator monotone functions satisfying $tb(t) + \frac{1}{f_t(1)} > 0$.

In our case, however, it is not necessary to exploit the full level of generality of Kumagai's work. It suffices to find a Riemannian metric tensor $\hat{G}_f$ on $\mathcal{P}(\mathcal{H})$ such that

$$j^*\hat{G}_f = G_f \tag{52}$$

where $j\colon \mathscr{S}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ is the canonical immersion and G_f is a monotone quantum metric tensor as in Equation (46). Accordingly, we consider $\hat{G}_f$ as given by

$$\left(\hat{G}_f\right)_\omega(\mathbf{v}_\omega, \mathbf{w}_\omega) = \kappa\,\mathrm{Tr}\left(\mathbf{v}_\omega \left(\mathbf{K}_\omega^f\right)^{-1}(\mathbf{w}_\omega)\right), \tag{53}$$

where f is the operator monotone function appearing in Equation (46) (and thus satisfying Equation (43)), $\omega \in \mathscr{P}$, $\mathbf{v}_\omega, \mathbf{w}_\omega \in T_\omega\mathcal{P}(\mathcal{H}) \cong \mathcal{B}_{sa}(\mathcal{H})$, and $\mathbf{K}_\omega^f$ is as in Equation (45). Equation (53) corresponds to the choice $b = 0$ and $f_t = f$ in Kumagai's classification.

If we introduce the operators $\mathbf{e}_{lm}^\omega$ diagonalizing ω, we can proceed as in the normalized case to obtain an equation analogous to Equation (49) so that, recalling Equation (43), we immediately obtain

$$\mathbf{K}_\omega^f(\Delta(\omega)) = \mathbf{K}_\omega^f(\omega) = \omega^2 \Longrightarrow \left(\mathbf{K}_\omega^f\right)^{-1}(\Delta(\omega)) = \mathbb{I}. \tag{54}$$

4. Lie Groups and Monotone Quantum Metric Tensors

We are interested in classifying all those actions of $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ that behave in the way described in the Introduction with respect to suitable monotone quantum metric tensors. Specifically, we want to find all those actions, say δ, of either $\mathcal{GL}(\mathcal{H})$ or

$T^*\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ for which there is a monotone metric tensor G_f on $\mathscr{S}(\mathcal{H})$ such that the fundamental vector fields X_b of the standard action α of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ together with the gradient vector fields Y_a^f associated with the expectation value functions l_a close on a representation of the Lie algebra of either $\mathcal{GL}(\mathcal{H})$ or $T^*\mathcal{U}(\mathcal{H})$ that integrates to the action δ. From the results in [6], we know that there are at least 3 monotone metric tensors for which this construction is possible for any finite-level quantum system. Moreover, from the results in [27], we know that in the case of two-level quantum systems, the Lie groups $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$ are the only Lie groups for which the construction described above is actually possible. Here, we want to understand if the group actions of $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$ found in [27] can be extended from a 2-level quantum system to a system with an arbitrary, albeit finite, number of levels.

For this purpose, it is important to recall all those properties, shared by $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$ and by their actions, that are at the heart of the results of [6,27]. First of all, both $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$ contain the Lie group $\mathcal{U}(\mathcal{H})$ as a Lie subgroup, and contain the elements $\lambda\mathbb{I}$ with $\lambda > 0$ and $\mathbb{I}$ the identity operator on $\mathcal{H}$. Then, all the (transitive) actions of both $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ appearing in the analysis of [6,27] arise as a sort of normalization of suitable (transitive) actions on $\mathcal{P}(\mathcal{H})$. Specifically, if G denotes either $\mathcal{GL}(\mathcal{H})$ or $T^*\mathcal{U}(\mathcal{H})$, then every G-action δ on $\mathscr{S}(\mathcal{H})$ can be written as

$$\delta(\mathrm{g},\rho) = \frac{\hat{\delta}(\mathrm{g},\rho)}{\mathrm{Tr}\big(\hat{\delta}(\mathrm{g},\rho)\big)} \tag{55}$$

with $\hat{\delta}$ a G-action on $\mathcal{P}(\mathcal{H})$ satisfying

$$\hat{\delta}(\mathrm{g},\lambda\omega) = \lambda\hat{\delta}(\mathrm{g},\omega) \tag{56}$$

for every $\mathrm{g} \in G$, for every $\omega \in \mathcal{P}(\mathcal{H})$, and for every $\lambda > 0$. Moreover, among all those actions $\hat{\delta}$ satisfying the properties discussed above, there is a preferred action $\hat{\delta}_0$ (the action $\hat{\beta}$ in Equation (10) for $\mathcal{GL}(\mathcal{H})$, and the action $\hat{\gamma}$ in Equation (19) for $T^*\mathcal{U}(\mathcal{H})$) such that every relevant action $\hat{\delta}$ can be written as

$$\hat{\delta}_\phi = \phi^{-1} \circ \hat{\delta}_0 \circ (\mathrm{Id}_G \times \phi) \tag{57}$$

with $\phi\colon \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ a smooth diffeomorphism arising from a smooth diffeomorphism $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ by means of functional calculus and such that

$$\phi\Big(\mathbf{U}\omega\mathbf{U}^\dagger\Big) = \mathbf{U}\phi(\omega)\mathbf{U}^\dagger \tag{58}$$

and

$$\phi(\omega) = \sum_{j=1}^{n} \phi(\omega_j)|e_j\rangle\langle e_j| \tag{59}$$

where $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ is a basis of $\mathcal{H}$ made of eigenvectors of ω.

Equation (57) implies that the map ϕ is equivariant with respect to the action $\hat{\delta}_\phi$ and $\hat{\delta}_0$, which in turn implies that the fundamental vector fields of $\hat{\delta}_\phi$ are ϕ-related with that of $\hat{\delta}_0$ (*cfr.* chapter 5 in [40]). By the very definition of ϕ-relatedness (*cfr.* Chapter 4 in [40]), denoting with ζ^ϕ a fundamental vector field of $\hat{\delta}_\phi$ and with ζ a fundamental vector field of $\hat{\delta}_0$, it follows that

$$\zeta^\phi = T(\phi^{-1}) \circ \zeta \circ \phi. \tag{60}$$

We exploit Equation (60) to explicitly describe how the fundamental vector fields $\hat{Y}_{\mathbf{a}}$ of the action $\hat{\beta}$ of $\mathcal{GL}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ (*cfr.* Equations (10) and (16)) transform under ϕ. We then equate the result with the gradient vector field associated with the expectation value function $\hat{l}_{\mathbf{a}}$ by means of the metric tensor $\hat{G}_f$ as in Equation (53), thus obtaining an explicit characterization of the diffeomorphism ϕ and the operator monotone function f compatible with the equality. Finally, with this choice of ϕ and f, we prove that the gradient vector fields $Y^f_{\mathbf{a}}$ associated with the expectation value functions $l_{\mathbf{a}}$ on $\mathscr{S}(\mathcal{H})$ by means of the monotone quantum metric G_f as in Equation (46) correspond to the fundamental vector fields $Z^{\phi}_{\mathbf{a}\,\mathbf{0}}$ of the action β_ϕ of $\mathcal{GL}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ associated with the action $\hat{\beta}_\phi$ on $\mathcal{P}(\mathcal{H})$.

A similar procedure is then applied to the fundamental vector fields $\hat{W}_{\mathbf{a}\,\mathbf{0}}$ of the action $\hat{\gamma}$ of $T^*\mathcal{U}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ (*cfr.* Equations (19) and (22)).

4.1. The General Linear Group

Following [6,27], when considering the general linear group $\mathcal{GL}(\mathcal{H})$, the reference action $\hat{\delta}_0$ appearing in Equation (57) is the action $\hat{\beta}$ in Equation (10). Therefore, denoting with $\hat{Z}_{\mathbf{a}\,\mathbf{b}}$ a fundamental vector field of $\hat{\delta}_0$ and with $\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{b}}$ a fundamental vector field of $\hat{\delta}_\phi$, from Equations (14) and (60), and [61] (Theorem 5.3.1), it follows that

$$\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = (\phi^{-1})^{[1]}(\phi(\omega))\Box\{\mathbf{a}, \phi(\omega)\}, \tag{61}$$

where $\Box$ denotes the Schur product with respect to the basis of eigenvectors of $\phi(\omega)$, and

$$(\phi^{-1})^{[1]}(\phi(\omega)) = \sum_{\omega_j=\omega_k} \frac{1}{\phi'(\omega_j)}\,|e_j\rangle\langle e_k| + \sum_{\omega_j\neq\omega_k} \frac{\omega_j-\omega_k}{\phi(\omega_j)-\phi(\omega_k)}|e_j\rangle\langle e_k|, \tag{62}$$

with $\phi(\omega_j)$ the eigenvalues of $\phi(\omega)$ and with $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ the basis of $\mathcal{H}$ of eigenvectors of $\phi(\omega)$ and ω (*cfr.* Equation (59)). Moreover, a direct computation shows that

$$\{\phi(\omega), \mathbf{a}\} = \frac{1}{2}\sum_{j,k}(\phi(\omega_j)+\phi(\omega_k))a_{jk}|e_j\rangle\langle e_k|, \tag{63}$$

where a_{jk} are the components of $\mathbf{a}$ in the basis given by the eigenvectors of ω. We thus conclude that

$$\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = \sum_{\omega_j=\omega_k} a_{jk}\frac{\phi(\omega_j)}{\phi'(\omega_j)}|e_j\rangle\langle e_k| + \frac{1}{2}\sum_{\omega_j\neq\omega_k} a_{jk}(\phi(\omega_j)+\phi(\omega_k))\frac{\omega_j-\omega_k}{\phi(\omega_j)-\phi(\omega_k)}|e_j\rangle\langle e_k|. \tag{64}$$

Now, we require that $\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}$ is the gradient vector field of the expectation value function $\hat{l}_{\mathbf{a}}$ with respect to the metric tensor $\hat{G}_f$ defined as in Section 3 in order to characterize the function f. From the very definition of the gradient vector field, it follows that

$$dl_{\mathbf{a}}(\Gamma)\big|_\omega = \left(G_f\right)_\omega(\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega), \Gamma(\omega)) = \kappa\,\mathrm{Tr}\left(\Gamma(\omega)\left(\mathbf{K}^f_\omega\right)^{-1}\left(\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega)\right)\right) \tag{65}$$

holds for any vector field Γ on $\mathcal{P}(\mathcal{H})$. On the other hand, it also holds that

$$dl_{\mathbf{a}}(\Gamma)\big|_\omega = \Gamma(l_{\mathbf{a}})|_\omega = \mathrm{Tr}\{(\mathbf{a}\Gamma(\omega))\}, \tag{66}$$

so that, comparing Equation (65) with Equation (66), we obtain

$$\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = \kappa^{-1}\mathbf{K}^f_\omega(\mathbf{a}) = \kappa^{-1}f(\mathbf{L}_\omega\mathbf{R}^{-1}_\omega)\mathbf{R}_\omega(\mathbf{a}). \tag{67}$$

Exploiting Equation (49), it follows that Equation (67) becomes

$$\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = \sum_{\omega_j=\omega_k} \frac{\omega_j a_{jk}}{\kappa}|e_j\rangle\langle e_k| + \sum_{\omega_j\neq\omega_k}\frac{\omega_k}{\kappa}f\left(\frac{\omega_j}{\omega_k}\right)|e_j\rangle\langle e_k|. \tag{68}$$

Comparing Equation (64) with Equation (68), we obtain

$$\frac{\phi(\omega_j)}{\phi'(\omega_j)} = \kappa^{-1}\omega_j \tag{69}$$

and

$$(\phi(\omega_j)+\phi(\omega_k))\frac{\omega_j-\omega_k}{\phi(\omega_j)-\phi(\omega_k)} = \frac{\omega_k}{\kappa} f\left(\frac{\omega_j}{\omega_k}\right). \tag{70}$$

Equation (69) implies

$$\phi(x) = c\,x^{\kappa}, \tag{71}$$

with $c>0$, so that, because of Equation (70), the function f in $\hat{G}_f$ must be of the form

$$f(x) = \frac{\kappa}{2}\frac{(x-1)(x^{\kappa}+1)}{x^{\kappa}-1}. \tag{72}$$

A direct check shows that the function f in Equation (72) satisfies the properties listed in Equation (43) for all $\kappa>0$, but we do not know if it is operator monotone for every $\kappa>0$. The following proposition shows that f is operator monotone if and only if $0<\kappa\leq 1$.

Proposition 1. *The function f in Equation (72) is operator monotone if and only if $0<\kappa<1$.*

Proof. When $\kappa=1$ it is $f(x)=\frac{1+x}{2}$ which is known to be operator monotone and to be associated with the Bures–Helstrom metric tensor [1].

The function f as in Equation (72) is clearly C^1 in $(0,+\infty)$ and it is continuous in $[0,+\infty)$. When $\kappa>1$, it holds

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}\frac{\kappa}{2}\left(-\frac{1+x^{\kappa}}{1-x^{\kappa}} + 2\kappa\frac{x^{\kappa-1}(1-x)}{(x^{\kappa}-1)^2}\right) = -\frac{\kappa}{2} < 0 \tag{73}$$

which means that there is $\epsilon>0$ such that $f(x)$ is decreasing for $x\in(0,\epsilon)$, and thus, f cannot be operator monotone. Note that (73) is no longer valid when $0<\kappa<1$ because of the term $x^{\kappa-1}$

When $\kappa\in(0,1)$, we consider the rational case $\kappa=\frac{k}{n}$ with $k<n$ since the passage to an irrational $\kappa\in(0,1)$ is obtained by continuity just as in [62] (Proposition 3.1). Following [62] (Proposition 3.1), we write

$$1-x^{\frac{k}{n}} = (1-x^{\frac{1}{n}})\sum_{l=0}^{k-1} x^{\frac{l}{n}} \tag{74}$$

so that

$$\begin{aligned} f(x) &= \frac{k}{2n}\frac{(x-1)(x^{\frac{k}{n}}+1)}{x^{\frac{k}{n}}-1} = \frac{k(x^{\frac{k}{n}}+1)}{2n}\left(\frac{\sum_{l=0}^{n-1}x^{\frac{l}{n}}}{\sum_{j=0}^{k-1}x^{\frac{j}{n}}}\right) = \frac{k(x^{\frac{k}{n}}+1)}{2n}\left(1+\frac{\sum_{l=k}^{n-1}x^{\frac{l}{n}}}{\sum_{j=0}^{k-1}x^{\frac{j}{n}}}\right) = \\ &= \frac{k}{2n}\left(x^{\frac{k}{n}}+1+\sum_{l=k}^{n-1}\left(\sum_{j=0}^{k-1}x^{\frac{j-l}{n}}\right)^{-1} + \sum_{l=k}^{n-1}\left(\sum_{j=0}^{k-1}x^{\frac{j-l-k}{n}}\right)^{-1}\right). \end{aligned} \tag{75}$$

Since $j<k<l<n$, the functions

$$g(x) = \left(\sum_{j=0}^{k-1}x^{\frac{j-l}{n}}\right)^{-1} \quad \text{and} \quad h(x) = \left(\sum_{j=0}^{k-1}x^{\frac{j-l-k}{n}}\right)^{-1} \tag{76}$$

are operator monotone according to [62] (Theorem LH-1), and thus, the function f in Equation (75) is operator monotone because it is the sum of operator monotone functions. □

Finally, when ϕ is as in Equation (69) and f is as in Equation (72), we prove that the fundamental vector fields $Z^{\phi}_{\mathbf{a}\,\mathbf{0}}$ of the normalized action β_{ϕ} of $\mathcal{GL}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ associated with $\hat{\beta}_{\phi}$ by means of Equation (55) are indeed the gradient vector fields associated with the expectation value functions $l_{\mathbf{a}}$ by means of the monotone metric tensor G_f. Indeed, from Equation (55), it follows that

$$\begin{aligned} Z^{\phi}_{\mathbf{a}\,\mathbf{0}}(\rho) &= \frac{d}{dt}\beta_{\phi}\left(\exp\left(\frac{t}{2}(\mathbf{a},\mathbf{0})\right),\rho\right)\Big|_{t=0} = \frac{d}{dt}\,\frac{\hat{\beta}_{\phi}(\exp(\frac{t}{2}(\mathbf{a},\mathbf{0})),j(\rho))}{\mathrm{Tr}(\hat{\beta}_{\phi}(\exp(\frac{t}{2}(\mathbf{a},\mathbf{0})),j(\rho)))}\Big|_{t=0} = \\ &= \hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(j(\rho)) - \mathrm{Tr}\left(\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(j(\rho))\right)\Delta(j(\rho)). \end{aligned} \tag{77}$$

Equation (77) is equivalent to

$$T_{\rho}i\left(Z^{\phi}_{\mathbf{a}\,\mathbf{0}}(\rho)\right) = \hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(j(\rho)) - \mathrm{Tr}\left(\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}(j(\rho))\right)\Delta(j(\rho)) \tag{78}$$

for all $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$ and all $\rho \in \mathscr{S}(\mathcal{H})$, and this last instance is equivalent to the fact that $Z^{\phi}_{\mathbf{a}\,\mathbf{0}}$ is i-related with the vector field $\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}} - (\mathcal{L}_{\hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}}\hat{l}_{\mathbb{I}})\Delta$ for all $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$.

We now set

$$\hat{Y}^{\phi}_{\mathbf{a}} := \hat{Z}^{\phi}_{\mathbf{a}\,\mathbf{0}}, \quad \text{and} \quad Y^{\phi}_{\mathbf{a}} := Z^{\phi}_{\mathbf{a}\,\mathbf{0}}. \tag{79}$$

To finish the proof of the proposition, we need to prove that $Y^{\phi}_{\mathbf{a}}$ is actually the gradient vector field of the expectation value function $l_{\mathbf{a}}$ for every $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$. For this purpose, we compute

$$\begin{aligned} (G_f(Y^{\phi}_{\mathbf{a}},V))(\rho) &= (j^*\hat{G}_f(Y^{\phi}_{\mathbf{a}},V))(\rho) = (\hat{G}_f)_{j(\rho)}(T_{\rho}j(Y_{\mathbf{a}}(\rho)),T_{\rho}j(V(\rho))) = \\ &\overset{(78)}{=} (\hat{G}_f)_{j(\rho)}\left(\hat{Y}^{\phi}_{\mathbf{a}}(j(\rho)) - \mathrm{Tr}\left(\hat{Y}^{\phi}_{\mathbf{a}}(j(\rho))\right)\Delta(j(\rho)),T_{\rho}j(V(\rho))\right). \end{aligned} \tag{80}$$

Since we proved that $\hat{Y}^{\phi}_{\mathbf{a}}$ is be the gradient vector field associated with $\hat{l}_{\mathbf{a}}$ by means of $\hat{G}_f$, Equation (80) becomes

$$(G_f(Y^{\phi}_{\mathbf{a}},V))(\rho) = (\mathcal{L}_V l_{\mathbf{a}})(\rho) - \mathrm{Tr}\left(\hat{Y}^{\phi}_{\mathbf{a}}(j(\rho))\right)(\hat{G}_f)_{j(\rho)}(\Delta(j(\rho)),T_{\rho}j(V(\rho))). \tag{81}$$

The second term on the right-hand-side of Equation (81) vanishes. Indeed, Equation (53) implies that

$$(\hat{G}_f)_{j(\rho)}(\Delta(j(\rho)),T_{\rho}j(V(\rho))) = \mathrm{Tr}\left(T_{\rho}j(V(\rho))\left(\mathbf{K}^f_{j(\rho)}\right)^{-1}(\Delta(j(\rho)))\right). \tag{82}$$

From Equation (54), we conclude that Equation (82) becomes

$$(\hat{G}_f)_{j(\rho)}(\Delta(j(\rho)),T_{\rho}j(V(\rho))) = \mathrm{Tr}(T_{\rho}j(V(\rho))) = 0, \tag{83}$$

where the last equality follows from Equation (27). Inserting Equation (83) in Equation (81), we obtain

$$(G_f(Y^{\phi}_{\mathbf{a}},V))(\rho) = (\mathcal{L}_V l_{\mathbf{a}})(\rho) \tag{84}$$

for every fundamental vector field of β_{ϕ} of the type $Y^{\phi}_{\mathbf{a}}$, for every vector field V on $\mathscr{S}(\mathcal{H})$, and for and every $\rho \in \mathscr{S}(\mathcal{H})$. Equation (84) is equivalent to the fact that $Y^{\phi}_{\mathbf{a}}$ is the gradient vector field associated with the expectation value function $l_{\mathbf{a}}$ by means of G_f for every $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$ as desired.

Collecting the results proved in this subsection, we obtain the following proposition.

Proposition 2. *The function f given by*

$$f(x) = \frac{\kappa}{2}\frac{(x-1)(x^{\kappa}+1)}{x^{\kappa}-1} \tag{85}$$

is operator monotone and satisfies Equation (43) if and only if $0 < \kappa \leq 1$. In these cases, denoting with $\{X_{\mathbf{b}}\}_{\mathbf{b}\in\mathcal{B}_{sa}(\mathcal{H})}$ the fundamental vector fields of the canonical action α of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ as in Equation (4), if G_f is the associated monotone quantum metric tensor on $\mathscr{S}(\mathcal{H})$ as in Equation (46) and $Y^f_{\mathbf{a}}$ is the gradient vector field associated with the expectation value function $l_{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$, the family $\{Y^f_{\mathbf{a}}, X_{\mathbf{b}}\}_{\mathbf{a},\mathbf{b}\in\mathcal{B}_{sa}(\mathcal{H})}$ of vector fields on $\mathscr{S}(\mathcal{H})$ close an anti-representation of the Lie algebra of the general linear group $\mathcal{GL}(\mathcal{H})$ integrating to the group action

$$\beta^{\kappa}(\mathrm{g},\rho) = \frac{\left(\mathrm{g}\rho^{\sqrt{\kappa}}\mathrm{g}^{\dagger}\right)^{\frac{1}{\sqrt{\kappa}}}}{\mathrm{Tr}\left(\left(\mathrm{g}\rho^{\sqrt{\kappa}}\mathrm{g}^{\dagger}\right)^{\frac{1}{\sqrt{\kappa}}}\right)}. \tag{86}$$

The action β^{κ} in Equation (86) is transitive on $\mathscr{S}(\mathcal{H})$ for every $0 < \kappa \leq 1$. In particular, when $\kappa = 1$, we recover the Bures–Helstrom metric tensor and the action β in Equation (1), while when $\kappa = \frac{1}{4}$, we recover the Wigner–Yanase metric tensor and the action β_{WY} in Equation (2).

4.2. The Cotangent Group of the Unitary Group

Following what is done in Section 4.1, we consider an action $\hat{\gamma}_{\phi}$ associated with the action $\hat{\gamma}$ (*cfr.* Equation (19)) by means of Equation (55) with $\hat{\delta}_0 \equiv \hat{\gamma}$. The fundamental vector fields $\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}$ of $\hat{\gamma}_{\phi}$ are obtained as follows. From Equations (22) and (60), and [61] (Theorem 5.3.1), it follows that

$$\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = (\phi^{-1})^{[1]}(\phi(\omega))\square\,\hat{W}_{\mathbf{a}\,\mathbf{0}}(\phi(\omega)), \tag{87}$$

where $\square$ denotes the Schur product with respect to the basis of eigenvectors of $\phi(\omega)$, and

$$(\phi^{-1})^{[1]}(\phi(\omega)) = \sum_{\omega_j=\omega_k}\frac{1}{\phi'(\omega_j)}|e_j\rangle\langle e_k| + \sum_{\omega_j\neq\omega_k}\frac{\omega_j-\omega_k}{\phi(\omega_j)-\phi(\omega_k)}|e_j\rangle\langle e_k|, \tag{88}$$

with $\phi(\omega_j)$ the eigenvalues of $\phi(\omega)$ and with $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ the basis of $\mathcal{H}$ of eigenvectors of $\phi(\omega)$ and ω (*cfr.* Equation (59)). On the other hand, from Equations (22) and (59), it follows that

$$\begin{aligned}\hat{W}_{\mathbf{a}\,\mathbf{0}}(\phi(\omega)) &= \int_0^1 \mathrm{d}\lambda\left((\phi(\omega))^{\lambda}\,\mathbf{a}\,(\phi(\omega))^{1-\lambda}\right) = \\ &= \sum_{\omega_j=\omega_k}\phi(\omega_j)a_{jk}|e_j\rangle\langle e_k| + \sum_{\omega_j\neq\omega_k}\frac{\phi(\omega_j)-\phi(\omega_k)}{\ln\left(\frac{\phi(\omega_j)}{\phi(\omega_k)}\right)}a_{jk}|e_j\rangle\langle e_k|,\end{aligned} \tag{89}$$

so that, exploiting Equations (88) and (89), Equation (87) becomes

$$\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = \sum_{\omega_j=\omega_k}\frac{\phi(\omega_j)}{\phi'(\omega_j)}a_{jk}|e_j\rangle\langle e_k| + \sum_{\omega_j\neq\omega_k}\frac{\omega_j-\omega_k}{\ln\left(\frac{\phi(\omega_j)}{\phi(\omega_k)}\right)}a_{jk}|e_j\rangle\langle e_k| \tag{90}$$

In analogy with what is done in Section 4.1, we now require that $\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}$ is the gradient vector field of the expectation value function $\hat{l}_{\mathbf{a}}$ with respect to a metric tensor $\hat{G}_f$ defined as in Section 3 in order to characterize the function f. From the very definition of gradient vector field, it follows that

$$dl_{\mathbf{a}}(\Gamma)\big|_{\omega} = \left(G_f\right)_{\omega}(\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega), \Gamma(\omega)) = \kappa\,\mathrm{Tr}\left(\Gamma(\omega)\left(\mathbf{K}^f_{\omega}\right)^{-1}\left(\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega)\right)\right) \tag{91}$$

holds for any vector field Γ on $\mathcal{P}(\mathcal{H})$. On the other hand, it also holds that

$$dl_{\mathbf{a}}(\Gamma)\big|_{\omega} = \Gamma(l_{\mathbf{a}})|_{\omega} = \mathrm{Tr}\{(\mathbf{a}\Gamma(\omega))\}, \tag{92}$$

so that, comparing Equation (91) with Equation (92), we obtain

$$\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = \kappa^{-1}\mathbf{K}^{f}_{\omega}(\mathbf{a}) = \kappa^{-1}f(\mathbf{L}_{\omega}\mathbf{R}^{-1}_{\omega})\mathbf{R}_{\omega}(\mathbf{a}). \tag{93}$$

Exploiting Equation (49), it follows that Equation (93) becomes

$$\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(\omega) = \sum_{\omega_j=\omega_k} \frac{\omega_j a_{jk}}{\kappa} |e_j\rangle\langle e_k| + \sum_{\omega_j\neq\omega_k} \frac{\omega_k}{\kappa} f\left(\frac{\omega_j}{\omega_k}\right) |e_j\rangle\langle e_k|. \tag{94}$$

Comparing Equation (90) with Equation (94), we obtain

$$\frac{\phi(\omega_j)}{\phi'(\omega_j)} = \kappa^{-1}\omega_j \tag{95}$$

and

$$\frac{\omega_j - \omega_k}{\ln\left(\frac{\phi(\omega_j)}{\phi(\omega_k)}\right)} = \frac{\omega_k}{\kappa} f\left(\frac{\omega_j}{\omega_k}\right). \tag{96}$$

Equation (95) implies

$$\phi(x) = c\, x^{\kappa}, \tag{97}$$

with $c > 0$, and it is worth noting that the family of diffeomorphisms found here is the same as that found in Section 4.1 in the case of the general linear group $\mathcal{GL}(\mathcal{H})$ (*cfr.* Equation (71)). Because of Equations (96) and (97), the function f in $\hat{G}_f$ must be of the form

$$f(x) = \kappa\, \frac{x-1}{\ln(x)}, \tag{98}$$

which is precisely the operator monotone function associated with the Bogoliubov–Kubo–Mori metric tensor up to the constant κ [1]. Note that the positive constant κ is here arbitrary differently from what happens for $\mathcal{GL}(\mathcal{H})$ (*cfr.* Section 4.1).

It is a matter of direct computation to check that the form of ϕ in Equation (97) implies that the action $\hat{\gamma}_{\phi}$ associated with the action $\hat{\gamma}$ (*cfr.* Equation (19)) by means of Equation (55) with $\hat{\delta}_0 \equiv \hat{\gamma}$ reads

$$\hat{\gamma}_{\phi}((\mathbf{U},\mathbf{a}),\omega) = \hat{\gamma}\left(\left(\mathbf{U},\frac{\mathbf{a}}{\kappa}\right),\omega\right), \tag{99}$$

so that

$$\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}} = \hat{W}_{\frac{\mathbf{a}}{\kappa}\,\mathbf{0}} = \frac{1}{\kappa}\,\hat{W}_{\mathbf{a}\,\mathbf{0}} \tag{100}$$

(*cfr.* Equations (89), (90) and (97)). Consequently, the fundamental vector fields $W^{\phi}_{\mathbf{a},\mathbf{0}}$ of the normalized action γ_{ϕ} associated with $\hat{\gamma}_{\phi}$ by means of Equation (55) read

$$\begin{aligned} W^{\phi}_{\mathbf{a}\,\mathbf{0}}(\rho) &= \frac{d}{dt}\frac{\hat{\gamma}_{\phi}\left(\exp\left(\frac{t}{2}(\mathbf{a},\mathbf{0})\right),j(\rho)\right)}{\mathrm{Tr}\left(\hat{\gamma}_{\phi}\left(\exp\left(\frac{t}{2}(\mathbf{a},\mathbf{0})\right),j(\rho)\right)\right)}\Bigg|_{t=0} = \\ &= \hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(j(\rho)) - \mathrm{Tr}\left(\hat{W}^{\phi}_{\mathbf{a}\,\mathbf{0}}(j(\rho))\right)\Delta(j(\rho)) = \\ &= \frac{1}{\kappa}\left(\hat{W}_{\mathbf{a}\,\mathbf{0}}(j(\rho)) - \mathrm{Tr}\left(\hat{W}_{\mathbf{a}\,\mathbf{0}}(j(\rho))\right)\Delta(j(\rho))\right). \end{aligned} \tag{101}$$

Equation (101) is equivalent to

$$T_{\rho}i\left(W^{\phi}_{\mathbf{a}\,\mathbf{0}}(\rho)\right) = \frac{1}{\kappa}\left(\hat{W}_{\mathbf{a}\,\mathbf{0}}(j(\rho)) - \mathrm{Tr}\left(\hat{W}_{\mathbf{a}\,\mathbf{0}}(j(\rho))\right)\Delta(j(\rho))\right) \tag{102}$$

for all $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$ and all $\rho \in \mathscr{S}(\mathcal{H})$, and this last instance is equivalent to the fact that $W^{\phi}_{\mathbf{a}\,\mathbf{0}}$ is j-related with the vector field $\frac{1}{\kappa}\left(\hat{W}_{\mathbf{a}\,\mathbf{0}} - (\mathcal{L}_{\hat{W}_{\mathbf{a}\,\mathbf{0}}}\hat{l}_{\mathbb{I}})\Delta\right)$ for all $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$.

Now, proceeding in complete analogy with what is done in Section 4.1, it is possible to prove that, when ϕ and f are as in Equations (97) and (98), respectively, then the fundamental vector field $W^{\phi}_{\mathbf{a}\,\mathbf{0}}$ is the gradient vector field associated with the expectation value function $l_{\mathbf{a}}$ by means of the monotone quantum metric tensor G_f (coinciding with the Bogoliubov–Kubo–Mori metric tensor up to the constant κ) for all $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$. Collecting the results in this subsection, we obtain the following proposition.

Proposition 3. *Given the operator monotone function*

$$f(x) = \kappa\,\frac{x-1}{\ln(x)} \tag{103}$$

satisfying Equation (43) and associated with the Bogoliubov–Kubo–Mori metric tensor $G_f \equiv \kappa G_{BKM}$ (up to the constant factor $\kappa > 0$) through Equation (46) [1], denoting with $\{X_{\mathbf{b}}\}_{\mathbf{b}\in\mathcal{B}_{sa}(\mathcal{H})}$ the fundamental vector fields of the canonical action α of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ as in Equation (4), and denoting with $V^{f}_{\mathbf{a}}$ the gradient vector field associated with the expectation value function $l_{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$ by means of G_f, the family $\{V^{f}_{\mathbf{a}}, X_{\mathbf{b}}\}_{\mathbf{a},\mathbf{b}\in\mathcal{B}_{sa}(\mathcal{H})}$ of vector fields on $\mathscr{S}(\mathcal{H})$ closes an anti-representation of the Lie algebra of the cotangent group $T^{*}\mathcal{U}(\mathcal{H})$, integrating to the group action

$$\gamma^{\kappa}((\mathbf{U},\mathbf{a}),\rho) := \frac{\mathrm{e}^{U\ln(\rho)U^{\dagger}+\frac{\mathbf{a}}{\kappa}}}{\mathrm{Tr}\left(\mathrm{e}^{U\ln(\rho)U^{\dagger}+\frac{\mathbf{a}}{\kappa}}\right)} \tag{104}$$

The action γ^{κ} in Equation (104) is transitive on $\mathscr{S}(\mathcal{H})$ for every $\kappa > 0$.

5. Conclusions

There are several ways in which the results presented here can be further developed in order to fully understand how the 2-dimensional picture discussed in [27] extends to arbitrary finite dimensions.

First of all, concerning the Lie group $\mathcal{GL}(\mathcal{H})$, it is necessary to understand if there exist smooth transitive actions on $\mathcal{P}(\mathcal{H})$ that are not of the form β_{ϕ} (*cfr.* Equations (10) and (57)). Then, it is necessary to understand if there exist smooth transitive actions on $\mathscr{S}(\mathcal{H})$ that do not arise from smooth actions of $\mathcal{GL}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ as in Equation (55). If the answer to both these questions are negative, then it follows that the only actions of $\mathcal{GL}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ whose associated Lie algebra anti-representations can be described in terms of the fundamental vector fields of the standard action of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ (*cfr.* Equations (4) and (29)) and the gradient vector fields $Y^{f}_{\mathbf{a}}$ associated with the expectation value functions $l_{\mathbf{a}}$ by means of a suitable monotone quantum metric tensor are those found in this work.

Concerning the group $T^{*}\mathcal{U}(\mathcal{H})$, it is necessary to understand if there exist smooth transitive actions on $\mathscr{S}(\mathcal{H})$ that do not arise from smooth actions of $T^{*}\mathcal{U}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$ as in Equation (55). If the answer to this question is negative, then it follows that the only action of $T^{*}\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ whose associated Lie algebra anti-representations can be described in terms of the fundamental vector fields of the standard action of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ (*cfr.* Equations (4) and (29)) and the gradient vector fields associated with the expectation value functions $l_{\mathbf{a}}$ by means of a suitable monotone quantum metric tensor are the ones found in this work, that is, the one associated with the Bogoliubov–Kubo–Mori metric tensor.

Besides the cases involving the Lie groups $\mathcal{GL}(\mathcal{H})$ and $T^{*}\mathcal{U}(\mathcal{H})$, it is also necessary to understand if, for a quantum system whose Hilbert space $\mathcal{H}$ has dimension greater than 2, there exists other Lie groups acting smoothly and transitively on $\mathscr{S}(\mathcal{H})$ and whose Lie algebra anti-representation can be described in terms of the fundamental vector fields of the standard action of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ (*cfr.* Equations (4) and (29)) and the gradient vector fields associated with the expectation value functions $l_{\mathbf{a}}$ by means of suitable monotone quantum metric tensors. Concerning this instance, something can be said on some general properties any such Lie group $\mathscr{G}$ must possess. First of all, the unitary group $\mathcal{U}(\mathcal{H})$ must appear as a subgroup of $\mathscr{G}$ and $\dim(\mathscr{G}) = 2\dim(\mathcal{U}(\mathcal{H}))$. This last condition follows from the fact that the gradient vector fields associated with the expectation value functions $l_{\mathbf{a}}$ are labeled by elements in $\mathcal{B}_{sa}(\mathcal{H})$, and thus, the dimension of the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$ is twice that of the Lie algebra of $\mathcal{U}(\mathcal{H})$. From this last observation, it also follows that

$$\mathfrak{g} \cong \mathfrak{u}(\mathcal{H}) \oplus \mathcal{B}_{sa}(\mathcal{H}) \tag{105}$$

as a vector space. Moreover, since $\mathcal{U}(\mathcal{H})$ must be a subgroup of $\mathscr{G}$, there must be a decomposition of $\mathfrak{g}$ as in Equation (105) for which $\mathfrak{u}(\mathcal{H}) \oplus \{\mathbf{0}\}$ is a Lie subalgebra isomorphic to $\mathfrak{u}(\mathcal{H})$. Then, as already argued in [6], the requirement that the fundamental vector fields $X_{\mathbf{b}}$ of the standard action α of $\mathcal{U}(\mathcal{H})$ on $\mathscr{S}(\mathcal{H})$ are Killing vector fields for every monotone quantum metric tensor G_f imposes additional constraints on the possible commutator between these vector fields and the gradient vector fields $Y^f_{\mathbf{a}}$ associated with the expectation value functions $l_{\mathbf{a}}$. Specifically, since $Y^f_{\mathbf{a}}$ is the gradient vector field associated with the expectation value function $l_{\mathbf{a}}$ for every $\mathbf{a} \in \mathcal{B}_{sa}(\mathcal{H})$, it follows that

$$\begin{aligned}
\mathcal{L}_{[X_{\mathbf{b}},Y^f_{\mathbf{a}}]} l_{\mathbf{c}} &= \mathcal{L}_{X_{\mathbf{b}}}\left(\mathcal{L}_{Y^f_{\mathbf{a}}} l_{\mathbf{c}}\right) - \mathcal{L}_{Y^f_{\mathbf{a}}}\left(\mathcal{L}_{X_{\mathbf{b}}} l_{\mathbf{c}}\right) = \\
&= \mathcal{L}_{X_{\mathbf{b}}}\left(G_f\left(Y^f_{\mathbf{a}}, Y^f_{\mathbf{c}}\right)\right) - \mathcal{L}_{Y^f_{\mathbf{a}}}\left(l_{[\mathbf{b},\mathbf{c}]}\right) = \\
&= G_f\left([X_{\mathbf{b}}, Y^f_{\mathbf{a}}], Y^f_{\mathbf{c}}\right) + G_f\left(Y^f_{\mathbf{a}}, [X_{\mathbf{b}}, Y^f_{\mathbf{c}}]\right) - \mathcal{L}_{Y^f_{\mathbf{a}}}\left(l_{[\mathbf{b},\mathbf{c}]}\right) = \\
&= \mathcal{L}_{[X_{\mathbf{b}},Y^f_{\mathbf{a}}]} l_{\mathbf{c}} + G_f\left(Y^f_{\mathbf{a}}, [X_{\mathbf{b}}, Y^f_{\mathbf{c}}]\right) - G_f\left(Y^f_{\mathbf{a}}, Y^f_{[\mathbf{b},\mathbf{c}]}\right)
\end{aligned} \tag{106}$$

where we used Equation (37), and the fact that $\mathcal{L}_{X_{\mathbf{a}}} G_f = 0$ because the fundamental vector fields of the action α of $\mathcal{U}(\mathcal{H})$ are Killing vector fields for all monotone quantum metric tensors. From Equation (106), we conclude that

$$G_f\left(Y^f_{\mathbf{a}}, [X_{\mathbf{b}}, Y^f_{\mathbf{c}}] - Y^f_{[\mathbf{b},\mathbf{c}]}\right) = 0 \tag{107}$$

for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{B}_{sa}(\mathcal{H})$. Then, since the differential of the expectation value functions provide a basis for the differential forms on $\mathscr{S}(\mathcal{H})$, Equation (107) is equivalent to

$$[X_{\mathbf{b}}, Y^f_{\mathbf{c}}] = Y^f_{[\mathbf{b},\mathbf{c}]}. \tag{108}$$

Equation (108) fixes the Lie bracket between elements of $\mathfrak{u}(\mathcal{H}) \oplus \{\mathbf{0}\}$ and its complement, thus leaving us with the freedom to only define the bracket among elements that lies in the complement of $\mathfrak{u}(\mathcal{H}) \oplus \{\mathbf{0}\}$ inside the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$.

We are currently investigating all the problems discussed in this section and we plan to address them in detail in the (hopefully not too distant) future.

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