Review

Recent Advances of Constrained Variational Problems Involving Second-Order Partial Derivatives: A Review

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Abstract: This paper comprehensively reviews the nonlinear dynamics given by some classes of constrained control problems which involve second-order partial derivatives. Specifically, necessary optimality conditions are formulated and proved for the considered variational control problems governed by integral functionals. In addition, the well-posedness and the associated variational inequalities are considered in the present review paper.

Keywords: multi-time controlled Lagrangian of second-order; isoperimetric constraints; Euler–Lagrange equations; multiple integral; differential 1-form; curvilinear integral; variational inequalities

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1. Introduction

We all know that Calculus of Variations and Optimal Control Theory are two strongly connected mathematical fields. In this direction, several researchers have investigated these areas, achieving remarkable results (see Friedman [1], Hestenes [2], Kendall [3], Udrişte [4], Petrat and Tumulka [5], Treanţă [6] and Deckert and Nickel [7]). The problems (in several time variables) studied by the aforementioned researchers have been continued, in the last period, in the study of multi-dimensional optimization problems. These studies have many applications in different branches of mathematical sciences, web access problems, management science, portfolio selection, engineering design, query optimization in databases, game theory, and so on. In this respect, we mention the papers conducted by Mititelu and Treanţă [8], Treanţă [9–18], and Jayswal et al. [19]. For other connected but different ideas on this topic, the reader can consult Arisawa and Ishii [20], Lai and Motta [21], Shi et al. [22], An et al. [23], Zhao et al. [24], Hung et al. [25], Chen et al. [26], Antonsev and Shmarev [27], Cekic et al. [28], Chen et al. [29], Diening et al. [30], and Zhikov [31].

This review article is structured as follows. Section 2 introduces the second-order PDE-constrained optimal control problem under study (see Theorem 1). This result formulates the necessary conditions of optimality for the considered PDE-constrained optimization problem. Section 3 states the associated necessary optimality conditions for a new class of isoperimetric constrained control problems governed by multiple and curvilinear integrals. In Section 4, by using the pseudomonotonicity, hemicontinuity, and monotonicity of the considered integral functionals, we present the well-posedness of some variational inequality problems determined by partial derivatives of a second-order. Section 5 formulates some very important open problems to be investigated in the near future. Section 6 contains the conclusions of the paper.

2. Second-Order PDE-Constrained Control Problem

Let $\mathcal{H}_\zeta\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big)$, $\zeta = \overline{1,m}$ be some functions of C^3-class, called *multi-time controlled Lagrangians of second order*, where $t = (t^\alpha) = (t^1, \cdots, t^m) \in \Lambda_{t_0,t_1} \subset \mathbb{R}_+^m$, $b = (b^i) = \big(b^1, \cdots, b^n\big) : \Lambda_{t_0,t_1} \to \mathbb{R}^n$ is a function of C^4-class (the *state variable*) and $u = (u^\vartheta) = \big(u^1, \cdots, u^k\big) : \Lambda_{t_0,t_1} \to \mathbb{R}^k$ is a piecewise continuous function (the *control variable*). In addition, denote $b_\alpha(t) := \dfrac{\partial b}{\partial t^\alpha}(t)$, $b_{\alpha\beta}(t) := \dfrac{\partial^2 b}{\partial t^\alpha \partial t^\beta}(t)$, $\alpha, \beta \in \{1, ..., m\}$ and consider $\Lambda_{t_0,t_1} = [t_0, t_1]$ (*multi-time interval* in $\mathbb{R}_+^m$) as a hyper-parallelepiped determined by the diagonally opposite points t_0, $t_1 \in \mathbb{R}_+^m$. Moreover, we assume that the previous multi-time controlled Lagrangians of second order determine a closed controlled Lagrange 1-form

$$\mathcal{H}_\zeta\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big) dt^\zeta$$

(see summation over the repeated indices), which provides the following curvilinear integral functional:

$$J(b(\cdot), u(\cdot)) = \int_{\Upsilon_{t_0,t_1}} \mathcal{H}_\zeta\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big) dt^\zeta, \tag{1}$$

where Υ_{t_0,t_1} is a smooth curve, included in Λ_{t_0,t_1}, joining t_0, $t_1 \in \mathbb{R}_+^m$.

Second-order PDE-constrained control problem. *Find the pair (b^*, u^*) that minimizes the aforementioned controlled path-independent curvilinear integral functional Equation (1), among all the pair functions (b, u) satisfying*

$$b(t_0) = b_0, \quad b(t_1) = b_1, \quad b_\gamma(t_0) = \tilde{b}_{\gamma 0}, \quad b_\gamma(t_1) = \tilde{b}_{\gamma 1}$$

and the partial speed-acceleration constraints:

$$g_\zeta^a\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big) = 0, \quad a = 1, 2, \cdots, r \le n, \ \zeta = 1, 2, \cdots, m.$$

In order to investigate the above controlled optimization problem in Equation (1), associated with the aforementioned partial speed-acceleration constraints, we introduce the Lagrange multiplier $p = (p_a(t))$ and build new multi-time-controlled second-order Lagrangians (see summation over the repeated indices):

$$\mathcal{H}_{1\zeta}\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), p(t), t\big) = \mathcal{H}_\zeta\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big)$$

$$+ p_a(t) g_\zeta^a\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big), \quad \zeta = \overline{1,m},$$

which change the initial controlled optimization problem (with second-order PDE constraints) into a partial speed-acceleration, unconstrained, controlled optimization problem:

$$\min_{(b(\cdot), u(\cdot), p(\cdot))} \int_{\Upsilon_{t_0,t_1}} \mathcal{H}_{1\zeta}\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), p(t), t\big) dt^\zeta \tag{2}$$

$$b(t_q) = b_q, \quad b_\gamma(t_q) = \tilde{b}_{\gamma q}, \quad q = 0, 1,$$

if the Lagrange 1-form $\mathcal{H}_{1\zeta}\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), p(t), t\big) dt^\zeta$ is completely integrable.

In accordance with Lagrange theory, an extreme point of Equation (1) is found among the extreme points of Equation (2).

To formulate the necessary optimality conditions associated with the aforementioned control problem, we shall introduce the Saunders's multi-index (Saunders [32], Treanţă [9–12]).

The following theorem represents the main result of this section (see Treanţă [12]). It establishes the necessary conditions of optimality associated with the considered second-order PDE-constrained control problem.

Theorem 1 (Treanţă [12]). *If $(b^*(\cdot),\ u^*(\cdot),\ p^*(\cdot))$ solves Equation (2), then*

$$(b^*(\cdot),\ u^*(\cdot),\ p^*(\cdot))$$

solves the following Euler–Lagrange system of PDEs:

$$\frac{\partial \mathcal{H}_{1\zeta}}{\partial b^i} - D_\gamma \frac{\partial \mathcal{H}_{1\zeta}}{\partial b^i_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{H}_{1\zeta}}{\partial b^i_{\alpha\beta}} = 0, \quad i = \overline{1,n},\ \zeta = \overline{1,m}$$

$$\frac{\partial \mathcal{H}_{1\zeta}}{\partial u^\vartheta} - D_\gamma \frac{\partial \mathcal{H}_{1\zeta}}{\partial u^\vartheta_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{H}_{1\zeta}}{\partial u^\vartheta_{\alpha\beta}} = 0, \quad \vartheta = \overline{1,k},\ \zeta = \overline{1,m}$$

$$\frac{\partial \mathcal{H}_{1\zeta}}{\partial p_a} - D_\gamma \frac{\partial \mathcal{H}_{1\zeta}}{\partial p_{a,\gamma}} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{H}_{1\zeta}}{\partial p_{a,\alpha\beta}} = 0, \quad a = \overline{1,r},\ \zeta = \overline{1,m},$$

where $p_{a,\gamma} := \dfrac{\partial p_a}{\partial t^\gamma},\ p_{a,\alpha\beta} := \dfrac{\partial^2 p_a}{\partial t^\alpha \partial t^\beta},\ u^\vartheta_{\alpha\beta} := \dfrac{\partial^2 u^\vartheta}{\partial t^\alpha \partial t^\beta},\ \alpha,\ \beta,\ \gamma \in \{1,2,...,m\}.$

Remark 1 (Treanţă [12]). *The system of Euler–Lagrange PDEs given in Theorem 1 becomes*

$$\frac{\partial \mathcal{H}_{1\zeta}}{\partial b^i} - D_\gamma \frac{\partial \mathcal{H}_{1\zeta}}{\partial b^i_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{H}_{1\zeta}}{\partial b^i_{\alpha\beta}} = 0, \quad i = \overline{1,n},\ \zeta = \overline{1,m}$$

$$\frac{\partial \mathcal{H}_{1\zeta}}{\partial u^\vartheta} - D_\gamma \frac{\partial \mathcal{H}_{1\zeta}}{\partial u^\vartheta_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{H}_{1\zeta}}{\partial u^\vartheta_{\alpha\beta}} = 0, \quad \vartheta = \overline{1,k},\ \zeta = \overline{1,m}$$

$$g^a_\zeta\big(b(t), b_\gamma(t), b_{\alpha,\beta}(t), u(t), t\big) = 0, \quad a = 1,2,\cdots,r \le n,\ \zeta = 1,2,\cdots,m.$$

Remark 2 (Treanţă [12]). *(i) The most general Lagrange 1-form that can be used in the previous problem is of the form:*

$$\mathcal{H}_{2\zeta}\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), p(t), t\big) = \mathcal{H}_\zeta\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big)$$

$$+ p^\lambda_{a\zeta}(t) g^a_\lambda\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big).$$

(ii) The closeness conditions $D_\theta \mathcal{H}_\zeta = D_\zeta \mathcal{H}_\theta$ associated with the Lagrange 1-form $\mathcal{H}_\zeta\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big) dt^\zeta$ are actually PDE constraints for the considered problem. The optimization problem of the controlled curvilinear integral cost functional $J(b(\cdot), u(\cdot))$, conditioned by $D_\theta \mathcal{H}_\zeta = D_\zeta \mathcal{H}_\theta$, can be studied by using the following Lagrange 1-form:

$$\mathcal{H}_{3\zeta}\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), p(t), t\big) = \mathcal{H}_\zeta\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big)$$

$$+ p^{\theta\lambda}_\zeta(t)(D_\theta \mathcal{H}_\lambda - D_\lambda \mathcal{H}_\theta).$$

Illustrative example. Minimize the following objective functional:

$$J(b(\cdot), u(\cdot)) = \int_{\Upsilon_{0,1}} \Big(b^2(t) + u^2(t)\Big) dt^1 + \Big(b^2(t) + u^2(t)\Big) dt^2$$

subject to $b_{t^1}(t) + b_{t^2}(t) = 0$, $b(0,0) = 0$, $b(1,1) = 0$, where $\Upsilon_{0,1}$ is a curve of C^1-class in $[0,1]^2$, joining $(0,0)$ and $(1,1)$.

Solution. The path-independence of the functional $J(b(\cdot), u(\cdot))$ gives:

$$b\Big(\frac{\partial b}{\partial t^2} - \frac{\partial b}{\partial t^1}\Big) = u\Big(\frac{\partial u}{\partial t^1} - \frac{\partial u}{\partial t^2}\Big).$$

Moreover, for the Lagrange 1-form (Remark 2), we obtain:

$$\Theta_{11} = b^2(t) + u^2(t) + \omega_1(t)(b_{t^1}(t) + b_{t^2}(t)),$$

$$\Theta_{12} = b^2(t) + u^2(t) + +\omega_2(t)(b_{t^1}(t) + b_{t^2}(t))$$

and the extreme points are formulated as below:

$$2s - \frac{\partial \omega_1}{\partial t^1} - \frac{\partial \omega_1}{\partial t^2} = 0, \quad 2s - \frac{\partial \omega_2}{\partial t^1} - \frac{\partial \omega_2}{\partial t^2} = 0,$$

$$2u = 0,$$

$$b_{t^1}(t) + b_{t^2}(t) = 0.$$

It follows that $(b^*, u^*) = (0,0)$ is the optimal point of the considered optimization problem, and satisfies $\frac{\partial \phi}{\partial t^1} + \frac{\partial \phi}{\partial t^2} = 0$, where $\phi := \omega_1 - \omega_2$.

3. Isoperimetric Constrained Controlled Optimization Problem

In this section, we use similar notations as in the previous section. We consider a C^3-class function $\mathcal{H}(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t)$, called *multi-time-controlled, second-order Lagrangian*, where $t = (t^\alpha) = (t^1, \cdots, t^m) \in \Lambda_{t_0,t_1} \subset \mathbb{R}_+^m$, $b = (b^i) = (b^1, \cdots, b^n) : \Lambda_{t_0,t_1} \to \mathbb{R}^n$ is a function of the C^4-class (the *state variable*), and $u = (u^\vartheta) = (u^1, \cdots, u^k) : \Lambda_{t_0,t_1} \to \mathbb{R}^k$ is a piecewise continuous function (the *control variable*). In addition, denote $b_\alpha(t) := \frac{\partial b}{\partial t^\alpha}(t)$, $b_{\alpha\beta}(t) := \frac{\partial^2 b}{\partial t^\alpha \partial t^\beta}(t)$, $\alpha, \beta \in \{1, ..., m\}$, and consider $\Lambda_{t_0,t_1} = [t_0, t_1]$ as a hyper-parallelepiped generated by the diagonally opposite points $t_0, t_1 \in \mathbb{R}_+^m$.

Isoperimetric constrained control problem. *Find the pair (b^*, u^*) that minimizes the following multiple integral functional:*

$$J(b(\cdot), u(\cdot)) = \int_{\Lambda_{t_0,t_1}} \mathcal{H}(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t) dt^1 \cdots dt^m \tag{3}$$

among all the pair functions (b, u) satisfying

$$b(t_0) = b_0, \quad b(t_1) = b_1, \quad b_\gamma(t_0) = \tilde{b}_{\gamma 0}, \quad b_\gamma(t_1) = \tilde{b}_{\gamma 1},$$

or

$$b(t)|_{\partial \Lambda_{t_0,t_1}} = given, \quad b_\gamma(t)|_{\partial \Lambda_{t_0,t_1}} = given$$

and the isoperimetric constraints (that is, constant level sets of some functionals) formulated as follows.

Isoperimetric Constraints Defined by Controlled Curvilinear Integral Functionals

Consider the isoperimetric constraints:

$$\int_{\Upsilon_{t_0,t_1}} g_\zeta^a(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t) dt^\zeta = l^a, \quad a = 1, 2, \cdots, r \le n,$$

where Υ_{t_0,t_1} is a smooth curve, included in Λ_{t_0,t_1}, joining the points $t_0, t_1 \in \mathbb{R}_+^m$, and

$$g_\zeta^a(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t) dt^\zeta, \quad a = 1, 2, \cdots, r$$

are completely integrable differential 1-forms, namely, $D_\gamma g_\zeta = D_\zeta g_\gamma, \gamma, \zeta \in \{1, \cdots, m\}, \gamma \neq \zeta$, with $D_\gamma := \frac{\partial}{\partial t^\gamma}, \gamma \in \{1, \cdots, m\}$.

In order to investigate the above controlled optimization problem in Equation (3), associated with the aforementioned isoperimetric constraints, we introduce the curve $Y_{t_0,t} \subset Y_{t_0,t_1}$ and the auxiliary variables:

$$y^a(t) = \int_{Y_{t_0,t}} g_\zeta^a\big(b(\tau), b_\gamma(\tau), b_{\alpha\beta}(\tau), u(\tau), \tau\big) d\tau^\zeta, \quad a = 1, 2, \cdots, r,$$

which satisfy $y^a(t_0) = 0$, $y^a(t_1) = l^a$. Consequently, the functions y^a fulfill the next first-order PDEs:

$$\frac{\partial y^a}{\partial t^\zeta}(t) = g_\zeta^a\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big), \quad y^a(t_1) = l^a.$$

Considering the Lagrange multiplier $p = \big(p_a^\zeta(t)\big)$ and by denoting $y = (y^a(t))$, we introduce a new multi-time-controlled Lagrangian of second order:

$$\mathcal{H}_1\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), y(t), y_\zeta(t), p(t), t\big) = \mathcal{H}\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big)$$

$$+ p_a^\zeta(t)\left(g_\zeta^a\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), t\big) - \frac{\partial y^a}{\partial t^\zeta}(t)\right)$$

that changes the initial control problem into an unconstrained control problem

$$\min_{b(\cdot), u(\cdot), y(\cdot), p(\cdot)} \int_{\Lambda_{t_0,t_1}} \mathcal{H}_1\big(b(t), b_\gamma(t), b_{\alpha\beta}(t), u(t), y(t), y_\zeta(t), p(t), t\big) dt^1 \cdots dt^m \quad (4)$$

$$b(t_q) = b_q, \quad b_\gamma(t_q) = \tilde{b}_{\gamma q}, \quad q = 0, 1$$

$$y(t_0) = 0, \quad y(t_1) = l.$$

In accordance with Lagrange theory, an extreme point of Equation (3) is found among the extreme points of Equation (4).

The following theorem (see Treanţă and Ahmad [13]) establishes the necessary conditions of optimality associated with the considered isoperimetric constrained control problem.

Theorem 2 (Treanţă and Ahmad [13]). *If* $(b^*(\cdot), u^*(\cdot), y^*(\cdot), p^*(\cdot))$ *solves Equation (4), then*

$$(b^*(\cdot), u^*(\cdot), y^*(\cdot), p^*(\cdot))$$

solves the following Euler–Lagrange system of PDEs:

$$\frac{\partial \mathcal{H}_1}{\partial b^i} - D_\gamma \frac{\partial \mathcal{H}_1}{\partial b_\gamma^i} + \frac{1}{\mu(\alpha,\beta)} D_{\alpha\beta}^2 \frac{\partial \mathcal{H}_1}{\partial b_{\alpha\beta}^i} = 0, \quad i = \overline{1, n}$$

$$\frac{\partial \mathcal{H}_1}{\partial u^\vartheta} - D_\gamma \frac{\partial \mathcal{H}_1}{\partial u_\gamma^\vartheta} + \frac{1}{\mu(\alpha,\beta)} D_{\alpha\beta}^2 \frac{\partial \mathcal{H}_1}{\partial u_{\alpha\beta}^\vartheta} = 0, \quad \vartheta = \overline{1, k}$$

$$\frac{\partial \mathcal{H}_1}{\partial y^a} - D_\zeta \frac{\partial \mathcal{H}_1}{\partial y_\zeta^a} + \frac{1}{\mu(\alpha,\beta)} D_{\alpha\beta}^2 \frac{\partial \mathcal{H}_1}{\partial y_{\alpha\beta}^a} = 0, \quad a = \overline{1, r}$$

$$\frac{\partial \mathcal{H}_1}{\partial p_a^\zeta} - D_\gamma \frac{\partial \mathcal{H}_1}{\partial p_{a,\gamma}^\zeta} + \frac{1}{\mu(\alpha,\beta)} D_{\alpha\beta}^2 \frac{\partial \mathcal{H}_1}{\partial p_{a,\alpha\beta}^\zeta} = 0,$$

where $p_{a,\gamma}^\zeta := \frac{\partial p_a^\zeta}{\partial t^\gamma}$, $p_{a,\alpha\beta}^\zeta := \frac{\partial^2 p_a^\zeta}{\partial t^\alpha \partial t^\beta}$, $u_{\alpha\beta}^\vartheta := \frac{\partial^2 u^\vartheta}{\partial t^\alpha \partial t^\beta}$, $y_{\alpha\beta}^a := \frac{\partial^2 y^a}{\partial t^\alpha \partial t^\beta}$, $\alpha, \beta, \gamma, \zeta \in \{1, 2, ..., m\}$.

Remark 3 (Treanță and Ahmad [13])**.** *The system of Euler–Lagrange PDEs given in Theorem* 2 *becomes*

$$\frac{\partial \mathcal{H}_1}{\partial b^i} - D_\gamma \frac{\partial \mathcal{H}_1}{\partial b^i_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{H}_1}{\partial b^i_{\alpha\beta}} = 0, \quad i = \overline{1,n}$$

$$\frac{\partial \mathcal{H}_1}{\partial u^\vartheta} - D_\gamma \frac{\partial \mathcal{H}_1}{\partial u^\vartheta_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{H}_1}{\partial u^\vartheta_{\alpha\beta}} = 0, \quad \vartheta = \overline{1,k}$$

$$\frac{\partial p^\zeta_a}{\partial t^\zeta} = 0, \quad a = \overline{1,r}, \quad \zeta \in \{1,2,\cdots,m\}$$

$$\frac{\partial y^a}{\partial t^\zeta}(t) = g^a_\zeta\big(b(t), b_\gamma(t), b_{\alpha,\beta}(t), u(t), t\big).$$

In consequence, the Lagrange matrix multiplier p has null total divergence. Moreover, it is well determined only if the optimal solution is not an extreme for at least one of the functionals $\int_{Y_{t_0,t_1}} g^a_\zeta\big(b(t), b_\gamma(t), b_{\alpha,\beta}(t), u(t), t\big) dt^\zeta, \ a = \overline{1,r}.$

4. Well-Posedness of Some Variational Inequalities Involving Second-Order Partial Derivatives

In the following, in accordance with Treanță [14–16], we consider: Λ_{s_1,s_2} as a compact set in $\mathbb{R}^m$; $\Lambda_{s_1,s_2} \ni s = (s^\zeta)$, $\zeta = \overline{1,m}$ as a multi-variate evolution parameter; $\Lambda_{s_1,s_2} \supset Y$ as a piecewise differentiable curve that links the points $s_1 = (s^1_1,\ldots,s^m_1)$, $s_2 = (s^1_2,\ldots,s^m_2)$ in Λ_{s_1,s_2}; $\mathcal{B}$ as the space of C^4-class *state* functions $b : \Lambda_{s_1,s_2} \to \mathbb{R}^n$; and $b_\kappa := \dfrac{\partial b}{\partial s^\kappa}$, $b_{\alpha\beta} := \dfrac{\partial^2 b}{\partial s^\alpha \partial s^\beta}$ denote the *partial speed* and *partial acceleration*, respectively. In addition, let $\mathbf{U}$ be the space of C^1-class *control* functions $u : \Lambda_{s_1,s_2} \to \mathbb{R}^k$ and assume that $B \times U$ is a (nonempty) convex and closed subset of $\mathcal{B} \times \mathbf{U}$, equipped with

$$\langle (b,u),(q,z)\rangle = \int_Y [b(s)\cdot q(s) + u(s)\cdot z(s)]ds^\zeta$$

$$= \int_Y [\sum_{i=1}^n b^i(s)q^i(s) + \sum_{j=1}^k u^j(s)z^j(s)]ds^\zeta$$

$$= \int_Y [\sum_{i=1}^n b^i(s)q^i(s) + \sum_{j=1}^k u^j(s)z^j(s)]ds^1 + \cdots + [\sum_{i=1}^n b^i(s)q^i(s) + \sum_{j=1}^k u^j(s)z^j(s)]ds^m,$$

$$\forall (b,u),(q,z) \in \mathcal{B} \times \mathbf{U}$$

and the norm induced by it.

Let $J^2(\mathbb{R}^m, \mathbb{R}^n)$ be the jet bundle of the second order of $\mathbb{R}^m$ and $\mathbb{R}^n$. Assume that the Lagrangians $w_\zeta : J^2(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{R}^k \to \mathbb{R}$, $\zeta = \overline{1,m}$ provide a closed controlled Lagrange 1-form

$$w_\zeta(s, b(s), b_\kappa(s), b_{\alpha\beta}(s), u(s))ds^\zeta,$$

which gives the following integral functional:

$$W : \mathcal{B} \times \mathbf{U} \to \mathbb{R}, \quad W(b,u) = \int_Y w_\zeta\big(s, b(s), b_\kappa(s), b_{\alpha\beta}(s), u(s)\big)ds^\zeta$$

$$= \int_Y w_1\big(s, b(s), b_\kappa(s), b_{\alpha\beta}(s), u(s)\big)ds^1 + \cdots + w_m\big(s, b(s), b_\kappa(s), b_{\alpha\beta}(s), u(s)\big)ds^m.$$

In order to state the problem under study, we introduce the Saunders's multi-index (Saunders [32]).

Now, we introduce the variational problem: find $(b, u) \in B \times U$ such that

$$\int_Y \left[\frac{\partial w_\zeta}{\partial b}(\Psi_{b,u}(s))(q(s) - b(s)) + \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{b,u}(s))D_\kappa(q(s) - b(s)) \right] ds^\zeta \quad (5)$$

$$+ \int_Y \left[\frac{1}{x(\alpha, \beta)} \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{b,u}(s))D^2_{\alpha\beta}(q(s) - b(s)) \right] ds^\zeta$$

$$+ \int_Y \left[\frac{\partial w_\zeta}{\partial u}(\Psi_{b,u}(s))(z(s) - u(s)) \right] ds^\zeta \geq 0, \quad \forall (q, z) \in B \times U,$$

where $D_\kappa := \dfrac{\partial}{\partial s^\kappa}$ is the total derivative operator, $D^2_{\alpha\beta} := D_\alpha(D_\beta)$, and $(\Psi_{b,u}(s)) := (s, b(s), b_\kappa(s), b_{\alpha\beta}(s), u(s))$.

Let Ω be the feasible solution set of (5):

$$\Omega = \Big\{ (b, u) \in B \times U : \int_Y [(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b}(\Psi_{b,u}(s))$$

$$+ D_\kappa(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{b,u}(s))$$

$$+ \frac{1}{x(\alpha, \beta)} D^2_{\alpha\beta}(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{b,u}(s))$$

$$+ (z(s) - u(s)) \frac{\partial w_\zeta}{\partial u}(\Psi_{b,u}(s))] ds^\zeta \geq 0,$$

$$\forall (q, z) \in B \times U \Big\}.$$

Assumption 1. *The next working hypothesis is assumed:*

$$dG := D_\kappa \left[\frac{\partial w_\zeta}{\partial b_\kappa}(b - q) \right] ds^\zeta \quad (6)$$

as a total exact differential, with $G(s_1) = G(s_2)$.

According to Equation (6) and considering the notion of monotonicity associated with variational inequalities, we formulate (see Treanţă et al. [14]) the monotonicity and pseudomonotonicity for W.

Definition 1. *The functional W is monotone on $B \times U$ if*

$$\int_Y \Big[(b(s) - q(s)) \Big(\frac{\partial w_\zeta}{\partial b}(\Psi_{b,u}(s)) - \frac{\partial w_\zeta}{\partial b}(\Psi_{q,z}(s)) \Big)$$

$$+ (u(s) - z(s)) \Big(\frac{\partial w_\zeta}{\partial u}(\Psi_{b,u}(s)) - \frac{\partial w_\zeta}{\partial u}(\Psi_{q,z}(s)) \Big)$$

$$+ D_\kappa(b(s) - q(s)) \Big(\frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{b,u}(s)) - \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{q,z}(s)) \Big)$$

$$+ \frac{1}{x(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - q(s)) \Big(\frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{b,u}(s)) - \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{q,z}(s)) \Big) \Big] ds^\zeta \geq 0,$$

$$\forall (b, u), (q, z) \in B \times U$$

is satisfied.

Definition 2. *The functional W is pseudomonotone on $B \times U$ if*

$$\int_Y [(b(s) - q(s)) \frac{\partial w_\zeta}{\partial b}(\Psi_{q,z}(s)) + (u(s) - z(s)) \frac{\partial w_\zeta}{\partial u}(\Psi_{q,z}(s))$$

$$+ D_\kappa (b(s) - q(s)) \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{q,z}(s))$$

$$+ \frac{1}{x(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - q(s)) \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{q,z}(s))] ds^\zeta \geq 0$$

$$\Rightarrow \int_Y [(b(s) - q(s)) \frac{\partial w_\zeta}{\partial b}(\Psi_{b,u}(s)) + (u(s) - z(s)) \frac{\partial w_\zeta}{\partial u}(\Psi_{b,u}(s))$$

$$+ D_\kappa (b(s) - q(s)) \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{b,u}(s))$$

$$+ \frac{1}{x(\alpha, \beta)} D^2_{\alpha\beta}(b(s) - q(s)) \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{b,u}(s))] ds^\zeta \geq 0,$$

$$\forall (b, u), (q, z) \in B \times U$$

is valid.

By using Usman and Khan [33], we introduce the following definition.

Definition 3. *W is hemicontinuous on $B \times U$ if*

$$\lambda \to \left\langle ((b(s), u(s)) - (q(s), z(s)), \left(\frac{\delta_\zeta W}{\delta b_\lambda}, \frac{\delta_\zeta W}{\delta u_\lambda} \right) \right\rangle, \quad 0 \leq \lambda \leq 1$$

is continuous at 0^+, for $\forall (b, u), (q, z) \in B \times U$, where

$$\frac{\delta_\zeta W}{\delta b_\lambda} := \frac{\partial w_\zeta}{\partial b}(\Psi_{b_\lambda, u_\lambda}(s)) - D_\kappa \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{b_\lambda, u_\lambda}(s)) + \frac{1}{x(\alpha, \beta)} D^2_{\alpha\beta} \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{b_\lambda, u_\lambda}(s)) \in B,$$

$$\frac{\delta_\zeta W}{\delta u_\lambda} := \frac{\partial w_\zeta}{\partial u}(\Psi_{b_\lambda, u_\lambda}(s)) \in U,$$

$$b_\lambda := \lambda b + (1 - \lambda)q, \quad u_\lambda := \lambda u + (1 - \lambda)z.$$

Lemma 1 (Treanţă et al. [14])**.** *Let the functional W be hemicontinuous and pseudomonotone on $B \times U$. A point $(b, u) \in B \times U$ solves Equation (5) if and only if $(b, u) \in B \times U$ solves:*

$$\int_Y [(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b}(\Psi_{q,z}(s)) + (z(s) - u(s)) \frac{\partial w_\zeta}{\partial u}(\Psi_{q,z}(s))$$

$$+ D_\kappa (q(s) - b(s)) \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{q,z}(s))$$

$$+ \frac{1}{x(\alpha, \beta)} D^2_{\alpha\beta}(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{q,z}(s))] ds^\zeta \geq 0, \quad \forall (q, z) \in B \times U.$$

Furthermore, according to Treanţă et al. [14], we present two well-posedness results associated with the considered variational inequality problem involving second-order PDEs.

Definition 4. *The sequence* $\{(b_n, u_n)\} \in B \times U$ *is called an approximating sequence of Equation (5) if there exists a sequence of positive real numbers* $\sigma_n \to 0$ *as* $n \to \infty$, *such that:*

$$\int_Y [(q(s) - b_n(s)) \frac{\partial w_\zeta}{\partial b}(\Psi_{b_n, u_n}(s)) + (z(s) - u_n(s)) \frac{\partial w_\zeta}{\partial u}(\Psi_{b_n, u_n}(s))$$

$$+ D_\kappa(q(s) - b_n(s)) \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{b_n, u_n}(s))$$

$$+ \frac{1}{x(\alpha, \beta)} D_{\alpha\beta}^2(q(s) - b_n(s)) \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{b_n, u_n}(s))] ds^\zeta + \sigma_n \geq 0, \quad \forall (q, z) \in B \times U.$$

Definition 5. *The problem Equation (5) is called well-posed if:*

(i) *The problem in Equation (5) has one solution* (b_0, u_0);

(ii) *Each approximating sequence of Equation (5) converges to* (b_0, u_0).

The approximating solution set of Equation (5) is given as follows:

$$\Omega_\sigma = \left\{ (b, u) \in B \times U : \int_Y [(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b}(\Psi_{b,u}(s)) + (z(s) - u(s)) \frac{\partial w_\zeta}{\partial u}(\Psi_{b,u}(s)) \right.$$

$$+ D_\kappa(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b_\kappa}(\Psi_{b,u}(s))$$

$$\left. + \frac{1}{x(\alpha, \beta)} D_{\alpha\beta}^2(q(s) - b(s)) \frac{\partial w_\zeta}{\partial b_{\alpha\beta}}(\Psi_{b,u}(s))] ds^\zeta + \sigma \geq 0, \forall (q, z) \in B \times U \right\}.$$

Remark 4. *We have:* $\Omega = \Omega_\sigma$, *when* $\sigma = 0$ *and* $\Omega \subseteq \Omega_\sigma$, $\forall \sigma > 0$.

Furthermore, for a set P, the diameter of P is defined as follows

$$diam \ P = \sup_{\phi, \eta \in P} \|\phi - \eta\|.$$

Theorem 3 (Treanţă et al. [14]). *Let the functional W be hemicontinuous and monotone on* $B \times U$. *The problem Equation (5) is well-posed if and only if:*

$$\Omega_\sigma \neq \varnothing, \forall \sigma > 0 \ and \ diam \ \Omega_\sigma \to 0 \ as \ \sigma \to 0.$$

Theorem 4 (Treanţă et al. [14]). *Let the functional W be hemicontinuous and monotone on* $B \times U$. *Then, Equation (5) is well-posed if and only if it has one solution.*

5. Open Problem

As in the previous sections, we start with $\mathcal{T}$ as a compact set in $\mathbb{R}^m$ and $\mathcal{T} \ni \zeta = (\zeta^\beta)$, $\beta = \overline{1, m}$, as a multi-variable. Let $\mathcal{T} \supset \mathsf{C} : \zeta = \zeta(\varsigma)$, $\varsigma \in [p, q]$ be a (piecewise) differentiable curve joining the following two fixed points $\zeta_1 = (\zeta_1^1, \dots, \zeta_1^m)$, $\zeta_2 = (\zeta_2^1, \dots, \zeta_2^m)$ in $\mathcal{T}$. In addition, we consider Λ as the space of (piecewise) smooth *state* functions $\sigma : \mathcal{T} \to \mathbb{R}^n$ and Ω as the space of *control* functions $\eta : \mathcal{T} \to \mathbb{R}^k$, which are considered to be piecewise continuous. Moreover, on the product space $\Lambda \times \Omega$, we consider the scalar product:

$$\langle (\sigma, \eta), (\pi, x) \rangle = \int_\mathsf{C} [\sigma(\zeta) \cdot \pi(\zeta) + \eta(\zeta) \cdot x(\zeta)] d\zeta^\beta$$

$$= \int_\mathsf{C} \left[\sum_{i=1}^n \sigma^i(\zeta) \pi^i(\zeta) + \sum_{j=1}^k \eta^j(\zeta) x^j(\zeta) \right] d\zeta^1$$

$$+ \cdots + \left[\sum_{i=1}^n \sigma^i(\zeta) \pi^i(\zeta) + \sum_{j=1}^k \eta^j(\zeta) x^j(\zeta) \right] d\zeta^m, \quad (\forall)(\sigma, \eta), (\pi, x) \in \Lambda \times \Omega$$

together with the norm induced by it.

In the following, we introduce the vector functional defined by curvilinear integrals:

$$\Psi : \Lambda \times \Omega \to \mathbb{R}^p, \quad \Psi(\sigma, \eta) = \int_C \psi_\beta(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta$$

$$= \left(\int_C \psi_\beta^1(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta, \cdots, \int_C \psi_\beta^p(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta \right),$$

where we used the vector-valued C^2-class functions $\psi_\beta = (\psi_\beta^l) : \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^k \to \mathbb{R}^p$, $\beta = \overline{1, m}$, $l = \overline{1, p}$. In addition, D_α, $\alpha \in \{1, \ldots, m\}$ represents the operator of total derivative, and the aforementioned 1-form densities

$$\psi_\beta = \left(\psi_\beta^1, \ldots, \psi_\beta^p \right) : \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^k \to \mathbb{R}^p, \quad \beta = \overline{1, m},$$

are closed ($D_\alpha \psi_\beta^l = D_\beta \psi_\alpha^l$, $\beta, \alpha = \overline{1, m}$, $\beta \neq \alpha$, $l = \overline{1, p}$). Throughout the paper, the following rules for equalities and inequalities are applied:

$$a = b \Leftrightarrow a^l = b^l, \quad a \leq b \Leftrightarrow a^l \leq b^l, \quad a < b \Leftrightarrow a^l < b^l, \quad a \preceq b \Leftrightarrow a \leq b, a \neq b, \quad l = \overline{1, p},$$

for all p-tuples, $a = \left(a^1, \cdots, a^p \right)$, $b = \left(b^1, \cdots, b^p \right)$ in $\mathbb{R}^p$.

Next, we formulate the partial differential equation/inequation constrained optimization problem:

$$(CP) \quad \min_{(\sigma, \eta)} \left\{ \Psi(\sigma, \eta) = \int_C \psi_\beta(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta \right\} \text{ subject to } (\sigma, \eta) \in \mathcal{S},$$

where

$$\Psi(\sigma, \eta) = \int_C \psi_\beta(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta$$

$$= \left(\int_C \psi_\beta^1(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta, \cdots, \int_C \psi_\beta^p(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta \right)$$

$$= \left(\Psi^1(\sigma, \eta), \ldots, \Psi^p(\sigma, \eta) \right)$$

and

$$\mathcal{S} = \Big\{ (\sigma, \eta) \in \Lambda \times \Omega \mid Z(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) = 0, \ Y(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) \leq 0,$$

$$\sigma|_{\zeta = \zeta_1, \zeta_2} = \text{given}, \ \sigma_\alpha|_{\zeta = \zeta_1, \zeta_2} = \text{given} \Big\}.$$

Above, we considered $Z = (Z^\iota) : \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^k \to \mathbb{R}^t$, $\iota = \overline{1, t}$, $Y = (Y^r) : \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^k \to \mathbb{R}^q$, $r = \overline{1, q}$ as C^2-class functions.

Definition 6. *A point $(\sigma^0, \eta^0) \in \mathcal{S}$ is called an efficient solution in (CP) if there exists no other $(\sigma, \eta) \in \mathcal{S}$ such that $\Psi(\sigma, \eta) \preceq \Psi(\sigma^0, \eta^0)$, or, equivalently, $\Psi^l(\sigma, \eta) - \Psi^l(\sigma^0, \eta^0) \leq 0$, $(\forall) l = \overline{1, p}$, with strict inequality for at least one l.*

Definition 7. *A point $(\sigma^0, \eta^0) \in \mathcal{S}$ is called a proper efficient solution in (CP) if $(\sigma^0, \eta^0) \in \mathcal{S}$ is an efficient solution in (CP) and there exists a positive real number M, such that, for all $l = \overline{1, p}$, we have*

$$\Psi^l(\sigma^0, \eta^0) - \Psi^l(\sigma, \eta) \leq M \Big(\Psi^s(\sigma, \eta) - \Psi^s(\sigma^0, \eta^0) \Big),$$

for some $s \in \{1, \cdots, p\}$ such that

$$\Psi^s(\sigma, \eta) > \Psi^s(\sigma^0, \eta^0),$$

whenever $(\sigma, \eta) \in \mathcal{S}$ and

$$\Psi^l(\sigma, \eta) < \Psi^l(\sigma^0, \eta^0).$$

Definition 8. *A point $(\sigma^0, \eta^0) \in \mathcal{S}$ is called a weak efficient solution in (CP) if there exists no other $(\sigma, \eta) \in \mathcal{S}$ such that $\Psi(\sigma, \eta) < \Psi(\sigma^0, \eta^0)$, or, equivalently, $\Psi^l(\sigma, \eta) - \Psi^l(\sigma^0, \eta^0) < 0$, $(\forall) l = \overline{1, p}$.*

According to Treanță [17,18], for $\sigma \in \Lambda$ and $\eta \in \Omega$, we consider the vector functional

$$K : \Lambda \times \Omega \to \mathbb{R}^p, \quad K(\sigma, \eta) = \int_C \kappa_\beta(\zeta, \sigma(\zeta), \sigma_\alpha(\zeta), \sigma_{ab}(\zeta), \eta(\zeta)) d\zeta^\beta$$

and define the concepts of invexity and pseudoinvexity associated with K.

For examples of invex and/or pseudoinvex curvilinear integral functionals, the reader can consult Treanță [17].

Definition 9 (Treanță [18]). *We say that $\mathsf{X} \times \mathsf{Q} \subset \Lambda \times \Omega$ is invex with respect to ϑ and v if*

$$(\sigma^0, \eta^0) + \lambda \left(\vartheta \left(\zeta, \sigma, \eta, \sigma^0, \eta^0 \right), v \left(\zeta, \sigma, \eta, \sigma^0, \eta^0 \right) \right) \in \mathsf{X} \times \mathsf{Q},$$

for all (σ, η), $(\sigma^0, \eta^0) \in \mathsf{X} \times \mathsf{Q}$ and $\lambda \in [0, 1]$.

Now, we introduce the following (weak) vector controlled variational inequalities:

I. Find $(\sigma^0, \eta^0) \in \mathcal{S}$ such that there exists no $(\sigma, \eta) \in \mathcal{S}$ satisfying

$$(VI) \quad \Bigg(\int_C \left[\frac{\partial \psi_\beta^1}{\partial \sigma} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) \vartheta + \frac{\partial \psi_\beta^1}{\partial \eta} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) v \right] d\zeta^\beta$$

$$+ \int_C \left[\frac{\partial \psi_\beta^1}{\partial \sigma_\alpha} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta$$

$$+ \frac{1}{x(a,b)} \int_C \left[\frac{\partial \psi_\beta^1}{\partial \sigma_{ab}} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_{ab}^2 \vartheta \right] d\zeta^\beta, \cdots,$$

$$\int_C \left[\frac{\partial \psi_\beta^p}{\partial \sigma} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) \vartheta + \frac{\partial \psi_\beta^p}{\partial \eta} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) v \right] d\zeta^\beta$$

$$+ \int_C \left[\frac{\partial \psi_\beta^p}{\partial \sigma_\alpha} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta$$

$$+ \frac{1}{x(a,b)} \int_C \left[\frac{\partial \psi_\beta^p}{\partial \sigma_{ab}} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_{ab}^2 \vartheta \right] d\zeta^\beta \Bigg) \leq 0;$$

II. Find $(\sigma^0, \eta^0) \in \mathcal{S}$ such that there exists no $(\sigma, \eta) \in \mathcal{S}$ satisfying

$$(WVI) \quad \Bigg(\int_C \left[\frac{\partial \psi_\beta^1}{\partial \sigma} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) \vartheta + \frac{\partial \psi_\beta^1}{\partial \eta} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) v \right] d\zeta^\beta$$

$$+ \int_C \left[\frac{\partial \psi_\beta^1}{\partial \sigma_\alpha} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta$$

$$+ \frac{1}{x(a,b)} \int_C \left[\frac{\partial \psi_\beta^1}{\partial \sigma_{ab}} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_{ab}^2 \vartheta \right] d\zeta^\beta, \cdots,$$

$$\int_C \left[\frac{\partial \psi_\beta^p}{\partial \sigma} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) \vartheta + \frac{\partial \psi_\beta^p}{\partial \eta} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) v \right] d\zeta^\beta$$

$$+ \int_{\mathsf{C}} \left[\frac{\partial \psi_\beta^p}{\partial \sigma_\alpha} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_\alpha \vartheta \right] d\zeta^\beta$$

$$+ \frac{1}{x(a,b)} \int_{\mathsf{C}} \left[\frac{\partial \psi_\beta^p}{\partial \sigma_{ab}} \left(\zeta, \sigma^0(\zeta), \sigma_\alpha^0(\zeta), \sigma_{ab}^0(\zeta), \eta^0(\zeta) \right) D_{ab}^2 \vartheta \right] d\zeta^\beta \right) < 0.$$

Note. *In the above formulation,* $\dfrac{1}{x(a,b)}$ *represents the Saunders's multi-index.*

Open Problem. *Taking into account the notion of an invex set with respect to some given functions, the Fréchet differentiability and invexity/pseudoinvexity of the considered curvilinear integral functionals (which are path-independent) state some relations between the solutions of the (weak) vector-controlled variational inequalities and (proper, weak) efficient solutions of the associated optimization problem.*

6. Conclusions

This paper presented the nonlinear dynamics generated by some classes of constrained controlled optimization problems involving second-order partial derivatives. More precisely, we have stated the necessary optimality conditions for the considered variational control problems given by integral functionals. In addition, the well-posedness and the associated variational inequalities have been considered in this review paper.

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