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Article

Generating Integrally Indecomposable Newton Polygons with Arbitrary Many Vertices

Petar Đapić [1], Ivan Pavkov [2,*], Siniša Crvenković [2] and Ilija Tanackov [3]

[1] Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 3, 21000 Novi Sad, Serbia; petar.djapic@dmi.uns.ac.rs
[2] Faculty of Mathematics and Computer Science, Alfa BK University, Palmira Toljatija 3, 11000 Belgrade, Serbia; sinisa.crvenkovic@alfa.edu.rs
[3] Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia; ilijat@uns.ac.rs
* Correspondence: ivan.pavkov@alfa.edu.rs

Abstract: In this paper we shall give another proof of a special case of Gao's theorem for generating integrally indecomposable polygons in the sense of Minkowski. The approach of proving this theorem will enable us to give an effective algorithm for construction integrally indecomposable convex integral polygons with arbitrary many vertices. In such a way, classes of absolute irreducible bivariate polynomials corresponding to those indecomposable Newton polygons are generated.

Keywords: Newton polygon; Minkowski sum; irreducible polynomial; geometrical approach

MSC: 11S05; 11R09

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1. Introduction

Geometrical approach to irreducibility testing of bivariate polynomials via associated polygons originates at geometrical generalization of some irreducibility criterions for univariate polynomials. Dumas [1], J. Kurschak [2], O. Ore [3–5] and T. Rella [6] have generalized Eisenstein irreducibility criterion [7] using Newton polygons and enabled geometrical approach in further research of irreducibility of multivariate polynomials. Lipkovski [8] associates a polynomial to an unbounded Newton polyhedron, which is a direct analogue of Newton polygon in higher dimensions. Schmidt [9] gives a method for constructing some classes of absolutely irreducible bivariate polynomials. A theoretical basis for geometrical approach in research into the irreducibility of polynomials in more than two variables is given by Ostrowski [10,11]. The relationship between the absolute irreducibility of multivariate polynomials and integral indecomposibility of associated Newton polytopes, in the sense of Minkowski, is fully explained by Gao in [12]. However, Erich Kaltofen [13] and Sturmfels [14] emphasized importance of solving the reverse problem, finding possible factorizations for multivariate polynomials with decomposable Newton polytopes. Koyuncu [15] discusses possible factorizations for bi- and tri-variate polynomials with decomposable Newton polytopes depending on the characteristic of the field. The necessary and sufficient condition for the existence of a non-trivial factorization of an arbitrary bivariate polynomial with integer coefficients into factor-polynomials with integer coefficients is given in [16]. An effective factorization algorithm based on this result is presented in [17].

Definition 1. *A polynomial over a field F is called* ***absolutely irreducible*** *if it remains irreducible over every algebraic extension of F.*

The ***convex hull*** of a set S in $\mathbb{R}^2$ (denoted by $conv(S)$) is the smallest convex set that contains the set S. Graham [18] presented an algorithm for construction of the convex hull of a finite planar set.

Definition 2. *Let $f(x,y) \in \mathbb{Z}[x,y]$:*

$$f(x,y) = \sum C_{e_1e_2} x^{e_1} y^{e_2}.$$

Consider an exponent vector (e_1, e_2) as a point in $\mathbb{Z}^2$. The ***Newton polygon of the polynomial*** *$f(x,y)$, denoted by P_f, is defined as the convex hull in $\mathbb{R}^2$ of all the points (e_1, e_2) with $C_{e_1e_2} \neq 0$.*

Definition 3. *For two arbitrary sets $A, B \subset \mathbb{R}^2$, the set $A + B = \{a + b \mid a \in A, b \in B\}$ is called the* ***Minkowski sum of sets A and B*** *.*

A point in $\mathbb{R}^2$ is called ***integral*** if both of its coordinates are integers. A polygon in $\mathbb{R}^2$ is called ***integral*** if all of its vertices are integral.

Definition 4. *An integral polygon C is called* ***integrally decomposable*** *if there exist integral polygons A and B such that $C = A + B$, where both A and B have at least two points. Otherwise, C is called* ***integrally indecomposable****.*

Theorem 1 ([10])**.** *Let F be a field. Let $f, g, h \in F[x,y]$ with $f = gh$. Then $P_f = P_g + P_h$.*

Theorem 2 ([12])**.** *Let $f(x,y)$ be a non-zero polynomial over an arbitrary field F, non-divisible either by x or by y. If the Newton polygon of the polynomial $f(x,y)$ is integrally indecomposable, then $f(x,y)$ is absolutely irreducible over F.*

For an arbitrary integral point $a = (a_x, a_y)$, $GCD(a_x, a_y)$ is denoted by $GCD(a)$. For arbitrary integral points a and b, $GCD(GCD(a), GCD(b))$, is denoted by $GCD(a,b)$. In [19] necessary and sufficient conditions for integral indecomposability in the sense of Minkowski for line segments and triangles are given.

Theorem 3 ([19])**.** *Let a_1 and a_2 be integer points from $\mathbb{R}^2$. Line segment $[a_1, a_2]$ is integrally indecomposable in the sense of Minkowski if and only if $GCD(a_2 - a_1) = 1$.*

Theorem 4 ([19])**.** *A triangle $conv(v_1, v_2, v_3)$ in $\mathbb{R}^2$ with integer vertices v_1, v_2, v_3 is integrally indecomposable in the sense of Minkowski if and only if:*

$$GCD(v_1 - v_2, v_1 - v_3) = 1.$$

Definition 5. *Let $f(x,y) \in \mathbb{Z}[x,y]$. The* ***non-extended lattice of nodes*** *of the polynomial $f(x,y)$ consists of all the points $(e_1, e_2)_i$, $i = 1, \ldots, k$ corresponding to the monomials of $f(x,y)$ with non-zero coefficients. If the Newton polygon of $f(x,y)$ contains some integer points in its inner area or on its edges different from $(e_1, e_2)_i$, $i = 1, \ldots, k$, some of these points, together with $(e_1, e_2)_i$, $i = 1, \ldots, k$, form an* ***extended lattice of nodes*** *of the polynomial $f(x,y)$.*

Definition 6. *Let $f(x,y) \in \mathbb{Z}[x,y]$ and let $P = \{A_1, A_2, \ldots, A_n\}$ be the lattice of nodes of the polynomial $f(x,y)$ possibly extended by some integer points that lay inside Newton polygon of the polynomial $f(x,y)$ or on its edges. Without loss of generality, we can assume that after the construction of the Newton polygon of $f(x,y)$, $A_1, A_2, \ldots, A_k$, $k \geq 2$, become its vertices, and $A_{k+1}, \ldots, A_n$ do not. We say that the grouping $G_1, \ldots, G_l$, $l \geq 2$, of the set P is a* ***super-covering of*** *P if:*

1. *Each group G_i, $i = 1, \ldots, l$, contains the same number of points not less than two,*
2. $\bigcup_{i=1}^{l} G_i = P$,
3. *Points $A_1, A_2, \ldots, A_k$ appear in exactly one of the sets $G_1, \ldots, G_l$,*

4. *Points $A_{k+1},\ldots,A_n$ appear in at least one of the sets $G_1,\ldots,G_l$,*
5. *Sets $G_2,\ldots,G_l$ are obtained from G_1 by translation.*

Definition 7. *Let $f(x,y)\in\mathbb{Z}[x,y]$ and let $P=\{A_1,A_2,\ldots,A_n\}$ be the lattice of nodes of the polynomial $f(x,y)$ possibly extended by some integer points that lay inside the Newton polygon of the polynomial $f(x,y)$ or on its edges. Let*

$$G_1=conv(A_{i_{1,1}},\ldots,A_{i_{1,k}}),\ldots,G_l=conv(A_{i_{l,1}},\ldots,A_{i_{l,k}}),l\geq 2,$$

$\{i_{1,1},\ldots,i_{1,k},\ldots,i_{l,1},\ldots,i_{l,k}\}=\{1,\ldots,n\}$, be a super-covering of P by l congruent $k-$gons. Furtherly, let $G_2=\tau_2(G_1)$,...,$G_l=\tau_l(G_1)$. Then :

$$conv(A_{i_{1,1}},\tau_2(A_{i_{1,1}}),\ldots,\tau_l(A_{i_{1,1}})),\ldots,conv(A_{i_{1,k}},\tau_2(A_{i_{1,k}}),\ldots,\tau_l(A_{i_{1,k}}))$$

is also a super-covering of P by k congruent $l-$gons, called dual super-covering of the aforementioned super-covering.

Remark 1. *Notions of extended lattice of nodes, non-extended lattice of nodes, super-covering and dual super-covering are not necessary related to the Newton polygon of a bivariate polynomial. These notions can be defined completely analogously for an arbitrary integral convex polygon.*

For a convex polygon in the Euclidean plane, there is a finite sequence of vectors associated with it in the following way. Let $v_0,v_1,\ldots,v_{m-1}$ be the vertices of the polygon in the counterclockwise direction. The edges of P are represented by the vectors $E_i=v_i-v_{i-1}=(a_i,b_i)$ for $1\leq i\leq m$, where a_i , $b_i\in\mathbb{Z}$ and the indices are taken modulo m. A vector E_i is called an ***edge vector***. A vector $v=(a,b)\in\mathbb{Z}^2$ is called a primitive vector if $gcd(a,b)=1$. Let $n_i=gcd(a_i,b_i)$ and let $e_i=(a_i/n_i,b_i/n_i)$. Then $E_i=n_ie_i$ where e_i is a primitive vector, $1\leq i\leq m$. Each edge E_i contains precisely n_{i+1} integral points including its endpoints. The sequence of vectors $\{n_ie_i\}_{1\leq i\leq m}$ is called the ***edge sequence*** or a polygonal sequence. By its edge sequence each polygon is determined uniquely up to translation determined by v_0. As the boundary of the polygon is a closed path, we have that $\sum_{1\leq i\leq m}n_ie_i=(0,0)$.

Lemma 1 ([20])**.** *Let P be a polygon with edge sequence $n_ie_{i1\leq i\leq m}$ where $e_i\in\mathbb{Z}^2$ are primitive vectors. Then an integral polygon is a summand of P iff its edge sequence is of the form $\{k_ie_i\}_{1\leq i\leq m}$, $0\leq k_i\leq n_i$, with $\sum_{1\leq i\leq m}n_ie_i=(0,0)$.*

2. Main Results

Lemma 2. *Each non-trivial decomposition of an integral polygon in the sense of Minkowski induces super-covering of the extended lattice of nodes of the polygon.*

Proof. Let P be an integral polygon and $P=Q+R$ its non-trivial decomposition, with polygons Q and R having at least two points. Suppose that $Q=conv(q_1,q_2,\ldots,q_{2+k})$, $k\geq 0$. It is obvious that $\tau_{q_1}(R),\tau_{q_2}(R),\ldots,\tau_{q_{2+k}}(R)$, $k\geq 0$, is a super-covering of the extended lattice of nodes of the polygon P. □

Lemma 3. *Let P be an integral polygon that has non-trivial decomposition. Each edge of P that does not contain integer points except its vertices is covered by each super-covering or its dual super-covering of the extended lattice of nodes of the polygon P by a line segment or as an edge of a polygon.*

Proof. Let $conv(x_k,x_{k+1})$ be an edge of P that does not contain integer points except x_k and x_{k+1} and consider an arbitrary super-covering of the extended lattice of nodes of the polygon P. If $conv(x_k,x_{k+1})$ in super-covering, the assertion holds. If $conv(x_k,x_{k+1})$ is not in some super-covering, let us prove that $conv(x_k,x_{k+1})$ is in the super-covering which is dual of that one. Suppose the opposite, i.e., $\tau_i(x_k)\neq x_{k+1}$, for any translation τ_i, determined by super-covering. It follows that $\tau_i(x_l) = x_{k+1}$, for some τ_i and some $l \neq k$. It is obvious that $\tau_i(x_k)$ is not on the same side of the line determined with x_k and x_{k+1} as the point x_l , that is contradiction to the construction of the convex hull of a finite planar set described in [18]. □

Definition 8. *Let $conv(x_1, x_2, \ldots, x_n)$ be an arbitrary convex polygon with integer vertices. Let x_1 be the vertex of the polygon with the smallest $x-$coordinate having simultaneously the largest $y-$coordinate and let x_k be the vertex of the polygon with the smallest $y-$coordinate having simultaneously the largest $x-$coordinate. Consider the line determined by x_1 and x_k. Vertices of the polygon that lay on the same side of the line as origin are called* ***inner vertices****. Other vertices of the polygon are called* ***outer vertices****.*

The following theorem is a direct consequence of Theorem 4.11 from [12]. We shall give two different proofs of the theorem that give an idea for the construction of an effective algorithm which generates integrally indecomposable Newton polygons with arbitrary many vertices.

Theorem 5. *Let $conv(x_1, x_2, \ldots, x_n)$ be an arbitrary convex polygon with integer vertices and let x_l be an arbitrary outer vertex of the polygon. Let x_{n+1} be an integer point such that:*

1. *Line segment $conv(x_l, x_{n+1})$ does not contain any integer points except x_l and x_{n+1},*
2. *All the points $x_1, \ldots, x_n$ except x_l are on the same side of the line determined by points x_l and x_{n+1},*
3. *Line segment $conv(x_l, x_{n+1})$ is larger than any parallel line segment whose vertices are integer points from $conv(x_1, x_2, \ldots, x_{n+1})$,*

Then polygon $conv(x_1, x_2, \ldots, x_{n+1})$ is integrally indecomposable in the sense of Minkowski.

First proof. Suppose the opposite, $conv(x_1, x_2, \ldots, x_{n+1})$ is integrally decomposable in the sense of Minkowski. From Lemma 2 it follows that extended lattice of nodes of the polygon has super-covering. Due to the point 2 in Theorem 5, line segment $conv(x_l, x_{n+1})$ is an edge of the polygon $conv(x_1, x_2, \ldots, x_{n+1})$. From point 1 in Theorem 5 and Lemma 3 it follows that line segment $conv(x_l, x_{n+1})$ is covered by that super-covering or dual super-covering. Consider the super-covering having $conv(x_l, x_{n+1})$. Then there exists a line segment whose vertices are from extended lattice of nodes of the polygon $conv(x_1, x_2, \ldots, x_{n+1})$ different from x_l and x_{n+1} that is congruent and parallel to the line segment $conv(x_l, x_{n+1})$, that is in contradiction to point 3 in Theorem 5. □

Second proof. Let $B_1, \ldots, B_k$ be vertices of the polygon $conv(x_l, \ldots, x_{n+1})$ in counterclockwise order, where B_1 and B_k correspond to the points x_l and x_{n+1}. Let $\{c_i e_i\}_{1 \leq i \leq k}$ be the edge sequence and the intersection of the lines $l(B_1, B_2)$ and $l(B_k, B_{k-1})$ is the point C. Let q be the line of symmetry of the angle $\angle B_1 C B_k$. Note thet, according to the condition 1, $c_k = 1$ holds. Denote with $pr_p^q(K)$ the set of points belonging to the line p which are projections of the points of a set K on the line p in the direction determined by q.

According to the point 2 in Theorem 5 and point 3 in Theorem 5, the point C exists and lays on the same side of the line $l(B_1, B_k)$ as the remaining vertices of the constructed polygon and $P = conv(x_1, x_2, \ldots, x_{n+1}) \subseteq conv(B_1, C, B_k)$. Let $F = \{pr_p^q(conv(B_i B_{i+1})) \mid 1 \leq i \leq n-1\}$. Then all the elements of F are real line segments (not points) and intersection of any two elements from F the one neighboring point at most. Hence,

$$\sum_{x \in K} \mid x \mid = \mid B_1 B_k \mid .$$

Suppose the opposite, i.e., that the polygon P is decomposable in the sense of Minkowski sum. Let $P = Q + R$. Without loss of generality, according to Lemma 1 it follows that $\{d_i e_i\}_{1 \leq i \leq k}$ is the edge sequence of Q different from the edge sequence of P, with $0 \leq d_i \leq c_i$ for $1 \leq i \leq k-1$ and $d_k = c_k = 1$. Then we will have $\mid B_1 B_k \mid \leq \sum_{1 \leq i \leq n-1} \mid d_i e_i \mid < \sum_{x \in K} \mid$

$x \mid=\mid B_1 B_k \mid$ (the inequality is strict due to the fact that $pr_p^q(e_i)$ are real line segments, not the points), that is an obvious contradiction. □

Remark 2. *Inner vertices of the polygon, as well as the vertex of the polygon with the smallest $x-$coordinate having simultaneously the largest $y-$coordinate and the vertex of the polygon with the smallest $y-$coordinate having simultaneously the largest $x-$coordinate, are also vertices of the polygon $conv(x_1, x_2, \ldots, x_{n+1})$. Therefore, absolute irreducible bivariate polynomials generated by $conv(x_1, x_2, \ldots, x_{n+1})$ have non-zero monomials corresponding to those vertices.*

Example 1. *Consider the line segment:*

$$conv((2,1),(0,9)).$$

It is obvious that:

$$GCD((0,9)-(2,1))) = GCD(-2,8) = 2.$$

From Theorem 3 it follows that line segment $conv((2,1),(0,9))$ shown in Figure 1 is integrally decomposable.

The point $(0,9)$ is outer vertex. Due to the fact that:

$$GCD((13,0)-(0,9)) = GCD(13,-9) = 1,$$

from Theorem 3 it follows that line segment $conv((0,9),(13,0))$ does not contain any integer points except $(0,9)$ and $(13,0)$. It is obvious that conditions of Theorem 5 are satisfied, so triangle $conv((13,0),(0,9),(2,1))$ (see Figure 2) is integrally indecomposable.

Therefore, the polynomial

$$f(x,y) = a_1 x^{13} + a_2 y^9 + a_3 x^2 y,$$

$a_1, a_2, a_3 \in F \backslash \{0\}$, is absolutely irreducible over an arbitrary field F. It remains absolutely irreducible over F if some monomials whose exponent vectors lay inside the triangle $conv((13,0),(0,9),(2,1))$ or on its edge are added. In other words, each polynomial:

$$f(x,y) = a_1 x^{13} + a_2 y^9 + a_3 x^2 y + \sum c_{ij} x^i y^j,$$

with $a_1, a_2, a_3 \in F \backslash \{0\}$, $(i,j) \in conv((13,0),(0,9),(2,1)) \setminus \{(13,0),(0,9),(2,1)\}$ is absolutely irreducible over F.

Analogously, e.g., by adding point (17,14) to the triangle $conv((13,0),(0,9),(2,1))$, integrally indecomposable quadrilateral $conv((13,0),(0,9),(2,1),(17,14))$ shown in Figure 3 is obtained.

Therefore, each polynomial:

$$f(x,y) = a_1 x^{13} + a_2 y^9 + a_3 x^2 y + a_4 x^{17} y^{14} + \sum c_{ij} x^i y^j,$$

with $a_1, a_2, a_3, a_4 \in F \backslash \{0\}$, $(i,j) \in conv((13,0),(0,9),(2,1),(17,14))$ is absolutely irreducible over F.

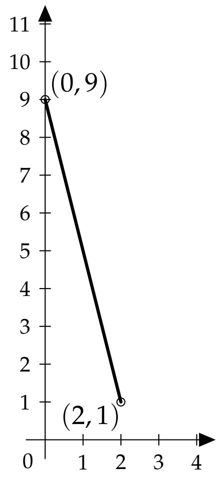

Figure 1. Integrally decomposable line segment $conv((2,1),(0,9))$.

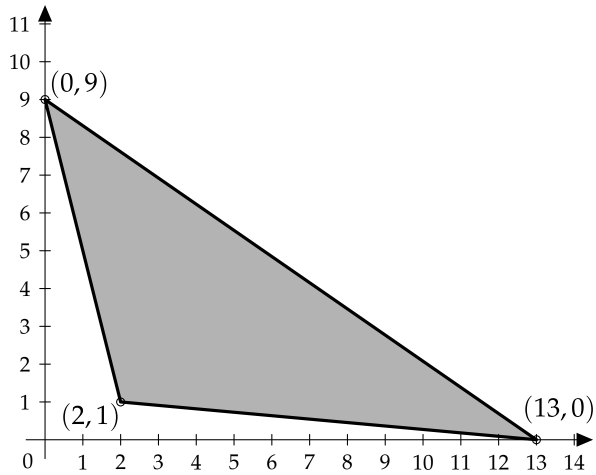

Figure 2. Integrally indecomposable triangle $conv((13,0),(0,9),(2,1))$.

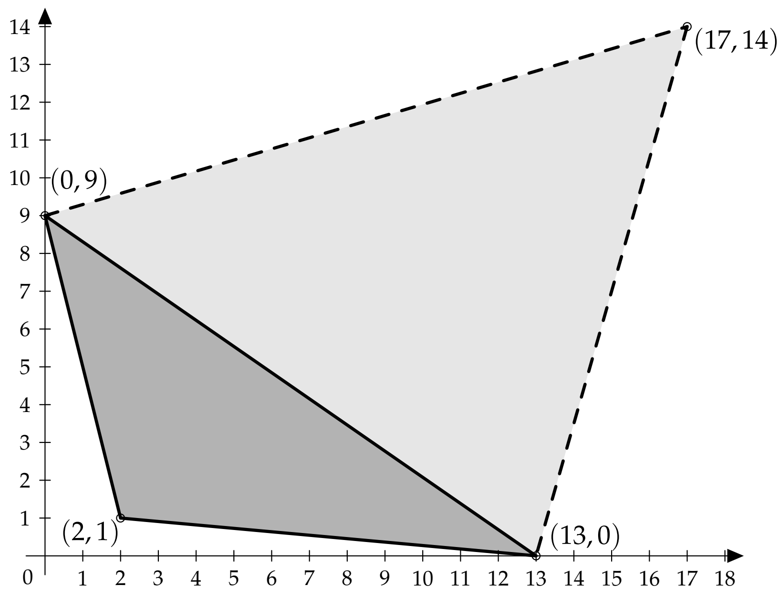

Figure 3. Integrally indecomposable quadrilateral $conv((13,0),(0,9),(2,1),(17,14))$.

3. Constructions of Integrally Indecomposable n-Gons

Lemma 4. *Let i, j be natural numbers, such that $GCD(i, j) = 1, i > 1$. Let p be the line determined by $iy - jx = 0$ and $A(c, d)$ be an integer point laying above the line p. Then, for an arbitrary natural number k, there exists an integer point $B(x_0, y_0)$ below the line p, such that the angle between the line p and the line containing line segment AB is less than* $\arctan 2^{-k} - \arctan 2^{-k-1}$ *and line segment AB contains no integer points except its vertices.*

Proof. As $i > 1$, it follows that there exists a natural number m such that the point $C(0, j - i^m)$ is at a greater distance from the line p than the point A (see Figure 4). The previous holds for the big enough natural number m.

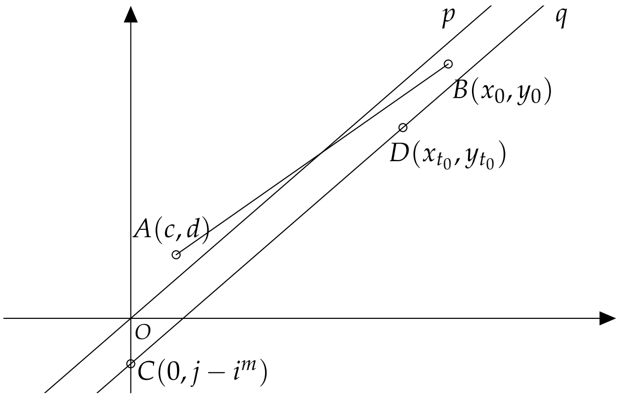

Figure 4. Line segment AB containing no integer points except its vertices.

Let q be the line determined by $iy - jx = i(j - i^m)$. Line q contains point C and is parallel with the line p. For each natural number t, line q contains the point $(x_t, y_t) = (i^t \cdot i, i^t \cdot j + j - i^m)$. Also, for each natural number t, $GCD(x_t, y_t) = 1$ holds. From Theorem 3 it follows that the line segment $conv((x_t, y_t), (0, 0))$ contains no integer points except its vertices. There exists big enough natural number t_0 such that for the point $D(x_{t_0}, y_{t_0})$, $\angle pOD < \arctan 2^{-k} - \arctan 2^{-k-1}$ holds.

Let $x_0 = x_{t_0} + c$, $y_0 = y_{t_0} + d$ and let $B(x_0, y_0)$. Due to the fact that the angle between the line p and the line containing the line segment AB is less than $\arctan 2^{-k} - \arctan 2^{-k-1}$, point B lays below the line p and the line segment AB contains no integer points except its vertices. □

The following corollary is a consequence of Lemma 4.

Corollary 1. *Let p be the line determined by $iy - jx = if - je$, with i, j, e, f natural numbers such that $GCD(i, j) = 1$ and $i > 1$. Let $A(c, d)$ be an integer point that lays above the line p. Then for an arbitrary natural number k there exists an integer point $B(x_0, y_0)$ under the line p, such that the angle between the line p and the line containing the line segment AB is less than* $\arctan 2^{-k} - \arctan 2^{-k-1}$ *and the line segment AB contains no integer points except its vertices.*

Theorem 6. *There exist an infinite sequence of monomials $p_1, p_2, \ldots, p_n, \ldots$ such that the polynomial $f_n = \sum_{j=1}^{n} p_i$ is absolutely irreducible over F for each natural number $n > 1$ and the corresponding Newton polygon has n vertices.*

Proof. Let us denote by A_i vertex corresponding to the monomial p_i, $i = 1, 2, \ldots$. From Theorems 3 and 4 it follows that there exist monomials p_1, p_2, p_3 such that A_1 lays on the x-axis and A_2 lays on the y-axis, polynomials f_2 and f_3 are absolutely irreducible over F and slope coefficient of the line that contains the line segment A_2A_3 is less than 1 (see Figure 5). There exists natural number k such that 2^{-k} is less than the slope coefficient of the line that contains the line segment A_2A_3.

Let us prove by induction, for $n > 3$, that there exist integer points $A_4, A_5, \ldots, A_n$ in the first quadrant that satisfy conditions of the Corollary 1 and slope coefficient of the line that contains the line segment $A_{n-1}A_n$ is greater than 2^{-k-n+3}, for each $n > 3$.

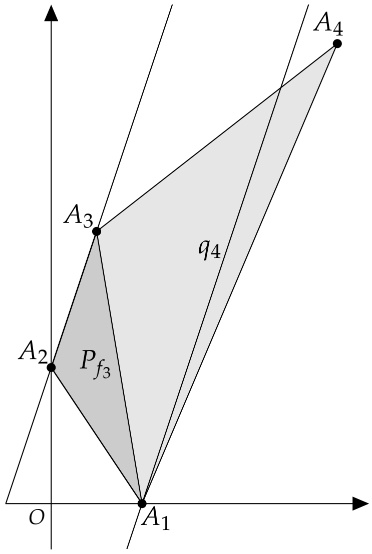

Figure 5. Integrally indecomposable polygon $conv(A_1, A_2, A_3, A_4)$ in the sense of Minkowski.

Consider the line q_4 that contains the point A_1 and has the same slope coefficient as the line that contains the line segment A_2A_3. From Corollary 1 it follows that there exists point A_4 that lays on the other side of the line q_4 than the polygon P_{f_3}, A_3A_4 has no integer points except its vertices and the angle between the line q_4 and the line containing the line segment A_3A_4 is less than $\arctan 2^{-k} - \arctan 2^{-k-1}$. The slope coefficient of the line that contains the line segment A_3A_4 is greater than 2^{-k-1}. The projection of the polygon P_{f_3} in direction of the line q_4 on the line containing the line segment A_3A_4 is contained in the line segment A_3A_4. From Theorem 5 it follows that polygon $conv(A_1, A_2, A_3, A_4)$ is integrally indecomposable in the sense of Minkowski and has four vertices.

Let us suppose that there exist integer points $A_4, A_5, \ldots, A_{n-1}$ in the first quadrant that satisfy conditions of the induction hypothesis and slope coefficients of the lines that contain line segments $A_{i-1}A_i$ are bigger than 2^{-k-i+3}, for each $3 < i < n$ (see Figure 6). Let q_n be the line that contains point A_1 and has the same slope coefficient as the line segment $A_{n-2}A_{n-1}$ bigger than 2^{-k-n+4}. From Corollary 1 it follows that there exists a point A_n that lays on the other side of the line q_n than the polygon $P_{f_{n-1}}$, $A_{n-1}A_n$ has no integer points except its vertices and the angle between the line q_n and the line containing the line segment $A_{n-1}A_n$ is less than $\arctan 2^{-k-n+4} - \arctan 2^{-k-n+3}$. The slope coefficient of the line that contains the line segment $A_{n-1}A_n$ is greater than 2^{-k-n+3}. The projection of polygon $P_{f_{n-1}}$ in the direction of the line q_n on the line $A_{n-1}A_n$ is contained in the line segment $A_{n-1}A_n$. According to Theorem 5 polygon $conv(A_1, A_2, \ldots, A_n)$ is integrally indecomposable in the sense of Minkowski and has n vertices. $\square$

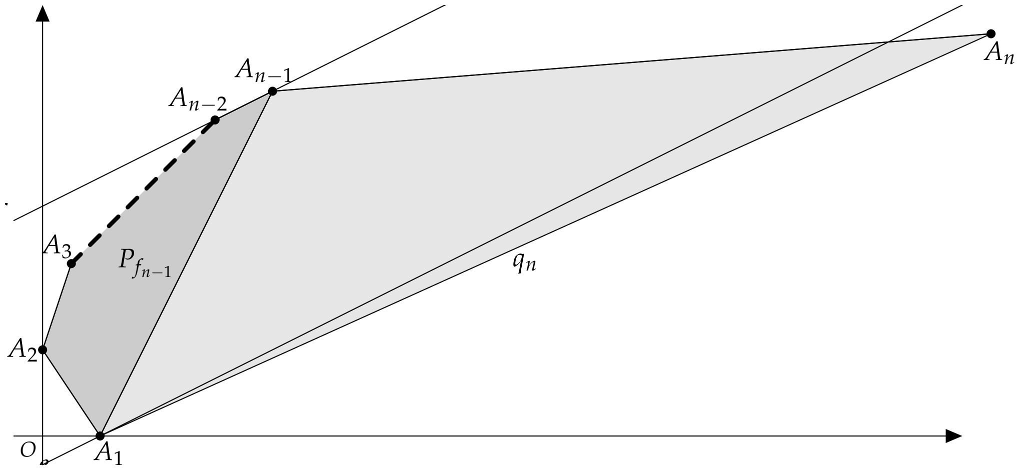

Figure 6. Integrally indecomposable polygon $conv(A_1, A_2, \ldots, A_n)$ in the sense of Minkowski.

4. Conclusions

Proving a special case of Gao's theorem presented in the paper enabled construction of an effective algorithm for building integrally indecomposable convex integral polygons with arbitrary many vertices and thus generating classes of absolute irreducible bivariate polynomials.

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