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Estimation of Reliability Indices for Alpha Power Exponential Distribution Based on Progressively Censored Competing Risks Data

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Abstract: In reliability analysis and life testing studies, the experimenter is frequently interested in studying a specific risk factor in the presence of other factors. In this paper, the estimation of the unknown parameters, reliability and hazard functions of alpha power exponential distribution is considered based on progressively Type-II censored competing risks data. We assume that the latent cause of failures has independent alpha power exponential distributions with different scale and shape parameters. The maximum likelihood method is considered to estimate the model parameters as well as the reliability and hazard rate functions. The approximate and two parametric bootstrap confidence intervals of the different estimators are constructed. Moreover, the Bayesian estimation method of the unknown parameters, reliability and hazard rate functions are obtained based on the squared error loss function using independent gamma priors. To get the Bayesian estimates as well as the highest posterior credible intervals, the Markov Chain Monte Carlo procedure is implemented. A comprehensive simulation experiment is conducted to compare the performance of the proposed procedures. Finally, a real dataset for the relapse of multiple myeloma with transplant-related mortality is analyzed.

Keywords: alpha power exponential distribution; progressive Type-II censoring; competing risks; maximum likelihood; bayesian estimation; loss function

MSC: 62N05; 62N02; 62F10; 62F15; 62F40

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1. Introduction

Numerous parametric probability distributions have been proposed in the literature to generalize the exponential distributions to add more flexibility in investigating real datasets. One of the most significant probability distributions recently introduced by Mahdavi and Kundu [1] is the alpha power exponential (APE) distribution. The APE distribution contains one scale parameter and an extra shape parameter which adds more flexibility to the distribution rather than the traditional exponential (Exp) distribution. Mahdavi and Kundu [1] declared that the APE distribution has many desirable features and can be considered a competitive model to some classic models such as Weibull, gamma and exponentiated exponential distributions. Its hazard rate function (HRF) can be constant, decreasing and increasing, depending on the shape parameter value. One of the important properties of the APE distribution is that its cumulative distribution function (CDF) takes an explicit form. Therefore, it can be derived relatively conveniently for analyzing censored data. Some studies investigated the statistical inference of the APE distribution by considering different procedures. Nassar et al. [2] explored the different estimation techniques of the APE distribution.

In life testing and reliability investigations, there are numerous situations where items are lost or eliminated from the experimentation before failure. The researcher may not guarantee complete information on failure times for all experimental items. Data collected from such experiments are called censored data. The most familiar censoring schemes are Type-I and Type-II censoring schemes. These schemes are widely employed in the literature. Nevertheless, one of the drawbacks of the aforementioned censoring schemes is that the removal of the tested items is not permitted at any time point other than the stop point of the test. To overcome this disadvantage, a progressive censoring scheme is suggested. Recently, the progressive Type-II censoring scheme has acquired significant awareness in the literature because of its wide-scale applicability. Using the progressively Type-II censored data, many authors investigated the estimation problems of some lifetime models. Kundu [3] studied the Bayesian estimation for the parameters of a Weibull distribution. Pradhan and Kundu [4] considered the inference of the unknown parameters of the generalized exponential distribution. Ahmed [5] studied the estimation of the generalized Gompertz distribution. Dey et al. [6] studied the estimation and prediction of the Marshall–Olkin extended exponential distribution. For an extensive review of progressively censoring, see Balakrishnan [7] and Balakrishnan and Cramer [8]. Under the assumption of APE distribution for lifetime observation, Salah [9] studied the estimation of the APE distribution using progressively Type-II censored data. Salah et al. [10] investigated the APE distribution using Type-II hybrid censored samples. Alotabi et al. [11] investigated the maximum likelihood and Bayesian estimates (BEs) of the APE distribution under progressively Type-II censored data.

On the other hand, the failure of the tested items may be caused by more than one cause, such as the different incentives for cancer recurrence. In life testing investigations, an experimenter is usually inquisitive about the examination of distinct risks in the existence of different risk factors. In the literature, this problem is known as the competing risks model. In studying the competing risks data, the data consist of the lifetime of the failed item and an indicator variable indicating the reason for failure. In this study, we utilize the latent failure times model, as introduced by Cox [12]. Employing the latent failure times model, it is supposed that the failure times are independent. The investigations on competing risks models using censored data have been becoming popular. Kundu et al. [13] considered the competing risks model using Exp distributions based on progressive Type-II censored samples. Pareek et al. [14] studied the estimation problems of Weibull distributions using progressive Type-II censored samples. Cramer and Schmiedt [15] considered the competing risks model using Lomax distributions under progressive Type-II censoring. Ashour and Nassar [16] investigated the competing risks model using Weibull distributions based on adaptive Type-I progressive censoring. Ren and Gui [17] studied the competing risks model from Weibull distributions using adaptive progressive Type-II censoring.

The major focus of this study is to investigate the competing risks model when the data is progressively Type-II censored with the assumption that the latent failure times are independent and follow an APE distribution with different scale and shape parameters. The maximum likelihood as a classical estimation procedure is employed to obtain the point and interval estimates of the unknown parameters. Moreover, the point and interval estimates of the reliability function (RF) and HRF are acquired. Two parametric bootstrap confidence intervals are considered. We consider the Bayesian estimation method to obtain the point and the highest posterior density (HPD) credible intervals of the unknown parameters, RF and HRF. To compare the efficiency of the different methods, a simulation study is performed using different scenarios for the sample size, the effective number of failures and censoring schemes. We also analyze one real dataset for the relapse of multiple myeloma, with transplant-related mortality belonging to patients treated at the Clinic for Stem Cell Transplantation, University Hospital Hamburg-Effendorf, Hamburg, Germany.

The organization of the paper is as follows: The model and the notation used in this paper are presented in Section 2. The maximum likelihood procedure is considered in Section 3. Two bootstrap confidence intervals are discussed in Section 4. Bayesian estimation is considered in Section 5. In Section 6, we perform a simulation study, and a simulated dataset is analyzed to acquire and compare the performance of the various estimates. One real dataset is investigated in Section 7. Section 8 extends the competing risks model when the causes of failure are unknown. Finally, some conclusions from our work are presented in Section 9.

2. Model Description and Notation

Without a loss of generality, assume that we have a lifetime experiment with n identical items and there are only two causes of failure, then $X_i = min\{X_{1i}, X_{2i}\}, i = 1, \ldots, n$ and $X_{ti}, t = 1, 2$ refers to the latent failure time of the ith unit under the tth cause of failure. Further, we assume that $(X_{1i}, X_{2i}), i = 1, \ldots, n$, are n independent and identically distributed random variables. In this study, it is assumed that X_{1i} and $X_{2i}, i = 1, \ldots, n$, follow the APE distribution with the following probability density function (PDF)

$$f_t(x) = \begin{cases} \frac{\theta_t \log(\alpha_t)}{\alpha_t - 1} e^{-\theta_t x} \alpha_t^{1-e^{-\theta_t x}}, & \theta_t, \alpha_t > 0, \alpha_t \neq 1, x > 0, \\ \theta_t e^{-\theta_t x}, & \alpha_t = 1. \end{cases} \quad (1)$$

and the corresponding CDF is

$$F_t(x) = \begin{cases} \frac{\alpha_t^{1-e^{-\theta_t x}} - 1}{\alpha_t - 1}, & \theta_t, \alpha_t > 0, \alpha_t \neq 1, x > 0, \\ 1 - e^{-\theta_t x}, & \alpha_t = 1. \end{cases} \quad (2)$$

where θ_t and α_t are the scale and shape parameters, respectively. Using (1) and (2), we can derive the CDF and PDF of X_i, respectively, as follows

$$F(x) = 1 - \frac{\alpha_1 \alpha_2}{(\alpha_1 - 1)(\alpha_2 - 1)} \left(1 - \alpha_1^{-e^{-\theta_1 x}}\right)\left(1 - \alpha_2^{-e^{-\theta_2 x}}\right), \ x > 0 \quad (3)$$

and

$$\begin{aligned} f(x) &= \frac{\alpha_1 \alpha_2}{(\alpha_1 - 1)(\alpha_2 - 1)} \left[\theta_1 \log(\alpha_1) e^{-\theta_1 x} \alpha_1^{-e^{-\theta_1 x}} \left(1 - \alpha_2^{-e^{-\theta_2 x}}\right)\right. \\ &+ \left.\theta_2 \log(\alpha_2) e^{-\theta_2 x} \alpha_2^{-e^{-\theta_2 x}} \left(1 - \alpha_1^{-e^{-\theta_1 x}}\right)\right]. \end{aligned} \quad (4)$$

Similarly, the RF and HRF of X_i are, respectively, given by

$$R(x) = \frac{\alpha_1 \alpha_2}{(\alpha_1 - 1)(\alpha_2 - 1)} \left(1 - \alpha_1^{-e^{-\theta_1 x}}\right)\left(1 - \alpha_2^{-e^{-\theta_2 x}}\right), \ x > 0 \quad (5)$$

and

$$h(x) = \left[\frac{\theta_1 \log(\alpha_1) e^{-\theta_1 x} \alpha_1^{-e^{-\theta_1 x}}}{1 - \alpha_1^{-e^{-\theta_1 x}}} + \frac{\theta_2 \log(\alpha_2) e^{-\theta_2 x} \alpha_2^{-e^{-\theta_2 x}}}{1 - \alpha_2^{-e^{-\theta_2 x}}}\right]. \quad (6)$$

It is noted that when $\alpha = 1$, the functions in (3)–(6) reduce to the case of the Exp distribution. Assume $m < n$ is determined before the test. Furthermore, m other integers, $R_1, \ldots, R_m$ are also prefixed so that $n = m + \sum_{i=1}^{m} R_i$. At the time of the initial failure $X_{1:m:n}$, R_1 of the surviving items are randomly discarded. Likewise, at the time of the second failure $X_{2:m:n}$, R_2 of the surviving items are randomly eliminated and so on. Lastly, at the time of the mth failure $X_{m:m:n}$, the remainder of the R_m items are withdrawn. In the presence of Type-II progressively censored competing risks data, we have the following observation

$$(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{m:m:n}, \delta_m, R_m),$$

where $R_m = n - m - \sum_{i=1}^{m-1} R_i$ and $\delta_i \in (1, 2)$ refer to the cause of failure of the ith individual. Let

$$I(\delta_i = 1) = \begin{cases} 1, & \delta_i = 1 \\ 0 & otherwise \end{cases} \quad \text{and} \quad I(\delta_i = 2) = \begin{cases} 1, & \delta_i = 2 \\ 0 & otherwise \end{cases}$$

then $m_1 = \sum_{i=1}^{m} I(\delta_i = 1)$ and $m_2 = \sum_{i=1}^{m} I(\delta_i = 2)$ are the number of failures due to cause one and cause two, respectively, and $m = m_1 + m_2$. Given a progressive censoring scheme $(R_1, \ldots, R_m)$, the likelihood function of the observed data $(x_{1:m:n}, \delta_1), \ldots, (x_{m:m:n}, \delta_m)$ takes the form, see Kundu et al. [13],

$$L = c \prod_{i=1}^{m} [f_1(x_i)\bar{F}_2(x_i)]^{I(\delta_i=1)} [f_2(x_i)\bar{F}_1(x_i)]^{I(\delta_i=2)} [\bar{F}_1(x_i)\bar{F}_2(x_i)]^{R_i}, \tag{7}$$

where $\bar{F}_t(x) = 1 - F_t(x), t = 1, 2, x_i = x_{i:m:n}$ for simplicity of notation and $c = n(n - R_1 - 1)$ $(n - R_1 - R_2 - 2) \ldots (n - m - R_1 - \ldots - R_{m-1} + 1)$. Another form of the likelihood function in (7) can be obtained using the identity $f_t(x) = h_t(x)\bar{F}_t(x)$ as follows

$$L = c \prod_{i=1}^{m} [h_1(x_i)]^{I(\delta_i=1)} [h_2(x_i)]^{I(\delta_i=2)} [\bar{F}_1(x_i)\bar{F}_2(x_i)]^{1+R_i}, \tag{8}$$

where $h_t(x) = f_t(x)/\bar{F}_t(x)$ and $\bar{F}_t(x) = 1 - F_t(x), t = 1, 2$.

3. Maximum Likelihood Estimation

In this section, the maximum likelihood method is considered to obtain maximum likelihood estimates (MLEs) of the unknown parameters and the reliability indices, including RF and HRF of the APE distribution under progressively Type-II censored competing risks data. Furthermore, the associated approximate confidence intervals (ACIs) are obtained based on the asymptotic normality of the MLEs.

3.1. MLEs

Using the observations as mentioned in the previous section and based on Equations (1), (2) and (8), the log-likelihood function, ignoring the normalized constant, can be written as follows

$$\begin{aligned} L(\varpi) &= \sum_{t=1}^{2} m_t \log(\theta_t \log(\alpha_t)) + n\sum_{t=1}^{2}[\log(\alpha_t) - \log(\alpha_t - 1)] - \sum_{t=1}^{2}\sum_{i=1}^{m_t}\left[\theta_t x_i + \log\left(\alpha_t^{e^{-\theta_t x_i}} - 1\right)\right] \\ &+ \sum_{t=1}^{2}\sum_{i=1}^{m}(1 + R_i)\log\left(1 - \alpha_t^{-e^{-\theta_t x_i}}\right), \end{aligned} \tag{9}$$

where $\varpi = (\theta_1, \theta_2, \alpha_1, \alpha_2)^\top$. The MLEs of the parameters θ_t and $\alpha_t, t = 1, 2$ can be obtained by maximizing the log-likelihood function in (9) with respect to θ_t and $\alpha_t, t = 1, 2$. Another way to acquire these estimates is to find the normal equations by getting the first partial derivatives of (9) with respect to θ_t and α_t and equating them to zero. The MLEs denoted by $\hat{\theta}_t$ and $\hat{\alpha}_t, t = 1, 2$, in this case, are obtained by solving the following normal equations simultaneously

$$\frac{\partial L(\varpi)}{\partial \theta_t} = \frac{m_t}{\theta_t} - \sum_{i=1}^{m_t}\left[x_i + \frac{x_i v_i \log(\alpha_t)}{w_i^* - 1}\right] - \log(\alpha_t)\sum_{i=1}^{m}(1 + R_i)\frac{x_i v_i}{w_i - 1} = 0 \tag{10}$$

and

$$\frac{\partial L(\varpi)}{\partial \alpha_t} = \frac{m_t}{\alpha_t \log(\alpha_t)} - \frac{n}{\alpha_t(\alpha_t - 1)} + \frac{1}{\alpha_t}\sum_{i=1}^{m_t}\frac{v_i}{w_i^* - 1} + \frac{1}{\alpha_t}\sum_{i=1}^{m}(1 + R_i)\frac{v_i}{w_i - 1} = 0, \tag{11}$$

where $w_i \equiv w(\theta_t, \alpha_t) = \alpha_t^{e^{-\theta_t x_i}}$, $w_i^* = w_i^{-1}$ and $v_i \equiv v(\theta_t) = e^{-\theta_t x_i}, t = 1, 2, i = 1, \ldots, m$. Due to the complicated expressions of (10) and (11), the MLEs cannot be computed in explicit forms. Accordingly, numerical iterative techniques must be performed to calculate the wanted estimates. Hence, owing to the invariance property of the MLEs, the MLEs of RF and HRF can be obtained from (5) and (6), respectively, as follows

$$\hat{R}(x) = \frac{\hat{\alpha}_1\hat{\alpha}_2}{(\hat{\alpha}_1-1)(\hat{\alpha}_2-1)}\left(1-\hat{\alpha}_1^{-e^{-\hat{\theta}_1 x}}\right)\left(1-\hat{\alpha}_2^{-e^{-\hat{\theta}_2 x}}\right),\ x>0 \tag{12}$$

and

$$\hat{h}(x) = \left[\frac{\hat{\theta}_1\log(\hat{\alpha}_1)e^{-\hat{\theta}_1 x}\hat{\alpha}_1^{-e^{-\hat{\theta}_1 x}}}{1-\hat{\alpha}_1^{-e^{-\hat{\theta}_1 x}}}+\frac{\hat{\theta}_2\log(\hat{\alpha}_2)e^{-\hat{\theta}_2 x}\hat{\alpha}_2^{-e^{-\hat{\theta}_2 x}}}{1-\hat{\alpha}_2^{-e^{-\hat{\theta}_2 x}}}\right]. \tag{13}$$

3.2. ACIs

In this subsection, the ACIs of the unknown parameters θ_t and $\alpha_t, t=1,2$, RF and HRF are constructed using the asymptotic normality of MLEs. Under some regularity conditions, it is known that $\hat{\varpi}=(\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha}_1,\hat{\alpha}_2)^\top$ is approximately normally distributed with mean $\varpi=(\theta_1,\theta_2,\alpha_1,\alpha_2)^\top$ and variance-covariance matrix $I(\varpi)\equiv I(\theta_1,\theta_2,\alpha_1,\alpha_2)$. Practically, $I(\varpi)$ can be approximated by taking the inverse of the observed Fisher information matrix, thus

$$\begin{aligned} I(\hat{\varpi}) &= \left(-\frac{\partial^2 L(\varpi)}{\partial\varpi_i\partial\varpi_j}\right)^{-1}_{(\varpi=\hat{\varpi})},\ i,j=1,2,3,4 \\ &= \begin{pmatrix} var(\hat{\theta}_1) & 0 & cov(\hat{\theta}_1,\hat{\alpha}_1) & 0 \\ 0 & var(\hat{\theta}_2) & 0 & cov(\hat{\theta}_2,\hat{\alpha}_2) \\ cov(\hat{\alpha}_1,\hat{\theta}_1) & 0 & var(\hat{\alpha}_1) & 0 \\ 0 & cov(\hat{\alpha}_2,\hat{\theta}_2) & 0 & var(\hat{\alpha}_2) \end{pmatrix}, \end{aligned} \tag{14}$$

where the the second derivatives $\partial^2 L(\varpi)/\partial\varpi_i\partial\varpi_j, i,j=1,2,3,4$, are given by

$$\begin{aligned} \frac{\partial^2 L(\varpi)}{\partial\theta_t^2} &= -\frac{m_t}{\theta_t^2}+\sum_{i=1}^{m_t}\frac{x_i^2\log(\alpha_t)v_i}{w_i^*-1}+\sum_{i=1}^{m_t}\frac{x_i^2\log^2(\alpha_t)v_i^2}{w_i\left(w_i^*-1\right)^2}+\sum_{i=1}^{m}\frac{(1+R_i)x_i^2\log(\alpha_t)v_i}{w_i-1} \\ &- \sum_{i=1}^{m}\frac{(1+R_i)x_i^2\log^2(\alpha_t)v_i^2}{w_i^*(w_i-1)^2}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 L(\varpi)}{\partial\alpha_t^2} &= -\frac{1}{\alpha_t^2}\left[\frac{m_t[1+\log(\alpha_t)]}{\log^2(\alpha_t)}+\frac{n(1-2\alpha_t)}{(\alpha_t-1)^2}\right]-\frac{1}{\alpha_t^2}\sum_{i=1}^{m_t}\frac{v_i}{w_i^*-1}+\frac{1}{\alpha_t^2}\sum_{i=1}^{m_t}\frac{v_i^2}{w_i\left(w_i^*-1\right)^2} \\ &- \frac{1}{\alpha_t^2}\sum_{i=1}^{m}\frac{(1+R_i)v_i}{w_i-1}-\frac{1}{\alpha_t^2}\sum_{i=1}^{m}\frac{(1+R_i)v_i^2}{w_i^*(w_i-1)^2}, \end{aligned}$$

$$\frac{\partial^2 L(\varpi)}{\partial\theta_t\partial\alpha_t} = -\frac{1}{\alpha_t}\sum_{i=1}^{m_t}\frac{x_i v_i}{w_i^*-1}\left[1+\frac{v_i\log(\alpha_t)}{w_i(w_i^*-1)}\right]-\frac{1}{\alpha_t}\sum_{i=1}^{m}\frac{x_i v_i(1+R_i)}{w_i-1}\left[1-\frac{v_i\log(\alpha_t)}{w_i^*(w_i-1)}\right]$$

and

$$\frac{\partial^2 L(\varpi)}{\partial\theta_t\partial\theta_{3-t}} = \frac{\partial^2 L(\varpi)}{\partial\theta_t\partial\alpha_{3-t}} = \frac{\partial^2 L(\varpi)}{\partial\alpha_t\partial\theta_{3-t}} = \frac{\partial^2 L(\varpi)}{\partial\alpha_t\partial\alpha_{3-t}} = 0,\ t=1,2.$$

Then, the $100(1-\zeta)$ ACIs of the parameters θ_t and α_t, are as follows

$$\left[\hat{\theta}_t \pm z_{\zeta/2}\sqrt{var(\hat{\theta}_t)}\right] \text{ and } \left[\hat{\alpha}_t \pm z_{\zeta/2}\sqrt{var(\hat{\alpha}_t)}\right],\ t=1,2,$$

where $z_{\zeta/2}$ is the upper $\zeta/2$ percentile of the standard normal distribution.

To build the ACIs of RF and HRF, we require their variances to be determined. Here, we consider using the delta method to approximate the variances of $\hat{R}(x)$ and $\hat{h}(x)$. For more details about the delta method, see Greene [18]. To utilize this method, let Ψ_1 and Ψ_2 be two quantities that take the following forms

$$\Psi_1 = \left(\frac{\partial R(x)}{\partial \theta_1}, \frac{\partial R(x)}{\partial \theta_2}, \frac{\partial R(x)}{\partial \alpha_1}, \frac{\partial R(x)}{\partial \alpha_2}\right) \text{ and } \Psi_2 = \left(\frac{\partial h(x)}{\partial \theta_1}, \frac{\partial h(x)}{\partial \theta_2}, \frac{\partial h(x)}{\partial \alpha_1}, \frac{\partial h(x)}{\partial \alpha_2}\right),$$

where

$$\frac{\partial R(x)}{\partial \theta_t} = \frac{\alpha_t \alpha_{3-t}}{(\alpha_t - 1)(\alpha_{3-t} - 1)} x \log(\alpha_t) e^{-\theta_t x} \alpha_t^{-e^{-\theta_t x}} \left(\alpha_{3-t}^{-e^{-\theta_{3-t} x}} - 1\right),$$

$$\frac{\partial R(x)}{\partial \alpha_t} = \frac{\alpha_{3-t}\left(\alpha_{3-t}^{-e^{-\theta_{3-t} x}} - 1\right)}{(\alpha_t - 1)(\alpha_{3-t} - 1)} \left\{ \left(\alpha_t^{-e^{-\theta_t x}} - 1\right)\left[1 - \frac{\alpha_t}{(\alpha_t - 1)}\right] - e^{-\theta_t x} \alpha_t^{-e^{-\theta_t x}} \right\},$$

$$\frac{\partial h(x)}{\partial \theta_t} = \frac{e^{-\theta_t x} \log(\alpha_t)}{\alpha_t^{e^{-\theta_t x}} - 1} \left(1 - \theta_t x - \frac{\theta_t \log(\alpha_t) x e^{-\theta_t x}}{1 - \alpha_t^{-e^{-\theta_t x}}}\right)$$

and

$$\frac{\partial h(x)}{\partial \alpha_t} = \frac{\theta_t e^{-\theta_t x}}{\alpha_t \left(\alpha_t^{e^{-\theta_t x}} - 1\right)} \left[1 - \frac{e^{-\theta_t x} \log(\alpha_t)}{1 - \alpha_t^{-e^{-\theta_t x}}}\right].$$

Then, the approximate variances of RF and HRF can be obtained as follows

$$var(\hat{R}) = \left[\Psi_1 I(\hat{\omega}) \Psi_1^{\top}\right]\Big|_{(\omega = \hat{\omega})} \text{ and } var(\hat{h}) = \left[\Psi_2 I(\hat{\omega}) \Psi_2^{\top}\right]\Big|_{(\omega = \hat{\omega})}, \; t = 1, 2,$$

where $I(\hat{\omega})$ is given by (14). Then, the $100(1-\zeta)$ ACIs of $R(x)$ and $h(x)$ are, respectively, as follows

$$\left[\hat{R} \pm z_{\zeta/2}\sqrt{var(\hat{R})}\right] \text{ and } \left[\hat{h} \pm z_{\zeta/2}\sqrt{var(\hat{h})}\right], \; t = 1, 2.$$

4. Bootstrap Confidence Intervals

In this subsection, we consider using two parametric bootstrap confidence intervals. The primary one is the percentile bootstrap confidence intervals (PBCIs) based on the concept of Efron [19]. The other one is the studentized bootstrap confidence intervals (SBCIs) introduced by Hall [20]. To calculate these confidence intervals, we apply the next algorithms:

(A) PBCIs

(1) From the original data $(x_{1:m:n}, \delta_1), \ldots, (x_{m:m:n}, \delta_m)$ with censoring scheme $R_1, \ldots, R_m$ calculate the MLEs of θ_t and α_t, $t = 1, 2$.

(2) Apply the MLEs received in (1) to create a Type-II progressively censored competing risks sample.

(3) Use the bootstrap sample from (2) to compute bootstrap estimates, $\hat{\theta}_t^\star$ and $\hat{\alpha}_t^\star$. Moreover, obtain the bootstrap estimates of RF and HRF as $\hat{R}^\star$ and $\hat{h}^\star$.

(4) Redo steps (2) and (3) B times to get $(\hat{\theta}_t^{\star(1)}, \ldots, \hat{\theta}_t^{\star(B)})$, $(\hat{\alpha}_t^{\star(1)}, \ldots, \hat{\alpha}_t^{\star(B)})$, $(\hat{R}^{\star(1)}, \ldots, \hat{R}^{\star(B)})$ and $(\hat{h}^{\star(1)}, \ldots, \hat{h}^{\star(B)})$.

(5) Adjust the estimates in (4) in ascending order to get $(\hat{\theta}_t^{\star[1]}, \ldots, \hat{\theta}_t^{\star[B]})$, $(\hat{\alpha}_t^{\star([1]}, \ldots, \hat{\alpha}_t^{\star[B]})$, $(\hat{R}^{\star[1]}, \ldots, \hat{R}^{\star[B]})$ and $(\hat{h}^{\star[1]}, \ldots, \hat{h}^{\star[B]})$.

(6) The two-sided $100(1-\zeta)\%$ PBCIs of θ_t, α_t, RF and HRF are given, respectively, by

$$\left[\hat{\theta}_t^{\star[B\zeta/2]}, \hat{\theta}_t^{\star[B(1-\zeta/2)]}\right], \quad \left[\hat{\alpha}_t^{\star[B\zeta/2]}, \hat{\alpha}_t^{\star[B(1-\zeta/2)]}\right], t = 1, 2,$$

$$\left[\hat{R}^{\star[B\zeta/2]}, \hat{R}^{\star[B(1-\zeta/2)]}\right] \text{ and } \left[\hat{h}^{\star[B\zeta/2]}, \hat{h}^{\star[B(1-\zeta/2)]}\right].$$

(B) SBCIs

(1–3) Same as in PBCIs.

(4) Compute $T^{\star}_{\theta(t)} = \frac{\hat{\theta}^{\star}_t - \hat{\theta}_t}{\sqrt{var(\hat{\theta}^{\star}_t)}}$, $T^{\star}_{\alpha(t)} = \frac{\hat{\alpha}^{\star}_t - \hat{\alpha}_t}{\sqrt{var(\hat{\alpha}^{\star}_t)}}$, $t = 1, 2$, $T^{\star}_R = \frac{\hat{R}^{\star} - \hat{R}}{\sqrt{var(\hat{R}^{\star})}}$ and $T^{\star}_h = \frac{\hat{h}^{\star} - \hat{h}}{\sqrt{var(\hat{h}^{\star})}}$. Here, the variances are obtained based on the bootstrap sample, where $var(\hat{\theta}^{\star}_t)$ and $var(\hat{\alpha}^{\star}_t)$ are computed from the asymptotic variance–covariance matrix in (14), and $var(\hat{R}^{\star})$ and $var(\hat{h}^{\star})$ are obtained by employing the delta method.

(5) Redo step 2-4 B times to compute $(T^{\star(1)}_{\theta(t)}, \ldots, T^{\star(B)}_{\theta(t)})$, $(T^{\star(1)}_{\alpha(t)}, \ldots, T^{\star(B)}_{\alpha(t)})$, $(T^{\star(1)}_R, \ldots, T^{\star(B)}_R)$ and $(T^{\star(1)}_h, \ldots, T^{\star(B)}_h)$.

(6) Arrange the quantities in (5) in ascending order to get $(T^{\star[1]}_{\theta(t)}, \ldots, T^{\star[B]}_{\theta(t)})$, $(T^{\star[1]}_{\alpha(t)}, \ldots, T^{\star[B]}_{\alpha(t)})$, $(T^{\star[1]}_R, \ldots, T^{\star[B]}_R)$ and $(T^{\star[1]}_h, \ldots, T^{\star[B]}_h)$.

(7) The two-sided $100(1-\zeta)$ SBCIs are given, respectively, by

$$\left[\hat{\theta}_t + T^{\star[B\zeta/2]}_{\theta(t)}\sqrt{var(\hat{\theta}_t)}, \hat{\theta}_t + T^{\star[B(1-\zeta/2)]}_{\theta(t)}\sqrt{var(\hat{\theta}_t)}\right],$$

$$\left[\hat{\alpha}_t + T^{\star[B\zeta/2]}_{\alpha(t)}\sqrt{var(\hat{\alpha}_t)}, \hat{\alpha}_t + T^{\star[B(1-\zeta/2)]}_{\alpha(t)}\sqrt{var(\hat{\alpha}_t)}\right], t = 1, 2,$$

$$\left[\hat{R} + T^{\star[B\zeta/2]}_{R}\sqrt{var(\hat{R})}, \hat{R} + T^{\star[B(1-\zeta/2)]}_{R}\sqrt{var(\hat{R})}\right]$$

and

$$\left[\hat{h} + T^{\star[B\zeta/2]}_{h}\sqrt{var(\hat{h})}, \hat{h} + T^{\star[B(1-\zeta/2)]}_{h}\sqrt{var(\hat{h})}\right],$$

where the variances $var(\hat{\theta}_t), var(\hat{\alpha}_t), var(\hat{R})$ and $var(\hat{h})$ are computed based on the MLEs obtained from the original sample.

5. Bayesian Estimation

The Bayesian approach appears as an exceptional alternative to the conventional estimation, allowing us to consider the unknown parameter as a random variable and mix the prior information with the sample data. The difference between the Bayesian approach and the traditional estimation approaches is that the prior distribution of parameter information is collected throughout the practical background. If there is no appropriate knowledge, it can be displaced by a non-information prior distribution. In this section, we study Bayesian estimation for the unknown parameters, RF and HRF of the APE distribution based on progressively Type-II censored competing risks data following the assumption that θ_t and $\alpha_t, t = 1, 2,$ have gamma prior distributions with the resulting forms

$$p(\theta_t) \propto \theta_t^{\tau_t - 1} e^{-\kappa_t \theta_t}, \quad \theta_t > 0, \quad \tau_t, \kappa_t > 0 \tag{15}$$

and

$$p(\alpha_t) \propto \alpha_t^{\varepsilon_t - 1} e^{-\upsilon_t \alpha_t}, \quad \alpha_t > 0, \quad \varepsilon_t, \upsilon_t > 0, t = 1, 2. \tag{16}$$

Using (15) and (16), the joint prior distribution takes the form

$$p(\varpi) \propto \theta_1^{\tau_1 - 1} \theta_2^{\tau_2 - 1} \alpha_1^{\varepsilon_1 - 1} \alpha_2^{\varepsilon_2 - 1} e^{-(\kappa_1 \theta_1 + \kappa_2 \theta_2 + \upsilon_1 \alpha_1 + \upsilon_2 \alpha_2)}. \tag{17}$$

Based on (9) and (17), the joint posterior distribution is as follows

$$
\begin{aligned}
g(\varpi|\underline{x}) &= A\, e^{L(\varpi)} p(\varpi) \\
&= A\, \theta_1^{\tau_1+m_1-1} \theta_2^{\tau_2+m_2-1} \alpha_1^{\varepsilon_1+n-1} \alpha_2^{\varepsilon_2+n-1} \tfrac{[\log(\alpha_1)]^{m_1}[\log(\alpha_2)]^{m_2}}{[(\alpha_1-1)(\alpha_2-1)]^n} \\
&\times \exp\Big\{-(\kappa_1\theta_1+\upsilon_1\alpha_1) - \textstyle\sum_{i=1}^{m_1}\Big[\theta_1 x_i + \log\Big(\alpha_1^{e^{-\theta_1 x_i}}-1\Big)\Big]\Big\} \prod_{i=1}^{m}\Big(1-\alpha_1^{-e^{-\theta_1 x_i}}\Big)^{1+R_i} \\
&\times \exp\Big\{-(\kappa_2\theta_2+\upsilon_2\alpha_2) - \textstyle\sum_{i=1}^{m_2}\Big[\theta_2 x_i + \log\Big(\alpha_2^{e^{-\theta_2 x_i}}-1\Big)\Big]\Big\} \prod_{i=1}^{m}\Big(1-\alpha_2^{-e^{-\theta_2 x_i}}\Big)^{1+R_i},
\end{aligned} \tag{18}
$$

where $L(\varpi)$ is given by (9) and A is the normalized constant expressed as

$$
A = \left(\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{L(\theta_1,\theta_2,\alpha_1,\alpha_2)} p(\theta_1,\theta_2,\alpha_1,\alpha_2) d\theta_1 d\theta_2 d\alpha_1 d\alpha_2\right)^{-1}.
$$

We see that (18) is analytically intractable, and the BE of any parametric function of (θ_t, α_t) includes a ratio of two integrals. Therefore, to obtain the BEs under any loss function, say $\phi(\theta_1, \theta_2, \alpha_1, \alpha_2)$, some approximation approaches should be applied to solve the corresponding ratio of integrals, such as the MCMC methods. Here, we employ the well-known Metropolis–Hastings (MH) algorithm to get the BEs and the associated HPD-credible intervals. The MH algorithm can be worked to return samples from the posterior density function (18) and, in turn, to compute the BEs as well as the corresponding HPD-credible intervals. To generate samples from (18), we need to derive the full conditional distributions of θ_t and α_t as

$$
g(\theta_t|\alpha_t,\underline{x}) \propto \theta_t^{\tau_t+m_t-1} \exp\left\{-\kappa_t\theta_t - \sum_{i=1}^{m_t}\Big[\theta_t x_i + \log\Big(\alpha_t^{e^{-\theta_t x_i}}-1\Big)\Big]\right\} \prod_{i=1}^{m}\Big(1-\alpha_t^{-e^{-\theta_t x_i}}\Big)^{1+R_i} \tag{19}
$$

and

$$
g(\alpha_t|\theta_t,\underline{x}) \propto \alpha_t^{\varepsilon_t+n-1} \frac{[\log(\alpha_t)]^{m_t}}{(\alpha_t-1)^n} \exp\left[-\upsilon_t\alpha_t - \sum_{i=1}^{m_t}\log\Big(\alpha_t^{e^{-\theta_t x_i}}-1\Big)\right] \prod_{i=1}^{m}\Big(1-\alpha_t^{-e^{-\theta_t x_i}}\Big)^{1+R_i}. \tag{20}
$$

It is observed that the distributions in (19) and (20) cannot be reduced to any familiar distribution. To solve this challenge, we suggest applying the MH algorithm to generate random samples from (19) and (20), respectively, using the steps shown as follows

Step 1. Set $k = 1$ and put $(\theta_t^{(0)}, \alpha_t^{(0)}) = (\hat{\theta}_t, \hat{\alpha}_{kt})$, $t = 1, 2$.

Step 2. Generate $\theta_t^{(k)}$ from (19) using MH steps using normal distribution $N[\theta^{(k-1)}, var(\hat{\theta}_t)]$.

Step 3. Generate $\alpha_t^{(k)}$ from (20) using MH steps using normal distribution $N[\alpha^{(k-1)}, var(\hat{\alpha}_t)]$.

Step 4. Put $k = k + 1$.

Step 5. Redo steps 2-4 M times and compute $(\theta_t^{(1)}, \ldots, \theta_t^{(M)})$ and $(\alpha_t^{(1)}, \ldots, \alpha_t^{(M)})$.

Step 6. Based on the values $\theta_t^{(k)}$ and $\alpha_t^{(k)}$, $k = 1, \ldots, M$, the BEs of $\theta_t, \alpha_t, R(x)$ and $h(x)$ under squared error loss function can be obtained as

$$
\tilde{\theta}_t = \frac{\sum_{k=Q+1}^{M} \theta_t^{(k)}}{M-Q}, \quad \tilde{\alpha}_t = \frac{\sum_{k=Q+1}^{M} \alpha_t^{(k)}}{M-Q}, \quad \tilde{R} = \frac{\sum_{k=Q+1}^{M} R^{(k)}}{M-Q}, \quad \text{and } \tilde{h} = \frac{\sum_{k=Q+1}^{M} h^{(k)}}{M-Q},
$$

where $R^{(k)}$ and $h^{(k)}$ are obtained from (5) and (6), respectively, and Q is the burn-in period.

Step 7. Follow the approach considered by Chen and Shao [21] to compute the HPD-credible intervals of $\theta_t, \alpha_t, R(x)$ and $h(x)$ as follows:

- Arrange the MCMC samples of $\theta_t, \alpha_t, R(x)$ and $h(x)$ after the burn-in period to get
  $(\theta_t^{[Q+1]}, \ldots, \theta_t^{[M]})$, $(\alpha_t^{[Q+1]}, \ldots, \alpha_t^{[M]})$, $(R^{[Q+1]}, \ldots, R^{[M]})$ and $(h^{[Q+1]}, \ldots, h^{[M]})$.

- The two-sided $100(1-\zeta)$ HPD-credible intervals of $\theta_t, \alpha_t, R(x)$ and $h(x)$, say μ, can be obtained in this case as follows

$$\left[\mu^{[k^\star]},\, \mu^{[k^\star+(1-\zeta)(M-Q)]}\right],$$

where $k^\star = Q+1, Q+2, \ldots, M$ is selected so that

$$\mu^{[k^\star+(1-\zeta)(M-Q)]} - \mu^{[k^\star]} = \min_{1\leqslant j\leqslant \zeta(M-Q)}\left(\mu^{[j+(1-\zeta)(M-Q)]} - \mu^{[j]}\right).$$

6. Numerical Analysis

In this section, a simulation study is conducted to compare the performance of the different point and intervals estimates. Further, a simulated dataset is investigated for illustration purposes.

6.1. Simulation Study

For illustration, we conduct a simulation study of a competing risks model with specified competing risk distribution under the progressive censoring to numerically estimate the model parameters. The competing risks data are simulated using the cause-specific hazard driven approach by Beyersmann [22] when $(\alpha_1, \theta_1, \alpha_2, \theta_2)$ are assumed to be $(1.5, 2.0, 2.0, 3.0)$. To compare the performance of parameter estimates, we choose the bias, mean square error (MSE) for point estimates, the interval length (IL) and coverage probability (CP) for interval estimates. We generated $N = 500$ competing risks datasets under progressive censoring. For each simulated dataset, we determine $B = 500$ bootstrap samples and $M = 3000$ samples, and the first 25% portion of MCMC samples as burn-in times within the Gibbs–MH algorithm. Denote n as the number of identical items.

For the l-th simulated dataset for $l = \{1, 2, \ldots, N\}$, the maximum likelihood estimates of the parameters can be obtained by solving the likelihood equations in Equations (10) and (11), and the Bayes estimates of parameters can be calculated using the steps of the MH algorithm in Section 5. We then get the estimates of reliability characteristics $R(t)$ and $h(t)$ given t and the interval estimates based on the parameter estimates. For simplicity, we denote the estimate of a parameter or a reliability characteristic as β. The point estimate and interval estimate of β using the l-th simulated dataset are given as $\hat{\beta}_l$ and $(\hat{\beta}_l^{\text{lower}}, \hat{\beta}_l^{\text{upper}})$, respectively. The indicator function is defined as

$$I_l = \begin{cases} 1, & \text{if} \quad \hat{\beta}_l^{\text{lower}} \leq \beta \leq \hat{\beta}_l^{\text{upper}}, \\ 0, & \qquad \text{otherwise}. \end{cases}$$

Thus, the performance measures are of the following forms:

$$\text{Estimate} = \frac{\sum_{l=1}^N \hat{\beta}_l}{N}, \quad \text{Bias} = \frac{\sum_{l=1}^N (\hat{\beta}_l - \beta)}{N}, \quad \text{MSE} = \frac{\sum_{l=1}^N (\hat{\beta}_l - \beta)^2}{N}$$
$$\text{IL} = \frac{\sum_{l=1}^N (\hat{\beta}_l^{\text{upper}} - \hat{\beta}_l^{\text{lower}})}{N}, \quad \text{CP} = \frac{\sum_{l=1}^N I_l}{N}.$$

We choose $n = \{50, 100, 200\}$, $p = 8\%$ surviving items discarded under the progressive censoring scheme and three specific schemes as follow:

Scheme 1: $R_1 = r, R_2 = R_3 = \ldots = R_m = 0.$

Scheme 2: $R_1 = r/2, R_2 = R_3 = \ldots = R_{m-1} = 0, R_m = r/2.$

Scheme 3: $R_{i_1} = R_{i_2} = \ldots = R_{i_r} = 1$, and the others $R_i = 0$ for $i = \{1, 2, \ldots, m\}/\{i_1, i_2, \ldots, i_r\}$.

In the schemes above, $m = n \times (1-p), r = n \times p$ and $i_1, i_2, \ldots, i_r$ are randomly selected from the set $\{1, 2, \ldots, m\}$.

Under the assumptions of bivariate competing failure causes (X_1, X_2) and APE distribution, the procedures to generate the competing risks data $(x_1, \delta_1), (x_2, \delta_2), \dots, (x_m, \delta_m))$ under the progressive censoring scheme $(R_1, R_2, \dots, R_m)$ are given as follows:

1. Generate the uniform progressive censoring data $(u_1, \dots, u_m)$ by specified scheme $(R_1, \dots, R_m)$ using the sampling algorithm in Balakrishnan and Sandhu [23].
2. Compute the roots x_i of the equations $F(x_i) = u_i, i = 1, 2, \dots, m$ where $F(x_i)$ is the survival function of the competing risks random variable $X = \min(X_1, X_2)$ given in Equation (3).
3. Simulate binomial samples δ_i with probability $\frac{h_1(x_i)}{h_1(x_i)+h_2(x_i)}$ on failure cause 1, and probability $\frac{h_2(x_i)}{h_1(x_i)+h_2(x_i)}$ on failure cause 2.

In Bayesian inference, independent gamma priors are often chosen for the model parameter. To determine the hyper-parameters of Gamma prior distribution, for simplicity we assume the scale parameters are equal to 1, and the MLEs $(\hat{\alpha}_1, \hat{\theta}_1, \hat{\alpha}_2, \hat{\theta}_2)$ are the expectations of prior distributions. Therefore, the priors are given as

$$\begin{aligned} p(\alpha_t) &\propto \alpha_t^{\hat{\alpha}_t - 1} e^{-\alpha_t}, \quad \alpha_t > 0, \\ p(\theta_t) &\propto \theta_t^{\hat{\theta}_t - 1} e^{-\theta_t}, \quad \theta_t > 0, t = 1, 2. \end{aligned} \tag{21}$$

Under the setting above, we obtain the MLEs and BEs of unknown parameters for $n = (50, 100, 200)$ and the three schemes in Tables 1–3, and the ACIs, PBCI and SBCIs and the HPD-credible intervals of unknown parameters are given in Tables 4–6.

Table 1. Point estimates of $(\alpha_t, \theta_t), t = 1, 2$ under Scheme 1.

n	Parameter	MLE			BE		
		Estimate	**Bias**	**MSE**	**Estimate**	**Bias**	**MSE**
50	α_1	2.2008	0.7008	0.9388	1.5001	0.0001	0.0082
	θ_1	2.3030	0.3030	0.3428	1.9820	−0.0180	0.0382
	α_2	3.1164	1.1164	2.0590	2.0008	0.0008	0.0111
	θ_2	3.4548	0.4548	0.5391	2.9694	−0.0306	0.0576
100	α_1	2.1824	0.6824	0.7295	1.4999	−0.0001	0.0092
	θ_1	2.2645	0.2645	0.1902	1.9799	−0.0201	0.0364
	α_2	3.0110	1.0110	1.4645	1.9856	−0.0144	0.0152
	θ_2	3.4089	0.4089	0.3506	2.9957	−0.0043	0.0545
200	α_1	2.1255	0.6255	0.5700	1.4960	−0.0040	0.0114
	θ_1	2.2883	0.2883	0.1523	1.9847	−0.0153	0.0252
	α_2	2.9488	0.9488	1.1027	1.9952	−0.0048	0.0196
	θ_2	3.4240	0.4240	0.2710	2.9739	−0.0261	0.0367

Table 2. Point estimates of $(\alpha_t, \theta_t), t = 1, 2$ under Scheme 2.

n	Parameter	MLE			BE		
		Estimate	**Bias**	**MSE**	**Estimate**	**Bias**	**MSE**
50	α_1	2.2990	0.7990	1.0707	1.5319	0.0319	0.0082
	θ_1	2.0085	0.0085	0.2097	1.8981	−0.1019	0.0444
	α_2	3.2216	1.2216	2.3895	2.0460	0.0460	0.0133
	θ_2	2.9947	−0.0053	0.2300	2.8616	−0.1384	0.0760
100	α_1	2.2799	0.7799	0.8965	1.5539	0.0539	0.0111
	θ_1	2.0255	0.0255	0.1140	1.8886	−0.1114	0.0417
	α_2	3.2979	1.2979	2.0914	2.0780	0.0780	0.0198
	θ_2	3.0767	0.0767	0.1260	2.8501	−0.1499	0.0642
200	α_1	2.3165	0.8165	0.8929	1.5912	0.0912	0.0178
	θ_1	2.0031	0.0031	0.0590	1.8556	−0.1444	0.0407
	α_2	3.2708	1.2708	1.8308	2.1292	0.1292	0.0324
	θ_2	3.0272	0.0272	0.0620	2.8056	−0.1944	0.0725

Table 3. Point estimates of $(\alpha_t, \theta_t), t = 1, 2$ under Scheme 3.

n	Parameter	MLE			BE		
		Estimate	Bias	MSE	Estimate	Bias	MSE
50	α_1	2.0884	0.5884	0.7528	1.5326	0.0326	0.0093
	θ_1	1.8887	−0.1113	0.2044	1.7975	−0.2025	0.0782
	α_2	2.8145	0.8145	1.4133	2.0354	0.0354	0.0118
	θ_2	2.8981	−0.1019	0.2638	2.7273	−0.2727	0.1287
100	α_1	1.8670	0.3670	0.3684	1.5735	0.0735	0.0158
	θ_1	1.8120	−0.1880	0.1259	1.7485	−0.2515	0.0921
	α_2	2.5574	0.5574	0.7110	2.0776	0.0776	0.0225
	θ_2	2.7544	−0.2456	0.1819	2.6310	−0.3690	0.1882
200	α_1	1.7279	0.2279	0.2028	1.5939	0.0939	0.0242
	θ_1	1.7697	−0.2303	0.1020	1.6998	−0.3002	0.1145
	α_2	2.3291	0.3291	0.2841	2.1053	0.1053	0.0337
	θ_2	2.6819	−0.3181	0.1695	2.5685	−0.4315	0.2250

Table 4. Interval estimates of $(\alpha_t, \theta_t), t = 1, 2$ under Scheme 1.

n	Parameter	ACI		PBCI		SBCI		HPD	
		IL	CP	IL	CP	IL	CP	IL	CP
50	α_1	9.7556	1.0000	1.4924	1.0000	1.5879	0.9540	1.3628	1.0000
	θ_1	4.6868	1.0000	1.7521	0.9240	1.8926	0.8660	1.2085	0.9980
	α_2	10.2280	1.0000	2.1764	1.0000	2.3790	0.9460	1.5658	1.0000
	θ_2	4.4634	1.0000	2.1155	0.9080	2.2250	0.8760	1.4432	0.9980
100	α_1	7.0482	1.0000	1.1913	1.0000	1.3116	0.9420	1.3390	1.0000
	θ_1	3.3404	1.0000	1.2760	0.9480	1.3459	0.9140	1.0412	0.9940
	α_2	7.7217	1.0000	1.6173	1.0000	1.8307	0.9660	1.5234	1.0000
	θ_2	3.0677	0.9980	1.5204	0.9340	1.5862	0.9100	1.2444	0.9920
200	α_1	5.2778	1.0000	1.0148	1.0000	1.1615	0.9640	1.3042	1.0000
	θ_1	2.3285	1.0000	0.9584	0.9320	1.0032	0.9260	0.8838	0.9900
	α_2	6.1778	1.0000	1.2492	1.0000	1.3832	0.9680	1.4861	1.0000
	θ_2	2.1783	0.9940	1.1260	0.9380	1.1600	0.9260	1.0307	0.9920

Table 5. Interval estimates of $(\alpha_t, \theta_t), t = 1, 2$ under Scheme 2.

n	Parameter	ACI		PBCI		SBCI		HPD	
		IL	CP	IL	CP	IL	CP	IL	CP
50	α_1	12.8080	1.0000	7.6623	1.0000	5.1672	0.9880	1.3814	1.0000
	θ_1	4.7067	1.0000	0.9526	1.0000	1.5020	0.9980	1.1629	0.9980
	α_2	11.0431	1.0000	9.0435	1.0000	6.0705	0.9880	1.5822	1.0000
	θ_2	4.1499	1.0000	1.1633	1.0000	1.8517	0.9980	1.3922	0.9880
100	α_1	8.0342	1.0000	4.3686	1.0000	3.4588	1.0000	1.3710	1.0000
	θ_1	3.3382	1.0000	0.8280	1.0000	1.1314	1.0000	0.9975	0.9820
	α_2	8.6272	1.0000	5.1059	1.0000	4.1223	0.9920	1.5563	1.0000
	θ_2	2.8882	1.0000	0.9407	1.0000	1.1044	0.9980	1.1856	0.9860
200	α_1	6.1535	1.0000	3.9199	1.0000	3.6637	1.0000	1.3583	1.0000
	θ_1	2.3102	1.0000	0.7508	1.0000	1.0114	1.0000	0.8308	0.9620
	α_2	6.9898	1.0000	4.5146	1.0000	4.4621	1.0000	1.5312	1.0000
	θ_2	2.0230	1.0000	0.7947	1.0000	0.9698	1.0000	0.9701	0.9260

Table 6. Interval estimates of $(\alpha_t, \theta_t), t = 1, 2$ under Scheme 3.

n	Parameter	ACI		PBCI		SBCI		HPD	
		IL	CP	IL	CP	IL	CP	IL	CP
50	α_1	9.8140	1.0000	8.1916	1.0000	5.3750	0.9920	1.3791	1.0000
	θ_1	4.1220	1.0000	0.9589	1.0000	2.0626	0.9480	1.1085	0.9480
	α_2	9.5198	1.0000	9.3754	1.0000	7.2460	0.9960	1.5735	1.0000
	θ_2	4.0537	1.0000	1.2505	1.0000	3.1918	0.9280	1.3282	0.9520
100	α_1	6.0852	1.0000	1.7548	1.0000	1.9489	0.9960	1.5827	1.0000
	θ_1	2.8252	1.0000	1.1748	1.0000	1.2765	0.9900	1.0509	0.9600
	α_2	6.7858	0.9980	2.1559	1.0000	2.3152	0.9760	1.7786	1.0000
	θ_2	2.7632	0.9980	1.3793	1.0000	1.4341	0.9500	1.2295	0.9080
200	α_1	4.3455	1.0000	1.4327	1.0000	1.6447	0.9940	1.4417	1.0000
	θ_1	1.9226	1.0000	0.8850	1.0000	0.9465	0.9700	0.8323	0.8580
	α_2	5.0429	1.0000	1.6726	1.0000	1.9462	0.9860	1.5820	1.0000
	θ_2	1.9413	0.9840	1.0390	0.8700	1.0741	0.6280	0.9677	0.7680

From Tables 1–3, we see that

1. The MSEs of MLEs and BEs decrease with the increasing samples n under the three schemes. However, the MSEs of shape parameters are larger than scale parameters for the maximum likelihood method and the Bayes method, and the scale parameters have smaller MSEs and absolute Bias using the Bayes method than the maximum likelihood method. Under Scheme 1, the MSEs of MLEs are smaller than BEs. This indicates that the progressive censoring schemes have a small influence on the performance of point estimates.
2. The biases of scale parameters θ_1 and θ_2 are negative except for the MLEs for $n = 200$ under Scheme 2. For Bayes estimation, the scale parameters are underestimated in all cases. The shape parameters α_1 and α_2 are overestimated using the maximum likelihood method and Bayes method. This shows that the parameter role, i.e., shape and scale parameters in the competing risks model considering the APE distribution, has an impact on the estimation.

From Tables 4–6, we observe that

1. The ILs of ACIs, PBCIs, SBCIs and HPD-credible intervals decrease when n increases under all schemes. This implies that the progressive censoring schemes have little influence on the performance of ILs.
2. The coverage probabilities of the mentioned confidence intervals show that the shape parameters are covered in the confidence intervals under all the schemes and specified sample sizes. This does not hold for coverage probabilities of the scale parameters. However, the CPs of HPD-credible intervals for scale parameters are close to the nominal probability when n is increasing under Schemes 1 and 2.
3. In terms of ILs, SBCIs and HPD-credible intervals are better than ACIs and PBCIs. For the coverage probabilities, bootstrap methods perform better than the other two interval estimation methods.

We also present the point estimates (MLEs and BEs) and interval estimates (ACIs, PBCIs, SBCIs and HPD-credible intervals) of $R(x)$ and $h(x)$ at given point x under the three schemes. Here, we choose $x = \{0.3, 0.5, 0.8\}$ to show the performance of estimation for reliability characteristics. The MLEs and BEs of $R(x)$ and $h(x)$ are given in Table 7 for Scheme 1, Table 8 for Scheme 2 and Table 9 for Scheme 3. The interval estimates of $R(x)$ and $h(x)$ are given in Table 10 for Scheme 1, Table 11 for Scheme 2 and Table 12 for Scheme 3.

We see from Tables 7–12 that

1. The MSEs of MLEs of $R(x)$ and $h(x)$ at $x = \{0.3, 0.5, 0.8\}$ decrease with the increasing samples n under the three schemes. However, the MSEs of BEs have small changes with increasing n when RF and HRF are estimated.

2. R(x) is underestimated using maximum likelihood and Bayes methods, and $h(x)$ is overestimated under the specified schemes. This indicates that the progressive censoring schemes have little influence on the performance of MLES.
3. In Tables 10–12, the ILs of ACIs, PBCIs, SBCIs and HPD-credible interval estimates for $R(x)$ and $h(x)$ decrease when n increases under all schemes. Further, $R(x)$ and $h(x)$ are always covered in ACIs, but they have the largest interval lengths. The progressive schemes have little influence on ACIs, but they have an influence on the other intervals.
4. In terms of ILs and CPs, PBCIs and HPD-credible intervals are better than ACIs and SBCIs for all the schemes.

We also note that, from the coverage probabilities of interval estimation of parameters and reliability characteristics, it can be found that the coverage probabilities are 1 in most cases for ACIs and PBCIs, which are conducted based on the maximum likelihood method. Compared with SBCIs and HPD-credible intervals in terms of the occurrance of coverage probabilities equal to 1, this indicates that SBCIs and HPD-credible intervals perform better than ACIs and PBCIs. In our simulation study, the interval estimates $(\hat{R}^{\text{lower}}(t), \hat{R}^{\text{upper}}(t))$ of $R(t)$ obtained using the estimation procedures in Sections 3–5 lie in the range $(0, 1)$. It is noted that the asymptotic confidence interval of $R(t)$ may be outside of $(0, 1)$. In this case, transformation on $R(t)$ could be applied to avoid the occurrence of exceeding the range $(0, 1)$. On the other hand, though the simulated results for the interval estimates of $R(t)$ and $h(t)$ are the subsets of the domains of reliability characteristics, the transformation approach could also be suggested to improve the performance of interval estimation.

To summarize, based on the MSEs, ILs and CPs of parameter estimates, the performance is relatively good and stable when $n = 100$ for Scheme 1.

Table 7. Point estimates of $R(x)$ and $h(x)$ at $x = \{0.3, 0.5, 0.8\}$ under Scheme 1.

n	Parameter	MLE			BE		
		Estimate	Bias	MSE	Estimate	Bias	MSE
50	R (0.3)	0.2904	0.0036	0.0034	0.2972	−0.0032	0.0008
	R (0.5)	0.1116	0.0076	0.0015	0.1238	−0.0047	0.0004
	R (0.8)	0.0246	0.0041	0.0002	0.0320	−0.0033	0.0001
	h (0.3)	4.7508	−0.3680	0.7303	4.3728	0.0100	0.1080
	h (0.5)	5.1834	−0.5549	0.9477	4.6027	0.0258	0.1115
	h (0.8)	5.5063	−0.6804	1.1040	4.7861	0.0398	0.1099
100	R (0.3)	0.2943	−0.0003	0.0019	0.2944	−0.0004	0.0007
	R (0.5)	0.1130	0.0061	0.0009	0.1214	−0.0022	0.0004
	R (0.8)	0.0245	0.0042	0.0001	0.0308	−0.0020	0.0001
	h (0.3)	4.6619	−0.2791	0.3908	4.3977	−0.0149	0.1080
	h (0.5)	5.0928	−0.4642	0.5610	4.6291	−0.0005	0.1115
	h (0.8)	5.4186	−0.5927	0.7022	4.8129	0.0130	0.1097
200	R (0.3)	0.2888	0.0052	0.0009	0.2952	−0.0012	0.0005
	R (0.5)	0.1087	0.0104	0.0005	0.1213	−0.0022	0.0002
	R (0.8)	0.0227	0.0061	0.0001	0.0304	−0.0016	0.0000
	h (0.3)	4.7072	−0.3244	0.2615	4.3760	0.0068	0.0638
	h (0.5)	5.1386	−0.5101	0.4339	4.6094	0.0192	0.0663
	h (0.8)	5.4640	−0.6381	0.5856	4.7954	0.0305	0.0661

Table 8. Point estimates of $R(x)$ and $h(x)$ at $x = \{0.3, 0.5, 0.8\}$ under Scheme 2.

n	Parameter	MLE			BE		
		Estimate	Bias	MSE	Estimate	Bias	MSE
50	R (0.3)	0.3530	−0.0590	0.0069	0.3153	−0.0213	0.0012
	R (0.5)	0.1570	−0.0379	0.0034	0.1366	−0.0175	0.0007
	R (0.8)	0.0431	−0.0144	0.0006	0.0374	−0.0086	0.0001
	h (0.3)	3.9850	0.3978	0.5850	4.1638	0.2190	0.1396
	h (0.5)	4.3785	0.2500	0.5579	4.3932	0.2354	0.1504
	h (0.8)	4.7010	0.1249	0.5425	4.5809	0.2450	0.1535

Table 8. *Cont.*

n	Parameter	MLE			BE		
		Estimate	**Bias**	**MSE**	**Estimate**	**Bias**	**MSE**
100	R (0.3)	0.3472	−0.0533	0.0044	0.3187	−0.0248	0.0013
	R (0.5)	0.1499	−0.0308	0.0018	0.1384	−0.0192	0.0007
	R (0.8)	0.0385	−0.0098	0.0003	0.0377	−0.0090	0.0001
	h (0.3)	4.0420	0.3408	0.3066	4.1238	0.2590	0.1526
	h (0.5)	4.4585	0.1700	0.2493	4.3601	0.2684	0.1608
	h (0.8)	4.7973	0.0286	0.2352	4.5544	0.2715	0.1611
200	R (0.3)	0.3541	−0.0601	0.0044	0.3281	−0.0341	0.0016
	R (0.5)	0.1545	−0.0353	0.0017	0.1444	−0.0253	0.0009
	R (0.8)	0.0400	−0.0113	0.0002	0.0399	−0.0111	0.0002
	h (0.3)	3.9582	0.4247	0.2768	4.0205	0.3623	0.1867
	h (0.5)	4.3737	0.2548	0.1797	4.2635	0.3650	0.1931
	h (0.8)	4.7167	0.1092	0.1375	4.4658	0.3601	0.1904

Table 9. Point estimates of $R(x)$ and $h(x)$ at $x = \{0.3, 0.5, 0.8\}$ under Scheme 3.

n	Parameter	MLE			BE		
		Estimate	**Bias**	**MSE**	**Estimate**	**Bias**	**MSE**
50	R (0.3)	0.3541	−0.0602	0.0068	0.3349	−0.0409	0.0026
	R (0.5)	0.1609	−0.0417	0.0037	0.1520	−0.0328	0.0016
	R (0.8)	0.0462	−0.0175	0.0008	0.0447	−0.0159	0.0004
	h (0.3)	3.8998	0.4830	0.6112	3.9405	0.4423	0.3071
	h (0.5)	4.2347	0.3939	0.5972	4.1569	0.4716	0.3398
	h (0.8)	4.5128	0.3131	0.5790	4.3392	0.4867	0.3542
100	R (0.3)	0.3593	−0.0654	0.0058	0.3502	−0.0563	0.0039
	R (0.5)	0.1658	−0.0466	0.0031	0.1632	−0.0440	0.0024
	R (0.8)	0.0485	−0.0197	0.0006	0.0496	−0.0208	0.0006
	h (0.3)	3.7698	0.6131	0.5349	3.7744	0.6084	0.4508
	h (0.5)	4.0625	0.5660	0.5084	3.9925	0.6361	0.4934
	h (0.8)	4.3124	0.5136	0.4768	4.1804	0.6455	0.5091
200	R (0.3)	0.3584	−0.0645	0.0050	0.3624	−0.0684	0.0053
	R (0.5)	0.1664	−0.0473	0.0028	0.1723	−0.0532	0.0033
	R (0.8)	0.0493	−0.0206	0.0006	0.0537	−0.0250	0.0007
	h (0.3)	3.7308	0.6521	0.5164	3.6492	0.7337	0.6075
	h (0.5)	3.9916	0.6369	0.5153	3.8678	0.7608	0.6584
	h (0.8)	4.2170	0.6089	0.4976	4.0593	0.7666	0.6738

Table 10. Interval estimates of $R(x)$ and $h(x)$ at $x = \{0.3, 0.5, 0.8\}$ under Scheme 1.

n	Parameter	ACI		PBCI		SBCI		HPD	
		IL	**CP**	**IL**	**CP**	**IL**	**CP**	**IL**	**CP**
50	R (0.3)	0.6295	1.0000	0.2194	0.9300	0.2452	0.9180	0.1711	0.9960
	R (0.5)	0.3211	0.9920	0.1509	0.9220	0.1704	0.8980	0.1133	0.9960
	R (0.8)	0.0948	0.9560	0.0646	0.9160	0.0558	0.8500	0.0440	0.9980
	h (0.3)	15.1937	1.0000	2.6979	0.9100	9.9158	0.8540	2.0098	0.9980
	h (0.5)	16.3907	1.0000	2.7699	0.9100	9.6123	0.8720	2.0910	0.9980
	h (0.8)	16.2365	1.0000	2.7892	0.9100	9.0189	0.8760	2.1337	1.0000
100	R (0.3)	0.4684	1.0000	0.1573	0.9500	0.1666	0.9300	0.1217	0.9780
	R (0.5)	0.2581	1.0000	0.1082	0.9440	0.1222	0.9100	0.0848	0.9760
	R (0.8)	0.0727	0.9720	0.0446	0.9420	0.0459	0.8760	0.0352	0.9800
	h (0.3)	10.2479	1.0000	1.9063	0.9340	7.6873	0.9120	1.4799	0.9800
	h (0.5)	11.2567	1.0000	1.9784	0.9340	7.4875	0.9200	1.5603	0.9800
	h (0.8)	11.0322	1.0000	2.0159	0.9360	7.1573	0.9380	1.6174	0.9840
200	R (0.3)	0.3240	1.0000	0.1117	0.9300	0.1154	0.9260	0.0947	0.9680
	R (0.5)	0.1946	0.9980	0.0769	0.9260	0.0829	0.9180	0.0661	0.9640
	R (0.8)	0.0538	0.9660	0.0312	0.9340	0.0349	0.8980	0.0274	0.9660
	h (0.3)	7.6700	1.0000	1.3565	0.9340	3.0475	0.9200	1.1517	0.9640
	h (0.5)	8.4498	1.0000	1.4246	0.9340	2.9934	0.9340	1.2441	0.9740
	h (0.8)	8.1572	1.0000	1.4711	0.9340	2.8896	0.9400	1.3213	0.9820

Table 11. Interval estimates of $R(x)$ and $h(x)$ at $x = \{0.3, 0.5, 0.8\}$ under Scheme 2.

n	Parameter	ACI		PBCI		SBCI		HPD	
		IL	CP	IL	CP	IL	CP	IL	CP
50	R (0.3)	0.7687	1.0000	0.2385	1.0000	0.3091	0.9960	0.1736	1.0000
	R (0.5)	0.4635	1.0000	0.1521	1.0000	0.2446	0.9700	0.1201	1.0000
	R (0.8)	0.1715	0.9980	0.0555	1.0000	0.0683	0.9980	0.0496	1.0000
	h (0.3)	12.3926	1.0000	1.9553	1.0000	5.5064	0.7920	1.9201	0.9980
	h (0.5)	13.7575	1.0000	1.7022	1.0000	6.7450	0.9480	2.0079	0.9980
	h (0.8)	13.9777	1.0000	1.6027	1.0000	9.6289	0.9960	2.0561	1.0000
100	R (0.3)	0.5354	1.0000	0.1686	1.0000	0.1831	0.9980	0.1430	0.9940
	R (0.5)	0.3392	1.0000	0.1033	1.0000	0.1352	0.9680	0.1008	0.9940
	R (0.8)	0.1135	1.0000	0.0363	1.0000	0.0396	0.9940	0.0423	0.9960
	h (0.3)	8.6210	1.0000	1.4166	1.0000	4.6827	0.8640	1.5567	0.9840
	h (0.5)	9.7277	1.0000	1.2680	1.0000	5.8272	0.9840	1.6519	0.9900
	h (0.8)	9.7485	1.0000	1.2530	1.0000	8.7395	1.0000	1.7175	0.9940
200	R (0.3)	0.3750	1.0000	0.1239	0.6580	0.1353	0.8360	0.1130	0.9660
	R (0.5)	0.2638	1.0000	0.0803	0.7580	0.0942	0.8740	0.0821	0.9640
	R (0.8)	0.0932	1.0000	0.0305	0.9440	0.0356	0.9920	0.0357	0.9640
	h (0.3)	6.0311	1.0000	1.0045	0.5260	2.4496	0.5080	1.1778	0.9040
	h (0.5)	6.8928	1.0000	0.9962	0.9980	2.6756	0.9460	1.2850	0.9360
	h (0.8)	6.8438	1.0000	1.0756	1.0000	3.6085	1.0000	1.3718	0.9580

Table 12. Interval estimates of $R(x)$ and $h(x)$ at $x = \{0.3, 0.5, 0.8\}$ under Scheme 3.

n	Parameter	ACI		PBCI		SBCI		HPD	
		IL	CP	IL	CP	IL	CP	IL	CP
50	R (0.3)	0.7253	1.0000	0.2582	1.0000	0.4573	0.9400	0.1524	0.8800
	R (0.5)	0.4396	1.0000	0.1831	0.9980	0.2902	0.9460	0.1171	0.8580
	R (0.8)	0.1674	1.0000	0.0785	1.0000	0.1077	0.9500	0.0561	0.8600
	h (0.3)	15.0507	1.0000	2.0378	0.9880	8.5405	0.5760	1.6314	0.8580
	h (0.5)	16.2536	1.0000	1.8913	1.0000	7.9381	0.7700	1.7047	0.8660
	h (0.8)	16.4884	1.0000	1.8356	1.0000	9.0854	0.9400	1.7509	0.8700
100	R (0.3)	0.5170	1.0000	0.1596	0.9880	0.1641	0.9980	0.1488	0.9240
	R (0.5)	0.3550	1.0000	0.1254	0.9920	0.1349	1.0000	0.1130	0.9220
	R (0.8)	0.1329	1.0000	0.0608	0.9960	0.0658	1.0000	0.0529	0.9320
	h (0.3)	11.1842	1.0000	1.5971	0.9160	3.0044	0.6920	1.4537	0.8160
	h (0.5)	12.0948	1.0000	1.7303	0.9740	3.1644	0.8580	1.5438	0.8260
	h (0.8)	12.3465	1.0000	1.8382	0.9920	3.5004	0.9560	1.6139	0.8560
200	R (0.3)	0.3618	1.0000	0.1158	0.6740	0.1242	0.8180	0.1215	0.6860
	R (0.5)	0.2668	1.0000	0.0913	0.7160	0.1078	0.9300	0.0962	0.6980
	R (0.8)	0.1093	1.0000	0.0448	0.7560	0.0554	0.9720	0.0469	0.7360
	h (0.3)	10.5825	1.0000	1.1902	0.6320	2.0651	0.6460	1.5501	0.8400
	h (0.5)	11.3103	1.0000	1.3214	0.7780	2.0850	0.7640	1.7166	0.8840
	h (0.8)	11.5046	1.0000	1.4394	0.9220	2.0677	0.8700	1.8304	0.8980

6.2. Numerical Example

In the simulation study, we discuss the performance of the competing risks model parameter estimates comparing the maximum likelihood method, bootstrap method and Bayes method considering the APE distribution with $(\alpha_1, \theta_1, \alpha_2, \theta_2) = (1.5, 2.0, 2.0, 3.0)$. The numerical results above show the performance of parameter estimates is affected by the schemes. For an illustration of the reliability estimates, here we choose the setting of $n = 100$ and Scheme 1 because the parameter estimates perform well under this setting, as mentioned above. The generated dataset of competing risks data under progressive censoring of Scheme 1 is presented in Table 13.

For the dataset in Table 13, 92 observations with 8 units censored in Scheme 3 are listed where 29 samples are from competing risk X_1, and the other 63 samples are from competing risk X_2. The parameter estimates are given in Table 14. In this table, ACI-IL indicates the ILs of interval estimates for ACIs. This is similar to PBCI-IL, SBCI-IL and HPD-IL-credible intervals. From the numerical estimates, HPD-credible intervals and PBCIs have smaller ILs than ACIs and SBCIs. Figure 1 shows the log-likelihood functions of $(\alpha_1, \theta_1, \alpha_2, \theta_2)$, which reveals that the MLEs exist and are unique. Moreover, to show the convergence of MCMC iterations, the trace plots are presented in Figure 2.

Table 13. Simulated competing risks dataset.

Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)
1	(0.0034,2,8)	17	(0.0593,2,0)	33	(0.1227,2,0)	49	(0.2212,2,0)	65	(0.3066,2,0)	81	(0.495,2,0)
2	(0.0097,2,0)	18	(0.0602,2,0)	34	(0.1264,1,0)	50	(0.2238,2,0)	66	(0.3262,2,0)	82	(0.5079,2,0)
3	(0.0105,1,0)	19	(0.0649,1,0)	35	(0.1351,1,0)	51	(0.2246,1,0)	67	(0.3403,2,0)	83	(0.5168,1,0)
4	(0.0141,2,0)	20	(0.0651,2,0)	36	(0.1373,1,0)	52	(0.2249,2,0)	68	(0.3463,2,0)	84	(0.5746,2,0)
5	(0.019,2,0)	21	(0.0668,1,0)	37	(0.1409,1,0)	53	(0.2372,1,0)	69	(0.3594,2,0)	85	(0.6539,2,0)
6	(0.0226,2,0)	22	(0.0707,2,0)	38	(0.1539,1,0)	54	(0.2378,1,0)	70	(0.3667,2,0)	86	(0.6551,2,0)
7	(0.0261,1,0)	23	(0.072,1,0)	39	(0.1599,2,0)	55	(0.2378,2,0)	71	(0.3687,1,0)	87	(0.7682,2,0)
8	(0.0275,1,0)	24	(0.0901,2,0)	40	(0.16,2,0)	56	(0.2418,1,0)	72	(0.3699,2,0)	88	(0.7683,1,0)
9	(0.0278,2,0)	25	(0.0927,2,0)	41	(0.1626,2,0)	57	(0.2506,1,0)	73	(0.3701,2,0)	89	(0.7725,1,0)
10	(0.031,2,0)	26	(0.0935,1,0)	42	(0.1707,2,0)	58	(0.2631,2,0)	74	(0.3713,1,0)	90	(0.7959,2,0)
11	(0.0327,2,0)	27	(0.0936,1,0)	43	(0.1709,2,0)	59	(0.2696,1,0)	75	(0.401,2,0)	91	(1.1449,2,0)
12	(0.0351,2,0)	28	(0.1038,2,0)	44	(0.1815,1,0)	60	(0.2736,2,0)	76	(0.4034,2,0)	92	(1.1716,2,0)
13	(0.0354,1,0)	29	(0.1066,1,0)	45	(0.1971,2,0)	61	(0.2962,2,0)	77	(0.4313,2,0)		
14	(0.0374,1,0)	30	(0.1083,2,0)	46	(0.2029,2,0)	62	(0.2963,2,0)	78	(0.4343,2,0)		
15	(0.0494,1,0)	31	(0.1191,2,0)	47	(0.2051,1,0)	63	(0.2991,2,0)	79	(0.4488,1,0)		
16	(0.0554,2,0)	32	(0.1216,2,0)	48	(0.2133,2,0)	64	(0.3018,2,0)	80	(0.4894,1,0)		

Table 14. Parameter estimates for the competing risks dataset.

Estimate	Method	α_1	θ_1	α_2	θ_2
Point	MLE	2.8694	2.4158	2.9509	3.6347
	BE	1.7035	1.6144	2.1310	3.1349
Interval	ACI-IL	9.6804	3.8258	6.9790	2.8800
	PBCI-IL	1.9411	1.4615	2.7206	1.5318
	SBCI-IL	3.2717	1.6997	2.3947	1.5352
	HPD-IL	1.6879	0.9725	1.8569	1.3334

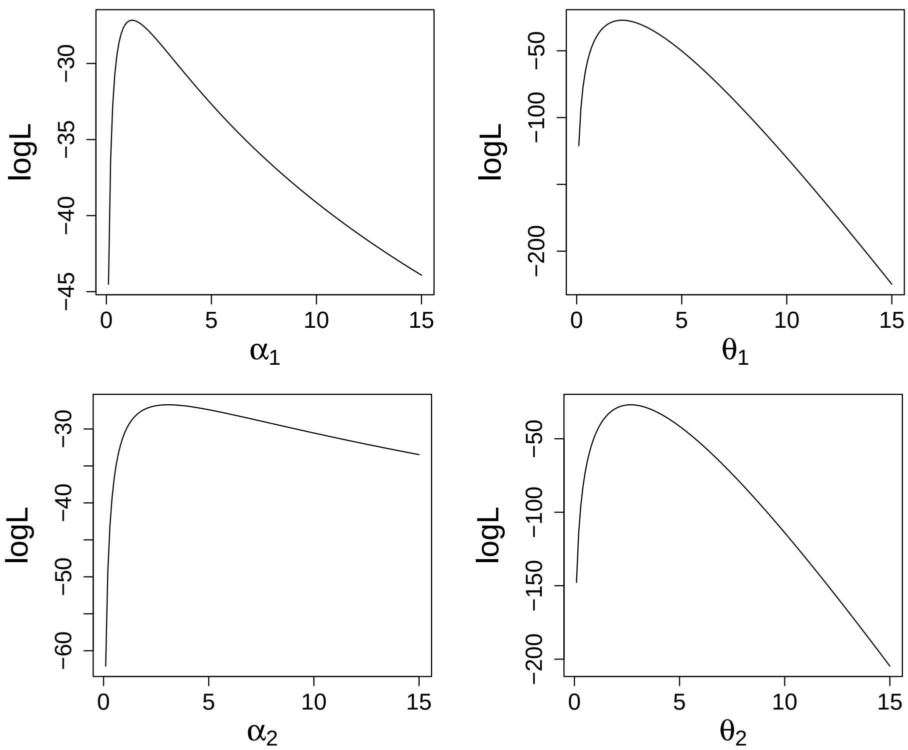

Figure 1. The log-likelihood functions of $(\alpha_1, \theta_1, \alpha_2, \theta_2)$ for the simulated dataset.

Further, using the MLEs, bootstrap samples and MCMC, we can compute the estimates of the reliability and hazard rate given a specified argument x in Equations (5) and (6).

For the continuous argument x, we present the curves of the MLEs and BEs in Figure 3, and ranges of ACIs using the delta method, bootstrap confidence intervals (PBCIs and SBCIs) and HPD-credible intervals in Figures 4 and 5.

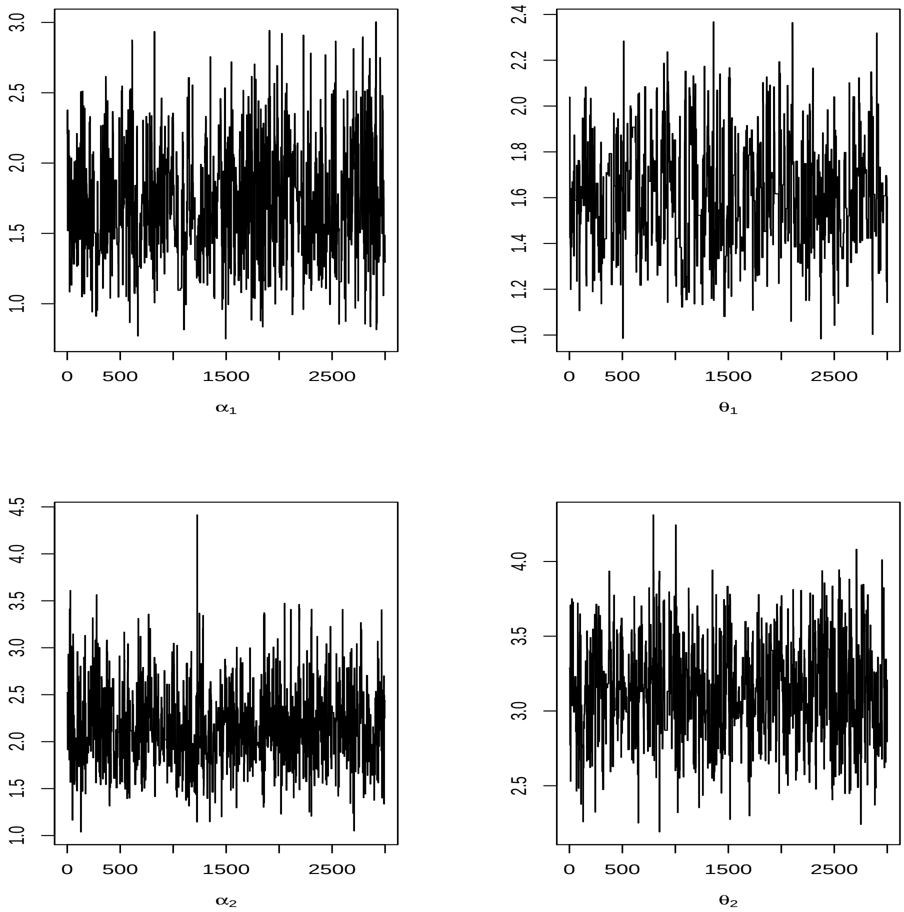

Figure 2. Trace plots of distribution parameters $(\alpha_1, \theta_1, \alpha_2, \theta_2)$.

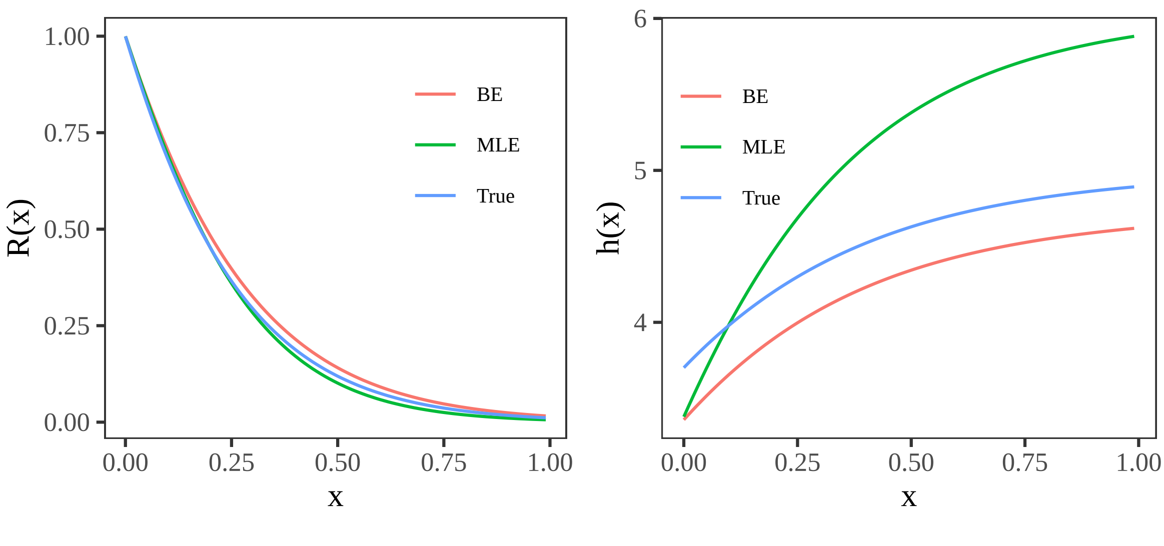

Figure 3. (**a**) Point estimates of $R(x)$ (**left**) and (**b**) $h(x)$ (**right**) for the simulated dataset.

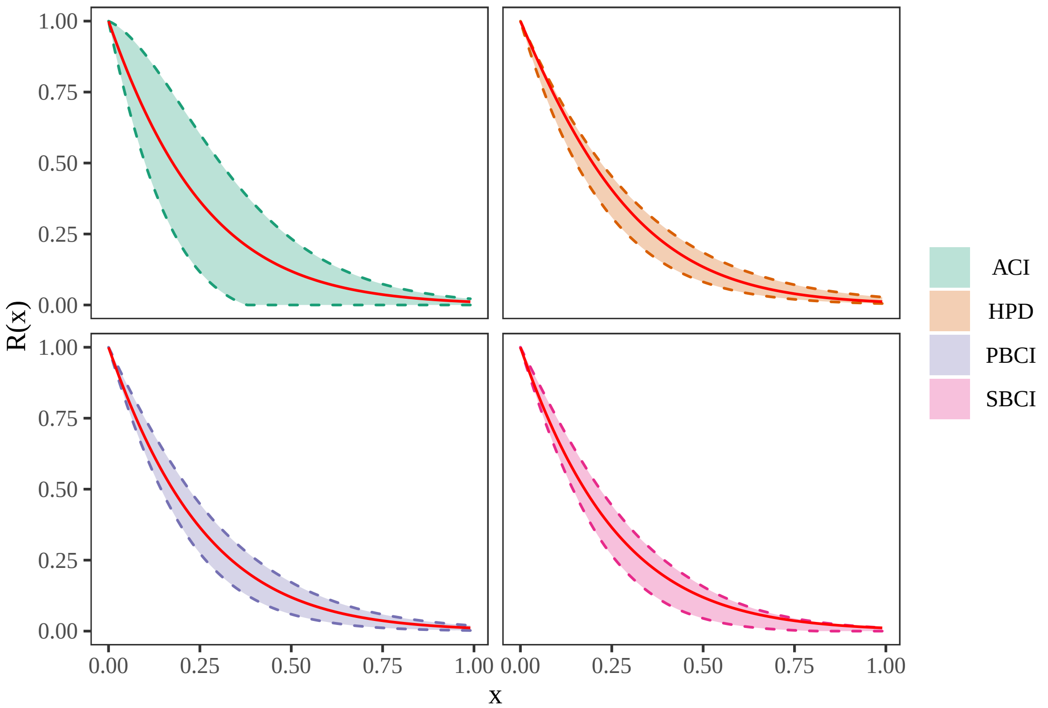

Figure 4. Interval estimates of RF $R(x)$ where the true values of $R(x)$ are marked in red.

We see from Figure 3a that the estimates of $R(x)$ are close to the true values of RF. This implies that the maximum likelihood and Bayes methods are effective and applicable to evaluate RF in the competing risks model with a progressive censoring scheme when the lifetime distribution of the individual is the APE distribution. We also see that BE is higher than the true values but MLE is lower than the true values. In Figure 3b, the MLE numerical errors between the estimated HRF and the true HRF are higher than the estimated RF when x is increasing. The BEs are lower than the true values, and a crossing point occurs between the BEs and the true values. We also find that the errors become larger when x is increasing, and, meanwhile, the MLEs perform better than BEs.

For the interval estimation of $R(x)$ in Figure 4, we find that PBCI, SBCI and HPD-credible intervals perform better than ACI, where ACI has a larger IL. The subtle differences among the PBCI, SBCI and HPD-credible intervals are that the lower limits of PBCI and SBCI and the upper limits of the HPD-credible intervals are closer to the true values. The interval estimates are asymmetric for the Bayes and bootstrap methods. As shown in Figure 5, we see that ACI has the largest errors in hazard rate estimates among the four interval estimate methods, while PBCI has the smallest error. That is, the Bayes and bootstrap methods are better than ACI in terms of the ILs. When x is increasing, the Bayes method has closer lower limits than bootstrap estimates.

In this numerical example, we focus on the comparison of estimates for RF and HRF. We suggest percentile bootstrap and Bayes methods for interval estimation of $R(x)$ and $h(x)$. Considering the sampling complexity and time-consuming features of MCMC, the maximum likelihood method is a good choice for point estimation.

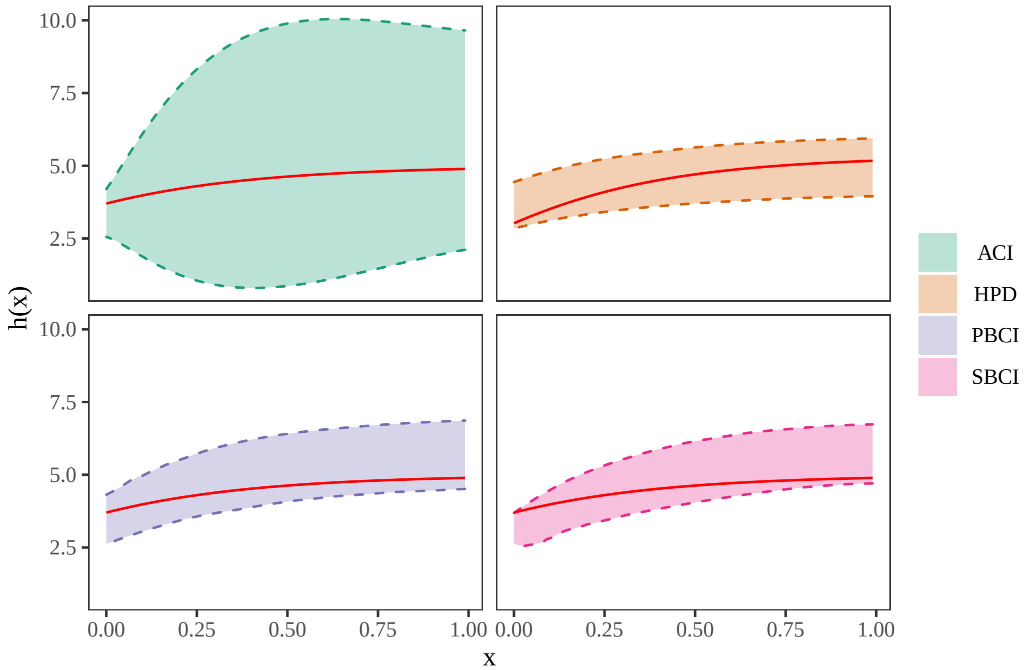

Figure 5. Interval estimates of HRF $h(x)$ where the true values of $h(x)$ are marked in red.

7. Real Data Analysis

We analyze a real competing risks dataset under the progressive censoring to illustrate the feasibility of parameter estimation under the underlying APE distribution. The dataset consists of 19 instances of relapse of multiple myeloma and 10 instances of transplant-related mortality as a competing risk from 35 patients treated at the Clinic for Stem Cell Transplantation, University Hospital Hamburg-Effendorf, Hamburg, Germany. The details of patients receiving transplants from donors with type AA haplotypes and type AB or BB are given in Donoghoe and Gebski [24], where a proportional subdistribution hazards model was applied to fit this data and to determine the improvement in time to relapse if donors were with group B killer immunoglobulin-like receptor haplotypes. Here, we focus on the statistical inference using the competing risks model for this data. The times with the pattern of events and censoring are given as (x_i, δ_i, R_i) for $i = 1, 2, \ldots, 35$ in Table 15, where $\delta_i = 1$ indicates the time to relapse; $\delta_i = 2$ indicates the transplant-related mortality; $\delta_i = 0$ indicates the censoring time together with the removal $R_i = 1$.

Table 15. Real dataset.

Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)	Id	(x_i, δ_i, R_i)
1	(14.82,2,0)	7	(41.17,1,0)	13	(9.33,1,0)	19	(12.35,1,0)	25	(1.97,2,0)	31	(6.7,2,0)
2	(3.91,2,0)	8	(15.74,1,0)	14	(56.57,0,1)	20	(5.03,1,0)	26	(9.92,1,0)	32	(28.29,1,0)
3	(3.45,1,0)	9	(22.31,1,0)	15	(6.21,2,0)	21	(41.17,1,0)	27	(3.81,1,0)	33	(4.14,1,0)
4	(89.89,0,1)	10	(80.46,1,0)	16	(14.72,1,0)	22	(0.26,2,0)	28	(23.79,0,1)	34	(1.94,2,0)
5	(45.96,1,0)	11	(4.57,1,0)	17	(53.55,0,1)	23	(46.55,0,1)	29	(1.81,2,0)	35	(10.68,1,0)
6	(0.66,2,0)	12	(17.31,1,0)	18	(44.02,0,1)	24	(3.58,1,0)	30	(3.55,2,0)		

The APE distribution discussed in this paper is a generalization of an Exp distribution. Thus we consider Exp and APE distributions for each event in the model selection. The models are listed with the same Exp and APE distributions and the hybrid Exp and APE distributions. For simplicity, they are written as (Exp, Exp), (Exp, APE), (APE, Exp) and (APE, APE), respectively. The Akaike information criterion (AIC) and the Bayesian information criterion (BIC) are used to select the parametric model. The AICs and BICs for the four models are presented in Table 16, and the selected model (APE, APE) for this dataset is in bold.

Table 16. AICs and BICs for model selection.

Model	(Exp, Exp)	(Exp, APE)	(APE, Exp)	(APE, APE)
AIC	320.4981	319.5292	319.5879	**318.6538**
BIC	322.0535	321.0845	321.1432	**320.4981**

Using the parameter estimation methods mentioned before and the selected model (APE, APE), the parameter estimates of the APE distribution based on the competing risks model are shown in Table 17. Figure 6 displays the log-likelihood functions of $(\alpha_1, \theta_1, \alpha_2, \theta_2)$, which indicates that the MLEs exist and are unique. Similarly, with the numerical example, we present the estimates of RF and HRF in Figure 7. We see that the reliability estimate using the Bayes method is lower than the maximum likelihood method. This implies that the hazard rate estimation using the maximum likelihood method is higher than Bayes, which is in line with Figure 7.

Table 17. Parameter estimates for the real dataset.

Estimate	Method	α_1	θ_1	α_2	θ_2
Point	MLE	0.1677	0.0074	0.2107	0.0041
	BE	0.1147	0.0059	0.1312	0.0032
Interval	ACI-IL	0.9583	0.0186	0.9898	0.0086
	PBCI-IL	4.4155	0.0171	5.2786	0.0112
	SBCI-IL	0.5029	0.0179	0.4966	0.0072
	HPD-IL	0.5040	0.0121	0.4202	0.0058

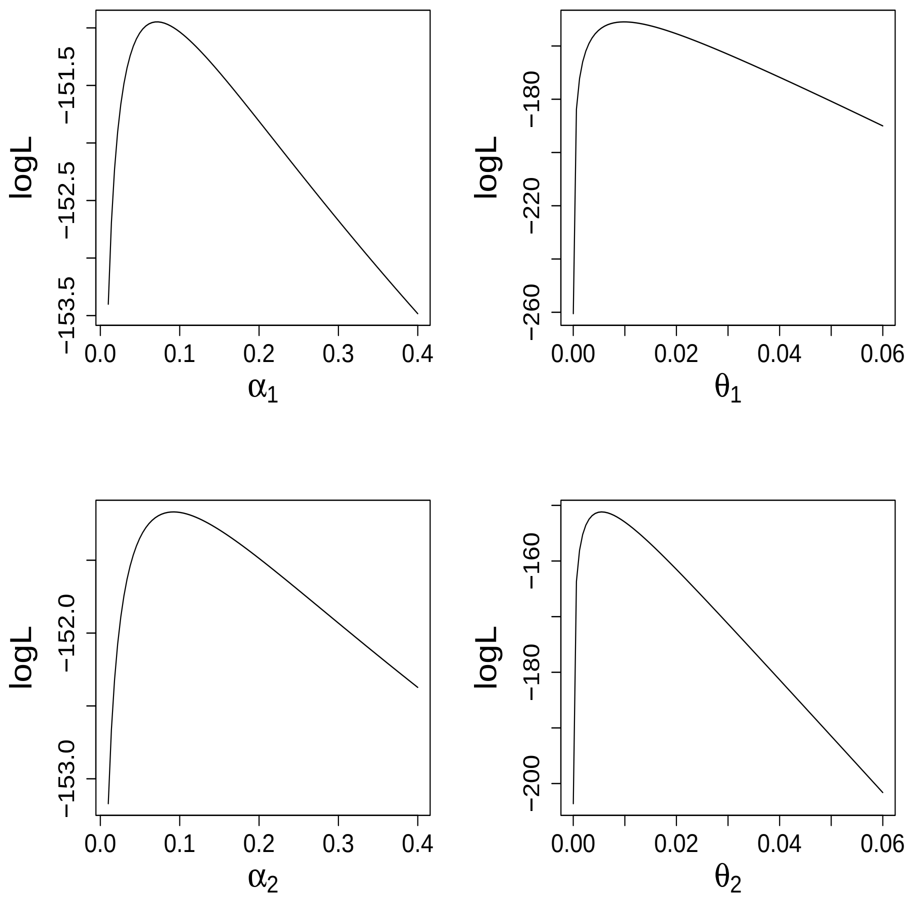

Figure 6. The log-likelihood functions of $(\alpha_1, \theta_1, \alpha_2, \theta_2)$ for the real dataset.

The interval estimates of $R(x)$ are shown in Figure 8 for ACI, PBCI, SBCI and HPD-credible intervals. We observe that the lower limit of SBCI is closer to 0 when x is increasing, and SBCI has the largest IL among the interval estimates. The ACI, PBCI and HPD-credible intervals have a good performance for this real dataset. For the interval estimates of $h(x)$ in Figure 9, we see that ACI and SBCI have larger Ils than HPD-credible interval and PBCI. This also indicates the estimate errors. The upper limits for HPD-credible interval and PBCI are lower than ACI and SBCI when x is small.

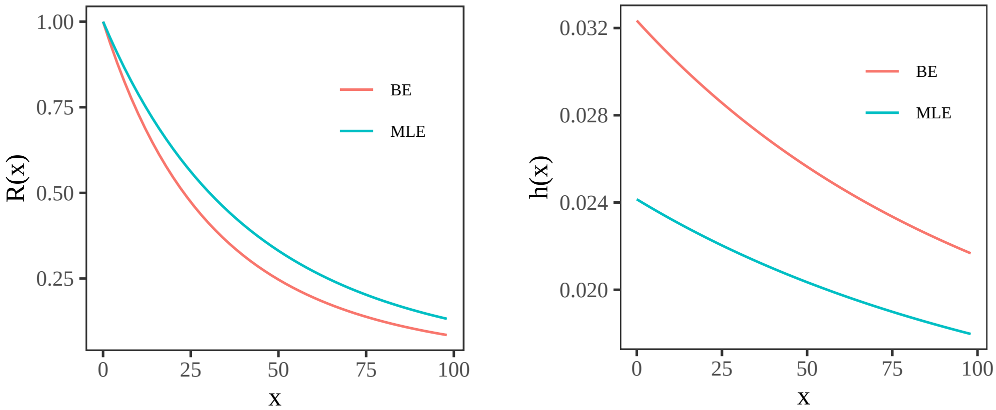

Figure 7. Point estimates of $R(x)$ (**left**) and $h(x)$ (**right**) for the real dataset.

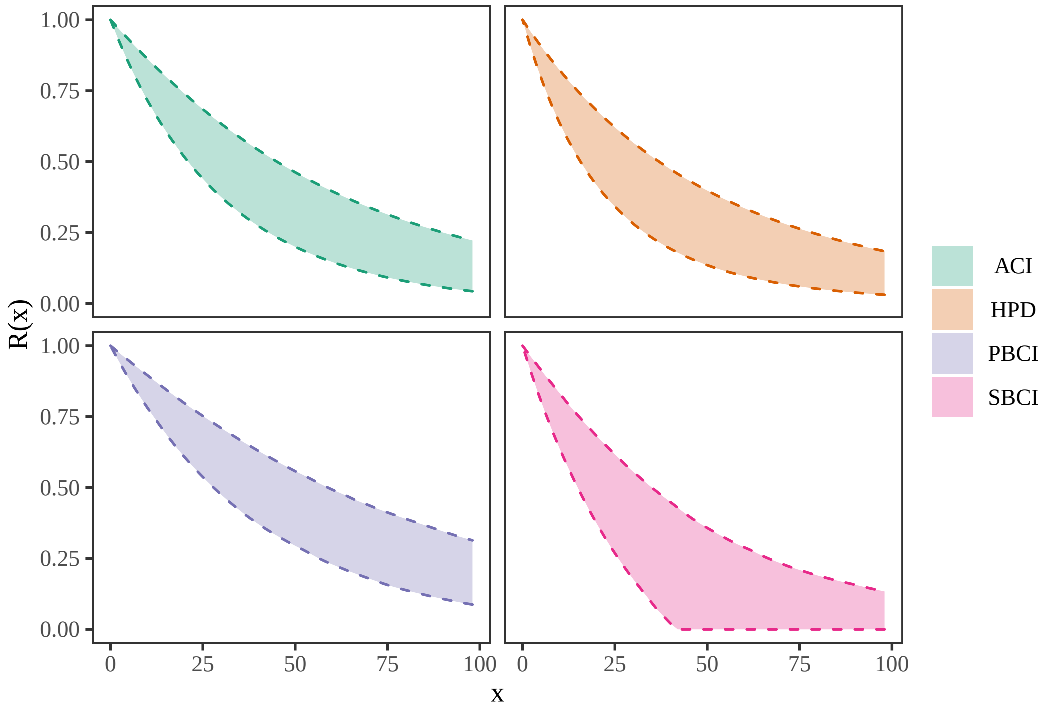

Figure 8. Interval estimates of RF $R(x)$ for the real dataset.

In the end, we note that the trace plots of $(\alpha_1, \theta_1, \alpha_2, \theta_2)$ for this real dataset are omitted here, but the convergence of MCMC samples is checked. The convergence of MCMC samples is monitored with the Gelman–Rubin diagnosis [25], which can be conducted with the `R` package `coda` [26]. It is to be mentioned here that the simulated and real datasets and all codes used for simulation and real data analysis are available as Supplementary materials.

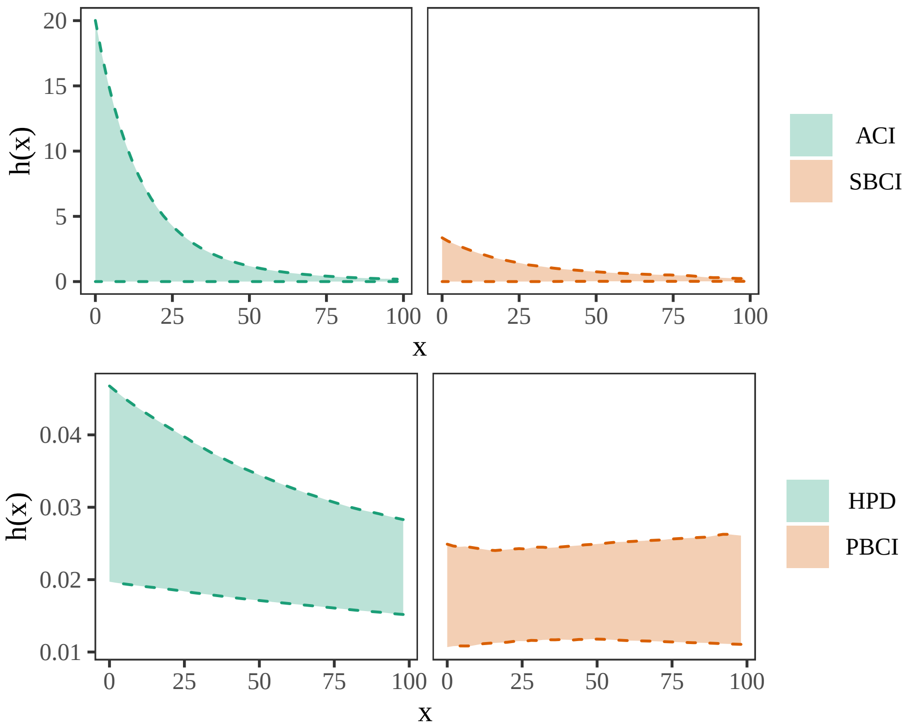

Figure 9. Interval estimates of HRF $h(x)$ for the real dataset.

8. Competing Risks with Unknown Cause of Failure

In all techniques discussed in this paper, we consider that the cause of failure for all items is known. In practice, the cause of failure for some items may be unknown or uncertain. For example, the cause of disease for some patients may be hard to decide. An investigation employing just the failures with known causes may direct to significant bias; for more detail, see Lu and Liang [27]. In this section, we consider the competing risks model with an unknown cause of failure when the data are progressively Type-II censored from the APE distribution as an extension of the model discussed in this paper. Suppose that we have only two causes of failure and

$$I(\delta_i = 1) = \begin{cases} 1, & \delta_i = 1 \\ 0 & otherwise \end{cases},$$

$$I(\delta_i = 2) = \begin{cases} 1, & \delta_i = 2 \\ 0 & otherwise \end{cases} \quad \text{and} \quad I(\delta_i = *) = \begin{cases} 1, & \delta_i = * \\ 0 & otherwise \end{cases},$$

where $m_1 = \sum_{i=1}^{m} I(\delta_i = 1)$, $m_2 = \sum_{i=1}^{m} I(\delta_i = 2)$ are the number of failures due to cause one and cause two, respectively, and $m_3 = \sum_{i=1}^{m} I(\delta_i = *)$ is the number of failures having failure times but related causes of failure are unidentified. In this case, the likelihood function of the observed data can be expressed as follows

$$\begin{aligned} L &= c\prod_{i=1}^{m}[f_1(x_i)\bar{F}_2(x_i)]^{I(\delta_i=1)}[f_2(x_i)\bar{F}_1(x_i)]^{I(\delta_i=2)}\,[f_1(x_i)\bar{F}_2(x_i)+f_2(x_i)\bar{F}_1(x_i)]^{I(\delta_i=*)} \\ &\times\ [\bar{F}_1(x_i)\bar{F}_2(x_i)]^{R_i}. \end{aligned} \tag{22}$$

where $m = m_1 + m_2 + m_3$. Using the relation $h_t(x) = f_t(x)\bar{F}_t(x), t = 1,2$, the likelihood function in (22) can be written as

$$L = c\prod_{i=1}^{m}[h_1(x_i)]^{I(\delta_i=1)}[h_2(x_i)]^{I(\delta_i=2)}\,[h_1(x_i)+h_2(x_i)]^{I(\delta_i=*)}[\bar{F}_1(x_i)\bar{F}_2(x_i)]^{1+R_i}, \tag{23}$$

From (1), (2) and (23), we can write the log-likelihood function, ignoring the normalized constant, as follows

$$L^{*}(\varpi) = L(\varpi) + \sum_{i=1}^{m_3} \log(D_i), \tag{24}$$

where $L(\varpi)$ is given by (9), $\varpi = (\theta_1, \theta_2, \alpha_1, \alpha_2)^{\top}$ and

$$D_i \equiv D(\varpi) = \left[\frac{\theta_1 \log(\alpha_1) e^{-\theta_1 x_i}}{\alpha_1^{e^{-\theta_1 x_i}} - 1} + \frac{\theta_2 \log(\alpha_2) e^{-\theta_2 x_i}}{\alpha_2^{e^{-\theta_2 x_i}} - 1} \right]. \tag{25}$$

By maximizing (24) with respect to θ_t and $\alpha_t, t = 1,2$, one can get the MLEs of θ_t and α_t. Alternatively, the MLEs can be obtained by solving the following normal equations

$$\frac{\partial L^{*}(\varpi)}{\partial \theta_t} = \frac{\partial L(\varpi)}{\partial \theta_t} + \sum_{i=1}^{m_3} \frac{D_i^{*}}{D_i} = 0 \tag{26}$$

and

$$\frac{\partial L^{*}(\varpi)}{\partial \alpha_t} = \frac{\partial L(\varpi)}{\partial \alpha_t} + \sum_{i=1}^{m_3} \frac{D_i^{\star}}{D_i} = 0, \tag{27}$$

where $\partial L(\varpi)/\partial \theta_t$ and $\partial L(\varpi)/\partial \alpha_t$ are given by (10) and (11), respectively, and

$$D_i^{*} = \frac{e^{-\theta_t x_i} \log(\alpha_t)}{\alpha_t^{e^{-\theta_t x_i}} - 1} \left(1 - \theta_t x_i - \frac{\theta_t \log(\alpha_t) x_i e^{-\theta_t x_i}}{1 - \alpha_t^{-e^{-\theta_t x_i}}} \right)$$

and

$$D_i^{\star} = \frac{\theta_t e^{-\theta_t x_i}}{\alpha_t \left(\alpha_t^{e^{-\theta_t x_i}} - 1 \right)} \left[1 - \frac{e^{-\theta_t x_i} \log(\alpha_t)}{1 - \alpha_t^{-e^{-\theta_t x_i}}} \right], t = 1,2.$$

Upon the MLEs of θ_t and $\alpha_t, t = 1,2$ obtained, one can simply compute the MLEs of RF and HRF using the invariance property. Further, the Bayesian estimation can be simply applied using the same approach discussed before, which requires the use of the MH algorithm.

9. Conclusions

This paper considers the competing risks model under the progressive Type-II censoring scheme when the lifetimes under the various causes have independent alpha power exponential distributions. The maximum likelihood and Bayes methods are used for the estimation of distribution parameters and reliability characteristics, and the bootstrap method is also applied to obtain the interval estimates. We discuss the performance of model parameter estimates in the simulation study considering three chosen schemes, and a numerical example and a real data analysis are shown for further illustration of the estimates of reliability and hazard rate functions. We conclude that the bootstrap and Bayes methods perform better than the maximum likelihood method for interval estimates, and the progressive censoring schemes have an impact on the estimation. In future work, we propose to consider the alpha power exponential distribution using different censoring schemes as adaptive Type-II progressively censors in the presence of a competing risks model. Another future work is to investigate the same procedures considered in this paper based on dependent competing risks.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/math10132258/s1. The simulated and real datasets saved in the excel documents, and the source codes for numerical analysis in R are uploaded.

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