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Linear Diophantine Fuzzy Set Theory Applied to *BCK/BCI*-Algebras

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Abstract: In this paper, we apply the concept of linear Diophantine fuzzy sets in BCK/BCI-algebras. In this respect, the notions of linear Diophantine fuzzy subalgebras and linear Diophantine fuzzy (commutative) ideals are introduced and some vital properties are discussed. Additionally, characterizations of linear Diophantine fuzzy subalgebras and linear Diophantine fuzzy (commutative) ideals are considered. Moreover, the associated results for linear Diophantine fuzzy subalgebras, linear Diophantine fuzzy ideals and linear Diophantine fuzzy commutative ideals are obtained.

Keywords: BCI-algebra; BCK-algebra; linear Diophantine fuzzy set ($LDFS$); LDF-subalgebra; LDF-ideal; LDF-commutative ideal

MSC: 06F35; 03G25; 08A72

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1. Introduction

Fuzzy set theory was launched in 1965 by Zadeh [1] as a generalization of the theory of classical sets. In a classical set, an element is either a member of the set or it is not a member of it, whereas in a fuzzy set, the membership of an element is a real number of the closed unit interval. So, in a fuzzy set, the sum of degree of belongingness of an element with its degree of non-belongingness is equal to one. Soon after their launch, fuzzy sets became an object of extensions by themselves. In 1983, Atanassov [2] generalized fuzzy sets to intuitionistic fuzzy sets (IFS). An IFS has two non-negative functions: the membership function and the non-membership function in a way that the sum of the degree of membership of an element with its degree of non-membership is in the unit real interval. Both fuzzy sets and intuitionistic fuzzy sets have their own restrictions related to the functions of membership and non-membership. To eliminate these restrictions by using reference parameters, Riaz and Hashmi [3] in 2019 found a new extension of fuzzy sets and called it *linear Diophantine fuzzy sets* ($LDFS$). Using the corresponding reference parameters to the membership and non-membership fuzzy relations, S. Ayub et al. [4] established a robust fusion of $LDFS$s and binary relations and introduced linear Diophantine fuzzy relations.

Imai and Iséki [5,6] introduced BCK/BCI-algebras in 1966 as an extension of the principles of set-theoretic difference and propositional calculus. Later, detailed study on the theory of BCK/BCI-algebras was published, with specific focus appearing to be placed on the ideal theory of BCK/BCI-algebras. For example, Khalid and Ahmad [7] studied h-ideals of BCI-algebras and Muhiuddin et al. [8,9] studied hybrid ideals of BCK/BCI-algebras.

In 1971, Rosenfeld [10] studied the first connection between the theories of algebraic structures and fuzzy sets. He introduced the concepts of fuzzy subgroups of a group. Since then, fuzzy algebraic structures have been firmly established as a fruitful area of research. Fuzzification was applied to BCK/BCI-algebras. For example, Jun et al. [11,12] investigated soft ideals of BCK/BCI-algebras, Al-Masarwah and Ahmad [13,14] discussed multipolar fuzzy ideals of BCK/BCI-algebras. Some applications of BCK-algebras can be found, e.g., in [15,16]. For more related details, we refer to [17–20].

The connection between algebraic structures and linear Diophantine fuzzy sets was launched by Kamaci [21] in 2021. He studied finite linear Diophantine fuzzy substructures of some algebraic structures such as groups, rings, and fields. In 2022, Al-Tahan et al. [22] studied linear Diophantine fuzzy subpolygroups of a polygroup. Inspired by the recent work on linear Diophantine fuzzy substructures (subhyperstructures) and by the previous work related to fuzzy algebraic structures, our paper studies linear Diophantine fuzzy sets in BCK/BCI-algebras. The remainder of it is structured as follows. In Section 2, we present basic definitions related to BCK/BCI-algebra and to LDFSs. In Section 3, we define linear Diophantine fuzzy subalgebras and linear Diophantine fuzzy ideals in *BCK/BCI*-algebras, present some examples, and investigate their properties. In Section 4, we define the notion of *LDF* commutative ideal of *BCK*-algebras and study some connections between *LDF* subalgebras, *LDF* ideals and *LDF* commutative ideals.

2. Preliminaries

In this section, we present some basic results and examples related to linear Diophantine fuzzy sets [3,4] and to *BCK/BCI*-algebras [23].

An algebra $(\mathcal{A}; *, 0)$ of type $(2, 0)$ is said to be a *BCI-algebra* if $\forall\ \vartheta, \varrho, \ell \in \mathcal{A}$, the following conditions hold:

(K_1) $((\vartheta * \ell) * (\vartheta * \varrho)) * (\varrho * \ell) = 0$,
(K_2) $(\vartheta * (\vartheta * \ell)) * \ell = 0$,
(K_3) $\vartheta * \vartheta = 0$,
(K_4) $\vartheta * \ell = 0$ and $\ell * \vartheta = 0 \Rightarrow \vartheta = \ell$.

If a BCI-algebra $\mathcal{A}$ satisfies the condition: (K_5) $0 * \vartheta = 0$, $\forall\ \vartheta \in \mathcal{A}$, then $\mathcal{A}$ is a BCK-algebra.

Every BCK/BCI-algebra $\mathcal{A}$ satisfies the following properties:

(π_1) $\vartheta * 0 = \vartheta$,
(π_2) $(\vartheta * \ell) * \varrho = (\vartheta * \varrho) * \ell$,
(π_3) $\vartheta \leq \ell \Rightarrow \vartheta * \varrho \leq \ell * \varrho$ and $\varrho * \ell \leq \varrho * \vartheta$,
(π_4) $0 * (\vartheta * \ell) = (0 * \vartheta) * (0 * \ell)$,
(π_5) $0 * (0 * (\vartheta * \ell)) = 0 * (\ell * \vartheta)$,
(π_6) $(\vartheta * \varrho) * (\ell * \varrho) \leq (\vartheta * \ell)$,
(π_7) $\vartheta * (\vartheta * (\vartheta * \ell)) = \vartheta * \ell$,
(π_8) $0 * (0 * ((\vartheta * \varrho) * (\ell * \varrho))) = (0 * \ell) * (0 * \vartheta)$,
(π_9) $0 * (0 * (\vartheta * \ell) = (0 * \ell) * (0 * \vartheta)$,

where $\vartheta \leq \ell \Leftrightarrow \vartheta * \ell = 0\ \forall\ \vartheta, \varrho, \ell \in \mathcal{A}$. Note that $(\mathcal{A}, \leq)$ is a partially ordered set.

A subset $Z (\neq \emptyset)$ of $\mathcal{A}$ is said to be a *subalgebra* of $\mathcal{A}$ if $\vartheta * \ell \in Z\ \forall\ \vartheta, \ell \in \mathcal{A}$ and it is called an *ideal* of Z if $0 \in Z$ and $\forall\ \vartheta, \varrho \in \mathcal{A}, \vartheta * \varrho \in Z, \varrho \in Z$ implies $\vartheta \in Z$. Furthermore, Z is called commutative ideal of $\mathcal{A}$ if $0 \in Z$ and $\forall\ \vartheta, \varrho, \omega \in Z$, $(\vartheta * \omega) * \varrho \in Z$ and $\varrho \in Z$ implies $\vartheta * (\omega * (\omega * \vartheta)) \in Z$.

Zadeh [1], in 1965, introduced the fuzzy set as an extension of the crisp set. In 1983, Atanassov [2] extended fuzzy set to intuitionistic fuzzy set. Recently, Riaz and Hashmi [3] introduced linear Diophantine fuzzy set ($LDFS$) as a new extension of fuzzy set. Due to the use of reference parameters in $LDFS$, the proposed model of $LDFS$ has more efficiency and flexibility in comparison to other generalizations of the fuzzy set.

Definition 1 ([1]). *Let E be a universal set, $I = [0,1]$, and $\mu : E \to I$ be a membership function. Then $A = \{(x, \mu(x)) : x \in E\}$ is a fuzzy set.*

Definition 2 ([2]). *Let E be a universal set, $I = [0,1]$, and $\mu, v : E \to I$ be the membership and non-membership functions, respectively. Then $A = \{(x, \mu(x), v(x)) : x \in E\}$ is an intuitionistic fuzzy set. Here, $\mu(x) + v(x) \in I$ for all $x \in E$.*

Definition 3 ([3]). *Let E be a universal set, $I = [0,1]$, $\mathcal{U}_{\mathcal{L}}(x), \mathcal{V}_{\mathcal{L}}(x) \in I$ are degrees of membership and non-membership respectively, and $\alpha^{\mathcal{L}}(x), \beta^{\mathcal{L}}(x) \in I$ are reference parameters. The degrees satisfy $\alpha^{\mathcal{L}}(x) + \beta^{\mathcal{L}}(x) \in I$ and $\alpha^{\mathcal{L}}(x)\mathcal{U}_{\mathcal{L}}(x) + \beta^{\mathcal{L}}(x)\mathcal{V}_{\mathcal{L}}(x) \in I$ for all $x \in E$. Then a linear Diophantine fuzzy set (LDFS) $\mathcal{L}_{\mathcal{D}}$ on E is described as follows.*

$$\mathcal{L}_{\mathcal{D}} = \{(x, \langle \mathcal{U}_{\mathcal{L}}(x), \mathcal{V}_{\mathcal{L}}(x)\rangle, \langle \alpha^{\mathcal{L}}(x), \beta^{\mathcal{L}}(x)\rangle) : x \in E\}.$$

Example 1. *Let $E_1 = \{w_1, w_2, w_3, w_4\}$ be a universal set and define $\mathcal{L}_{\mathcal{D}}$ on E_1 as follows: $\mathcal{L}_{\mathcal{D}}(w_1) = (\langle 0.9, 0.3\rangle, \langle 0.25, 0.45\rangle)$, $\mathcal{L}_{\mathcal{D}}(w_2) = (\langle 0.452, 0.99\rangle, \langle 0.234, 0.009\rangle\ )$, $\mathcal{L}_{\mathcal{D}}(w_3) = (\langle 0.678, 0.124\rangle, \langle 0.21, 0.35\rangle)$, and $\mathcal{L}_{\mathcal{D}}(w_4) = (\langle 0.36, 0.251\rangle, \langle 0.43, 0.57\rangle)$. Then $\mathcal{L}_{\mathcal{D}}$ is an LDFS on E_1.*

Remark 1. *A fuzzy set A on a universal set E with a membership function μ is a special case of linear Diophantine fuzzy set. This is easily seen as*

$$A = \{(x, \langle \mu(x), 0\rangle, \langle 1, 0\rangle) : x \in E\}$$

is an LDFS on E.

Definition 4 ([3]). *Let E be a universal set and $\mathcal{L}_{\mathcal{D}1}, \mathcal{L}_{\mathcal{D}2}$ be LDFSs on E. Then*

(1) The intersection $\mathcal{L}_{\mathcal{D}1} \cap \mathcal{L}_{\mathcal{D}2}$ of $\mathcal{L}_{\mathcal{D}1}$ and $\mathcal{L}_{\mathcal{D}2}$ is defined as

$$\{(x, \langle \mathcal{U}_{\mathcal{L}_1}(x) \wedge \mathcal{U}_{\mathcal{L}_2}(x), \mathcal{U}_{\mathcal{L}_1}(x) \vee \mathcal{U}_{\mathcal{L}_2}(x)\rangle, \langle \alpha_1^{\mathcal{L}}(x) \wedge \alpha_2^{\mathcal{L}}(x), \beta_1^{\mathcal{L}}(x) \vee \beta_2^{\mathcal{L}}(x)\rangle) : x \in E\},$$

(2) The union $\mathcal{L}_{\mathcal{D}1} \cup \mathcal{L}_{\mathcal{D}2}$ of $\mathcal{L}_{\mathcal{D}1}$ and $\mathcal{L}_{\mathcal{D}2}$ is defined as

$$\{(x, \langle \mathcal{U}_{\mathcal{L}_1}(x) \vee \mathcal{U}_{\mathcal{L}_2}(x), \mathcal{V}_{\mathcal{L}_1}(x) \wedge \mathcal{V}_{\mathcal{L}_2}(x)\rangle, \langle \alpha_1^{\mathcal{L}}(x) \vee \alpha_2^{\mathcal{L}}(x), \beta_1^{\mathcal{L}}(x) \wedge \beta_2^{\mathcal{L}}(x)\rangle) : x \in E\},$$

(3) $\mathcal{L}_{\mathcal{D}1}$ is subset of $\mathcal{L}_{\mathcal{D}2}$, denoted by $\mathcal{L}_{\mathcal{D}1} \subseteq \mathcal{L}_{\mathcal{D}2}$, if $\mathcal{L}_{\mathcal{D}1}(x) \leq \mathcal{L}_{\mathcal{D}2}(x)$ for all $x \in E$. i.e., $\mathcal{U}_{\mathcal{L}_1}(x) \leq \mathcal{U}_{\mathcal{L}_2}(x)$, $\mathcal{V}_{\mathcal{L}_1}(x) \geq \mathcal{V}_{\mathcal{L}_2}(x)$, $\alpha_1^{\mathcal{L}}(x) \leq \alpha_2^{\mathcal{L}}(x)$, and $\beta_1^{\mathcal{L}}(x) \geq \beta_2^{\mathcal{L}}(x)$ for all $x \in E$,

(4) $\mathcal{L}_{\mathcal{D}1} = \mathcal{L}_{\mathcal{D}2}$ if $\mathcal{L}_{\mathcal{D}1} \subseteq \mathcal{L}_{\mathcal{D}2}$ and $\mathcal{L}_{\mathcal{D}2} \subseteq \mathcal{L}_{\mathcal{D}1}$,

(5) The complement $\mathcal{L}_{\mathcal{D}1}^c$ of $\mathcal{L}_{\mathcal{D}1}$ is defined as

$$\mathcal{L}_{\mathcal{D}1}^c = \{(x, \langle \mathcal{V}_{\mathcal{L}_1}(x), \mathcal{U}_{\mathcal{L}_1}(x)\rangle, \langle \beta_1^{\mathcal{L}}(x), \alpha_1^{\mathcal{L}}(x)\rangle) : x \in E\}.$$

Here, "$\vee$" and "$\wedge$" represent the maximum and minimum respectively.

Example 2. *Let $E_2 = \{w_5, w_6\}$ be a universal set and define the LDFSs $\mathcal{L}_{\mathcal{D}}', \mathcal{L}_{\mathcal{D}}''$ on E_2 respectively as follows:*

$$\mathcal{L}_{\mathcal{D}}' = \{(w_5, \langle 0.3, 0.1\rangle, \langle 0.5, 0.4\rangle), (w_6, \langle 0.4, 0.1\rangle, \langle 0.3, 0.6\rangle)\},$$

$$\mathcal{L}_{\mathcal{D}}'' = \{(w_5, \langle 0.2, 0.04\rangle, \langle 0.4, 0.6\rangle), (w_6, \langle 0.3, 0.2\rangle, \langle 0.4, 0.4\rangle)\}.$$

Then the LDFS $\mathcal{L}_{\mathcal{D}} = \mathcal{L}_{\mathcal{D}}' \cap \mathcal{L}_{\mathcal{D}}''$ on E_2 is defined as follows:
$\mathcal{L}_{\mathcal{D}}(w_5) = (\langle 0.2, 0.1\rangle, \langle 0.4, 0.6\rangle)$ and $\mathcal{L}_{\mathcal{D}}(w_6) = (\langle 0.3, 0.2\rangle, \langle 0.3, 0.6\rangle)$.

3. Linear Diophantine Fuzzy Ideals

In this section, linear Diophantine fuzzy subalgebras and linear Diophantine fuzzy ideals in *BCK/BCI*-algebras are described and characterized.

Definition 5. *A LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ of $\mathcal{A}$ is called a LDF subalgebra (briefly, LDFSub) if $\forall\, \vartheta, \varrho \in \mathcal{A}$:*

(L1) $\mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \geq \mathcal{U}_{\mathcal{L}}(\vartheta) \wedge \mathcal{U}_{\mathcal{L}}(\varrho)$,
(L2) $\mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \leq \mathcal{V}_{\mathcal{L}}(\vartheta) \vee \mathcal{V}_{\mathcal{L}}(\varrho)$,
(L3) $\alpha^{\mathcal{L}}(\vartheta * \varrho) \geq \alpha^{\mathcal{L}}(\vartheta) \wedge \alpha^{\mathcal{L}}(\varrho)$,
(L4) $\beta^{\mathcal{L}}(\vartheta * \varrho) \leq \beta^{\mathcal{L}}(\vartheta) \vee \beta^{\mathcal{L}}(\varrho)$.

Example 3. *Consider a BCK-algebra $\mathcal{A} = \{0, \vartheta, \varrho, \ell\}$ defined by Table 1:*
Now define a LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ on $\mathcal{A}$ as:

$$\mathcal{L}_{\mathcal{D}}(x) = \begin{cases} (\langle 0.7, 0 \rangle, \langle 0.8, 0 \rangle) & \text{if } x = 0, \\ (\langle 0.5, 0 \rangle, \langle 0.6, 0 \rangle) & \text{if } x = \vartheta, \\ (\langle 0, 0.1 \rangle, \langle 0, 0.1 \rangle) & \text{if } x = \varrho \text{ or } x = \ell. \end{cases}$$

It is straightforward to show that $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is a LDFSub of $\mathcal{A}$.

Table 1. Cayley's table for $*$-operation.

$*$	**0**	$\boldsymbol{\vartheta}$	$\boldsymbol{\varrho}$	$\boldsymbol{\ell}$
0	0	0	0	0
ϑ	ϑ	0	0	ϑ
ϱ	ϱ	ϑ	0	ϱ
ℓ	ℓ	ℓ	ℓ	0

Lemma 1. *If $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is a LDFSub of $\mathcal{A}$, then*

$$\mathcal{L}_{\mathcal{D}}(0) \geq \mathcal{L}_{\mathcal{D}}(\vartheta),\ \forall\, \vartheta \in \mathcal{A}.$$

Proof. Let $\vartheta \in \mathcal{A}$. Then we have

$$\mathcal{U}_{\mathcal{L}}(0) = \mathcal{U}_{\mathcal{L}}(\vartheta * \vartheta) \geq \mathcal{U}_{\mathcal{L}}(\vartheta) \wedge \mathcal{U}_{\mathcal{L}}(\vartheta) = \mathcal{U}_{\mathcal{L}}(\vartheta),$$

$$\mathcal{V}_{\mathcal{L}}(0) = \mathcal{V}_{\mathcal{L}}(\vartheta * \vartheta) \leq \mathcal{V}_{\mathcal{L}}(\vartheta) \vee \mathcal{V}_{\mathcal{L}}(\vartheta) = \mathcal{V}_{\mathcal{L}}(\vartheta),$$

$$\alpha^{\mathcal{L}}(0) = \alpha^{\mathcal{L}}(\vartheta * \vartheta) \geq \alpha^{\mathcal{L}}(\vartheta) \wedge \alpha^{\mathcal{L}}(\vartheta) = \alpha^{\mathcal{L}}(\vartheta),$$

and

$$\beta^{\mathcal{L}}(0) = \beta^{\mathcal{L}}(\vartheta * \vartheta) \leq \beta^{\mathcal{L}}(\vartheta) \vee \beta^{\mathcal{L}}(\vartheta) = \beta^{\mathcal{L}}(\vartheta).$$

Therefore, $\mathcal{L}_{\mathcal{D}}(0) \geq \mathcal{L}_{\mathcal{D}}(\vartheta)$. □

Definition 6. *A LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ of $\mathcal{A}$ is called a LDF ideal ($LDFI$) if $\forall\, \vartheta, \varrho \in \mathcal{A}$, the following conditions hold.*

(L5) $\mathcal{U}_{\mathcal{L}}(0) \geq \mathcal{U}_{\mathcal{L}}(\vartheta), \mathcal{V}_{\mathcal{L}}(0) \leq \mathcal{V}_{\mathcal{L}}(\vartheta), \alpha^{\mathcal{L}}(0) \geq \alpha^{\mathcal{L}}(\vartheta)$ *and* $\beta^{\mathcal{L}}(0) \leq \beta^{\mathcal{L}}(\vartheta)$,
(L6) $\mathcal{U}_{\mathcal{L}}(\vartheta) \geq \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\varrho)$ *and* $\mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \leq \mathcal{V}_{\mathcal{L}}(\vartheta) \vee \mathcal{V}_{\mathcal{L}}(\varrho)$,
(L7) $\alpha^{\mathcal{L}}(\vartheta) \geq \alpha^{\mathcal{L}}(\vartheta * \varrho) \wedge \alpha^{\mathcal{L}}(\varrho)$ *and* $\beta^{\mathcal{L}}(\vartheta * \varrho) \leq \beta^{\mathcal{L}}(\vartheta) \vee \beta^{\mathcal{L}}(\varrho)$.

Example 4. *Consider a BCI-algebra $\mathcal{A} = \{0, \vartheta, \varrho, \ell\}$ defined by Table 2:*

Now define a LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ *on* $\mathcal{A}$ *as:*

$$\mathcal{L}_{\mathcal{D}}(x) = \begin{cases} (\langle 0.6, 0.3 \rangle, \langle 0.7, 0.1 \rangle) & \text{if } x = 0, \\ (\langle 0.5, 0.3 \rangle, \langle 0.6, 0.2 \rangle) & \text{if } x = 1, \\ (\langle 0.2, 0.3 \rangle, \langle 0.4, 0.3 \rangle) & \text{if } x = \vartheta, \\ (\langle 0.3, 0.6 \rangle, \langle 0.4, 0.4 \rangle) & \text{if } x = \varrho, \\ (\langle 0.2, 0.6 \rangle, \langle 0.4, 0.4 \rangle) & \text{if } x = \ell. \end{cases}$$

It is easy to show that $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ *is a LDFI of* $\mathcal{A}$.

Table 2. Cayley's table for $*$-operation.

$*$	**0**	**1**	$\boldsymbol{\vartheta}$	$\boldsymbol{\varrho}$	$\boldsymbol{\ell}$
0	0	0	ϑ	ϱ	ℓ
1	1	0	ϑ	ϱ	ℓ
ϑ	ϑ	ϑ	0	ℓ	ϱ
ϱ	ϱ	ϱ	ℓ	0	ϑ
ℓ	ℓ	ℓ	ϱ	ϑ	0

Lemma 2. *Let* $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ *be a LDFI of* $\mathcal{A}$ *and* $\vartheta, \varrho \in \mathcal{A}$ *such that* $\vartheta \leq \varrho$. *Then*

$$\mathcal{L}_{\mathcal{D}}(\vartheta) \geq \mathcal{L}_{\mathcal{D}}(\varrho).$$

Proof. Let $\vartheta, \varrho \in \mathcal{A}$ such that $\vartheta \leq \varrho$. Then we have

$$\mathcal{U}_{\mathcal{L}}(\vartheta) \geq \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\varrho) = \mathcal{U}_{\mathcal{L}}(0) \wedge \mathcal{U}_{\mathcal{L}}(\varrho) = \mathcal{U}_{\mathcal{L}}(\varrho),$$

$$\mathcal{V}_{\mathcal{L}}(\vartheta) \leq \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \vee \mathcal{V}_{\mathcal{L}}(\varrho) = \mathcal{V}_{\mathcal{L}}(0) \vee \mathcal{V}_{\mathcal{L}}(\varrho) = \mathcal{V}_{\mathcal{L}}(\varrho),$$

$$\alpha^{\mathcal{L}}(\vartheta) \geq \alpha^{\mathcal{L}}(\vartheta * \varrho) \wedge \alpha^{\mathcal{L}}(\varrho) = \alpha^{\mathcal{L}}(0) \wedge \alpha^{\mathcal{L}}(\varrho) = \alpha^{\mathcal{L}}(\varrho),$$

and

$$\beta^{\mathcal{L}}(\vartheta) \leq \beta^{\mathcal{L}}(\vartheta * \varrho) \vee \beta^{\mathcal{L}}(\varrho) = \beta^{\mathcal{L}}(0) \vee \beta^{\mathcal{L}}(\varrho) = \beta^{\mathcal{L}}(\varrho).$$

Therefore, $\mathcal{L}_{\mathcal{D}}(\vartheta) \geq \mathcal{L}_{\mathcal{D}}(\varrho)$. $\square$

Lemma 3. *Let* $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ *be a LDFI of* $\mathcal{A}$ *and* $\vartheta, \varrho, \hbar \in \mathcal{A}$ *such that* $\vartheta * \varrho \leq \hbar$. *Then*

$$\mathcal{L}_{\mathcal{D}}(\vartheta) \geq \mathcal{L}_{\mathcal{D}}(\varrho) \wedge \mathcal{L}_{\mathcal{D}}(\hbar).$$

Proof. Let $\vartheta, \varrho, \hbar \in \mathcal{A}$ such that $\vartheta * \varrho \leq \hbar$. Then we have

$$\begin{aligned} \mathcal{U}_{\mathcal{L}}(\vartheta) &\geq \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\varrho) \\ &\geq \mathcal{U}_{\mathcal{L}}((\vartheta * \varrho) * \hbar) \wedge \mathcal{U}_{\mathcal{L}}(\hbar) \wedge \mathcal{U}_{\mathcal{L}}(\varrho) \\ &= \mathcal{U}_{\mathcal{L}}(0) \wedge \mathcal{U}_{\mathcal{L}}(\hbar) \wedge \mathcal{U}_{\mathcal{L}}(\varrho) \\ &= \mathcal{U}_{\mathcal{L}}(\hbar) \wedge \mathcal{U}_{\mathcal{L}}(\varrho), \end{aligned}$$

$$\begin{aligned} \mathcal{V}_{\mathcal{L}}(\vartheta) &\leq \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \vee \mathcal{V}_{\mathcal{L}}(\varrho) \\ &\leq \{\mathcal{V}_{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \mathcal{V}_{\mathcal{L}}(\hbar)\} \vee \mathcal{V}_{\mathcal{L}}(\varrho) \\ &= \mathcal{V}_{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \mathcal{V}_{\mathcal{L}}(\hbar) \vee \mathcal{V}_{\mathcal{L}}(\varrho) \\ &= \mathcal{V}_{\mathcal{L}}(0) \vee \mathcal{V}_{\mathcal{L}}(\hbar) \vee \mathcal{V}_{\mathcal{L}}(\varrho) \\ &= \mathcal{V}_{\mathcal{L}}(\hbar) \vee \mathcal{V}_{\mathcal{L}}(\varrho), \end{aligned}$$

$$\begin{aligned}
\alpha^{\mathcal{L}}(\vartheta) &\geq \alpha^{\mathcal{L}}(\vartheta * \varrho) \wedge \alpha^{\mathcal{L}}(\varrho) \\
&\geq \alpha^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \wedge \alpha^{\mathcal{L}}(\hbar) \wedge \alpha^{\mathcal{L}}(\varrho) \\
&= \alpha^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \wedge \alpha^{\mathcal{L}}(\hbar) \wedge \alpha^{\mathcal{L}}(\varrho) \\
&= \alpha^{\mathcal{L}}(0) \wedge \alpha^{\mathcal{L}}(\hbar) \wedge \alpha^{\mathcal{L}}(\varrho) \\
&= \alpha^{\mathcal{L}}(\hbar) \wedge \alpha^{\mathcal{L}}(\varrho)
\end{aligned}$$

and

$$\begin{aligned}
\beta^{\mathcal{L}}(\vartheta) &\leq \beta^{\mathcal{L}}(\vartheta * \varrho) \vee \beta^{\mathcal{L}}(\varrho) \\
&\leq \{\beta^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \beta^{\mathcal{L}}(\hbar)\} \vee \beta^{\mathcal{L}}(\varrho) \\
&= \beta^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \beta^{\mathcal{L}}(\hbar) \vee \beta^{\mathcal{L}}(\varrho) \\
&= \beta^{\mathcal{L}}(0) \vee \beta^{\mathcal{L}}(\hbar) \vee \beta^{\mathcal{L}}(\varrho) \\
&= \beta^{\mathcal{L}}(\hbar) \vee \beta^{\mathcal{L}}(\varrho).
\end{aligned}$$

Therefore, $\mathcal{L}_{\mathcal{D}}(\vartheta) \geq \mathcal{L}_{\mathcal{D}}(\varrho) \wedge \mathcal{L}_{\mathcal{D}}(\hbar)$. □

Theorem 1. *Every LDFI of BCK-algebra $\mathcal{A}$ is a LDFSub of $\mathcal{A}$.*

Proof. Let $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ be any *LDFI* of $\mathcal{A}$ and $\vartheta, \varrho \in \mathcal{A}$. Since $(\vartheta * \varrho) * \vartheta = (\vartheta * \vartheta) * \varrho = 0 * \varrho = 0$, it follows that $\vartheta * \varrho \leq \vartheta$ in $\mathcal{A}$. Lemma 2 asserts that $\mathcal{U}_{\mathcal{L}}(\vartheta) \leq \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho), \mathcal{V}_{\mathcal{L}}(\vartheta) \geq \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho), \alpha^{\mathcal{L}}(\vartheta) \leq \alpha^{\mathcal{L}}(\vartheta * \varrho)$ and $\beta^{\mathcal{L}}(\vartheta) \geq \beta^{\mathcal{L}}(\vartheta * \varrho)$. Thus, we have

$$\mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \geq \mathcal{U}_{\mathcal{L}}(\vartheta) \geq \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\varrho) \geq \mathcal{U}_{\mathcal{L}}(\vartheta) \wedge \mathcal{U}_{\mathcal{L}}(\varrho),$$

$$\mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \leq \mathcal{V}_{\mathcal{L}}(\vartheta) \leq \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \vee \mathcal{V}_{\mathcal{L}}(\varrho) \leq \mathcal{V}_{\mathcal{L}}(\vartheta) \vee \mathcal{V}_{\mathcal{L}}(\varrho),$$

$$\alpha^{\mathcal{L}}(\vartheta * \varrho) \geq \alpha^{\mathcal{L}}(\vartheta) \geq \alpha^{\mathcal{L}}(\vartheta * \varrho) \wedge \alpha^{\mathcal{L}}(\varrho) \geq \alpha^{\mathcal{L}}(\vartheta) \wedge \alpha^{\mathcal{L}}(\varrho)$$

and

$$\beta^{\mathcal{L}}(\vartheta * \varrho) \leq \beta^{\mathcal{L}}(\vartheta) \leq \beta^{\mathcal{L}}(\vartheta * \varrho) \vee \beta^{\mathcal{L}}(\varrho) \leq \beta^{\mathcal{L}}(\vartheta) \vee \beta^{\mathcal{L}}(\varrho).$$

Therefore, $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is a *LDFSub* of $\mathcal{A}$. □

Remark 2. *The converse of Theorem 1 is not true in general. See Example 5.*

Example 5. *Consider a BCK-algebra $\mathcal{A} = \{0, \vartheta, \varrho, \ell\}$ with Table 3:*
Now define a LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ on $\mathcal{A}$ as:

$$\mathcal{L}_{\mathcal{D}}(x) = \begin{cases} (\langle 0.7, 0.3 \rangle, \langle 0.8, 0.1 \rangle) & \text{if } x = 0, \\ (\langle 0.5, 0.3 \rangle, \langle 0.6, 0.2 \rangle) & \text{if } x = \vartheta, \\ (\langle 0.3, 0.3 \rangle, \langle 0.4, 0.3 \rangle) & \text{if } x = \varrho, \\ (\langle 0.6, 0.6 \rangle, \langle 0.7, 0.4 \rangle) & \text{if } x = \ell. \end{cases}$$

It is easy to verify that $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is a LDFSub of $\mathcal{A}$ but it is not a LDFI of $\mathcal{A}$ because $0.5 = \mathcal{U}_{\mathcal{L}}(\vartheta) \ngeq \mathcal{U}_{\mathcal{L}}(\vartheta * \ell) \wedge \mathcal{U}_{\mathcal{L}}(\ell) = 0.6$.

Table 3. Cayley's table for $*$-operation.

$*$	$\mathbf{0}$	$\boldsymbol{\vartheta}$	$\boldsymbol{\varrho}$	$\boldsymbol{\ell}$
0	0	0	0	0
ϑ	ϑ	0	ϑ	0
ϱ	ϱ	ϱ	0	0
ℓ	ℓ	ℓ	ℓ	0

Theorem 2. *Let $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ be a LDFSub of $\mathcal{A}$. Then $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is a LDFI $\Leftrightarrow \forall\ \vartheta, \varrho, \hbar \in \mathcal{A}$ such that $\vartheta * \varrho \leq \hbar$ implies $\mathcal{U}_{\mathcal{L}}(\vartheta) \geq \mathcal{U}_{\mathcal{L}}(\varrho) \wedge \mathcal{U}_{\mathcal{L}}(\hbar), \mathcal{V}_{\mathcal{L}}(\vartheta) \leq \mathcal{V}_{\mathcal{L}}(\varrho) \vee \mathcal{V}_{\mathcal{L}}(\hbar), \alpha^{\mathcal{L}}(\vartheta) \geq \alpha^{\mathcal{L}}(\varrho) \wedge \alpha^{\mathcal{L}}(\hbar)$ and $\beta^{\mathcal{L}}(\vartheta) \leq \beta^{\mathcal{L}}(\varrho) \vee \beta^{\mathcal{L}}(\hbar)$.*

Proof. ($\Rightarrow$) Follows from Lemma 3.

($\Leftarrow$) Let $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ be a *LDFSub* of $\mathcal{A}$ such that $\forall\ \vartheta, \varrho, \hbar \in \mathcal{A}$, $\vartheta * \varrho \leq \hbar$ implies $\mathcal{U}_{\mathcal{L}}(\vartheta) \geq \mathcal{U}_{\mathcal{L}}(\varrho) \wedge \mathcal{U}_{\mathcal{L}}(\hbar), \mathcal{V}_{\mathcal{L}}(\vartheta) \leq \mathcal{V}_{\mathcal{L}}(\varrho) \vee \mathcal{V}_{\mathcal{L}}(\hbar), \alpha^{\mathcal{L}}(\vartheta) \geq \alpha^{\mathcal{L}}(\varrho) \wedge \alpha^{\mathcal{L}}(\hbar)$ and $\beta^{\mathcal{L}}(\vartheta) \leq \beta^{\mathcal{L}}(\varrho) \vee \beta^{\mathcal{L}}(\hbar)$. As $\vartheta * (\vartheta * \varrho) \leq \varrho$, so by hypothesis $\mathcal{U}_{\mathcal{L}}(\vartheta) \geq \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\varrho), \mathcal{V}_{\mathcal{L}}(\vartheta) \leq \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \vee \mathcal{V}_{\mathcal{L}}(\varrho), \alpha^{\mathcal{L}}(\vartheta) \geq \alpha^{\mathcal{L}}(\vartheta * \varrho) \wedge \alpha^{\mathcal{L}}(\varrho)$ and $\beta^{\mathcal{L}}(\vartheta) \leq \beta^{\mathcal{L}}(\vartheta * \varrho) \vee \beta^{\mathcal{L}}(\varrho)$. Moreover, Lemma 1 asserts that $\mathcal{L}_{\mathcal{D}}(0) \geq \mathcal{L}_{\mathcal{D}}(\vartheta)\ \forall\ \vartheta \in \mathcal{A}$. Therefore, $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is a *LDFI* of $\mathcal{A}$. $\square$

4. Linear Diophantine Fuzzy Commutative Ideals

In this section, we define the notion of *LDF* commutative ideal of *BCK*-algebras. Moreover, we study some connections between *LDF* subalgebras, *LDF* ideals, and *LDF* commutative ideals.

In this section, $\mathcal{A}$ will stand for a BCK-algebra unless it is otherwise specified.

Definition 7. *A LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is called a LDF commutative ideal LDFCI if it satisfies (L5) and the following conditions $\forall\ \vartheta, \varrho, \hbar \in \mathcal{A}$:*

(L8) $\mathcal{U}_{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \mathcal{U}_{\mathcal{L}}((\vartheta * \varrho) * \hbar) \wedge \mathcal{U}_{\mathcal{L}}(\hbar)$ *and* $\mathcal{V}_{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \mathcal{V}_{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \mathcal{V}_{\mathcal{L}}(\hbar)$,

(L9) $\alpha^{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \alpha^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \wedge \alpha^{\mathcal{L}}(\hbar)$, *and* $\beta^{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \beta^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \beta^{\mathcal{L}}(\hbar)$.

Example 6. *Consider a BCK-algebra $\mathcal{A}$ of Example 3. Now define a LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ on $\mathcal{A}$ as:*

$$\mathcal{L}_{\mathcal{D}}(x) = \begin{cases} (\langle 0.6, 0.3 \rangle, \langle 0.5, 0.1 \rangle) & \text{if } x = 0, \\ (\langle 0.5, 0.3 \rangle, \langle 0.4, 0.2 \rangle) & \text{if } x = \vartheta, \\ (\langle 0.4, 0.3 \rangle, \langle 0.3, 0.3 \rangle) & \text{if } x = \varrho, \\ (\langle 0.4, 0.6 \rangle, \langle 0.3, 0.4 \rangle) & \text{if } x = \ell. \end{cases}$$

Some computations show that $\mathcal{L}_{\mathcal{D}}$ is a LDFCI of $\mathcal{A}$.

Theorem 3. *Every LDFCI of BCK-algebra $\mathcal{A}$ is a LDFI of $\mathcal{A}$.*

Proof. Let $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ be any *LDFCI* of $\mathcal{A}$ and $\vartheta, \varrho, \hbar \in \mathcal{A}$. Having $\mathcal{A}$ a *BCK*-algebra implies that $0 * \hbar = 0$ and hence, $\vartheta = \vartheta * (0 * (0 * \vartheta))$. Then we obtain

$$\begin{aligned} \mathcal{U}_{\mathcal{L}}(\vartheta) &= \mathcal{U}_{\mathcal{L}}(\vartheta * (0 * (0 * \vartheta))) \\ &\geq \mathcal{U}_{\mathcal{L}}((\vartheta * 0) * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\varrho) \\ &= \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\varrho), \end{aligned}$$

$$\begin{aligned} \mathcal{V}_{\mathcal{L}}(\vartheta) &= \mathcal{V}_{\mathcal{L}}(\vartheta * (0 * (0 * \vartheta))) \\ &\leq \mathcal{V}_{\mathcal{L}}((\vartheta * 0) * \varrho) \vee \mathcal{V}_{\mathcal{L}}(\varrho) \\ &= \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \vee \mathcal{V}_{\mathcal{L}}(\varrho), \end{aligned}$$

$$\begin{aligned}\alpha^{\mathcal{L}}(\vartheta) &= \alpha^{\mathcal{L}}(\vartheta * (0 * (0 * \vartheta))) \\ &\geq \alpha^{\mathcal{L}}((\vartheta * 0) * \varrho) \wedge \alpha^{\mathcal{L}}(\varrho) \\ &= \alpha^{\mathcal{L}}(\vartheta * \varrho) \wedge \alpha^{\mathcal{L}}(\varrho)\end{aligned}$$

and

$$\begin{aligned}\beta^{\mathcal{L}}(\vartheta) &= \beta^{\mathcal{L}}(\vartheta * (0 * (0 * \vartheta))) \\ &\leq \beta^{\mathcal{L}}((\vartheta * 0) * \varrho) \vee \beta^{\mathcal{L}}(\varrho) \\ &= \beta^{\mathcal{L}}(\vartheta * \varrho) \vee \beta^{\mathcal{L}}(\varrho).\end{aligned}$$

Therefore, $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ is a *LDFI* of $\mathcal{A}$. □

Corollary 1. *Every LDFCI of $\mathcal{A}$ is a LDFSub of $\mathcal{A}$.*

Proof. The proof follows from Theorems 1 and 3. □

Remark 3. *In general, the converse of Theorem 3 does not hold. See Example 7.*

Example 7. *Consider a BCK-algebra $\mathcal{A} = \{0, \vartheta, \jmath, \varrho, \ell\}$ defined by Table 4:*
Now define a LDFS $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ on $\mathcal{A}$ as:

$$\mathcal{L}_{\mathcal{D}}(x) = \begin{cases} (\langle 0.5, 0.1 \rangle, \langle 0.4, 0.3 \rangle) & \text{if } x = 0, \\ (\langle 0.4, 0.2 \rangle, \langle 0.3, 0.4 \rangle) & \text{if } x = \vartheta, \\ (\langle 0.3, 0.3 \rangle, \langle 0.2, 0.6 \rangle) & \text{if } x \in \{\jmath, \varrho, \ell\}. \end{cases}$$

It is easy to verify that $\mathcal{L}_{\mathcal{D}}$ is a LDFI of $\mathcal{A}$ but it is not a LDFCI of $\mathcal{A}$ because $0.3 = \mathcal{U}_{\mathcal{L}}(\jmath) = \mathcal{U}_{\mathcal{L}}(\jmath * (\varrho * (\varrho * \jmath))) \ngeq \mathcal{U}_{\mathcal{L}}((\jmath * \varrho) * 0) \wedge \mathcal{U}_{\mathcal{L}}(0) = \mathcal{U}_{\mathcal{L}}(0) = 0.5$.

Table 4. Cayley's table for $*$-opertaion.

$*$	$\mathbf{0}$	$\boldsymbol{\vartheta}$	$\boldsymbol{\jmath}$	$\boldsymbol{\varrho}$	$\boldsymbol{\ell}$
0	0	0	0	0	0
ϑ	ϑ	0	ϑ	0	0
$\jmath$	$\jmath$	$\jmath$	0	0	0
ϱ	ϱ	ϱ	ϱ	0	0
ℓ	ℓ	ℓ	ϱ	$\jmath$	0

Theorem 4. *Let $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ be a LDFI of $\mathcal{A}$. Then $\mathcal{L}_{\mathcal{D}}$ is a LDFCI $\Leftrightarrow \forall\, \vartheta, \varrho \in \mathcal{A}$,*

$$\mathcal{L}_{\mathcal{D}}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \mathcal{L}_{\mathcal{D}}(\vartheta * \varrho).$$

Proof. ($\Rightarrow$) Let $\mathcal{L}_{\mathcal{D}} = (\langle \mathcal{U}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}} \rangle, \langle \alpha^{\mathcal{L}}, \beta^{\mathcal{L}} \rangle)$ be a *LDFCI* of $\mathcal{A}$. Then $\forall\, \vartheta, \varrho, \hbar \in \mathcal{A}$, we have $\mathcal{U}_{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \mathcal{U}_{\mathcal{L}}((\vartheta * \varrho) * \hbar) \wedge \mathcal{U}_{\mathcal{L}}(\hbar)$, $\mathcal{V}_{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \mathcal{V}_{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \mathcal{V}_{\mathcal{L}}(\hbar)$, $\alpha^{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \alpha^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \wedge \alpha^{\mathcal{L}}(\hbar)$ and $\beta^{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \beta^{\mathcal{L}}((\vartheta * \varrho) * \hbar) \vee \beta^{\mathcal{L}}(\hbar)$. Taking $\hbar = 0$, so

$$\begin{aligned}\mathcal{U}_{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta)) &\geq \mathcal{U}_{\mathcal{L}}((\vartheta * \varrho) * 0) \wedge \mathcal{U}_{\mathcal{L}}(0) \\ &\geq \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \wedge \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho) \\ &= \mathcal{U}_{\mathcal{L}}(\vartheta * \varrho),\end{aligned}$$

$$\begin{aligned}\mathcal{V}_{\mathcal{L}}(\vartheta * (\varrho * (\varrho * \vartheta)) &\leq \mathcal{V}_{\mathcal{L}}((\vartheta * \varrho) * 0) \vee \mathcal{V}_{\mathcal{L}}(0) \\ &\leq \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \vee \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho) \\ &= \mathcal{V}_{\mathcal{L}}(\vartheta * \varrho),\end{aligned}$$

$$\begin{aligned}\alpha^{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) &\geq \alpha^{\mathcal{L}}((\vartheta*\varrho)*0)\wedge\alpha^{\mathcal{L}}(0)\\ &\geq \alpha^{\mathcal{L}}(\vartheta*\varrho)\wedge\alpha^{\mathcal{L}}(\vartheta*\varrho)\\ &= \alpha^{\mathcal{L}}(\vartheta*\varrho)\end{aligned}$$

and

$$\begin{aligned}\beta^{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) &\leq \beta^{\mathcal{L}}((\vartheta*\varrho)*0)\vee\beta^{\mathcal{L}}(0)\\ &\leq \beta^{\mathcal{L}}(\vartheta*\varrho)\vee\beta^{\mathcal{L}}(\vartheta*\varrho)\\ &= \beta^{\mathcal{L}}(\vartheta*\varrho).\end{aligned}$$

($\Leftarrow$) Let $\mathcal{L}_{\mathcal{D}} = (\langle\mathcal{U}_{\mathcal{L}},\mathcal{V}_{\mathcal{L}}\rangle,\langle\alpha^{\mathcal{L}},\beta^{\mathcal{L}}\rangle)$ be a *LDFI* such that $\mathcal{L}_{\mathcal{D}}(\vartheta*(\varrho*(\varrho*\vartheta))) \geq \mathcal{L}_{\mathcal{D}}(\vartheta*\varrho), \forall\, \vartheta,\varrho,\hbar\in\mathcal{A}$. By assumption, we have

$$\begin{aligned}\mathcal{U}_{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) &\geq \mathcal{U}_{\mathcal{L}}(\vartheta*\varrho)\\ &\geq \mathcal{U}_{\mathcal{L}}((\vartheta*\varrho)*\hbar)\wedge\mathcal{U}_{\mathcal{L}}(\hbar),\\ \mathcal{V}_{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) &\leq \mathcal{V}_{\mathcal{L}}(\vartheta*\varrho)\\ &\leq \mathcal{V}_{\mathcal{L}}((\vartheta*\varrho)*\hbar)\vee\mathcal{V}_{\mathcal{L}}(\hbar),\\ \alpha^{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) &\geq \alpha^{\mathcal{L}}(\vartheta*\varrho)\\ &\geq \alpha^{\mathcal{L}}((\vartheta*\varrho)*\hbar)\wedge\alpha^{\mathcal{L}}(\hbar)\end{aligned}$$

and

$$\begin{aligned}\beta^{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) &\leq \beta^{\mathcal{L}}(\vartheta*\varrho)\\ &\leq \beta^{\mathcal{L}}((\vartheta*\varrho)*\hbar)\vee\beta^{\mathcal{L}}(\hbar).\end{aligned}$$

Therefore, $\mathcal{L}_{\mathcal{D}} = (\langle\mathcal{U}_{\mathcal{L}},\mathcal{V}_{\mathcal{L}}\rangle,\langle\alpha^{\mathcal{L}},\beta^{\mathcal{L}}\rangle)$ is a *LDFCI* of $\mathcal{A}$. $\square$

Theorem 5. *Every LDFI of a commutative BCK-algebra $\mathcal{A}$ is a LDFCI.*

Proof. Let $\mathcal{L}_{\mathcal{D}} = (\langle\mathcal{U}_{\mathcal{L}},\mathcal{V}_{\mathcal{L}}\rangle,\langle\alpha^{\mathcal{L}},\beta^{\mathcal{L}}\rangle)$ be a *LDFS* of $\mathcal{A}$. Then $\forall\, \vartheta,\varrho,\hbar\in\mathcal{A}$, we have

$$\begin{aligned}\Big((\vartheta*(\varrho*(\varrho*\vartheta)))*((\vartheta*\varrho)*\hbar)\Big)*\hbar &= \Big((\vartheta*(\varrho*(\varrho*\vartheta)))*\hbar\Big)*((\vartheta*\varrho)*\hbar)\\ &\leq \big(\vartheta*(\varrho*(\varrho*\vartheta))\big)*(\vartheta*\varrho)\\ &= \big(\vartheta*(\vartheta*\varrho)\big)*\big(\varrho*(\varrho*\vartheta)\big)\\ &= 0.\end{aligned}$$

It follows that $\Big((\vartheta*(\varrho*(\varrho*\vartheta)))*((\vartheta*\varrho)*\hbar)\Big) \leq \hbar$. As $\mathcal{L}_{\mathcal{D}} = (\langle\mathcal{U}_{\mathcal{L}},\mathcal{V}_{\mathcal{L}}\rangle,\langle\alpha^{\mathcal{L}},\beta^{\mathcal{L}}\rangle)$ is a *LDFI* of $\mathcal{A}$, so by Lemma 3, $\mathcal{U}_{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) \geq \mathcal{U}_{\mathcal{L}}((\vartheta*\varrho)*\hbar)\wedge\mathcal{U}_{\mathcal{L}}(\hbar), \mathcal{V}_{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) \leq \mathcal{V}_{\mathcal{L}}((\vartheta*\varrho)*\hbar)\vee\mathcal{V}_{\mathcal{L}}(\hbar)$ and $\alpha^{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) \geq \alpha^{\mathcal{L}}((\vartheta*\varrho)*\hbar)\wedge\alpha^{\mathcal{L}}(\hbar), \beta^{\mathcal{L}}(\vartheta*(\varrho*(\varrho*\vartheta))) \leq \beta^{\mathcal{L}}((\vartheta*\varrho)*\hbar)\vee\beta^{\mathcal{L}}(\hbar)$. Hence, $\mathcal{L}_{\mathcal{D}} = (\langle\mathcal{U}_{\mathcal{L}},\mathcal{V}_{\mathcal{L}}\rangle,\langle\alpha^{\mathcal{L}},\beta^{\mathcal{L}}\rangle)$ is a *LDFCI* of $\mathcal{A}$. $\square$

5. Conclusions

Linear Diophantine fuzzification of algebraic structures is a new field that generalizes fuzzy algebraic structures. In our paper, we applied linear Diophantine fuzzifications in BCK/BCI-algebras. We introduced the notions $LDFSub$, $LDFI$ and $LDFCI$ of a BCK-algebra. Moreover, we discussed some of their properties and investigated some relationships among them. Our main results are presented in Sections 3 and 4. Since every fuzzy set can be viewed as a LDFS, it follows that our results of LDF-substructures of BCK/BCI-algebras are generalizations of fuzzy substructures of BCK/BCI-algebras.

For future work, we raise the following problems.

1. Introduce LDF-level subalgebras/ideals/commutative ideals of BCK-algebra and investigate the relationship between them and subalgebras/ideals/commutative ideals of BCK algebra.
2. Define LDF substructures of other types of algebras.

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