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Branching Solutions of the Cauchy Problem for Nonlinear Loaded Differential Equations with Bifurcation Parameters

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Abstract: The Cauchy problem for a nonlinear system of differential equations with a Stieltjes integral (loads) of the desired solution is considered. The equation contains bifurcation parameters where the system has a trivial solution for any values. The necessary and sufficient conditions are derived for those parameter values (bifurcation points) in the neighborhood of which the Cauchy problem has a non-trivial real solution. The constructive method is proposed for the solution of real solutions in the neighborhood of those points. The method uses successive approximations and builds asymptotics of the solution. The theoretical results are illustrated by example. The Cauchy problem with loads and bifurcation parameters has not been studied before.

Keywords: Cauchy problem; loaded differential equation; bifurcation; homotopy; Newton diagram

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1. Introduction

The development of advanced methods and models for nonlinear dynamical systems control needs the theory of loaded systems of equations [1–4]. Indeed, there is the great variety of types of loads that must be taken into account both in the design of technical systems and in the creation of adequate models of complex energy systems [5]. Loaded ordinary differential equations model heat transfer phenomena and are solved by the finite difference method in [6].

There are many publications in this field; see, e.g., [1–4,7–9]. It is to be noted that loaded differential and loaded integro-differential equations are directly relevant to the non-local problem for integro-differential equations; see [10,11] and the references therein. Nevertheless, the problem of analyzing loaded systems of differential equations with bifurcation parameters is still an open problem despite the significant progress in nonlinear analysis [12].

The purpose of this paper is to prove the general existence theorems and construct approximate methods for solutions of nonlinear loaded Cauchy problem with bifurcation parameters.

Let us consider the following Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} = \Phi(x(t), x_\alpha, t, \lambda), \\ x(0) = 0. \end{cases} \tag{1}$$

The desired vector-function $x(t) := (x_1(t), \dots, x_n(t))^T$ is continuous for $t \in [0, T]$, the parameter $\lambda \in \Omega \subset E^m$, where E^m is a normed space. Load $x_\alpha := (x_{\alpha_1}, \dots, x_{\alpha_n})^T$ is given using the linear Riemann–Stieltjes functionals

$$x_{\alpha_i} := \int_{a_1}^{b_i} x_i(t)\, d\alpha_i(t),\ i = 1, \ldots, n,$$

where $\alpha_i(t)$ is a limited variation function, $[a_i, b_i] \subset (0, T)$. Such functionals, x_α, can be called *integral loads*. Likewise, the functional $x_\alpha := (x(\alpha_1), \ldots, x(\alpha_n))^T$, where $a_i \in (0, T)$, can be called a *local load*.

The function Φ in system (1) is continuous and defined as $\Phi(x(t), x_\alpha, t, \lambda) = (\Phi_1(x(t), x_\alpha, t, \lambda), \ldots, \Phi_n(x(t), x_\alpha, t, \lambda))^T$, where $\Phi_i = \sum_{j=1}^{n} a_{ij}(t, \lambda) x_{\alpha_j} + \sum_{j=l}^{N} F_{ij}(x(t), t) + o(||x||^N)$, where $i = 1, \ldots, n,\ \ l \geq 2$, $F_{ij}(x(t), t)$ are j-homogeneous (normal) forms of vector-function $x(t)$. System (1) for an arbitrary parameter λ has the trivial solution $x(t) = 0,\ x_\alpha = 0$.

The goal is to find the points λ^0, such that the Cauchy problem (1) has a nontrivial real solution in neighborhood of λ^0. Such values of the parameter are usually called bifurcation points (branching points) of solutions. In applications, the parameter λ may be associated with an external influence imposed on the system. Therefore, bifurcation points are of particular interest in mathematical modeling. Note that loaded equation model dynamic processes with trajectories at any fixed points in time (or local time intervals) influence a whole trajectory; however, there is no practical (technical) way to perform measurements at these points or intervals.

There are publications dealing with the studies of loaded differential equations [1–3]. The nonlinear Volterra integral equation with local and/or integral loads on the desired solution given by the Stieltjes integral is considered in [7]. The Cauchy problem (1) contains a bifurcation parameter λ and has a trivial solution for any of the parameter values. Necessary and sufficient conditions are obtained for those values of the parameter (bifurcation points), in the neighborhood of which, the problem (1) has nontrivial continuous real solutions. In this paper, we continue these studies since the bifurcation analysis for Cauchy problems for systems of differential equations with loads has not been conducted yet.

Definition 1. *The arbitrary bounded domain $S \subset E^m$ is called the sectorial neighborhood of the point λ^0 if $\lambda^0 \in \partial S$.*

Definition 2. *Point λ^0 is called the bifurcation point of the initial problem* (1) *if, for arbitrary $\varepsilon > 0$, $\delta > 0$ there exist $x(t)$ and λ in the sectorial neighborhood of point λ^0 satisfying problem* (1) *such that* $0 < ||x|| \leq \varepsilon,\ \ ||\lambda - \lambda^0|| \leq \delta$.

In this paper, the necessary and sufficient conditions for problem (1) to have the bifurcation point are derived. In such a case, one can construct asymptotics of non-trivial real branches of small solutions of system (1) in the vectorial domain of this point. By a branch, we mean a continuous real solution $x(t)$ such that $x(t) \to 0$ as $S \ni \lambda \to \lambda^0$.

Clearly, each solution $x(t)$ in the problem (1) is assigned a single load using a given linear functional. We obtain a solution to the problem by constructing an equation with respect to the load with parameter λ and investigate it using the Lyapunov and Schmidt ascending approach based on a combination of methods:

1. Analytical method of successive approximations and Nekrasov–Nazarov method of uncertain coefficients;
2. The geometric method based on the Kronecker–Poincaré index and the power geometry of Newton diagrams (polygon);
3. The homotopy perturbation method.

The solution methodology is constructive, does not employ any complex generalizations and is accessible to a wide range of specialists in applied fields. This paper employs the methods from the monographs [5,13–17].

Paper Overview

The Cauchy problem for a single equation is considered in Sections 2 and 3. Necessary and sufficient conditions are given under which the point λ^0 will be a bifurcation point, and a method for constructing a real solution is proposed. The theoretical results are illustrated by a solution of the differential equation. In Section 4, we present two theorems on the existence of bifurcation points in the Cauchy problem for systems of equations.

2. Construction of the Equation with Respect to the Load: The Necessary Conditions for Bifurcation and Solution Existence

Fore sake of clarity, let us start with the simple case of the single equation

$$\begin{cases} \frac{dx}{dt} = a(t,\lambda)x_\alpha + \sum\limits_{n=l}^{\infty} F_n(t)x^n(t) \\ x(0) = 0. \end{cases} \tag{2}$$

with single load x_α and parameter $\lambda \in \mathbb{R}^1$. As we are interested in the small solutions, then desired solution $x(t)$ as function of the load can be constructed as following series

$$x(t) = a_1(t,\lambda)x_\alpha + \sum_{i=l}^{\infty} a_i(t,\lambda)x_\alpha^i. \tag{3}$$

In a nonanalytic case, the method of successive approximations can be applied. The series coefficients can be calculated using the method of undetermined coefficients by substitution of the series (3) into the equation

$$x(t) = \int_0^t a(t,\lambda)\,dt x_\alpha + \sum_{n=l}^{\infty} \int_0^t F_n(t)x^n(t)\,dt.$$

Then, clearly we find the recursive formulas

$$a_1(t,\lambda) = \int_0^t a(t,\lambda)\,dt,$$

$$a_l(t,\lambda) = \int_0^t F_l(t)a_1^l(t,\lambda)\,dt,$$

$$\dots$$

and uniformly converging for small enough $|x_\alpha|$ series (3) can be constructed. Using the standard notion $\langle x, \alpha \rangle$ and by application of the functional to both sides of the Formula (3), the following equation with respect to load x_α can be derived

$$L(x_\alpha,\lambda) := L_1(\lambda)x_\alpha + \sum_{i=l}^{\infty} L_i(\lambda)x_\alpha^i = 0, \tag{4}$$

where $L_1(\lambda) = \langle a_1(t,\lambda), \alpha \rangle - 1$, $L_i(\lambda) = \langle a_i(t,\lambda), \alpha \rangle$, $i = l, l+1, \dots$. Equation (4) is called a *branching equation* with respect to the load.

Thus, the following statement is true.

Lemma 1. *Load x_α in problem (2) must satisfies Equation (4) with parameter λ. Bifurcation points of Cauchy problem (2) will be bifurcation points of branching Equation (4).*

Then, from Lemma 1 and implicit function theorem, there follows two corollaries.

Corollary 1. *(Necessary conditions for a bifurcation) In order for point λ^0 be the bifurcation point in problem (2), it is necessary $L_1(\lambda^0) = 0$ in Equation (4).*

Corollary 2. *Let, in Equation (4), all the coefficients $L_i(\lambda)$, $i = 1, l, l+1, \dots$ in point λ^0 are zeros. Then, λ^0 is the bifurcation point of Equation (2). Moreover, Equation (2) for $\lambda = \lambda^0$ will enjoy c-parametric nontrivial solution $x(t,c) = a_1(t,\lambda^0)c + \sum_{i=l}^{\infty} a_i(t,\lambda^0)c^i$ depending on sufficiently small parameter c. For $0 < |\lambda - \lambda^0| < \delta$, where δ is sufficiently small and there are no other small solutions for problem (2).*

3. Sufficient Conditions for Existence of the Bifurcation Points and Solution Asymptotics

By Lemma 1, to construct solutions of Equation (2), in Equation (4) we must use $\lambda = \lambda^0 + \mu$, where μ is a small real parameter and construct the asymptotics of the small solution $x_\alpha \to 0$ as $\mu \to 0$ of Equation (4). For this purpose, we introduce the conditions.

I. λ^0 is an odd root of equation $L_1(\lambda) = 0$ of multiplicity p, $L_l(\lambda^0) \neq 0$.

Then, for $\lambda = \lambda^0 + \mu$, Equation (4) can be transformed as follows $\big(\frac{1}{p!}L_1^{(p)}(\lambda^0)\mu^p + \mathcal{O}(|\mu|^{p+1})\big)x_\alpha + \big(L_l(\lambda^0) + \mathcal{O}(|\mu|)x_\alpha^l + \mathcal{O}(|x_\alpha|^{l+1})\big) = 0$ in the neighborhood of points $x_\alpha = 0$, $\mu = 0$. Application of the Newton diagram [14] makes it possible to construct the load as $x_\alpha = (c_0 + \mathcal{O}(|\mu|))\mu^{\frac{p}{l-1}}$. If p is odd, then c_0 satisfies, for $\mu > 0$, the following equation

$$\frac{1}{p!}L_1^{(p)}(\lambda^0) + L_l(\lambda^0)c_0^{l-1} = 0,$$

and, for $\mu < 0$, it satisfies

$$-\frac{1}{p!}L_1^{(p)}(\lambda^0) + L_l(\lambda^0)c_0^{l-1} = 0.$$

One of these equations for odd l has a simple nontrivial real solution.Then, real c_0 can be always constructed explicitly, and one can construct the main term of asymptotics of the solution of the Cauchy problem (2).

Then, it follows:

Theorem 1. *Let condition* **I** *be satisfied. Then, λ^0 is the bifurcation point of Equation (2). Moreover, for even l and arbitrary p Equation (2) enjoys a unique small real solution $x(t,\lambda)$ in the neighborhood of point λ^0 and*

$$x(t,\lambda) \sim \int_0^t a(t,\lambda^0)\,dt \sqrt[l-1]{-\frac{1}{p!}\frac{L_1^{(p)}(\lambda^0)}{L_l(\lambda^0)}(\lambda - \lambda^0)^p}.$$

For odd l and odd p, Equation (2) has real solution with a similar asymptotic in one half neighborhood of point λ^0.

Example 1. *Let us cinside the following problem:*

$$\begin{cases} \frac{dx(t)}{dt} = (1-\lambda)x(\alpha) + bx^3(t), & t \in \mathbb{R}^1, \\ x(0) = 0. \end{cases} \tag{5}$$

Real value α is given. This example satisfies the conditions of Theorem 1 for $p = 1$, $l = 3$.

For sufficiently small $|x(\alpha)|$, its solution according to decomposition (3) is represented as a function of the load $x(\alpha)$ in the form of a series $x(t) = t(1-\lambda)x(\alpha) + \frac{b}{4}t^4(1-\lambda)^3x(\alpha)^3 + \mathcal{O}(|x(\alpha)|^4)$. Then, branching Equation (4) in this example follows

$$[\alpha(1-\lambda) - 1]x(\alpha) + \frac{b}{4}\alpha^4(1-\lambda)^3x(\alpha)^3 + \mathcal{O}(|x(\alpha)|^4) = 0.$$

The load x_α is defined from Equation (4) with $L_1(\lambda) = \alpha(1-\lambda) - 1$, $L_3(\lambda) = \frac{b}{4}\alpha^4(1-\lambda)^3$. Then, due to Corollary 1 $\lambda^0 = 1 - \frac{1}{\alpha}$ is the unique bifurcation point.

Using the Newton diagram method, the branching Equation (4) has the solution in the form of a power series by powers of $\mu^{1/2}$. Therefore, using Formula (3), two branches of nontrivial real solution of Cauchy problem should be sought in the form of a series

$$x(t) = \sum_{n=1}^{\infty} c_n(t)\mu^{n/2},\ c_n(0) = 0,\ n = 1, 2, \ldots. \tag{6}$$

To determine the coefficients $c_n(t)$, we obtain the following recurrent sequence of initial problems

$$\begin{cases} \frac{dc_1(t)}{dt} = \frac{1}{\alpha}c_1(\alpha), \\ c_1(0) = 0, \end{cases}$$

$$\begin{cases} \frac{dc_2(t)}{dt} = \frac{1}{\alpha}c_2(\alpha), \\ c_2(0) = 0, \end{cases}$$

$$\begin{cases} \frac{dc_3(t)}{dt} = \frac{1}{\alpha}c_3(\alpha) - c_1(\alpha) + bc_1^3(t), \\ c_3(0) = 0, \end{cases}$$

$$\ldots$$

Then, $c_1(t) = \frac{1}{\alpha}c_1(\alpha)t$, $c_2(t) = \frac{1}{\alpha}c_2(\alpha)t$. The constant value $c_1(\alpha)$ can be uniquely defined from a third level solvability condition. Indeed, the third equation follows

$$\begin{cases} \frac{dc_3(t)}{dt} = \frac{1}{\alpha}c_3(\alpha) - c_1(\alpha) + \frac{b}{\alpha^3}c_1^3(t)t^3, \\ c_3(0) = 0. \end{cases}$$

Then, $c_3(t) = \frac{t}{\alpha}c_3(\alpha) - c_1(\alpha)t + \frac{bc_1(\alpha)^3}{\alpha^3}\frac{t^4}{4}$. The solvability condition is as follows $-c_1(\alpha)\alpha + bc_1(\alpha)^3\frac{\alpha}{4} = 0$. Then, $c_1(\alpha) = \pm\frac{2}{\sqrt{b}}$, and Cauchy problem (5) due to (6) have two real solutions in the semi-neighborhood of point $\lambda^0 = 1 - \frac{1}{\alpha}$ with asymptotics

$$x_{1,2}(t) \sim \pm\frac{2t}{\alpha}\sqrt{\frac{\lambda - \lambda^0}{b}}.$$

The sign of the semicircle in which the solution is real coincides with the sign of the coefficient b, i.e., for $(\lambda - \lambda^0)/b > 0$.

Asymptotics can be specified by determining the coefficients of series (6). To determine the arbitrary constants appearing in the solution of the nth linear boundary value problem, the solvability conditions of $n + 2$ equations of the recurrence chain of equations are used. Thus, as in the Nekrasov–Nazarov method [8,14] developed for mechanics problems, in our case, the calculation methodology is two-step and uses the Newton– Puiseux expansion. This method, considering recent results of power geometry [15] and methods of nonlinear analysis [14,16], can also be applied to the study of the Cauchy problem with load vector and vector bifurcation parameters.

In Section 4, we present a general theorem for the existence of bifurcation points of the system (1) in the vector case. Theorem 2 is a justification for the possible linearization in the obtained sufficient condition for the existence of bifurcation points. Namely, Theorem 2 provides a method for checking for bifurcation using only the linearization of the Cauchy problem (1). A similar result for other nonlinear problems was first proven by M.A. Krasnoselsky [18].

4. Necessary and Sufficient Conditions for the Existence of a Nontrivial Solution of the Cauchy Problem for System in the Vector Case

As in the case of a single equation, the desired solution of system (1) will be plotted as a function of the load vector. In this section, $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$, $x_\alpha = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n})^T$.

Let us build the following sequence $x_n = \int_0^t \Phi(x_{n-1}(t), \lambda, x_\alpha)\, dt,\ n = 1, 2, \ldots$ at initial approach

$$x_0(t) = \sum_{j=1}^{n} \int_0^t a_{ij}(s, \lambda)\, ds x_{\alpha_j}\big|_{j=1}^{n}.$$

For a sufficiently small norm $||x_\alpha||$ this sequence converges, and its limit represents the solution of the Cauchy problem (1) expressed in terms of the load vector x_α. The load vector x_α can be found from a system of small implicit functions

$$L(x_\alpha, \lambda) := L_1(\lambda) x_\alpha + \mathcal{O}(||x_\alpha||^l) = 0, \tag{7}$$

where

$$L_1(\lambda) = \left[\int_{a_i}^{b_i} \int_0^t a_{ij}(s, \lambda)\, ds\, d\alpha_i(t) - \delta_{ij} \right]_{i,j=1}^{n}$$

is a square matrix.

For the convenience of calculations, we use the notation

$$L_1(\lambda) := \left[A_{ij}(\lambda) \right]_{i,j=1}^{n}.$$

Bifurcation points of system (7) will be the desired bifurcation points of the Cauchy problem (1).

Let us introduce set $D = \{\lambda |\, \det L_1(\lambda) = 0\} \subset \Omega$. Based on the implicit mapping theorem, the following Lemma is valid.

Lemma 2. *(Necessary bifurcation conditions of the Cauchy problem) For the point λ to be a bifurcation point of the Cauchy problem (1), it is necessary to fulfill the inclusion $\lambda^0 \in D$.*

To obtain sufficient bifurcation conditions in problem (1), we will need the following conditions.

(A) Let $\lambda^0 \in D$, and there exists its sectorial neighborhood $S = S_+ \cup S_-$ and $\lambda^0 \in \partial S_+ \cap \partial S_-$. Let l be fixed vector from the set Ω and let exists continuous mapping $\lambda(\mu) := \lambda^0 + l\mu$, where $\mu = (2\theta - 1)\delta$, where δ is a small enough value, $\delta > 0$, $\theta \in [0, 1]$.

$$\det[L_1(\lambda)] = \alpha(\mu),$$

where $\alpha(\mu)$ is a continuous function, $\alpha(\mu) < 0$ for $\mu \in [-\delta, 0)$, $\alpha(\mu) > 0$ for $\mu \in (0, +\delta]$, $\alpha(0) = 0$.

It is not noted that μ is a small parameter here. The next Lemma formulates the sufficient conditions providing condition **(A)** fulfillment.

Lemma 3. *Let λ^0 and l be fixed vectors from normed space E^m. μ is small real parameter,*

$$A_{ik}(\lambda^0 + \mu l) = b_{ik}\mu^{p_k} + o(|\mu|^{p_k})$$

for $i, k = 1, \ldots, n$, and $p_k \in \mathbb{N}$, $\det[b_{ik}]_{i,k=1}^{n} \neq 0$, $p_1 + \cdots + p_n$ are odd numbers. Then, condition **(A)** *is fulfilled, and*

$$\alpha(\mu) \sim \mu^{p_1 + \cdots + p_n} \det[b_{ik}]_{i,k=1}^{n}$$

for $\mu \to 0$.

Proof. Since $p_k > 0$, $k = 1, \ldots, n$, then $\lambda^0 \in D$ and, under Lemma 3, the following estimate is valid

$$||L_1(\lambda^0 + \mu l) - [b_{ik}\mu^{p_k}]_{i,k=1}^{n}|| = o(|\mu|^{p_1 + \cdots + p_n}).$$

Matrix $[b_{ik}\mu^{p_k}]_{i,k=1}^n$ is product of the non-degenerate matrix $[b_{ik}]_{i,k=1}^n$ and diagonal matrix $[\delta_{ik}\mu^{p_k}]_{i,k=1}^n$. Since the determinant of two matrices' product is the product of the determinants of these matrices, then, due to the above outlined estimate, the following estimate is true

$$\det[A_{ij}(\mu)]_{i,j=1}^n \sim \mu^{p_1+\cdots+p_n} \det[b_{ik}]_{i,k=1}^n.$$

The lemma is proven. □

Remark 1. *Lemma 3 strengthens the results presented in the paper [13] for the case of equations with loads.*

Using Lemma 3 and the methods of paper [13] we obtain the following result.

Theorem 2. *Let condition* **(A)** *be satisfied. Then, λ^0 is the bifurcation point of problem* (1).

Proof. Let us define arbitrary small $\varepsilon > 0, \delta > 0$ and introduce a continuous vector field

$$H(x_\alpha, \theta) := L(x_\alpha, \lambda^0 + l(2\theta - 1)\delta) : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n,$$

defined for $x_\alpha, \theta \in M$, where

$$M = \{x_\alpha, \theta \mid ||x_\alpha|| = \varepsilon, 0 \le \theta \le 1\}.$$

Let us assume that theorem is not valid, i.e., $H(x_\alpha, \theta) \neq 0$ for arbitrary x_α, θ from set M. Then, fields $H(x_\alpha, 0)$, $H(x_\alpha, 1)$ are homotopic on sphere $||x_\alpha|| = \varepsilon$. Moreover, there are homotopic for sufficiently small ε with respect to their linearization and non-homogeneous case. However, that is impossible due to condition **(A)** the inequality $\alpha(\mu) < 0$ for $\mu \in [-\delta, 0)$ and $\alpha(\mu) > 0$ for $\mu \in (0, +\delta]$. Then, in set M, there exists a pair (x_α^*, θ^*), such as $H(x_\alpha^*, \theta^*) = 0$, and λ^0 is bifurcation point. □

5. Conclusions

Many works are devoted to the study of bifurcation points of nonlinear equations. From the computational point of view, the bifurcation problem is ill-posed in the Tikhonov–Lavrent'ev sense. A small perturbation of the original equation can lead to significant changes in the structure of the solution. In particular, the perturbed problem may have no real solutions at all in a sufficiently small neighborhood of the bifurcation point. Note that the presence of a bifurcation parameter in the equation significantly complicates the problem of constructing stable computations, since we must talk about "parameter-equal regularization". To solve this problem in the Cauchy problem with loadings and bifurcation parameters, one can use the methods proposed in works [13] and developed in monographs [5,16,17].

The above monographs provide a systematic account of a number of sections and applications of the modern theory of branching nonlinear equations. The presentation is based on the reduction of the original nonlinear problem to an equivalent finite-dimensional problem. In this paper, this approach, going back to Lyapunov and Schmidt, using a combination of some recent results of nonlinear analysis, is effectively applied to constructing solutions of classes of nonlinear differential equations with stresses. In the considered Cauchy problem, the role of such a finite-dimensional problem is played by the system with respect to loads.

The works [5,16,17] outline a number of sections and applications of the modern theory of branching solutions of nonlinear equations with parameters. In this paper, based on the methods outlined in [17], the existence theorems of bifurcation points for solutions of Cauchy problems for differential equations with loadings given by the Stieltjes functions are proven. A method for constructing the solutions of such a problem in the neighborhood of bifurcation points is presented. As footnote, let us outline that proposed methods can be useful for the various classes of nonlinear PDE equations; see, e.g., [19–21] and the references therein.

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