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Hilfer Fractional Neutral Stochastic Volterra Integro-Differential Inclusions via Almost Sectorial Operators

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Abstract: In our paper, we mainly concentrate on the existence of Hilfer fractional neutral stochastic Volterra integro-differential inclusions with almost sectorial operators. The facts related to fractional calculus, stochastic analysis theory, and the fixed point theorem for multivalued maps are used to prove the result. In addition, an illustration of the principle is provided.

Keywords: Hilfer fractional (HF) system; neutral system; stochastic system; integro-differential system; almost sectorial operators; multivalued maps

MSC: 26A33; 34A08; 47D09; 60H30

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1. Introduction

In 1695, fractional calculus was presented as a major field of mathematics. It happened approximately simultaneously with the development of classical calculus. Researchers have discovered that fractional calculus may accurately portray a range of nonlocal phenomena in the fields of natural science and technology, and the notion of fractional calculus has recently been successfully applied to a variety of sectors. The most common fields of fractional calculus are rheology, dynamical cycles in identity and heterogeneous structures, diffusive transport equivalent to dispersion, liquid stream, optics, viscoelasticity, and others. As diagnostic arrangements can be tough to come by in many fields, the successful use of fractional systems has prompted many investigators to reconsider their mathematical estimation methods. In [1–13], readers can find some interesting conclusions related to fractional dynamical systems and research articles related to fractional differential systems theory. In particular, partial neutral structures with or without delays serve as a summary affiliation of a large number of partial neutral structures that emerge in problems involving heat flow in ingredients, viscoelasticity, and a range of natural phenomena. Furthermore, the most successful neutral structures have received much interest in the present population, with readers able to review books [8,10–12,14,15] and research papers [16–18].

Throughout the past decade, fractional calculus has been one of the most important frameworks for analysing brief operations. Such models pique the interests of architects, scientists, and pure mathematicians alike. The most essential of these models are fractional equations with fractional-order derivatives. Furthermore, in [15,19–21] there is focus on qualitative behaviours such as fractional dynamical systems, stability, existence, and controllability. In practical use, since stochastic fluctuation is unavoidable, we must investigate deterministic problems for stochastic differential equations [22,23]. Due to their applicability in several disciplines of science and engineering, stochastic differential equations have piqued people's curiosity. Furthermore, it should be noted that in nature, even in artificial systems, noise or stochastic discomfort cannot be prohibited. Stochastic differential systems have sparked interest as a result of their wide application in presenting a variety of sophisticated dynamical systems in scientific, physical, and pharmaceutical domains; one can check [24–26]. Differential inclusion tools make it easier to study numerical solutions that have kinematics that are not even solely governed by the system's state.

Other fractional-order derivatives, such as the R-L derivatives and Caputo fractional derivatives, were started by Hilfer [27–33]. Furthermore, theoretical simulations of thermoelastic in crystal compounds, chemical processing, rheological constitutive modelling, engineering, and other domains have uncovered the usefulness and applicability of the Hilfer fractional derivative. Gu and Trujillo [34] recently employed a noncompact measure approach and a fixed point technique to show that there is an integral solution to the Hilfer fractional derivative evolution problem. To designate the derivative's order, they developed the latest variable, $\mu \in [0,1]$, as well as a fractional variable, λ, so that $\mu = 0$ provides the R-L derivative and $\lambda = 1$ yields the Caputo derivative. Hilfer fractional calculus [7,25,34–36] has been the subject of several articles. In [37–40], researchers revealed the existence of a mild solution for *HF* differential systems via almost sectorial operators applying a fixed point approach. In [41–43], the authors explored the solvability and controllability of differential systems using a fixed point technique.

A growing number of researchers are advancing fractional existence for fractional calculus using almost sectorial operators. For the system under examination, the investigators established a new technique for identifying mild solutions. Furthermore, the investigators developed a theory to derive various properties of related semigroups created by almost sectorial operators using fractional calculus, semigroups, multivalued analysis, a measure of noncompactness, the Laplace transform, Wright-type function, and fixed point theorem. We refer to [44–49]. Furthermore, in [4] researchers studied fractional differential inclusion papers using Bohnenblust–Karlin's fixed point theorem for multivalued maps. As a result of these findings, we extend the literature's earlier findings to a class of HF stochastic Volterra–Fredholm integro-differential inclusions in which the closed operator is almost sectorial.

In this paper, we will look at the following topic: *HF* neutral stochastic Volterra integro-differential inclusions containing almost sectorial operators

$$ {}^{H}D_{0^+}^{\eta,\zeta}\left[\mathfrak{u}(\xi) - \mathcal{G}(\xi,\mathfrak{u}(\xi))\right] \in \mathtt{A}\mathfrak{u}(\xi) + \mathcal{H}\left(\xi,\mathfrak{u}(\xi),\int_0^{\xi} f(\xi,s,\mathfrak{u}(s))ds\right)\frac{dW(\xi)}{d\xi}, \quad \xi \in \mathcal{I}' = (0,d], \tag{1} $$

$$ I_{0^+}^{(1-\eta)(1-\zeta)}\mathfrak{u}(0) = \mathfrak{u}_0, \tag{2} $$

where $\mathtt{A}$ is an almost sectorial operator of the analytic derivative $\{T(\xi), \xi \geq 0\}$ on Y. ${}^{H}D_{0^+}^{\eta,\zeta}$ denotes the *HFD* of order $\eta \in (0,1)$ and type $\zeta \in [0,1]$, with the condition $\mathfrak{u}(\cdot)$ taking the value in a Hilbert space Y with norm $\|\cdot\|$. Let $\mathcal{I} = [0,d]$ be the interval, $\mathcal{H} : \mathcal{I} \times Y \times Y \to 2^{Y}\backslash\{\emptyset\}$ be a nonempty, bounded, closed convex multivalued map, $\mathcal{G} : \mathcal{I} \times Y \to Y$, $f : \mathcal{I} \times \mathcal{I} \times Y \to Y$ be the appropriate functions and the function $W(\xi)$ be a one-dimensional standard Brownian motion in Y defined on the filtered probability space $(\Omega, \mathscr{E}, \mathscr{P})$. For brevity, we take

$$ (F\mathfrak{u})(\xi) = \int_0^{\xi} f(\xi,s,\mathfrak{u}(s))ds. $$

The structure of the article is broken down as follows. The principles of fractional calculus, semigroup theory, sectorial operators, stochastic analysis theory, and the fixed point theorem for multivalued maps are covered in Section 2. The required hypotheses and the existence of the mild solution are established in Section 3. We provide an illustration in Section 4 to demonstrate our main ideas. Lastly, some recommendations are made.

2. Preliminaries

This section introduces the required principles and facts that will be needed to obtain the new results throughout the paper.

Two real separable Hilbert spaces are denoted by $(Y, \|\cdot\|)$ and $(U, \|\cdot\|)$. Assume $(\Omega, \mathscr{E}, \mathscr{P})$ is a complete probability space connected with a proper set of right continuous increasing sub-σ-algebras $\{\mathscr{E}_\xi : \xi \in \mathcal{I}\}$ satisfying $\mathscr{E}_\xi \subset \mathscr{E}$. Let $W = (W_\xi)_{\xi \geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathscr{E}, \mathscr{P})$ with the correlation operator Q such that $Tr(Q) < \infty$. We suppose that there exists a proper orthonormal system $e_m,\ m \geq 1$ in U, a limited sequence of nonnegative real numbers δ_m such that $Qe_m = \delta_m e_m,\ m = 1, 2, \cdots$ and $\{\widehat{\beta}_m\}$ of isolated Brownian motions such that

$$(W(\xi), e)_U = \sum_{m=1}^{\infty} \sqrt{\delta_m}(e_m, e)\widehat{\beta}_m(\xi),\ e \in U,\ \xi \geq 0.$$

Assume that $L_2^0 = L_2\big(Q^{\frac{1}{2}}U, Y\big)$ stands for the space of all Q-Hilbert–Schmidt operators $\phi : Q^{\frac{1}{2}}U \to Y$ with the inner product $\|\phi\|_Q^2 = \langle \phi, \phi \rangle = Tr(\phi Q \phi)$ being a Hilbert space. Let us consider $0 \in \rho(\mathtt{A})$, the resolvent set of $\mathtt{A}$, where $\mathcal{S}(\cdot)$ is uniformly bounded, i.e., $\|\mathcal{S}(\xi)\| \leq M,\ M \geq 1$ and $\xi \geq 0$. The fractional power operator $\mathtt{A}^\lambda$ on its domain $D(\mathtt{A}^\lambda)$ may then be determined for $\lambda \in (0, 1]$. In addition, $D(\mathtt{A}^\lambda)$ is dense in Y.

The following are the fundamental properties of $\mathtt{A}^\lambda$.

Theorem 1 ([11])**.**
1. *Suppose* $0 < \lambda \leq 1$, *corresponding* $Y_\lambda = D(\mathtt{A}^\lambda)$ *is a Banach space with* $\|\mathfrak{u}\|_\lambda = \|\mathtt{A}^\lambda \mathfrak{u}\|,\ \mathfrak{u} \in Y_\lambda$.
2. *Suppose* $0 < \bar{\upsilon} < \lambda \leq 1$, *corresponding* $D(\mathtt{A}^\lambda) \to D(\mathtt{A}^{\bar{\upsilon}})$ *and the embedding is compact every time that* $\mathtt{A}$ *is compact.*
3. *For all* $\lambda \in (0, 1]$, *there exists* $C_\lambda > 0$ *such that*

$$\|\mathtt{A}^\lambda \mathcal{S}(\xi)\| \leq \frac{C_\lambda}{\xi^\lambda},\ 0 < \xi \leq d.$$

The set of all strongly measurable, square-integrable, Y-valued random variables, indicated by $L_2(\Omega, Y)$, is a Banach space connected with $\|\mathfrak{u}(\cdot)\|_{L_2(\Omega,Y)} = \big(E\|\mathfrak{u}(., W)\|^2\big)^{\frac{1}{2}}$ where E is classified as $E(\mathfrak{u}) = \int_\Omega \mathfrak{u}(W) d\mathscr{P}$. An essential subspace of $L_2(\Omega, Y)$ is provided by

$$L_2^0(\Omega, Y) = \{\mathfrak{u} \in L_2(\Omega, Y),\ \mathfrak{u} \text{ is } \mathcal{E}_0 - \text{measurable}\}.$$

For $d > 0$, let $\mathcal{I} = [0, d]$, and $\mathcal{I}' = (0, d]$. Denote $C(\mathcal{I}, Y) = \mathtt{C}$ as the Banach space of all continuous functions from $\mathcal{I} \to Y$ that satisfies the condition $\sup_{\xi \in \mathcal{I}} E\|\mathfrak{u}(\xi)\|^2 < \infty$. Let $\Delta = \big\{\mathfrak{u} \in C(\mathcal{I}', Y) : \lim_{\xi \to 0} \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\mathfrak{u}(\xi)$ exists and finite $\big\}$ be a Banach space with $\|\cdot\|_\Delta$ and $\|\mathfrak{u}\|_\Delta = (\sup_{\xi \in \mathcal{I}'} E\|\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\mathfrak{u}(\xi)\|^2)^{\frac{1}{2}}$. Set $B_r(\mathcal{I}) = \{x \in \mathtt{C}$ such that $\|x\| \leq r\}$ and $B_r^\Delta(\mathcal{I}) = \{\mathfrak{u} \in \Delta$ such that $\|\mathfrak{u}\|_\Omega \leq r\}$.

Definition 1 ([46])**.** *For* $0 < \vartheta < 1,\ 0 < \omega < \frac{\pi}{2}$, *we define* $\Theta_\omega^{-\vartheta}$ *as the set of closed linear operators, the sector* $S_\omega = \{v \in \mathbb{C} \backslash \{0\}$ *with* $|arg\ v| \leq \omega\}$ *and* $\mathtt{A} : D(\mathtt{A}) \subset Y \to Y$ *that satisfy*

(a) $\sigma(\mathtt{A}) \subseteq S_\omega$;

(b) $\|(vI - \mathtt{A})^{-1}\| \leq \mathbb{M}_\delta |v|^{-\vartheta}$, *offered for all* $\omega < \delta < \pi$ *and there exists* $\mathbb{M}_\delta$ *as a constant,*

then $\mathtt{A} \in \Theta_\omega^{-\vartheta}$ *is known as an almost sectorial operator on* Y.

Define the power of $\mathtt{A}$ as

$$\mathtt{A}^\theta = \frac{1}{2\pi i} \int_{\Gamma_\mu} v^\theta R(v; \mathtt{A}) dv,\ \theta > 1 - \vartheta,$$

where $\Gamma_\mu = \{\mathbb{R}^+ e^{i\mu}\} \bigcup \{\mathbb{R}^+ e^{-i\mu}\}$ is an appropriate path oriented counter-clockwise and $\omega < \mu < \delta$. Then, the linear power space $Y_\theta := D(\mathtt{A}^\theta)$ can be defined and Y_θ is a Banach space with the graph norm $\|\mathfrak{u}\|_\theta = |\mathtt{A}^\theta \mathfrak{u}|,\ \mathfrak{u} \in D(\mathtt{A}^\theta)$.

Next, let us introduce the semigroup associated with $\mathtt{A}$. We denote the semigroup associated with $\mathtt{A}$ by $\{T(\xi)\}_{\xi \geq 0}$. For $\xi \in S^0_{\frac{\pi}{2}-\omega}$

$$T(\xi) = e^{-\xi v}(\mathtt{A}) = \frac{1}{2\pi i}\int_{\Gamma_\mu} e^{-\xi v}R(v;\mathtt{A})dv,$$

where the integral contour $\Gamma_\mu = \{\mathbb{R}^+ e^{i\mu}\}\bigcup\{\mathbb{R}^+ e^{-i\mu}\}$ is oriented counter-clockwise and $\omega < \mu < \delta < \frac{\pi}{2} - |arg\ \xi|$, it forms an analytic semigroup of growth order $1 - \vartheta$.

Proposition 1 ([46]). *Let $\mathtt{A} \in \Theta^{-\vartheta}_{\omega}$ for $0 < \vartheta < 1$ and $0 < \omega < \frac{\pi}{2}$. Then, the following are satisfied:*

(a) $T(\xi + \nu) = T(\xi)T(\nu)$, *for all* $\nu, \xi \in S_{\frac{\pi}{2}-\omega}$;

(b) $\|T(\xi)\|_{L(Y)} \leq \kappa_0 \xi^{\vartheta-1}$, $\xi > 0$; *where* $\kappa_0 > 0$ *is the constant;*

(c) *the range* $R(T(\xi))$ *of* $T(\xi)$, $\xi \in S_{\frac{\pi}{2}-\omega}$ *is contained in* $D(\mathtt{A}^\infty)$. *Particularly,* $R(T(\xi)) \subset D(\mathtt{A}^\theta)$ *for all* $\theta \in \mathbb{C}$ *with* $Re(\theta) > 0$,

$$\mathtt{A}^\theta T(\xi)\mathtt{u} = \frac{1}{2\pi i}\int_{\Gamma_\zeta} v^\theta e^{-\xi v}R(v;\mathtt{A})\mathtt{u}dv, \text{ for all } \mathtt{u} \in Y,$$

and hence there exists a constant $C' = C'(\gamma, \theta) > 0$ *such that*

$$\|\mathtt{A}^\theta T(\xi)\|_{L(Y)} \leq C'\xi^{-\gamma - Re(\theta) - 1}, \text{ for all } \xi > 0;$$

(d) *if* $\Sigma_T = \{\mathtt{u} \in Y : \lim_{\xi \to 0^+} T(\xi)\mathtt{u} = \mathtt{u}\}$, *then* $D(\mathtt{A}^\theta) \subset \Sigma_T$ *if* $\theta > 1 - \vartheta$;

(e) $(vI - \mathtt{A})^{-1} = \int_0^\infty e^{-v\nu}T(\nu)d\nu$, $v \in \mathbb{C}$ *and* $Re(v) > 0$.

Definition 2 ([14]). *The left-sided R-L fractional integral of order η with the lower limit d for the function $\mathcal{H} : [d, \infty) \to \mathbb{R}$ is presented by*

$$I^\eta_{d^+}\mathcal{H}(\xi) = \frac{1}{\Gamma(\eta)}\int_d^\xi \frac{\mathcal{H}(\nu)}{(\xi - \nu)^{1-\eta}}d\nu,\ \xi > 0,\ \eta > 0,$$

provided the right side is point-wise determined on $[d, +\infty)$, $\Gamma(\cdot)$ is the gamma function.

Definition 3 ([14]). *The left-sided R-L fractional derivative of order $\eta > 0$, $m - 1 \leq \eta < m$, $m \in \mathbb{N}$, for a function $\mathcal{H} : [d, +\infty) \to \mathbb{R}$, is presented by*

$${}^{RL}D^\eta_{d+}\mathcal{H}(\xi) = \frac{1}{\Gamma(m-\eta)}\frac{d^m}{d\xi^m}\int_d^\xi \frac{\mathcal{H}(\nu)}{(\xi - \nu)^{\eta+1-m}}d\nu,\ \xi > d,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 4 ([14]). *The left-sided Caputo derivative of type of order $\eta > 0$, $m - 1 \leq \eta < m$, $m \in \mathbb{N}$, for a function $\mathcal{H} : [d, +\infty) \to \mathbb{R}$, is defined as*

$${}^{C}D^\eta_{d^+}\mathcal{H}(\xi) = \frac{1}{\Gamma(m-\eta)}\int_d^\xi \frac{\mathcal{H}^m(\nu)}{(\xi - \nu)^{\eta+1-m}}d\nu = I^{m-\eta}_{d+}\mathcal{H}^m(\xi),\ \xi > d,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 5 ([29]). *The left-sided HFD of order $0 < \eta < 1$ and type $\zeta \in [0, 1]$, of function $\mathcal{H} : [d, +\infty) \to \mathbb{R}$, is classified as*

$${}^{H}D^{\eta,\zeta}_{d^+}\mathcal{H}(\xi) = I^{(1-\eta)\zeta}_{d^+}\frac{d}{dt}I^{(1-\eta)(1-\zeta)}_{d^+}\mathcal{H}(\xi).$$

Remark 1.

1. *If $\zeta = 0$, $0 < \eta < 1$, and $d = 0$, then the HFD denotes to the classical R-L fractional derivative:*

$$^{H}D_{0^+}^{\eta,0}\mathcal{H}(\xi) = \frac{d}{d\xi} I_{0^+}^{1-\eta}\mathcal{H}(\xi) =^{L} D_{0^+}^{\eta}\mathcal{H}(\xi).$$

2. *If $\zeta = 1$, $0 < \eta < 1$ and $d = 0$, then the HFD equals the classical Caputo fractional derivative:*

$$^{H}D_{0^+}^{\eta,1}\mathcal{H}(\xi) = I_{0^+}^{1-\eta}\frac{d}{d\xi}\mathcal{H}(\xi) =^{C} D_{0^+}^{\eta}\mathcal{H}(\xi).$$

Definition 6 ([49])**.** *Define the Wright function $\varphi_\eta(\beta)$ by*

$$\varphi_\eta(\beta) = \sum_{m\in\mathbb{N}} \frac{(-\beta)^{m-1}}{\Gamma(1-\eta m)(m-1)!}, \qquad \beta \in \mathbb{C}, \tag{3}$$

with the following property

$$\int_0^\infty \theta^\iota \varphi_\eta(\theta)d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\eta\iota)}, \qquad \text{for } \iota \geq 0.$$

Definition 7 ([33])**.** *A multivalued map $\mathcal{H}$ is called u.s.c. on Y if for all $\mathtt{u}_0 \in Y$ the set $\mathcal{H}(\mathtt{u}_0)$ is a nonempty, closed subset of Y, and if for each open set $\mathcal{U}$ of Y containing $\mathcal{H}(\mathtt{u}_0)$, there exists an open neighbourhood $\mathcal{V}$ of $\mathtt{u}_0$ such that $\mathcal{H}(\mathcal{V}) \subseteq \mathcal{U}$.*

Definition 8 ([33])**.** *$\mathcal{H}$ is completely continuous if $\mathcal{H}(C)$ is relatively compact for each bounded subset C of Y. If a multivalued map $\mathcal{H}$ is completely continuous with nonempty compact values, then $\mathcal{H}$ is upper semicontinuous if $\mathcal{H}$ has a closed graph, i.e., $\mathtt{u}_m \to \mathtt{u}_0$, $z_m \to z_0$, $z_m \in \mathcal{H}(\mathtt{u}_m)$ implying $z_0 \in \mathcal{H}(\mathtt{u}_0)$.*

Lemma 1 ([34])**.** *Systems (1) and (2) are equivalent to an integral inclusion given by*

$$\mathtt{u}(\xi) \in \frac{\mathtt{u}_0 - \mathcal{G}(0,\mathtt{u}(0))}{\Gamma(\zeta(1-\eta)+\eta)}\xi^{-(1-\eta)(1-\zeta)} + \mathcal{G}(\xi,\mathtt{u}(\xi)) + \frac{1}{\Gamma(\eta)}\int_0^\xi (\xi-\nu)^{\eta-1}\big[\mathtt{A}\mathcal{G}(\nu,\mathtt{u}(\nu))d\nu + \mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)dW(\nu)\big].$$

Lemma 2 ([34])**.** *Let $\mathtt{u}(\xi)$ be a solution of the integral inclusions provide in Lemma 1, then $\mathtt{u}(\xi)$ satisfies*

$$\begin{aligned}\mathtt{u}(\xi) =& \mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0 - \mathcal{G}(0,\mathtt{u}(0))\big] + \mathcal{G}(\xi,\mathtt{u}(\xi)) + \int_0^\xi \mathcal{K}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}(\nu,\mathtt{u}(\nu))d\nu \\ &+ \int_0^\xi \mathcal{K}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)dW(\nu), \quad \xi \in \mathcal{I}',\end{aligned}$$

where

$$\mathcal{S}_{\eta,\zeta}(\xi) = I_0^{\zeta(1-\eta)}\mathcal{K}_\eta(\xi),\ \mathcal{K}_\eta(\xi) = \xi^{\eta-1}\mathcal{Q}_\eta(\xi),\ \text{and}\ \mathcal{Q}_\eta(\xi) = \int_0^\infty \eta\theta\varphi(\theta)T(\xi^\eta\theta)d\theta.$$

Definition 9. *An $\mathscr{E}_\xi$-adapted stochastic process $\mathtt{u}(\xi) \in C(\mathcal{I}', Y)$ is said to be a mild solution of the Cauchy problem, (1) and (2), given $I_0^{(1-\eta)(1-\zeta)}\mathtt{u}(0) = \mathtt{u}_0$; $\mathtt{u}_0 \in L_2^0(\Omega, Y)$ and there exists $h \in L^2(\Omega, Y)$ such that $h(\xi) \in \mathcal{H}(\xi, \mathtt{u}(\xi), (F\mathtt{u})(\xi))$ on $\xi \in \mathcal{I}'$ and that satisfies*

$$\begin{aligned}\mathtt{u}(\xi) =& \mathcal{S}_{\eta,\zeta}(\xi)\left[\mathtt{u}_0 - \mathcal{G}(0, \mathtt{u}(0))\right] + \mathcal{G}(\xi, \mathtt{u}(\xi)) + \int_0^\xi \mathcal{K}_\eta(\xi - \nu)\mathtt{A}\mathcal{G}(\nu, \mathtt{u}(\nu))d\nu \\ &+ \int_0^\xi \mathcal{K}_\eta(\xi - \nu)\mathcal{H}(\nu, \mathtt{u}(\nu), (F\mathtt{u})(\nu))dW(\nu), \quad \xi \in \mathcal{I}',\end{aligned}$$

where $(F\mathtt{u})(\nu) = \int_0^\nu f(\nu, s, \mathtt{u}(s))ds$.

Lemma 3 ([49])**.** *If $\{T(\xi)\}_{\xi>0}$ is a compact operator, then $\{\mathcal{S}_{\eta,\zeta}(\xi)\}_{\xi>0}$ and $\{\mathcal{Q}_\eta(\xi)\}_{\xi>0}$ are also compact operators.*

Lemma 4 ([49])**.** *For each fixed $\xi > 0$, $\mathcal{Q}_\eta(\xi)$, $\mathcal{K}_\eta(\xi)$ and $\mathcal{S}_{\eta,\zeta}(\xi)$ are linear operators, and for any $\mathtt{u} \in Y$,*

$$\begin{aligned}\|\mathcal{Q}_\eta(\xi)\mathtt{u}\| \leq& \kappa_p \xi^{\eta(\vartheta-1)}\|\mathtt{u}\|, \ \|\mathcal{K}_\eta(\xi)\mathtt{u}\| \leq \kappa_p \xi^{\eta\vartheta-1}\|\mathtt{u}\|, \text{ and} \\ \|\mathcal{S}_{\eta,\zeta}(\xi)\mathtt{u}\| \leq& \kappa_s \xi^{-1+\zeta-\eta\zeta+\eta\vartheta}\|\mathtt{u}\|,\end{aligned}$$

where

$$\kappa_p = \frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\eta\vartheta)}, \quad \kappa_s = \frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta) + \eta\vartheta)}.$$

Lemma 5 ([49])**.** *Assume that $\{T(\xi)\}_{\xi>0}$ is equicontinuous. Then, $\{\mathcal{Q}_\eta(\xi)\}_{\xi>0}$, $\{\mathcal{K}_\eta(\xi)\}_{\xi>0}$ and $\{\mathcal{S}_{\eta,\zeta}\}_{\xi>0}$ are strongly continuous, that is, for any $\mathtt{u} \in Y$ and $\xi'' > \xi' > 0$,*

$$\begin{aligned}&|\mathcal{Q}_\eta(\xi')\mathtt{u} - \mathcal{Q}_\eta(\xi'')\mathtt{u}| \to 0, \ |\mathcal{K}_\eta(\xi')\mathtt{u} - \mathcal{K}_\eta(\xi'')\mathtt{u}| \to 0, \\ &|\mathcal{S}_{\eta,\zeta}(\xi')\mathtt{u} - \mathcal{S}_{\eta,\zeta}(\xi'')\mathtt{u}| \to 0, \text{ as } \xi'' \to \xi'.\end{aligned}$$

Theorem 2 ([14])**.** *$\mathcal{S}_\eta(\xi)$ and $\mathcal{Q}_\eta(\xi)$ are continuous in the uniform operator topology, for $\xi > 0$, for all $d > 0$, the continuity is uniform on $[d, \infty)$.*

Proposition 2 ([50])**.** *Let $\eta \in (0,1)$, $\mu \in (0,1]$ and for all $\mathtt{u} \in D(\mathtt{A})$, then there exists a $\kappa_\mu > 0 \ni$*

$$\|\mathtt{A}^\mu \mathcal{Q}_\eta(\xi)\mathtt{u}\| \leq \frac{\eta\kappa_\mu\Gamma(2-\mu)}{\xi^{\eta\mu}\Gamma(1+\eta(1-\mu))}\|\mathtt{u}\|, \ 0 < \xi < d.$$

Lemma 6 ([51])**.** *Let $\mathcal{I}$ be a compact real interval, and $\mathcal{P}_{bd,cv,cl}(Y)$ be the family of all nonempty, bounded, convex and closed subsets of Y. Let $\mathcal{H}$ be the L^1-Caratheodory multivalued map, measurable to ξ for all $\mathtt{u} \in Y$, u.s.c. to $\mathtt{u}$ for all $\xi \in C(\mathcal{I}, Y)$, the set*

$$S_{\mathcal{H},\mathtt{u}} = \{h \in L^1(\mathcal{I}, Y) : h(\xi) \in \mathcal{H}(\xi, \mathtt{u}(\xi), (F\mathtt{u})(\xi)), \quad \xi \in \mathcal{I}\} \tag{4}$$

is nonempty. Let Υ be the linear continuous function from $L^1(\mathcal{I}, Y)$ to $\mathtt{C}$, then

$$\Upsilon \circ S_{\mathcal{H}} : \mathtt{C} \to BCC(\mathtt{C}), \quad \mathtt{u} \to (\Upsilon \circ S_{\mathcal{H}})(\mathtt{u}) = \Upsilon(S_{\mathcal{H},\mathtt{u}}) \tag{5}$$

is a closed graph operator in $\mathtt{C} \times \mathtt{C}$.

Lemma 7 ([4])**.** *[Bohnenblust–Karlin's fixed point theorem] Suppose that Y is a closed, bounded and convex subset Y of $\mathtt{u}$. Assume $\mathcal{D} : Y \to 2^Y \backslash \{\emptyset\}$ is upper semicontinuous with closed, convex values such that $\mathcal{D}(Y) \subset Y$ and $\mathcal{D}(Y)$ are compact, then $\mathcal{D}$ has a fixed point.*

3. Existence of Mild Solution

We require the following hypotheses:

(H_1) $\mathtt{A}$ is the almost sectorial operator, which generates an analytic semigroup $T(\xi),\ \xi \geq 0$ in Y such that $\|T(\xi)\| \leq M$, for all $M > 0$.

(H_2) The multivalued map $\mathcal{H} : \mathcal{I} \times Y \times Y \to BCC(Y)$ is measurable to ξ for any fixed $\mathfrak{u} \in Y$, u.s.c. to $\mathfrak{u}$ for all $\xi \in \mathcal{I}$ and for all $\mathfrak{u} \in \mathtt{C}$, the set

$$S_{\mathcal{H},\mathfrak{u}} = \{h \in L^1(\mathcal{I}, Y) : h(\xi) \in \mathcal{H}(\xi, \mathfrak{u}(\xi), (F\mathfrak{u})(\xi)), \quad \xi \in \mathcal{I}\}$$

is nonempty.

(H_3) For $\xi \in \mathcal{I}$, $\mathcal{H}(\xi, \cdot, \cdot) : Y \times Y \to Y$, $f(\xi, s, \cdot) : Y \to Y$ are continuous functions and for all $\mathfrak{u} \in \mathtt{C}$, $\mathcal{H}(\cdot, \mathfrak{u}, (F\mathfrak{u})) : \mathcal{I} \to \mathcal{I}$ and $f(\cdot, \cdot, \mathfrak{u}) : \mathcal{I} \times \mathcal{I} \to Y$ are strongly measurable.

(H_4) For $r > 0$, $\mathfrak{u} \in \mathtt{C}$ along with $\|\mathfrak{u}\|_{\mathtt{C}} \leq r$ and $L_{\mathcal{H},r}(\xi) \in L^1(\mathcal{I}', \mathbb{R}^+)$ such that

$$\lim_{\xi \to 0^+} \xi^{1-\zeta+\eta\zeta-\eta\vartheta} I_{0^+}^{\eta\vartheta} L_{\mathcal{H},r}(\xi) = 0,$$
$$\sup\{E\|h\|^2 : h(\xi) \in \mathcal{H}(\xi, \mathfrak{u}(\xi), (F\mathfrak{u})(\xi))\} \leq L_{\mathcal{H},r}(\xi),$$

for a.e. $\xi \in \mathcal{I}$.

(H_5) The function $\nu \to (\xi - \nu)^{2(\eta\vartheta-1)} L_{\mathcal{H},r}(\nu) \in L^1(\mathcal{I}, \mathbb{R}^+)$ and there exists a constant $\gamma > 0$ such that

$$\lim_{r \to \infty} \inf \frac{\int_0^\xi (\xi - \nu)^{2(\eta\vartheta-1)} L_{\mathcal{H},r}(\nu) d\nu}{r} = \gamma < \infty,$$

for almost everywhere $\xi \in \mathcal{I}$.

(H_6) The function $\mathcal{G} : \mathcal{I} \times Y \to Y$ is a continuous function and there exists $\mu \in (0,1)$ and $M_g,\ M_g' > 0$ such that $\mathtt{A}^\mu \mathcal{G}$ satisfies the following condition:

$$E\|\mathtt{A}^\mu \mathcal{G}(\xi, \mathfrak{u})\|^2 \leq M_g^2 (1 + \xi^{2(1-\zeta+\eta\zeta+\eta\vartheta)} \|\mathfrak{u}\|^2) \text{ and } \|\mathtt{A}^{-\mu}\| \leq M_0,\ (\xi, \mathfrak{u}) \in \mathcal{I} \times Y,$$
$$E\|\mathtt{A}^\mu [\mathcal{G}(\xi, \mathfrak{u}_1) - \mathcal{G}(\xi, \mathfrak{u}_2)]\|^2 \leq M_g'^2 (1 + \xi^{2(1-\zeta+\eta\zeta+\eta\vartheta)} \|\mathfrak{u}_1 - \mathfrak{u}_2\|^2),\ \mathfrak{u}_1, \mathfrak{u}_2 \in Y,\ \xi \in \mathcal{I}.$$

Theorem 3. *Assume that (H_1)–(H_6) hold. Then, the HF systems (1) and (2) have a mild solution on $\mathcal{I}$ provided*

$$4Tr(Q)\kappa_p^2 d^{2(1-\zeta+\eta\zeta-\eta\vartheta)} \gamma \geq 1.$$

Proof. We define the multivalued operator $\Psi : \mathtt{C} \to 2^{\mathtt{C}}$ by

$$\Psi(\mathfrak{u}(\xi)) = \Big\{ z \in \mathtt{C} : z(\xi) = \xi^{1-\zeta+\eta\zeta-\eta\vartheta} \Big[\mathcal{S}_{\eta,\zeta}(\xi)[\mathfrak{u}_0 - \mathcal{G}(0, \mathfrak{u}(0))] + \mathcal{G}(\xi, \mathfrak{u}(\xi))$$
$$+ \int_0^\xi (\xi - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi - \nu) \mathtt{A} \mathcal{G}(\nu, \mathfrak{u}(\nu)) d\nu$$
$$+ \int_0^\xi (\xi - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi - \nu) \mathcal{H}(\nu, \mathfrak{u}(\nu), (F\mathfrak{u})(\nu)) dW(\nu) \Big],\ \xi \in (0, d] \Big\}.$$

To prove that Ψ has a fixed point:

Step 1: $\Psi(\mathfrak{u})$ is convex for all $\mathfrak{u} \in \mathtt{C}$.

Let $z_1, z_2 \in \mathtt{C}$ and $h_1, h_2 \in S_{\mathcal{H},\mathfrak{u}}$ such that $\xi \in \mathcal{I}$, and we have

$$\begin{aligned} z_i =& \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathfrak{u}_0 - \mathcal{G}(0,\mathfrak{u}(0))\big] + \mathcal{G}\big(\xi,\mathfrak{u}(\xi)\big) \\ &+ \int_0^\xi (\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu \\ &+ \int_0^\xi (\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)h_i(\nu)dW(\nu)\bigg], \quad i = 1,2. \end{aligned}$$

Consider $\lambda \in [0,1]$, then for all $\xi \in \mathcal{I}$, we obtain

$$\begin{aligned} \lambda\big(z_1 + (1-\lambda)z_2\big)(\xi) =& \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg(\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathfrak{u}_0 - \mathcal{G}(0,\mathfrak{u}(0))\big] + \mathcal{G}\big(\xi,\mathfrak{u}(\xi)\big) \\ &+ \int_0^\xi (\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu\bigg) \\ &+ \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^\xi (\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\big[\lambda h_1(\nu) \\ &\qquad + (1-\lambda)h_2(\nu)\big]dW(\nu). \end{aligned}$$

We know that $\mathcal{H}$ has convex values, then $S_{\mathcal{H},\mathfrak{u}}$ is convex. Therefore, $\lambda h_1 + (1-\lambda)h_2 \in S_{\mathcal{H},\mathfrak{u}}$. Therefore,

$$\lambda z_1 + (1-\lambda)z_2 \in \Psi\mathfrak{u}(\xi),$$

hence Ψ is convex.

Step 2: On the space $\mathtt{C}$, consider $B_r = \{\mathfrak{u} \in \mathtt{C} : \|\mathfrak{u}\|^2_{\mathtt{C}} \leq r\}$, for $r > 0$. Clearly, B_r are bounded, closed and convex sets of $\mathtt{C}$. Now, we prove that there exists $r > 0$ such that $\Psi(B_r) \subseteq B_r$.

If not, then for all $r > 0$, there exists $\mathfrak{u}^r \in B_r$, but $\Psi(u^r) \notin B_r$, i.e.,

$$\|\Psi(\mathfrak{u}^r)\|_{\mathtt{C}} \equiv \sup\{\|z^r\|_{\mathtt{C}} : z^r \in (\Psi\mathfrak{u}^r)\} > r$$

and

$$\begin{aligned} z = & \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\big[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathfrak{u}_0 - \mathcal{G}(0,\mathfrak{u}(0))\big] + \mathcal{G}\big(\xi,\mathfrak{u}(\xi)\big) \\ &+ \int_0^\xi (\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu \\ &+ \int_0^\xi (\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu)\big)dW(\nu)\big], \end{aligned}$$

for some $h^r \in S_{\mathcal{H},\mathtt{u}^r}$.

$$
\begin{aligned}
r &\le E\|(\Psi\mathtt{u}^r)(\xi)\|^2 \\
&\le E\bigg\|\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg\{\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big) \\
&\quad+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\nu \\
&\quad+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)dW(\nu)\bigg\}\bigg\|^2 \\
&\le 4E\big\|\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]\big\|^2+4E\big\|\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big)\big\|^2 \\
&\quad+4E\Big\|\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\nu\Big\|^2 \\
&\quad+4E\Big\|\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)dW(\nu)\Big\|^2 \\
&\le 4\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\sup_{\xi\in\mathcal{I}}E\big\|\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]\big\|^2+4\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\sup_{\xi\in\mathcal{I}}E\big\|\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big)\big\|^2 \\
&\quad+4\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\sup_{\xi\in\mathcal{I}}\int_0^\xi(\xi-\nu)^{2\eta-2}\big\|\mathtt{A}^{1-\mu}\mathcal{Q}_\eta(\xi-\nu)\big\|^2E\big\|\mathtt{A}^\mu\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)\big\|^2d\nu \\
&\quad+4Tr(Q)\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\sup_{\xi\in\mathcal{I}}\int_0^\xi(\xi-\nu)^{2\eta-2}\big\|\mathcal{Q}_\eta(\xi-\nu)\big\|^2E\big\|\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)\big\|^2d\nu \\
&\le 4\sup_{\xi\in\mathcal{I}}\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\Big[\kappa_s^2\xi^{2(-1+\zeta-\eta\zeta+\eta\vartheta)}\big(\|\mathtt{u}_0\|^2-M_0^2M_g^2\big)\Big] \\
&\quad+4\sup_{\xi\in\mathcal{I}}\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\bigg[M_0^2M_g^2(1+P)+\kappa_{1-\mu}^2\frac{\xi^{2\eta\mu}\Gamma(1+\mu)}{\mu\Gamma(1+\eta\mu)}\big(M_g^2(1+P)\big) \\
&\quad+Tr(Q)\kappa_p^2\int_0^\xi(\xi-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu)d\nu\bigg].
\end{aligned}
$$

Dividing both sides by r and taking $r\to\infty$, we obtain that

$$4Tr(Q)\kappa_p^2d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\gamma\ge 1,$$

which is a contradiction to our assumption. Thus, for $\delta>0$, there exists $r>0$ and some $h\in S_{\mathcal{H},\mathtt{u}}$, $\Psi(B_r)\subset B_r$.

Step 3: Ψ mapping bounded sets into equicontinuous sets of $\mathtt{C}$.

For all $z\in\Psi(\mathtt{u})$ and $\mathtt{u}\in B_r$, there exists $\mathcal{H}\in S_{\mathcal{H},\mathtt{u}}$, and we define

$$
\begin{aligned}
z(\xi)=\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg(&\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big) \\
&+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\nu \\
&+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)dW(\nu)\bigg).
\end{aligned}
$$

Consider $0 < \xi_1 < \xi_2 \leq d$.

$$
\begin{aligned}
&E\Big\|z(\xi_2) - z(\xi_1)\Big\|^2 \\
&\leq E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(\mathcal{S}_{\eta,\zeta}(\xi_2)\big[\mathfrak{u}_0 - \mathcal{G}(0,\mathfrak{u}(0))\big] + \mathcal{G}\big(\xi_2,\mathfrak{u}(\xi_2)\big) \\
&\quad + \int_0^{\xi_2}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu \\
&\quad + \int_0^{\xi_2}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathcal{H}\big(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu)\big)dW(\nu)\Big) \\
&\quad - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(\mathcal{S}_{\eta,\zeta}(\xi_1)\big[\mathfrak{u}_0 - \mathcal{G}(0,\mathfrak{u}(0))\big] + \mathcal{G}\big(\xi_1,\mathfrak{u}(\xi_1)\big) \\
&\quad + \int_0^{\xi_1}(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_1-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu \\
&\quad + \int_0^{\xi_1}(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_1-\nu)\mathcal{H}\big(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu)\big)dW(\nu)\Big)\Big\|^2 \\
&\leq 4E\Big\|\big[\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\xi_2) - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\xi_1)\big]\big[\mathfrak{u}_0 - \mathcal{G}(0,\mathfrak{u}(0))\big]\Big\|^2 \\
&\quad + 4E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{G}(\xi_2,\mathfrak{u}(\xi_2)) - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{G}(\xi_1,\mathfrak{u}(\xi_1))\Big\|^2 \\
&\quad + 4E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu \\
&\quad + \xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu \\
&\quad - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_1-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu\Big\|^2 \\
&\quad + 4E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathcal{H}\big(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu)\big)dW(\nu) \\
&\quad + \xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathcal{H}\big(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu)\big)dW(\nu) \\
&\quad - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_1-\nu)\mathcal{H}\big(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu)\big)dW(\nu)\Big\|^2 \\
&\leq 4E\Big\|\big[\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\xi_2) - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\xi_1)\big]\big[\mathfrak{u}_0 - \mathcal{G}(0,\mathfrak{u}(0))\big]\Big\|^2 \\
&\quad + 4E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{G}(\xi_2,\mathfrak{u}(\xi_2)) - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{G}(\xi_1,\mathfrak{u}(\xi_1))\Big\|^2 \\
&\quad + 12E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu\Big\|^2 \\
&\quad + 12E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu \\
&\quad - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu\Big\|^2 \\
&\quad + 12E\Big\|\xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathfrak{u}(\nu)\big)d\nu
\end{aligned}
$$

$$
\begin{aligned}
&- \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\xi_1} (\xi_1 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_1 - \nu) \mathtt{A}\mathcal{G}\big(\nu, \mathrm{u}(\nu)\big) d\nu \Big\|^2 \\
&+ 12E\Big\| \xi_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\xi_1}^{\xi_2} (\xi_2 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_2 - \nu) \mathcal{H}\big(\nu, \mathrm{u}(\nu), (F\mathrm{u})(\nu)\big) dW(\nu) \Big\|^2 \\
&+ 12E\Big\| \xi_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\xi_1} (\xi_2 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_2 - \nu) \mathcal{H}\big(\nu, \mathrm{u}(\nu), (F\mathrm{u})(\nu)\big) dW(\nu) \\
&- \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\xi_1} (\xi_1 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_2 - \nu) \mathcal{H}\big(\nu, \mathrm{u}(\nu), (F\mathrm{u})(\nu)\big) dW(\nu) \Big\|^2 \\
&+ 12E\Big\| \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\xi_1} (\xi_1 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_2 - \nu) \mathcal{H}\big(\nu, \mathrm{u}(\nu), (F\mathrm{u})(\nu)\big) dW(\nu) \\
&- \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\xi_1} (\xi_1 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_1 - \nu) \mathcal{H}\big(\nu, \mathrm{u}(\nu), (F\mathrm{u})(\nu)\big) dW(\nu) \Big\|^2 \\
&= \sum_{i=1}^{8} I_i.
\end{aligned}
$$

By the strong continuity of $\mathcal{S}_{\eta,\zeta}(\xi)(\mathrm{u}_0 - \mathcal{G}(0, \mathrm{u}(0)))$, we obtain

$$I_1 \to 0 \text{ as } \xi_2 \to \xi_1.$$

The equicontinuity of $\mathcal{G}$ ensures that

$$I_2 \to 0 \text{ as } \xi_2 \to \xi_1.$$

$$
\begin{aligned}
I_3 &= 12E\Big\| \xi_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\xi_1}^{\xi_2} (\xi_2 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_2 - \nu) \mathtt{A}\mathcal{G}\big(\nu, \mathrm{u}(\nu)\big) d\nu \Big\|^2 \\
&\leq 12\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)} \kappa_{1-\mu}^2 M_g^2 (1+P) \left(\frac{\Gamma(1+\mu)}{\mu\Gamma(1+\eta\mu)} \right)^2 (\xi_2 - \xi_1)^{2\eta\mu}.
\end{aligned}
$$

Then, $I_3 \to 0$ as $\xi_2 \to \xi_1$.

$$
\begin{aligned}
I_4 &= 12E\Big\| \xi_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\xi_1} (\xi_2 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_2 - \nu) \mathtt{A}\mathcal{G}\big(\nu, \mathrm{u}(\nu)\big) d\nu \\
&\quad - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\xi_1} (\xi_1 - \nu)^{\eta-1} \mathcal{Q}_\eta(\xi_2 - \nu) \mathtt{A}\mathcal{G}\big(\nu, \mathrm{u}(\nu)\big) d\nu \Big\|^2 \\
&\leq 12E\Big\| \int_0^{\xi_1} \left(\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta} (\xi_2 - \nu)^{\eta-1} - \xi_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\xi_1 - \nu)^{\eta-1} \right) \\
&\quad \times \mathcal{Q}_\eta(\xi_2 - \nu) \mathtt{A}\mathcal{G}\big(\nu, \mathrm{u}(\nu)\big) d\nu \Big\|^2 \\
&\leq 12\eta^2 \kappa_{1-\mu}^2 M_g^2 (1+P) \left(\frac{\Gamma(1+\mu)}{\mu\Gamma(1+\eta\mu)} \right)^2 \Bigg[\int_0^{\xi_1} \Big(\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)} (\xi_2 - \nu)^{2\eta-2} \\
&\quad - \xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)} (\xi_1 - \nu)^{2\eta-2} \Big) (\xi_2 - \nu)^{2\eta(\mu-1)} d\nu \Bigg].
\end{aligned}
$$

We obtain $I_4 \to 0$ as $\xi_2 \to \xi_1$. Additionally,

$$
\begin{aligned}
I_5 &= 12E\Big\|\xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}\Big((\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathtt{A}\mathcal{G}(\nu,\mathtt{u}(\nu)) \\
&\qquad\qquad -(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_1-\nu)\mathtt{A}\mathcal{G}(\nu,\mathtt{u}(\nu))\Big)d\nu\Big\|^2 \\
&\le 12\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\xi_1}(\xi_1-\nu)^{2\eta-2}\|[\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)]\|^2E\|\mathtt{A}\mathcal{G}(\nu,\mathtt{u}(\nu))\|^2d\nu \\
&\le M_0^2M_g^2(1+P)\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\xi_1}(\xi_1-\nu)^{2\eta-2}\|[\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)]\|^2d\nu.
\end{aligned}
$$

By Theorem 2 and strong continuity of $\mathcal{Q}_\eta(\xi)$, $I_5 \to 0$ as $\xi_2 \to \xi_1$.

$$
\begin{aligned}
I_6 &= 12E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathcal{H}(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu))dW(\nu)\Big\|^2 \\
&\le 12Tr(Q)\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)^{2\eta-2}\|\mathcal{Q}_\eta(\xi_2-\nu)\|^2E\|\mathcal{H}(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu))\|^2d\nu \\
&\le 12Tr(Q)\kappa_p^2\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu)d\nu \\
&\le 12Tr(Q)\kappa_p^2\Big[\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\xi_2}(\xi_2-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu)d\nu \\
&\quad -\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\xi_1}(\xi_1-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu)d\nu\Big] \\
&\quad +12Tr(Q)\kappa_p^2\int_0^{\xi_1}\Big[\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_1-\nu)^{2(\eta\vartheta-1)} \\
&\quad -\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_2-\nu)^{2(\eta\vartheta-1)}\Big]L_{\mathcal{H},r}(\nu)d\nu.
\end{aligned}
$$

Then, $I_6 \to 0$ as $\xi_2 \to \xi_1$ by using (H_4) and the Lebesgue dominated convergence theorem.

$$
\begin{aligned}
I_7 &= 12E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_2-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathcal{H}(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu))dW(\nu) \\
&\quad -\xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi_1}(\xi_1-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi_2-\nu)\mathcal{H}(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu))dW(\nu)\Big\|^2 \\
&\le 12Tr(Q)\kappa_p^2\int_0^{\xi_1}(\xi_2-\nu)^{2\eta(\vartheta-1)}E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\xi_2-\nu)^{\eta-1} \\
&\quad -\xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\xi_1-\nu)^{\eta-1}\Big\|^2L_{\mathcal{H},r}(\nu)d\nu,
\end{aligned}
$$

consider

$$
\begin{aligned}
&(\xi_2-\nu)^{2\eta(\vartheta-1)}E\Big\|\xi_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\xi_2-\nu)^{\eta-1}-\xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\xi_1-\nu)^{\eta-1}\Big\|^2L_{\mathcal{H},r}(\nu) \\
&\le \Big[2\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_2-\nu)^{2(\eta\vartheta-1)} \\
&\quad +2\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_1-\nu)^{2(\eta-1)}(\xi_2-\nu)^{2\eta(\vartheta-1)}\Big]L_{\mathcal{H},r}(\nu) \\
&\le \Big[2\xi_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_2-\nu)^{2(\eta\vartheta-1)}+2\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_1-\nu)^{2(\eta\vartheta-1)}\Big]L_{\mathcal{H},r}(\nu) \\
&\le 4\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_1-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu),
\end{aligned}
$$

and $\int_0^{\xi_1} 4\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_1-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu)d\nu$ exists ($\nu \in (0,\xi_1]$), then by Lebesgue's dominated convergence theorem, we obtain

$$\int_0^{\xi_1}(\xi_2-\nu)^{2\eta(\vartheta-1)}E\left\|\xi_2^{(1+\eta\vartheta)(1-\zeta)}(\xi_2-\nu)^{\eta-1}-\xi_1^{(1+\eta\vartheta)(1-\zeta)}(\xi_1-\nu)^{\eta-1}\right\|^2L_{\mathcal{H},r}(\nu)d\nu \\ \to 0 \text{ as } \xi_2\to\xi_1,$$

so we conclude $\lim_{\xi_2\to\xi_1} I_7 = 0$.

For any $\epsilon > 0$, we have

$$\begin{aligned}
I_8 &= 12E\left\|\int_0^{\xi_1}\xi_1^{1-\zeta+\eta\zeta-\eta\vartheta}\left[\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)\right]\right.\\
&\quad\left.\times(\xi_1-\nu)^{\eta-1}\mathcal{H}(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu))dW(\nu)\right\|^2\\
&\le 12Tr(Q)\int_0^{\xi_1}\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\|\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)\|^2(\xi_1-\nu)^{2\eta-2}\\
&\quad\times E\|\mathcal{H}(\nu,\mathfrak{u}(\nu),(F\mathfrak{u})(\nu))\|^2d\nu\\
&\le 12Tr(Q)\int_0^{\xi_1}\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\|\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)\|^2(\xi_1-\nu)^{2\eta-2}L_{\mathcal{H},r}(\nu)d\nu\\
&\le 12Tr(Q)\left\{\int_0^{\xi_1-\epsilon}\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\|\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)\|^2(\xi_1-\nu)^{2(\eta-1)}L_{\mathcal{H},r}(\nu)d\nu\right.\\
&\quad\left.+\int_{\xi_1-\epsilon}^{\xi_1}\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\|\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)\|^2(\xi_1-\nu)^{2(\eta-1)}L_{\mathcal{H},r}(\nu)d\nu\right\}\\
&\le 12Tr(Q)\left\{\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\xi_1-\epsilon}(\xi_1-\nu)^{2(\eta-1)}L_{\mathcal{H},r}(\nu)d\nu\right.\\
&\quad\times\sup_{\nu\in[0,\xi_1-\epsilon]}\|\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)\|^2\\
&\quad\left.+\kappa_p^2\int_{\xi_1-\epsilon}^{\xi_1}\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}[(\xi_2-\nu)^{2\eta(\vartheta-1)}+(\xi_1-\nu)^{2\eta(\vartheta-1)}](\xi_1-\nu)^{2(\eta-1)}L_{\mathcal{H},r}(\nu)d\nu\right\}\\
&\le 12Tr(Q)\left\{\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)-2\eta(\vartheta-1)}\int_0^{\xi_1}(\xi_1-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu)d\nu\right.\\
&\quad\times\sup_{\nu\in[0,\xi_1-\epsilon]}\|\mathcal{Q}_\eta(\xi_2-\nu)-\mathcal{Q}_\eta(\xi_1-\nu)\|^2\\
&\quad\left.+4\kappa_p^2\int_{\xi_1-\epsilon}^{\xi_1}\xi_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}(\xi_1-\nu)^{2(\eta\vartheta-1)}L_{\mathcal{H},r}(\nu)d\nu\right\}.
\end{aligned}$$

From Theorem (2) and $\lim_{\xi_2\to\xi_1} I_6 = 0$, we obtain $I_8 \to 0$ independently of $\mathfrak{u} \in \mathsf{C}$ as $\xi_2 \to \xi_1$, $\epsilon \to 0$. Hence, $\|z(\xi_2) - z(\xi_1)\| \to 0$ independently of $\mathfrak{u} \in \mathsf{C}$ as $\xi_2 \to \xi_1$. This implies that $\{\Psi\mathfrak{u}(\xi) : \mathfrak{u} \in \mathsf{C}\}$ is equicontinuous on $\mathcal{I}$.

Step 4: Show that $V(\xi) = \{z(\xi) : z \in \Psi(B_P(\mathcal{I}))\}$ is relatively compact for $\xi \in \mathcal{I}$.

For $\alpha \in (0,\xi)$ and $q > 0$, consider the operator $z'(\xi)$ on $B_P(\mathcal{I})$ by

$$\begin{aligned}
z'_{\alpha,q}(\xi) &= \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big)\\
&\quad+\int_0^{\xi-\alpha}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\nu\\
&\quad+\int_0^{\xi-\alpha}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)dW(\nu)\Big]\\
&= \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big)\\
&\quad+\int_0^{\xi-\alpha}\int_q^\infty \eta\theta M_\eta(\theta)(\xi-\nu)^{\eta-1}T((\xi-\nu)^\eta\theta)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\theta d\nu\\
&\quad+\int_0^{\xi-\alpha}\int_q^\infty \eta\theta M_\eta(\theta)(\xi-\nu)^{\eta-1}T((\xi-\nu)^\eta\theta)\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)d\theta dW(\nu)\Big]\\
&= \xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big)\Big]\\
&\quad+\eta\xi^{1-\zeta+\eta\zeta-\eta\vartheta}T(\alpha^\eta q)\int_0^{\xi-q}\int_q^\infty \theta M_\eta(\theta)(\xi-\nu)^{\eta-1}\\
&\quad\times T((\xi-\nu)^\eta\theta-\alpha^\eta q)\Big[\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\theta d\nu+\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)d\theta dW(\nu)\Big].
\end{aligned}$$

Hence, $V_{\alpha,\vartheta}(\xi) = \{(z'_{\alpha,q}(\xi))\mathtt{u}(\xi) : \mathtt{u} \in B_P(\mathcal{I})\}$ is precompact in Y for all $\alpha \in (0,\xi)$ and $q > 0$ due to the compactness of $T(\alpha^\eta q)$. For every $\mathtt{u} \in B_P(\mathcal{I})$, we obtain

$$\begin{aligned}
&E\Big\|z(\xi)-z'_{\alpha,q}(\xi)\Big\|^2\\
&\leq E\Big\|\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}(\xi)\big)\\
&\quad+\int_0^{\xi}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\nu\\
&\quad+\int_0^{\xi}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)dW(\nu)\Big)\\
&\quad-\Big(\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\sqcap(\xi)\big)\Big]\\
&\quad+\eta\xi^{1-\zeta+\eta\zeta-\eta\vartheta}T(\alpha^\eta q)\int_0^{\xi-q}\int_q^\infty \theta M_\eta(\theta)(\xi-\nu)^{\eta-1}\\
&\quad\times T((\xi-\nu)^\eta\theta-\alpha^\eta q)\big[\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\theta d\nu+\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)d\theta dW(\nu)\big]\Big)\Big\|^2\\
&\leq 2E\Big\|\eta\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\xi}\int_0^{q}\theta M_\eta(\theta)(\xi-\nu)^{\eta-1}T((\xi-\nu)^\eta\theta)\\
&\qquad\qquad\big[\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\theta d\nu+\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)d\theta dW(\nu)\big]\Big\|^2\\
&\quad+2E\Big\|\eta\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\xi-\alpha}^{\xi}\int_q^\infty(\xi-\nu)^{\eta-1}\theta M_\eta(\theta)T((\xi-\nu)^\eta\theta)\\
&\qquad\qquad\big[\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}(\nu)\big)d\theta d\nu+\mathcal{H}\big(\nu,\mathtt{u}(\nu),(F\mathtt{u})(\nu)\big)d\theta dW(\nu)\big]\Big\|^2
\end{aligned}$$

$$\le 2\eta^2\kappa_0^2\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\bigg(\int_0^\xi\int_0^q \theta^2 M_\eta^2(\theta)(\xi-\nu)^{2(\eta-1)}(\xi-\nu)^{2\eta\vartheta-2\eta}\theta^{2\vartheta-2}$$
$$\times\big[M_0^2M_g^2(1+P)+Tr(Q)L_{\mathcal{H},r}(\nu)\big]d\theta d\nu$$
$$+\int_{\xi-\alpha}^\xi\int_q^\infty(\xi-\nu)^{2(\eta-1)}\theta^2M_\eta^2(\theta)(\xi-\nu)^{2\eta\vartheta-2\eta}\theta^{2\vartheta-2}\big[M_0^2M_g^2(1+P)$$
$$+Tr(Q)L_{\mathcal{H},r}(\nu)\big]d\theta d\nu\bigg)$$
$$\le 2\eta^2\kappa_0^2\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\bigg(\int_0^\xi(\xi-\nu)^{2(\eta\vartheta-1)}\big[M_0^2M_g^2(1+P)$$
$$+Tr(Q)L_{\mathcal{H},r}(\nu)\big]d\nu\int_0^q\theta^{2\vartheta}M_\eta^2(\theta)d\theta+\int_{\xi-\alpha}^\xi(\xi-\nu)^{2(\eta\vartheta-1)}\big[M_0^2M_g^2(1+P)$$
$$+Tr(Q)L_{\mathcal{H},r}(\nu)\big]d\nu\int_0^\infty\theta^{2\vartheta}M_\eta^2(\theta)d\theta\bigg)$$
$$\le 2\eta^2\kappa_0^2\xi^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\bigg(\int_0^\xi(\xi-\nu)^{2(\eta\vartheta-1)}\big[M_0^2M_g^2(1+P)$$
$$+Tr(Q)L_{\mathcal{H},r}(\nu)\big]d\nu\int_0^q\theta^{2\vartheta}M_\eta^2(\theta)d\theta$$
$$+\frac{\Gamma(1+2\vartheta)}{\Gamma(1+2\eta\vartheta)}\int_{\xi-\alpha}^\xi(\xi-\nu)^{2(\eta\vartheta-1)}\big[M_0^2M_g^2(1+P)+Tr(Q)L_{\mathcal{H},r}(\nu)\big]d\nu\bigg)$$
$$\to 0 \text{ as } \alpha\to 0,\ q\to 0.$$

Therefore, $V_{\alpha,q}(\xi)=\{z'_{\alpha,q}(\xi):\xi\in\mathcal{I}\}$ are arbitrary closed to $V(\xi)=\{z(\xi):\xi\in\mathcal{I}\}$. As a result of the Arzelà–Ascoli theorem, $\{z(\xi):\xi\in\mathcal{I}\}$ is relatively compact. As a result, $z(\xi)$ is a completely continuous operator due to the continuity of $z(\xi)$ and relative compactness of $\{z(\xi):\xi\in\mathcal{I}\}$.

Step 5: Ψ has closed graph.

Let $\mathtt{u}_m\to\mathtt{u}_*$ as $m\to\infty$, $z_m(\xi)\in\Psi(\mathtt{u}_m)$ and $z_m\to z_*$ as $m\to\infty$, and we need to prove that $z_*\in\Psi(\mathtt{u}_*)$. Since $z_m\in\Psi(\mathtt{u}_m)$ then there exists a function $\mathcal{H}_m\in S_{\mathcal{H},\mathtt{u}_m}$ such that

$$z_m(\xi)=\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}_m(\xi)\big)$$
$$+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}_m(\nu)\big)d\nu$$
$$+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}_m(\nu)dW(\nu)\bigg].$$

We have to show that there exists $\mathcal{H}_*\in S_{\mathcal{H},\mathtt{u}_*}$ such that

$$z_*(\xi)=\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg[\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathtt{u}_0-\mathcal{G}(0,\mathtt{u}(0))\big]+\mathcal{G}\big(\xi,\mathtt{u}_*(\xi)\big)$$
$$+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathtt{u}_*(\nu)\big)d\nu$$
$$+\int_0^\xi(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}_*(\nu)dW(\nu)\bigg].$$

Clearly,

$$\Bigg\|\Bigg[z_m(\xi)-\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Bigg(\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathrm{u}_0-\mathcal{G}\big(0,\mathrm{u}(0)\big)\big]-\mathcal{G}\big(\xi,\mathrm{u}_m(\xi)\big) \\ -\int_0^{\xi}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathrm{u}_m(\nu)\big)d\nu\Bigg)\Bigg] \\ -\Bigg[z_*(\xi)-\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Bigg(\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathrm{u}_0-\mathcal{G}\big(0,\mathrm{u}(0)\big)\big]-\mathcal{G}\big(\xi,\mathrm{u}_*(\xi)\big) \\ -\int_0^{\xi}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathrm{u}_*(\nu)\big)d\nu\Bigg)\Bigg]\Bigg\|\to 0 \text{ as } m\to\infty.$$

Now, we consider an operator $\Upsilon : L^2(\mathcal{I},Y)\to \mathtt{C}(\mathcal{I},Y)$,

$$\Upsilon(h)(\xi)=\int_0^{\xi}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathcal{H}\big(\nu,\mathrm{u}(\nu),(F\mathrm{u})(\nu)\big)dW(\nu).$$

We have by (6) that $\Upsilon\circ S_{\mathcal{H},\mathrm{u}}$ is closed graph operator. Therefore, by comparing Υ, we have

$$\Bigg[z_m(\xi)-\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Bigg(\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathrm{u}_0-\mathcal{G}\big(0,\mathrm{u}(0)\big)\big]-G\big(\xi,\mathrm{u}_m(\xi)\big) \\ -\int_0^{\xi}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathrm{u}_m(\nu)\big)d\nu\Bigg)\Bigg]\in\Upsilon(S_{\mathcal{H},\mathrm{u}_m}).$$

Since $\mathcal{H}_m\to\mathcal{H}_*$, it follows from (6) that

$$\Bigg[z_*(\xi)-\xi^{1-\zeta+\eta\zeta-\eta\vartheta}\Bigg(\mathcal{S}_{\eta,\zeta}(\xi)\big[\mathrm{u}_0-\mathcal{G}\big(0,\mathrm{u}(0)\big)\big]-\mathcal{G}\big(\xi,\mathrm{u}_*(\xi)\big) \\ -\int_0^{\xi}(\xi-\nu)^{\eta-1}\mathcal{Q}_\eta(\xi-\nu)\mathtt{A}\mathcal{G}\big(\nu,\mathrm{u}_*(\nu)\big)d\nu\Bigg)\Bigg]\in\Upsilon(S_{\mathcal{H},\mathrm{u}_*}).$$

Hence, Ψ is a closed graph.

As a result of applying the Arzelà–Ascoli theorem on **Step 1–5**, Ψ is a u.s.c. multivalued mapping because it is a completely continuous multivalued mapping with compact value. As a result of Lemma 7, Ψ has a fixed point $z(\cdot)$ on $B_r(\cdot)$, and $z(\cdot)$ is the mild solution of (1) and (2). □

4. Example

As an example of how our findings can be put to use, consider the following: an *HF* neutral stochastic Volterra integro-differential inclusion

$$\begin{cases} D_{0^+}^{\frac{4}{7},\zeta}\big[w(\xi,v)-\bar{\mathcal{G}}(\xi,w(\xi,v))\big]\in w_{\xi\xi}(\xi,v)+ & \bar{\mathcal{H}}\big(\xi,w(\xi,v),(Fw)(\xi,v)\big)\frac{dW(\xi)}{d\xi}, \\ & \xi\in(0,d],\ v\in[0,\pi], \\ w(\xi,0)=w(\xi,\pi)=0 & \xi\in[0,d], \\ I^{(1-\frac{4}{7})(1-\zeta)}\mathrm{u}(w,0)=\mathrm{u}_0(v), & v\in[0,\pi], \end{cases} \tag{6}$$

where $D_{0^+}^{\frac{4}{7},\zeta}$ is the *HFD* of order $\frac{4}{7}$ and type ζ, $I^{(1-\frac{4}{7})(1-\zeta)}$ is the R-L integral of order $\frac{3}{7}(1-\zeta)$, $\bar{\mathcal{H}}\big(\xi,w(\xi,v),(Fw)(\xi,v)\big)$, $(Fw)(\xi,v)$ and $\bar{\mathcal{G}}(\xi,w(\xi,v))$ are the required functions.

Let $W(\xi)$ be a one-dimensional standard Brownian motion in Y defined on the complete probability space $(\Omega,\mathscr{E},\mathscr{P})$ and with the norm $\|\cdot\|_Y$ to write the system (6) in the abstract form of (1) and (2). Define an almost sectorial operator $\mathtt{A}: D(\mathtt{A})\subset Y\times Y$ by $\mathtt{A}w=w_{\xi\xi}$ with the domain

$$D(\mathtt{A})=\{w\in Y: w_\xi,w_{\xi\xi}\in Y: w(\xi,0)=w(\xi,\pi)=0\}.$$

Then, $\mathtt{A}$ generates a compact semigroup $T(\xi)_{\xi \geq 0}$ that is the analytic and self-adjoint. In addition, $\mathtt{A}$ has a discrete spectrum, and the eigenvalues are $m^2, m \in \mathbb{N}$, with corresponding orthogonal eigenvectors $\mathfrak{e}_m(z) = \sqrt{\frac{2}{\pi}} \sin(mz)$. Then, $\mathtt{A}z = \sum_{m=0}^{\infty} m^2 \langle z, \mathfrak{e}_m \rangle \mathfrak{e}_m$. Furthermore, we know that for all $v \in Y$, $T(\xi)v = \sum_{m=1}^{\infty} e^{-m^2 \xi} \langle v, \mathfrak{e}_m \rangle \mathfrak{e}_m$. In particular, $T(\cdot)$ is a uniformly analytic semigroup and $\|T(\xi)\| \leq e^{-\xi}$.

$\mathfrak{u}(\xi)(v) = w(\xi, v)$, $\xi \in \mathcal{I} = [0, d]$, $v \in [0, \pi]$. Now, any $\mathfrak{u} \in Y = L^2[0, \pi]$, $v \in [0, \pi]$, and we define the function $\mathcal{H} : \mathcal{I} \times Y \times Y \to Y$,

$$\begin{aligned}\mathcal{H}\big(\xi, \mathfrak{u}(\xi), (F\mathfrak{u})(\xi)\big) &= \bar{\mathcal{H}}\big(\xi, w(\xi, v), (Fw)(\xi, v)\big) \\ &= \frac{e^{-\xi}}{1 + e^{-\xi}} \sin\left(w(\xi, v) + \int_0^{\xi} \cos(\xi s) w(s, v) ds\right),\end{aligned}$$

where

$$(F\mathfrak{u})(\xi)(v) = \int_0^{\xi} f(\xi, s, w(s, v)) ds = \int_0^{\xi} \cos(\xi s) w(s, v) ds.$$

Additionally, $\mathcal{G} : \mathcal{I} \times Y \to Y$ is completely continuous mapping, defined as $\mathcal{G}(\xi, \mathfrak{u}(\xi)) = \bar{\mathcal{G}}(\xi, w(\xi, v))$, which satisfies the required hypotheses. Therefore, fractional system (6) can be reformulated as the nonlocal Cauchy problem, (1) and (2). Obviously, $\bar{\mathcal{H}}\big(\xi, w(\xi, v), (Fw)(\xi, v)\big)$ is uniformly bounded. Then, by Theorem 3, the problem has a mild solution on $\mathcal{I}$.

5. Conclusions

The existence of a mild solution of an abstract *HF* neutral stochastic Volterra integro-differential inclusion via almost sectorial operators was investigated using the fixed point theorem for multivalued maps in this paper. The findings were subjected to a set of sufficient criteria that were met. In the future, we will use the fixed point approach to study the approximate controllability of the *HF* neutral stochastic derivative with almost sectorial operators.

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Abbreviations

The following abbreviations are used in this manuscript:

HF	Hilfer Fractional
HFD	Hilfer Fractional Derivative
R-L	Riemann–Liouville

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