mathematics

MDPI

Article

Stancu-Type Generalized q-Bernstein–Kantorovich Operators Involving Bézier Bases

Wen-Tao Cheng [1,*], Md Nasiruzzaman [2] and Syed Abdul Mohiuddine [3,4]

[1] School of Mathematics and Physics, Anqing Normal University, Anqing 246133, China
[2] Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia; nasir3489@gmail.com
[3] Department of General Required Courses, Mathematics, Faculty of Applied Studies, King Abdulaziz University, Jeddah 21589, Saudi Arabia; mohiuddine@gmail.com
[4] Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
* Correspondence: chengwentao_0517@163.com; Tel.: +86-139-6699-9238

Abstract: We construct the Stancu-type generalization of q-Bernstein operators involving the idea of Bézier bases depending on the shape parameter $-1 \leq \zeta \leq 1$ and obtain auxiliary lemmas. We discuss the local approximation results in term of a Lipschitz-type function based on two parameters and a Lipschitz-type maximal function, as well as other related results for our newly constructed operators. Moreover, we determine the rate of convergence of our operators by means of Peetre's K-functional and corresponding modulus of continuity.

Keywords: (ζ, q)-Bernstein operators; Bézier bases; uniform convergence; Lipschitz-type functions; rate of convergence

MSC: 41A25; 41A36

check for updates

Citation: Cheng, W.-T.; Nasiruzzaman, M.; Mohiuddine, S.A. Stancu-Type Generalized q-Bernstein–Kantorovich Operators Involving Bézier Bases. *Mathematics* **2022**, *10*, 2057. https://doi.org/10.3390/math10122057

Academic Editor: Jay Jahangiri

Received: 5 May 2022
Accepted: 11 June 2022
Published: 14 June 2022

1. Introduction and Preliminaries

The shortest (and an elegant) proof of the most famous Weierstrass approximation theorem was given by the mathematician S. N. Bernstein, who invented the sequence of positive linear operators $\{B_m\}_{m\geq 1}$ (see [1]). Bernstein, in his investigation, showed that for any continuous function Ψ on $[0,1]$, written in symbols as $\Psi \in C[0,1]$, is uniformly approximated by B_m. Thus, for $w \in [0,1]$, the famous Bernstein operators are given by

$$B_m(\Psi; w) = \sum_{\ell=0}^{m} \Psi\left(\frac{\ell}{m}\right) b_{m,\ell}(w),$$

where $m \in \mathbb{N}$ (the set of positive integers) and the Bernstein polynomials $b_{m,\ell}(u)$ of the degree of the most m are defined by

$$b_{m,\ell}(w) = \binom{m}{\ell} w^{\ell}(1-w)^{m-\ell} \qquad (\ell = 0, 1, \cdots, m;\ w \in [0,1])$$

and

$$b_{m,\ell}(w) = 0 \qquad (\ell < 0 \ \text{ or } \ \ell > m).$$

It is very easy to verify the recursive relation for the Bernstein polynomials $b_{m,\ell}(w)$, given by

$$b_{m,\ell}(w) = (1-w) b_{m-1,\ell}(w) + w b_{m-1,\ell-1}(w).$$

First, the operators $B_m(h; u)$ in the sense of q-calculus were introduced by Lupaş [2], in which the approximation and shape-preserving properties were investigated, while one more form of the q-analogue of $B_m(h;u)$ was given by Phillips [3]. The rate of convergence was discussed by Heping and Meng [4] and Dogrŭ et al. [5] for the q-Bernstein and q-Lupaş–Stancu operators, respectively. Nowak [6] and Mahmudov and Sabancigil [7] defined and studied the q-Bernstein operators with the view of q-differences and in the Kantorovich sense, respectively.

Mathematicians developed q-calculus as a novel and productive link between mathematics and physics. Here, we use some basic tools of q-calculus (see [8]).

Consider $q \in (0,1]$ and an integer $\tau > 0$. The q-integer "denoted as $[\tau]_q$" is defined as $[\tau]_q = \dfrac{1-q^\tau}{1-q}$ for the value of $q \neq 1$ and $[\tau]_q = \tau$ for $q = 1$. The formula for the q-factorial "$[\tau]_q!$" is defined as $[\tau]_q! = 1$ for $\tau = 0$ and $[\tau]_q! = [\tau]_q[\tau-1]_q \cdots [1]_q$ for the value of $\tau = 1, 2, \cdots$. For $0 \leq s \leq r$, the q-binomial coefficient $\begin{bmatrix} r \\ s \end{bmatrix}_q = \dfrac{[r]_q!}{[r-s]_q!\,[s]_q!}$, and

$$(1+\tau)_q^r = \begin{cases} (1+\tau)(1+q\tau)\cdots(1+q^{r-1}\tau) & (r = 1,2,\cdots) \\ 1 & (r = 0). \end{cases}$$

The formula for the q-Jackson of the improper integral is given by

$$\int_0^{\infty/K} f(w)d_qw = (1-q)\sum_{m\in\mathbb{N}} f\left(\frac{q^m}{K}\right)\frac{q^m}{K},\ K \in \mathbb{R} - \{0\}.$$

The classical q-Jackson integral formula ([8,9]) for 0 where $U \in \mathbb{R}$ is given by

$$\int_0^U f(w)d_qw = U(1-q)\sum_{s=0}^{\infty} f(Uq^s)q^s,$$

while the more general form of the above for $[U,V]$ is

$$\int_U^V f(w)d_qw = \int_0^V f(w)d_qw - \int_0^U f(w)d_qw.$$

In recent times, there have been many authors who constructed the Bernstein-type operators by use of various parameters, such as α-Bernstein operators [10], which were linear and positive for $\alpha \in [0,1]$ and their modifications in the sense of Kantorovich [11], Schurer [12], Kantorovich-Stancu [13], Durrmeyer [14], Lupaş–Durrmeyer [15], q-Bernstein–Stancu[16], and Kantorovich-type q-Bernstein operators [17,18] and the references therein. Recently, α-Baskakov operators and their Kantorovich and Kantorovich–Stancu variants were discussed in [19–21], respectively.

Cai et al. [22] defined and systematically studied the Bernstein kind operators by taking into their consideration Bézier bases associated with the shape parameter $-1 \leq \zeta \leq 1$, which were introduced by Ye et al. [23] and called ζ-Bernstein operators, in addition to their Kantorovic and Schurer variants (see [24,25]). In 2019, Srivastava et al. [26] defined the Stancu-type modification of the aforementioned operators and the studied direct approximation, uniform convergence, Voronovskaja-type theorems, rate of convergence and some other approximation results. Inspired by these studies, Cai again, together with Zhou and Li [27] (see also [28]), presented a modification of ζ-Bernstein operators by taking q-calculus into account by simply writing the (ζ,q)-Bernstein operators as follows. Let $w \in [0,1]$, $q \in (0,1]$, $-1 \leq \zeta \leq 1$, and $m \geq 2$. Then, for any function $\Psi \in C[0,1]$, the (ζ,q)-Bernstein operators are defined by

$$B_{m,q,\zeta}(\Psi;w) = \sum_{\ell=0}^{m} \xi_{m,\ell}(w;q,\zeta)\Psi\left(\frac{[\ell]_q}{[m]_q}\right). \quad (1)$$

In this case, the Bézier bases associated with the ζ- and q-integers are given by

$$\begin{aligned}
\xi_{m,0}(w;q,\zeta) &= \chi_{m,0}(w;q) - \frac{\zeta}{[m]_q+1}\chi_{1+m,1}(w;q), \\
\xi_{m,\ell}(w;q,\zeta) &= \chi_{m,\ell}(w;q) + \zeta\biggl(\frac{[m]_q-2[\ell]_q+1}{[m]_q^2-1}\chi_{1+m,\ell}(w;q) \\
&\quad - \frac{[m]_q-2[\ell]_q+1}{[m]_q^2-1}\chi_{1+m,\ell+1}(w;q)\biggr), \quad (1\leq \ell \leq m-1) \\
\xi_{m,m}(w;q,\zeta) &= \chi_{m,m}(w;q) - \frac{\zeta}{[m]_q+1}\chi_{1+m,m}(w;q),
\end{aligned}$$

where

$$\chi_{m,\ell}(w;q) = \begin{bmatrix} m \\ \ell \end{bmatrix}_q w^\ell \prod_{s=0}^{m-\ell-1}(1-q^s w).$$

The authors of [29] presented another generalization of $B_{m,q,\zeta}(\Psi;w)$ (i.e., the Chlodowsky-type (ζ,q)-Bernstein–Stancu operators) and investigated their approximation properties. The Kantorovich variant of $B_{m,q,\zeta}(\Psi;w)$ [30] has been recently defined as

$$K_{m,q,\zeta}(\Psi;w) = [1+m]_q \sum_{\ell=0}^{m} \xi_{m,\ell}(w;q,\zeta) q^{-\ell} \int_{\frac{[\ell]_q}{[1+m]_q}}^{\frac{[\ell+1]_q}{[1+m]_q}} \Psi(t) d_q t. \tag{2}$$

Lemma 1 ([27])**.** *Let $e_i(w) = w^i$, $i = 0,1,2$. Then, the operators $B_{m,q,\zeta}$ are defined by Equation (1) and satisfy $B_{m,q,\zeta}(1;w) = 1$ and the following equalities:*

$$\begin{aligned}
B_{m,q,\zeta}(t;w) &= w + \frac{[1+m]_q\zeta w(1-w^m)}{[m]_q([m]_q-1)} - \frac{2[1+m]_q\zeta w}{[m]_q^2-1}\left(\frac{1-w^m}{[m]_q} + qw(1-w^{m-1})\right) \\
&\quad + \frac{\zeta}{q[m]_q(1+[m]_q)}\left(1-\prod_{s=0}^{m}(1-q^s w) - w^{m+1} - [1+m]_q w(1-w^m)\right) \\
&\quad + \frac{\zeta}{[m]_q^2-1}\biggl\{2[1+m]_q w^2(1-w^{m-1}) - \frac{2[1+m]_q\zeta w(1-w^m)}{q[m]_q} \\
&\quad + \frac{2}{q[m]_q}\left(1-\prod_{s=0}^{m}(1-q^s w) - w^{m+1}\right)\biggr\};
\end{aligned}$$

$$\begin{aligned}
B_{m,q,\zeta}(t^2;w) &= w^2 + \frac{w(1-w)}{[m]_q} + \frac{[1+m]_q\zeta w}{[m]_q([m]_q-1)}\left(qw(1-w^{m-1}) + \frac{1-w^m}{[m]_q}\right) \\
&\quad - \frac{2[1+m]_q\zeta}{[m]_q([m]_q^2-1)}\biggl(\frac{w(1-w^m)}{[m]_q} + q(2+q)w^2(1-w^{m-1}) \\
&\quad + q^3[m-1]_q w^3(1-w^{m-2})\biggr) \\
&\quad - \frac{\zeta}{q[m]_q(1+[m]_q)}\biggl([1+m]_q w^2(1-w^{m-1}) - \frac{[1+m]_q w(1-w^m)}{q[m]_q} \\
&\quad + \frac{1-\prod_{s=0}^{m}(1-q^s w) - w^{m+1}}{q[m]_q}\biggr) \\
&\quad + \frac{2\zeta}{[m]_q([m]_q^2-1)}\biggl\{q[m-1]_q[1+m]_q w^3(1-w^{m-2}) \\
&\quad - \frac{(1-q)[1+m]_q w^2(1-w^{m-1})}{q}
\end{aligned}$$

$$+\frac{[1+m]_q w(1-w^m)}{q^2[m]_q}-\frac{1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}}{q^2[m]_q}\Bigg\}.$$

2. Construction of Operators and Basic Estimations

We introduce the following linear positive operators, determine their moments and central moments, and study the uniform convergence of our operators.

Suppose μ and ν are non-negative parameters such that $0 \leq \mu \leq \nu$, and suppose that $q \in (0,1]$ where $-1 \leq \zeta \leq 1$. Then, for $\Psi \in C[0,1]$, we define the Stancu variant of $K_{m,q,\zeta}(\Psi;w)$ (or, for example, by the Stancu-type (ζ,q)-Bernstein–Kantorovich operator) by

$$S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)=([1+m]_q+\nu)\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)q^{-\ell}\int_{\frac{[\ell]_q+\mu}{[1+m]_q+\nu}}^{\frac{[\ell+1]_q+\mu}{[1+m]_q+\nu}}\Psi(t)d_qt \quad (w\in[0,1]). \tag{3}$$

Lemma 2. *For $e_i(w)=w^i$, $i=0,1,2$, the operators $S^{\mu,\nu}_{m,q,\zeta}$ defined by Equation (3) satisfy $S^{\mu,\nu}_{m,q,\zeta}(1;w)=1$ and the following equalities:*

$$\begin{aligned}
S^{\mu,\nu}_{m,q,\zeta}(t;w) \;=\;& \frac{[m]_q}{([1+m]_q+\nu)}\Bigg[w+\frac{[1+m]_q\zeta w(1-w^m)}{[m]_q([m]_q-1)}\\
&-\frac{2[1+m]_q\zeta w}{[m]_q^2-1}\left(\frac{1-w^m}{[m]_q}+qw(1-w^{m-1})\right)\\
&+\frac{\zeta}{q[m]_q(1+[m]_q)}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}-[1+m]_q w(1-w^m)\right)\\
&+\frac{\zeta}{[m]_q^2-1}\Bigg\{2[1+m]_q w^2(1-w^{m-1})-\frac{2[1+m]_q\zeta w(1-w^m)}{q[m]_q}\\
&+\frac{2}{q[m]_q}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}\right)\Bigg\}\Bigg]+\frac{1}{[2]_q([1+m]_q+\nu)};
\end{aligned}$$

$$\begin{aligned}
S^{\mu,\nu}_{m,q,\zeta}(t^2;w) \;=\;& \frac{[m]_q^2}{([1+m]_q+\nu)^2}\Bigg[w^2+\frac{w(1-w)}{[m]_q}\\
&+\frac{[1+m]_q\zeta w}{[m]_q([m]_q-1)}\left(qw(1-w^{m-1})+\frac{1-w^m}{[m]_q}\right)\\
&-\frac{2[1+m]_q\zeta}{[m]_q([m]_q^2-1)}\Bigg(\frac{w(1-w^m)}{[m]_q}+q(2+q)w^2(1-w^{m-1})\\
&+q^3[m-1]_q w^3(1-w^{m-2})\Bigg)\\
&-\frac{\zeta}{q[m]_q(1+[m]_q)}\Bigg([1+m]_q w^2(1-w^{m-1})-\frac{[1+m]_q w(1-w^m)}{q[m]_q}\\
&+\frac{1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}}{q[m]_q}\Bigg)\\
&+\frac{2\zeta}{[m]_q([m]_q^2-1)}\Bigg\{q[m-1]_q[1+m]_q w^3(1-w^{m-2})\\
&-\frac{(1-q)[1+m]_q w^2(1-w^{m-1})}{q}\\
&+\frac{[1+m]_q w(1-w^m)}{q^2[m]_q}-\frac{1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}}{q^2[m]_q}\Bigg\}\Bigg]\\
&+\left(\frac{2q+1}{[3]_q}+\frac{3\mu[2]_q}{[3]_q}\right)\frac{[m]_q}{([1+m]_q+\nu)^2}\Bigg[w+\frac{[1+m]_q\zeta w(1-w^m)}{[m]_q([m]_q-1)}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2[1+m]_q\zeta w}{[m]_q^2-1}\left(\frac{1-w^m}{[m]_q}+qw(1-w^{m-1})\right)\\
&+\frac{\zeta}{q[m]_q(1+[m]_q)}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}-[1+m]_q w(1-w^m)\right)\\
&+\frac{\zeta}{[m]_q^2-1}\bigg\{2[1+m]_q w^2(1-w^{m-1})-\frac{2[1+m]_q\zeta w(1-w^m)}{q[m]_q}\\
&+\frac{2}{q[m]_q}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}\right)\bigg\}\bigg]\\
&+\left(\frac{3\mu^2+3\mu+1}{[3]_q}\right)\frac{1}{([1+m]_q+\nu)^2}.
\end{aligned}
$$

Proof. Here, we will use Lemma 1 and the equalities $[\ell+1]_q = q^\ell + [\ell]_q$, and $[\ell+1]_q = 1+q[\ell]_q$. Then, from the q-Jackson integral, we have

$$
\begin{aligned}
\int_{\frac{[\ell]_q+\mu}{[1+m]_q+\nu}}^{\frac{[\ell+1]_q+\mu}{[1+m]_q+\nu}} t^\beta d_q t &= \int_0^{\frac{[\ell+1]_q+\mu}{[1+m]_q+\nu}} t^\beta d_q t - \int_0^{\frac{[\ell]_q+\mu}{[1+m]_q+\nu}} t^\beta d_q t\\
&= \frac{(1-q)}{([1+m]_q+\nu)^{\beta+1}}\left(([\ell+1]_q+\mu)^{\beta+1}-([\ell]_q+\mu)^{\beta+1}\right)\sum_{m=0}^{\infty} q^{m(1+\beta)}.
\end{aligned}
$$

Therefore, the following is true:

$$
\int_{\frac{[\ell]_q+\mu}{[1+m]_q+\nu}}^{\frac{[\ell+1]_q+\mu}{[1+m]_q+\nu}} t^\beta d_q t = \begin{cases}
\frac{q^\ell}{[m+1]_q+\nu} & \text{for } \beta=0;\\
\frac{q^\ell}{([m+1]_q+\nu)^2}\left([\ell]_q+\frac{2\mu+1}{[2]_q}\right) & \text{for } \beta=1;\\
\frac{q^\ell}{([m+1]_q+\nu)^3}\left([\ell]_q^2+\left(\frac{2q+1}{[3]_q}+\frac{3\mu[2]_q}{[3]_q}\right)[\ell]_q+\frac{3\mu^2+3\mu+1}{[3]_q}\right) & \text{for } \beta=2.
\end{cases} \tag{4}
$$

Thus, in the light of Equation (4), the operators in Equation (3) give us

$$
\begin{aligned}
S_{m,q,\zeta}^{\mu,\nu}(1;w) &= ([1+m]_q+\nu)\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)q^{-\ell}\int_{\frac{[\ell]_q+\mu}{[1+m]_q+\nu}}^{\frac{[\ell+1]_q+\mu}{[1+m]_q+\nu}} d_q t,\\
&= ([1+m]_q+\nu)\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)q^{-\ell}\frac{q^\ell}{[1+m]_q+\nu}\\
&= B_{m,q,\zeta}(1;w)\\
&= 1.
\end{aligned}
$$

Now, the following is true:

$$
\begin{aligned}
S_{m,q,\zeta}^{\mu,\nu}(t;w) &= ([1+m]_q+\nu)\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)q^{-\ell}\int_{\frac{[\ell]_q+\mu}{[1+m]_q+\nu}}^{\frac{[\ell+1]_q+\mu}{[1+m]_q+\nu}} t d_q t,\\
&= \frac{1}{([1+m]_q+\nu)}\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)\left([\ell]_q+\frac{2\mu+1}{[2]_q}\right)\\
&= \frac{[m]_q}{([1+m]_q+\nu)}\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)\left(\frac{[\ell]_q}{[m]_q}\right)\\
&\quad+\frac{1}{[2]_q([1+m]_q+\nu)}\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)\\
&= \frac{[m]_q}{([1+m]_q+\nu)}B_{m,q,\zeta}(t;w)+\frac{1}{[2]_q([1+m]_q+\nu)}B_{m,q,\zeta}(1;w).
\end{aligned}
$$

Finally, the following is true:

$$\begin{aligned}
S^{\mu,\nu}_{m,q,\zeta}(t^2;w) &= ([1+m]_q+\nu)\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)q^{-\ell}\int_{\frac{[\ell]_q+\mu}{[1+m]_q+\nu}}^{\frac{[\ell+1]_q+\mu}{[1+m]_q+\nu}} t^2 d_q t,\\
&= \frac{1}{([1+m]_q+\nu)^2}\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)\Bigg([\ell]_q^2+\left(\frac{2q+1}{[3]_q}+\frac{3\mu[2]_q}{[3]_q}\right)[\ell]_q\\
&\quad+\frac{3\mu^2+3\mu+1}{[3]_q}\Bigg)\\
&= \frac{[m]_q^2}{([1+m]_q+\nu)^2}\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)\left(\frac{[\ell]_q}{[m]_q}\right)^2\\
&\quad+\left(\frac{2q+1}{[3]_q}+\frac{3\mu[2]_q}{[3]_q}\right)\frac{[m]_q}{([1+m]_q+\nu)^2}\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)\left(\frac{[\ell]_q}{[m]_q}\right)\\
&\quad+\left(\frac{3\mu^2+3\mu+1}{[3]_q}\right)\frac{1}{([1+m]_q+\nu)^2}\sum_{\ell=0}^{m}\xi_{m,\ell}(w;q,\zeta)\\
&= \frac{[m]_q^2}{([1+m]_q+\nu)^2}B_{m,q,\zeta}(t^2;w)+\left(\frac{2q+1}{[3]_q}+\frac{3\mu[2]_q}{[3]_q}\right)\frac{[m]_q}{([1+m]_q+\nu)^2}\\
&\quad\times B_{m,q,\zeta}(t;w)+\left(\frac{3\mu^2+3\mu+1}{[3]_q}\right)\frac{1}{([1+m]_q+\nu)^2}B_{m,q,\zeta}(1;w).
\end{aligned}$$

By using Lemma (1), we find the desired results. □

By simple calculation, and with the help of Lemma 2, we obtain the following lemma:

Lemma 3. *The operators $S^{\mu,\nu}_{m,q,\zeta}$ have the following central moments:*

$$\begin{aligned}
S^{\mu,\nu}_{m,q,\zeta}(t-w;w) &= \frac{[m]_q}{([1+m]_q+\nu)}\Bigg[w-\frac{[1+m]_q+\nu}{[m]_q}w+\frac{[1+m]_q\zeta w(1-w^m)}{[m]_q([m]_q-1)}\\
&\quad-\frac{2[1+m]_q\zeta w}{[m]_q^2-1}\left(\frac{1-w^m}{[m]_q}+qw(1-w^{m-1})\right)\\
&\quad+\frac{\zeta}{q[m]_q(1+[m]_q)}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}-[1+m]_q w(1-w^m)\right)\\
&\quad+\frac{\zeta}{[m]_q^2-1}\Bigg\{2[1+m]_q w^2(1-w^{m-1})-\frac{2[1+m]_q\zeta w(1-w^m)}{q[m]_q}\\
&\quad+\frac{2}{q[m]_q}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}\right)\Bigg\}\Bigg]+\frac{1}{[2]_q([1+m]_q+\nu)};
\end{aligned}$$

$$\begin{aligned}
&S^{\mu,\nu}_{m,q,\zeta}\Big((t-w)^2;w\Big)\\
&= \frac{[m]_q^2}{([1+m]_q+\nu)^2}\Bigg[w^2+\frac{w(1-w)}{[m]_q}+\frac{[1+m]_q\zeta w}{[m]_q([m]_q-1)}\left(qw(1-w^{m-1})+\frac{1-w^m}{[m]_q}\right)\\
&\quad-\frac{2[1+m]_q\zeta}{[m]_q([m]_q^2-1)}\Bigg(\frac{w(1-w^m)}{[m]_q}+q(2+q)w^2(1-w^{m-1})\\
&\quad+q^3[m-1]_q w^3(1-w^{m-2})\Bigg)-\frac{\zeta}{q[m]_q(1+[m]_q)}\Bigg([1+m]_q w^2(1-w^{m-1})\\
&\quad-\frac{[1+m]_q w(1-w^m)}{q[m]_q}+\frac{1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}}{q[m]_q}\Bigg)+\frac{2\zeta}{[m]_q([m]_q^2-1)}
\end{aligned}$$

$$
\begin{aligned}
&\times\left\{q[m-1]_q[1+m]_q w^3(1-w^{m-2})-\frac{(1-q)[1+m]_q w^2(1-w^{m-1})}{q}\right.\\
&\left.+\frac{[1+m]_q w(1-w^m)}{q^2[m]_q}-\frac{1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}}{q^2[m]_q}\right\}\Bigg]\\
&+\left(\frac{2q+1}{[3]_q}+\frac{3\mu[2]_q}{[3]_q}\right)\frac{[m]_q}{([1+m]_q+\nu)^2}\left[w+\frac{[1+m]_q\zeta w(1-w^m)}{[m]_q([m]_q-1)}\right.\\
&-\frac{2[1+m]_q\zeta w}{[m]_q^2-1}\left(\frac{1-w^m}{[m]_q}+qw(1-w^{m-1})\right)\\
&+\frac{\zeta}{q[m]_q(1+[m]_q)}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}-[1+m]_q w(1-w^m)\right)\\
&+\frac{\zeta}{[m]_q^2-1}\left\{2[1+m]_q w^2(1-w^{m-1})-\frac{2[1+m]_q\zeta w(1-w^m)}{q[m]_q}\right.\\
&\left.\left.+\frac{2}{q[m]_q}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}\right)\right\}\right]+\left(\frac{3\mu^2+3\mu+1}{[3]_q}\right)\frac{1}{([1+m]_q+\nu)^2}\\
&-\frac{[m]_q}{([1+m]_q+\nu)}\left[w^2+\frac{[1+m]_q\zeta w^2(1-w^m)}{[m]_q([m]_q-1)}-\frac{2[1+m]_q\zeta w^2}{[m]_q^2-1}\left(\frac{1-w^m}{[m]_q}\right.\right.\\
&\left.+qw(1-w^{m-1})\right)+\frac{\zeta w}{q[m]_q(1+[m]_q)}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}-[1+m]_q\right.\\
&\left.\times w(1-w^m)\right)+\frac{\zeta w}{[m]_q^2-1}\left\{2[1+m]_q w^2(1-w^{m-1})-\frac{2[1+m]_q\zeta w(1-w^m)}{q[m]_q}\right.\\
&\left.\left.+\frac{2}{q[m]_q}\left(1-\prod_{s=0}^{m}(1-q^s w)-w^{m+1}\right)\right\}\right]+\frac{w}{[2]_q([1+m]_q+\nu)}+w^2.
\end{aligned}
$$

Theorem 1. *Let $q=(q_m)$ be a sequence such that $0<q_m<1$, and let $\lim_{m\to\infty} q_m=1$. Then,*

$$\lim_{m\to\infty} S^{\mu,\nu}_{m,q_m,\zeta}(\Psi;w)=\Psi \qquad (\Psi\in C[0,1])$$

uniformly on $[0,1]$.

Proof. According to the Bohman–Korovkin theorem (see [31]), it is sufficient to show that

$$\lim_{m\to\infty} S^{\mu,\nu}_{m,q_m,\zeta}(e_i;w)=w^i, \qquad i=0,1,2, \tag{5}$$

uniformly on $[0,1]$. The assertion in Equation (5) follows by taking Lemma 2 and the limit $m\to\infty$ into account. □

3. Local Approximation of $S^{\mu,\nu}_{m,q,\zeta}$

Here, we obtain the local approximation of $S^{\mu,\nu}_{m,q,\zeta}$. Suppose $E_\phi=\{\phi|\phi\in C[0,1]\}$. For any $\delta^*>0$ and $\phi\in E_\phi$, the modulus of smoothness $\omega^*(\phi;\delta^*)$ of ϕ to the order of one is given by

$$\omega^*(\phi;\delta^*)=\sup_{|\mu_1-\mu_2|\le\delta^*}|\,\phi(\mu_1)-\phi(\mu_2)\,|, \qquad \mu_1,\mu_2\in[0,1], \tag{6}$$

$$|\,\phi(\mu_1)-\phi(\mu_2)\,|\le\left(1+\frac{|\,\mu_1-\mu_2\,|}{\delta^*}\right)\omega^*(\phi;\delta^*). \tag{7}$$

In addition, the following is true:

$$\lim_{\delta^*\to 0+}\omega^*(\phi;\delta^*)=0.$$

Theorem 2 ([32]). *For the sequence of positive linear operators $\{P\}_{m\geq 1} : C[w_1, w_2] \to C[y_1, y_2]$ such that $[y_1, y_2] \subseteq [w_1, w_2]$, one has the following:*

1. *For all $\phi \in C[w_1, w_2]$ and $w \in [y_1, y_2]$, it follows that*

$$\begin{aligned} |P_m(\phi; w) - \phi(w)| &\leq |\phi(w)||P_m(1; w) - 1| \\ &+ \left\{ P_m(1; w) + \frac{1}{\delta^*} \sqrt{P_m((t-w)^2; w)} \sqrt{P_m(1; w)} \right\} \omega^*(\phi; \delta^*). \end{aligned}$$

2. *For all $\varphi' \in C[w_1, w_2]$ and $w \in [y_1, y_2]$, it follows that*

$$\begin{aligned} |P_m(\varphi; w) - \varphi(w)| &\leq |\varphi(w)||P_m(1; w) - 1| + |\varphi'(w)||P_m(t-w; w)| \\ &+ P_m((t-w)^2; w) \left\{ \sqrt{P_m(1; w)} \right. \\ &\left. + \frac{1}{\delta^*} \sqrt{P_m((t-w)^2; w)} \right\} \omega^*(\varphi'; \delta^*). \end{aligned}$$

Theorem 3. *Let $\phi \in E_\phi$ and $w \in [0, 1]$. Then, the operators $S^{\mu,\nu}_{m,q,\zeta}$ satisfy*

$$|S^{\mu,\nu}_{m,q,\zeta}(\phi; w) - \phi(w)| \leq 2\omega^* \left(\phi; \sqrt{\delta^{\mu,\nu}_{m,q,\zeta}(w)} \right),$$

where $\delta^{\mu,\nu}_{m,q,\zeta}(w) = S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2; w \right)$.

Proof. We prove the inequality by taking into account (1) from Theorem 2 and the use of Lemmas 2 and 3 such that

$$\begin{aligned} |S^{\mu,\nu}_{m,q,\zeta}(\phi; w) - \phi(w)| &\leq |\phi(w)||S^{\mu,\nu}_{m,q,\zeta}(1; w) - 1| + \left\{ S^{\mu,\nu}_{m,q,\zeta}(1; w) \right. \\ &\left. + \frac{1}{\delta^*} \sqrt{S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2; w \right)} \sqrt{S^{\mu,\nu}_{m,q,\zeta}(1; w)} \right\} \omega^*(\phi; \delta^*). \end{aligned}$$

In this case, $\delta^* = \sqrt{S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2; w \right)} = \sqrt{\delta^{\mu,\nu}_{m,q,\zeta}(w)}$, which gives the desired result. □

Theorem 4. *For all $\varphi' \in C[0, 1]$ and $w \in [0, 1]$, we get the inequality*

$$|S^{\mu,\nu}_{m,q,\zeta}(\varphi; w) - \varphi(w)| \leq |\varphi'(w)|\mu^{\mu,\nu}_{m,q,\zeta} + 2\, \delta^{\mu,\nu}_{m,q,\zeta}(w)\, \omega^* \left(\varphi'; \sqrt{\delta^{\mu,\nu}_{m,q,\zeta}(w)} \right),$$

where $\mu^{\mu,\nu}_{m,q,\zeta} = \max_{w\in[0,1]} \left| S^{\mu,\nu}_{m,q,\zeta}((t-w); w) \right|$, and $\delta^{\mu,\nu}_{m,q,\zeta}$ is defined in Theorem 3.

Proof. If we consider (2) from Theorem 2 and the use of Lemmas 2 and 3, then

$$\begin{aligned} |S^{\mu,\nu}_{m,q,\zeta}(\varphi; w) - \varphi(w)| &\leq |\varphi(w)||S^{\mu,\nu}_{m,q,\zeta}(1; w) - 1| + |\varphi'(w)||S^{\mu,\nu}_{m,q,\zeta}(t-w; w)| \\ &+ S^{\mu,\nu}_{m,q,\zeta}((t-w)^2; w) \left\{ 1 + \frac{\sqrt{S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2; w \right)}}{\delta^*} \right\} \\ &\times \omega^*(\varphi'; \delta^*) \\ &\leq |\varphi'(w)|\mu^{\mu,\nu}_{m,q,\zeta} + 2\, \delta^{\mu,\nu}_{m,q,\zeta}(w)\, \omega^* \left(\varphi'; \sqrt{\delta^{\mu,\nu}_{m,q,\zeta}(w)} \right), \end{aligned}$$

where $\mu^{\mu,\nu}_{m,q,\zeta} = \max_{w\in[0,1]} \left| S^{\mu,\nu}_{m,q,\zeta}((t-w); w) \right|$. □

Now, we estimate the local approximation for our new operators $S^{\mu,\nu}_{m,q,\zeta}$ by use of the Lipschitz-type function. Thus, for any $0 < \chi \leq 1$, the Lipschitz-type function $Lip^{\chi}_{\mathcal{K}}$ is defined in the form of any positive real parameters $\mu_1,\ \mu_2$ such that (see [33])

$$Lip^{w,t}_{\mathcal{K}}(\chi) = \left\{ \Psi \in C[0,1] : |\Psi(t) - \Psi(w)| \leq \mathcal{K} \frac{|t-w|^{\chi}}{(\mu_1 w^2 + \mu_2 w + t)^{\frac{\chi}{2}}};\ w,\ t \in [0,1] \right\},$$

where $\mathcal{K}$ is a positive constant.

Theorem 5. *For any $\Psi \in Lip^{w,t}_{\mathcal{K}}(\chi)$, operators $S^{\mu,\nu}_{m,q,\zeta}$ we have the inequality*

$$|S^{\mu,\nu}_{m,q,\zeta}(\Psi; w) - \Psi(w)| \leq \mathcal{K} \left(\frac{\delta^{\mu,\nu}_{m,q,\zeta}(w)}{(\mu_1 w^2 + \mu_2 w)} \right)^{\frac{\chi}{2}},$$

where $\delta^{\mu,\nu}_{m,q,\zeta}(w)$ is given by Theorem 3.

Proof. Suppose $\Psi \in Lip^{w,t}_{\mathcal{K}}(\chi)$. Now, we first verify that the Theorem 5 holds for $\chi = 1$. Therefore, for any $\mu_1,\ \mu_2 \geq 0$, it is easy to find that $(\mu_1 w^2 + \mu_2 w + t)^{-1/2} \leq (\mu_1 w^2 + \mu_2 w)^{-1/2}$. By taking into account the Cauchy–Schwarz inequality, then it is easy to write

$$\begin{aligned}
|S^{\mu,\nu}_{m,q,\zeta}(\Psi; w) - \Psi(w)| &\leq |S^{\mu,\nu}_{m,q,\zeta}(|\Psi(t) - \Psi(w)|; w)| + \Psi(w)|S^{\mu,\nu}_{m,q,\zeta}(1; w) - 1| \\
&\leq S^{\mu,\nu}_{m,q,\zeta}\left(\frac{|t-w|}{(\mu_1 w^2 + \mu_2 w + t)^{\frac{1}{2}}}; w \right) \\
&\leq \mathcal{K}(\mu_1 w^2 + \mu_2 w)^{-1/2} S^{\mu,\nu}_{m,q,\zeta}(|t-w|; w) \\
&\leq \mathcal{K}(\mu_1 w^2 + \mu_2 w)^{-1/2} S^{\mu,\nu}_{m,q,\zeta}((t-w)^2; w)\Big|^{1/2}.
\end{aligned}$$

Therefore, we conclude that statement is valid for $\chi = 1$.

Next, we want to verify that the statement is also true whenever $\chi \in (0,1)$. We apply here the monotonicity property to the operators $S^{\mu,\nu}_{m,q,\zeta}$ and use Hölder's inequality. Thus, we find that

$$\begin{aligned}
\left| S^{\mu,\nu}_{m,q,\zeta}(\Psi; w) - \Psi(w) \right| &\leq S^{\mu,\nu}_{m,q,\zeta}\left(\left| \Psi(t) - \Psi(w) \right|; w \right) \\
&\leq \left(S^{\mu,\nu}_{m,q,\zeta}\left(\left| \Psi(t) - \Psi(w) \right|^{\frac{2}{\chi}}; w \right) \right)^{\frac{\chi}{2}} \left(S^{\mu,\nu}_{m,q,\zeta}(1; w) \right)^{\frac{2-\chi}{2}} \\
&\leq \mathcal{K} \left\{ \frac{S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2; w \right)}{t + \mu_1 w^2 + \mu_2 w} \right\}^{\frac{\chi}{2}} \\
&\leq \mathcal{K}(\mu_1 w^2 + \mu_2 w)^{-\chi/2} \left\{ S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2; w \right) \right\}^{\frac{\chi}{2}} \\
&= \mathcal{K} \left(\frac{\delta^{\mu,\nu}_{m,q,\zeta}(w)}{(\mu_1 w^2 + \mu_2 w)} \right)^{\frac{\chi}{2}}.
\end{aligned}$$

Thus, the statement is true for all $0 < \chi < 1$. This completes the proof. □

We obtain another local approximation result for the operators $S^{\mu,\nu}_{m,q,\zeta}$ by use of the Lipschitz maximal function. Suppose the class of all Lipschitz-type maximal functions $\Psi \in C[0,1]$ given by

$$\omega^*_{\chi}(\Psi; w) = \sup_{t \neq w,\ t \in [0,1]} \frac{|\Psi(t) - \Psi(w)|}{|t-w|^{\chi}} \qquad (t, w \in [0,1]) \tag{8}$$

where $0 < \chi \leq 1$ (see [34]).

Theorem 6. *Let $\Psi \in C[0,1]$ and $w \in [0,1]$. Then, $S^{\mu,\nu}_{m,q,\zeta}$ verifies the property*

$$\left| S^{\mu,\nu}_{m,q,\zeta}(\Psi;w) - \Psi(w) \right| \leq \left(\delta^{\mu,\nu}_{m,q,\zeta}(w) \right)^{\frac{\chi}{2}} \omega^*_{\chi}(\Psi;w),$$

where $\omega^*_{\chi}(\Psi;w)$ is defined by Equation (8) and $\delta^{\mu,\nu}_{m,q,\zeta}(w)$ is same as in Theorem 3.

Proof. It follows from the Hölder inequality that

$$\begin{aligned} \left| S^{\mu,\nu}_{m,q,\zeta}(\Psi;w) - \Psi(w) \right| &\leq S^{\mu,\nu}_{m,q,\zeta}(|\Psi(t) - \Psi(w)|;w) \\ &\leq \omega^*_{\chi}(\Psi;w) \mid S^{\mu,\nu}_{m,q,\zeta}(|t-w|^{\chi};w) \\ &\leq \omega^*_{\chi}(\Psi;w) \left(S^{\mu,\nu}_{m,q,\zeta}(1;w) \right)^{\frac{2-\chi}{2}} \left(S^{\mu,\nu}_{m,q,\zeta}(|t-w|^2;w) \right)^{\frac{\chi}{2}} \\ &= \omega^*_{\chi}(\Psi;w) \left(S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2;w \right) \right)^{\frac{\chi}{2}}, \end{aligned}$$

which gives the desired result. □

4. Rate of Convergence of $S^{\mu,\nu}_{m,q,\zeta}$

This section gives the rate of convergence for our new operators $S^{\mu,\nu}_{m,q,\zeta}$ defined by Equation (3) with the help of following definitions.

Suppose $C^2[0,1] = \{\Psi \in C[0,1] : \ \Psi', \Psi'' \in C[0,1]\}$. For $\Psi \in C[0,1]$ and any $\delta^* > 0$, Peetre's K-functional is defined as

$$K_2(\Psi;\delta^*) = \inf\left\{ \delta^* \|\Psi''\|_{C[0,1]} + \|\Psi - \Theta\|_{C[0,1]} : \Theta \in C^2[0,1] \right\}. \tag{9}$$

From [35], for any $\Psi \in C[0,1]$, for an absolute constant $C > 0$, we have

$$K_2(\Psi;\delta^*) \leq C\omega_2(\Psi;\sqrt{\delta^*}), \tag{10}$$

where $\omega_2(\Psi;\sqrt{\delta^*})$ denotes the second-order modulus of smoothness, given by

$$\omega_2(\Psi;\sqrt{\delta^*}) = \sup_{0<\mu<\sqrt{\delta^*}} \ \sup_{w,w+2\mu \in [0,1]} |\Psi(w+2\mu) - 2\Psi(w+\mu) + \Psi(w)|.$$

Note that the usual modulus of continuity is

$$\omega(\Psi;\delta^*) = \sup_{0<\mu\leq\delta^*} \ \sup_{w,w+\mu \in [0,1]} |\Psi(w+\mu) - \Psi(w)|.$$

Theorem 7. *By letting $\Psi \in C[0,1]$, then*

$$\begin{aligned} \left| S^{\mu,\nu}_{m,q,\zeta}(\Psi;w) - \Psi(w) \right| \leq\ & 4K_2\left(\Psi; \frac{\delta^{\mu,\nu}_{m,q,\zeta}(w) + \left(S^{\mu,\nu}_{m,q,\zeta}(t;w) - w \right)^2}{4} \right) \\ & + \omega\left(\Psi; S^{\mu,\nu}_{m,q,\zeta}(t;w) - w \right) \end{aligned}$$

for $w \in [0,1]$, where $\delta^{\mu,\nu}_{m,q,\zeta}(w) = S^{\mu,\nu}_{m,q,\zeta}\left((t-w)^2;w \right)$.

Proof. For $\Psi \in C[0,1]$ and $w \in [0,1]$, we define the auxiliary operators $T^{\mu,\nu}_{m,q,\zeta}$ as

$$T^{\mu,\nu}_{m,q,\zeta}(\Psi;w) = S^{\mu,\nu}_{m,q,\zeta}(\Psi;w) + \Psi(w) - \Psi\left(S^{\mu,\nu}_{m,q,\zeta}(t;w) \right). \tag{11}$$

Take $\Psi = \Psi_i = t^i$ for $i = 0, 1$. Then, we can verify that $T^{\mu,\nu}_{m,q,\zeta}(\Psi_0; w) = 1$:

$$T^{\mu,\nu}_{m,q,\zeta}(\Psi_1; w) = S^{\mu,\nu}_{m,q,\zeta}(\Psi_1; w) + w - S^{\mu,\nu}_{m,q,\zeta}(t; w) = w.$$

and $T^{\mu,\nu}_{m,q,\zeta}(t - w; w) = 0$. From the Taylor series expression, we know the equality

$$\Theta(t) = \Theta(w) + (t - w)\Theta'(w) + \int_w^t (t - \chi)\Theta''(\chi)d\chi, \ \ \Theta \in C^2[0, 1]. \tag{12}$$

Applying $T^{\mu,\nu}_{m,q,\zeta}$ to Equation (12) gives

$$\begin{aligned}
T^{\mu,\nu}_{m,q,\zeta}(\Theta; w) - \Theta(w) &= \Theta'(w)T^{\mu,\nu}_{m,q,\zeta}(t - w; w) + T^{\mu,\nu}_{m,q,\zeta}\left(\int_w^t (t - \chi)\Theta''(\chi)d\chi; w\right) \\
&= T^{\mu,\nu}_{m,q,\zeta}\left(\int_w^t (t - \chi)\Theta''(\chi)d\chi; w\right) \\
&= S^{\mu,\nu}_{m,q,\zeta}\left(\int_w^t (t - \chi)\Theta''(\chi)d\chi; w\right) + \int_w^w (w - \chi)\Theta''(\chi)d\chi; w \\
&\quad - \int_w^{S^{\mu,\nu}_{m,q,\zeta}(t;w)} \left(S^{\mu,\nu}_{m,q,\zeta}(t; w) - \chi\right)\Theta''(\chi)d\chi
\end{aligned}$$

which gives

$$\begin{aligned}
|T^{\mu,\nu}_{m,q,\zeta}(\Theta; w) - \Theta(w)| &\leq \left|S^{\mu,\nu}_{m,q,\zeta}\left(\int_w^t (t - \chi)\Theta''(\chi)d\chi; w\right)\right| \\
&\quad + \left|\int_w^{S^{\mu,\nu}_{m,q,\zeta}(t;w)} \left(S^{\mu,\nu}_{m,q,\zeta}(t; w) - \chi\right)\Theta''(\chi)d\chi\right|.
\end{aligned} \tag{13}$$

A simple calculation yields

$$\left|\int_w^t (t - \chi)\Theta''(\chi)d\chi\right| \leq (t - w)^2\|\Theta''\|$$

and

$$\left|\int_w^{S^{\mu,\nu}_{m,q,\zeta}(t;w)} \left(S^{\mu,\nu}_{m,q,\zeta}(t; w) - \chi\right)\Theta''(\chi)d\chi\right| \leq \left(S^{\mu,\nu}_{m,q,\zeta}(t; w) - w\right)^2\|\Theta''\|.$$

Thus, Equation (13) becomes

$$|T^{\mu,\nu}_{m,q,\zeta}(\Theta; w) - \Theta(w)| \leq \left\{S^{\mu,\nu}_{m,q,\zeta}\left((t - w)^2; w\right) + \left(S^{\mu,\nu}_{m,q,\zeta}(t; w) - w\right)^2\right\}\|\Theta''\|.$$

We deduce from Equation (3) that

$$|S^{\mu,\nu}_{m,q,\zeta}(\Psi; w)| \leq \|\Psi\|,$$

Thus, this yields

$$|T^{\mu,\nu}_{m,q,\zeta}(\Psi; w)| \leq |S^{\mu,\nu}_{m,q,\zeta}(\Psi; w)| + |\Psi(w)| + \left|\Psi\left(S^{\mu,\nu}_{m,q,\zeta}(t; w)\right)\right| \leq 3\|\Psi\|. \tag{14}$$

It follows from Equations (11) and (14) that

$$\begin{aligned}
\left|S^{\mu,\nu}_{m,q,\zeta}(\Psi; w) - \Psi(w)\right| &\leq \left|T^{\mu,\nu}_{m,q,\zeta}(\Psi - \Theta; w) - (\Psi - \Theta)(w)\right| \\
&\quad + \left|T^{\mu,\nu}_{m,q,\zeta}(\Theta; w) - \Theta(w)\right| + \left|\Psi(w) - \Psi\left(S^{\mu,\nu}_{m,q,\zeta}(t; w)\right)\right|
\end{aligned}$$

$$\begin{aligned} &\leq 4\|\Psi-\Theta\|+\omega\left(\Psi; S^{\mu,\nu}_{m,q,\zeta}(t;w)-w\right) \\ &\quad +\left\{\delta^{\mu,\nu}_{m,q,\zeta}(w)+\left(S^{\mu,\nu}_{m,q,\zeta}(t;w)-w\right)^2\right\}\|\Theta''\|. \end{aligned}$$

By letting $\inf_{\Theta\in C^2[0,1]}$ and using Equation (9), then we arrive at

$$\begin{aligned} \left|S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)-\Psi(w)\right| &\leq 4K_2\left(\Psi; \frac{\delta^{\mu,\nu}_{m,q,\zeta}(w)+\left(S^{\mu,\nu}_{m,q,\zeta}(t;w)-w\right)^2}{4}\right) \\ &\quad +\omega\left(\Psi; S^{\mu,\nu}_{m,q,\zeta}(t;w)-w\right). \end{aligned}$$

Thus, we obtain our desired inequality: □

Corollary 1. *Let $\Psi\in C[0,1]$. Then, the inequality*

$$\begin{aligned} \left|S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)-\Psi(w)\right| &\leq \mathcal{C}\omega_2\left(\Psi; \frac{\sqrt{\delta^{\mu,\nu}_{m,q,\zeta}(w)+\left(S^{\mu,\nu}_{m,q,\zeta}(t;w)-w\right)^2}}{2}\right) \\ &\quad +\omega\left(\Psi; S^{\mu,\nu}_{m,q,\zeta}(t;w)-w\right) \end{aligned}$$

holds for any $w\in[0,1]$, where $\mathcal{C}$ is a positive constant.

Proof. The proof follows from Theorem 7 and the inequality in Equation (10). □

5. Conclusions

In our discussion, we defined the Stancu-type modification of q-Bernstein–Kantorovich positive linear operators involving Bézier bases $\xi_{m,\ell}(w;q,\zeta)$ $(\ell:=0,1,\ldots,m)$. We discussed certain approximation results for the operators, namely the uniform convergence, local approximation, rate of convergence, as well as some other related results.

In the case of $\mu=\nu=0$, the operators $S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)$ become $K_{m,q,\zeta}(\Psi;w)$ ((ζ,q)-Bernstein–Kantorovich operators) [30]. In addition, if $\zeta=0$, then $S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)$ becomes a q-Bernstein–Kantorovich operator [36]. If $q=1$, together with above assumptions (i.e., $\mu=\nu=\zeta=0$), then $S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)$ becomes a classical Bernstein–Kantorovich operator [37]. Furthermore, the choice of $\mu=\nu=0$ and $q=1$ in $S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)$ gives $S^{0,0}_{m,1,\zeta}(\Psi;w)$, which was studied in [24]. Consequently, we conclude that $S^{\mu,\nu}_{m,q,\zeta}(\Psi;w)$ is a non-trivial modification of certain widely studied operators and therefore our approximation results too.

The detailed description of the application of Bernstein-type operators has been presented by Occorsio and Russo [38], wherein they constructed a stable and convergent cubature rule by means of their Bernstein operators and obtained the solutions of Fredholm integral equations with the help of the Nyström method, which was based on the aforementioned cubature rule. Based on the above observation, we suggest the application of our Stancu-type (ζ,q)-Bernstein–Kantorovich operators in this study to find the solution of the Fredholm and Volterra integral equations for researchers who are working on linking these two theories.

Author Contributions: Conceptualization, W.-T.C., M.N. and S.A.M.; methodology, W.-T.C., M.N. and S.A.M.; formal analysis, W.-T.C., M.N. and S.A.M.; investigation, W.-T.C., M.N. and S.A.M.; writing—original draft preparation, W.-T.C., M.N. and S.A.M.; writing—review and editing, W.-T.C., M.N. and S.A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the National Natural Science Foundation of China (Grant No. 11626031), the Key Natural Science Research Project in Universities of Anhui Province (Grant No. KJ2021A0648 and KJ2019A0572), the Philosophy and Social Sciences General Planning Project of Anhui Province of China (Grant No. AHSKYG2017D153), and the Natural Science Foundation of Anhui Province of China (Grant No. 1908085QA29).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees for their very valuable suggestions and remarks.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bernstein, S.N. Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités. *Commun. Soc. Math. Kharkow* **1913**, *2*, 1–2.
2. Lupaş, A. A q-analogue of the Bernstein operator. In *Seminar on Numerical and Statistical Calculus*; University of Cluj-Napoca: Cluj-Napoca, Rominia, 1987; Volume 9, pp. 85–92.
3. Phillips, G.M. Bernstein polynomials based on the q-integers. *Ann. Numer. Math.* **1997**, *4*, 511–518.
4. Heping, W.; Meng, F. The rate of convergence of q-Bernstein polynomials for $0 < q < 1$. *J. Approx. Theor.* **2005**, *136*, 151–158.
5. Dogrŭ, O.; İçöz, G.; Kanat, K. On the rates of convergence of the q-Lupas-Stancu operators. *Filomat* **2016**, *30*, 1151–1160. [CrossRef]
6. Nowak, G. Approximation properties for generalized q-Bernstein polynomials. *J. Math. Anal. Appl.* **2009** *350*, 50–55. [CrossRef]
7. Mahmudov, N.I.; Sabancigil, P. Approximation theorems for q-Bernstein-Kantorovich operators. *Filomat* **2013**, *27*, 721–730. [CrossRef]
8. Kac, V.; Cheung, P. *Quantum Calculus*; Springer, New York, NY, USA, 2002.
9. Jackson, F.H. On q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
10. Chen, X.; Tan, J.; Liu, Z.; Xie, J. Approximation of functions by a new family of generalized Bernstein operators. *J. Math. Anal. Appl.* **2017**, *450*, 244–261. [CrossRef]
11. Mohiuddine, S.A.; Acar, T.A. Alotaibi, Construction of a new family of Bernstein-Kantorovich operators. *Math. Meth. Appl. Sci.* **2017**, *40*, 7749–7759. [CrossRef]
12. Özger, F.; Srivastava, H.M.; Mohiuddine, S.A. Approximation of functions by a new class of generalized Bernstein-Schurer operators. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM* **2020**, *114*, 173. [CrossRef]
13. Mohiuddine, S.A.; Özger, F. Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter α. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM* **2020**, *114*, 70. [CrossRef]
14. Kajla, A.; Miclăuş, D. Blending type approximation by GBS operators of generalized Bernstein-Durrmeyer type. *Results Math.* **2017**, *73*, 1. [CrossRef]
15. Kajla, A.; Mohiuddine, S.A.; Alotaibi, A. Blending-type approximation by Lupaş-Durrmeyer-type operators involving Pólya distribution. *Math. Meth. Appl. Sci.* **2021**, *44*, 9407–9418. [CrossRef]
16. Mursaleen, M.; Ansari, K.J.; Khan, A. Approximation by a Kantorovich type q-Bernstein-Stancu operators. *Complex Anal. Oper. Theory* **2017**, *11*, 85–107. [CrossRef]
17. Mursaleen, M.; Khan, F.; Khan, A. Approximation properties for King's type modified q-Bernstein-Kantorovich operators. *Math. Meth. Appl. Sci.* **2015**, *38*, 5242–5252. [CrossRef]
18. Mursaleen, M.; Khan, F.; Khan, A. Approximation properties for modified q-Bernstein-Kantorovich operators. *Numer. Funct. Anal. Optim.* **2015**, *36*, 1178–1197. [CrossRef]
19. Aral, A.; Erbay, H. Parametric generalization of Baskakov operators. *Math. Commun.* **2019**, *24*, 119–131.
20. İlarslan, H.G.I.; Erbay, H.; Aral, A. Kantorovich-type generalization of parametric Baskakov operators. *Math. Meth. Appl. Sci.* **2019**, *42*, 6580–6587. [CrossRef]
21. Mohiuddine, S.A.; Ahmad, N.; Özger, F.; Alotaibi, A.; Hazarika, B. Approximation by the parametric generalization of Baskakov-Kantorovich operators linking with Stancu operators. *Iran. J. Sci. Technol. Trans. Sci.* **2021**, *45*, 593–605. [CrossRef]
22. Cai, Q.B.; Lian, B.Y.; Zhou, G. Approximation properties of λ-Bernstein operators. *J. Inequal. Appl.* **2018**, *2018*, 61. [CrossRef]
23. Ye, Z.; Long, X.; Zeng, X.M. Adjustment algorithms for Bézier curve and surface. In Proceedings of the International Conference on Computer Science and Education, Hefei, China, 24–27 August 2010; pp. 1712–1716.
24. Cai, Q.B. The Bézier variant of Kantorovich type λ-Bernstein operators. *J. Inequal. Appl.* **2018**, *2018*, 90. [CrossRef] [PubMed]
25. Özger, F. On new Bézier bases with Schurer polynomials and corresponding results in approximation theory. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2020**, *69*, 376–393. [CrossRef]
26. Srivastava, H.M.; Özger, F.; Mohiuddine, S.A. Construction of Stancu type Bernstein operators based on Bézier bases with shape parameter λ. *Symmetry* **2019**, *11*, 316. [CrossRef]

27. Cai, Q.B.; Zhou, G.; Li, J. Statistical approximation properties of λ-Bernstein operators based on q-integers. *Open Math.* **2019**, *17*, 487–498. [CrossRef]
28. Cai, Q.B.; Aslan, R. On a new construction of generalized q-Bernstein polynomials based on shape parameter λ. *Symmetry* **2021**, *13*, 1919. [CrossRef]
29. Mursaleen, M.; Al-Abied, A.A.H.; Salman, M. A. Chlodowsky type (λ, q)-Bernstein-Stancu operators. *Azerb. J. Math.* **2020**, *10*, 75–101.
30. Cai, Q.B.; Aslan, R. Note on a new construction of Kantorovich form q-Bernstein operators related to shape parameter λ. *CMES Comput. Model. Eng. Sci.* **2021**, *130*, 1479–1493. [CrossRef]
31. Cheney, E.W. *Introduction to Approximation Theory*; McGraw-Hil: New York, NY, USA, 1966.
32. Shisha, O.; Bond, B. The degree of convergence of sequences of linear positive operators. *Proc. Nat. Acad. Sci. USA* **1968**, *60*, 1196–1200. [CrossRef]
33. Ozarslan, M.A.; Aktuğlu, H. Local approximation for certain King type operators. *Filomat* **2013**, *27*, 173–181. [CrossRef]
34. Lenze, B. On Lipschitz type maximal functions and their smoothness spaces. *Nederl. Akad. Indag. Math.* **1988**, *50*, 53–63. [CrossRef]
35. DeVore, R.A.; Lorentz, G.G. *Constructive Approximation*; Springer, Berlin/Heidelberg, Germany, 1993.
36. Dalmanoglu, O. Approximation by Kantorovich type q-Bernstein operators. In Proceedings of the 12th WSEAS International Conference on Applied Mathematics, Cairo, Egypt, 29–31 December 2007; pp. 113–117.
37. Kantorovich, L.V. Sur certain d́eveloppements suivant les polyñomes de la forme de S. Bernstein I, II. *C. R. Acad. URSS* **1930**, *563–568*, 595–600.
38. Occorsio, D.; Russo, M.G. Bivariate generalized bernstein operators and their application to Fredholm integral equations. *Publ. Inst. Math.* **2016**, *100*, 141–162. [CrossRef]