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Stability and Stabilization of 2D Linear Discrete Systems with Fractional Orders Based on the Discrimination System of Polynomials

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Abstract: This paper considers the stability and stabilization of two-dimensional (2D) fractional-order systems described by state-space model based on the discrimination system of polynomials. Necessary and sufficient conditions of stability and stabilization are established. We change the criterion for checking the stability of linear discrete-time 2D fractional-order systems into an easy checking criterion whether some polynomials are positive. We use the discrimination system of polynomials to check the new conditions. For the stabilization problem, we get a stable gain matrix region. The unstable system with the gain parameters of the stable gain matrix region is stable. We give the method of stability analysis and stabilization for the general 2D fractional-order system. An example shows the validity of the proposed stability and stabilization methods.

Keywords: 2D fractional-order systems; stability; stabilization; the discrimination system for polynomials

MSC: 93D09; 93D15

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1. Introduction

In recent years, 2D systems have increasingly attracted attention and become more important in the theory and practice field with broad applications, such as for river pollution models, photogrammetric data, batch processing, iterative learning control, and multi-dimensional digital filtering [1]. This paper focuses on the two models introduced by Roesser in [2] and Fornasini and Marchesini in [3,4], which are popular with people for their concise description. Although there are fruitful methods concerning stability and stabilization of 2D systems which have been proposed, they mostly focus on integer-order systems. Integer-order 2D systems can't accurately describe many practical systems, such as circuit components, electro-magnetic systems, heat transfer processes, or viscoelastic systems. On the contrary, they are well characterized by the fractional-order 2D systems [5,6]. Many results have been given about the stability analysis of fractional-order systems in [7–11]. Specifically, the methods based on Lyapunov functions were derived in [7–9] for analyzing stability of fractional one-dimensional (1D) systems. Yang and Hou in [10,11] studied the fractional-order systems with perturbation via cylindrical algebraic decomposition method. These methods for studying fractional systems mostly focus on 1D systems.

Kaczorek in [12] firstly proposed the concept of fractional-order 2D discrete systems. For 2D fractional systems, some results have been investigated in [13–18]. Specifically, for the fractional-orders continuous 2D systems represented by the FM (Fornasini–Marchesini) second model, the general solution formula was obtained based on 2D Laplace transform in [13]. Ref. [14] proposed an asymptotic stability criterion of fractional 2D non-linear continuous-time system based on they Lyapunov function method. In [15], the concept, the practical stability of positive fractional 2D linear systems, was proposed by Tadeusz Kaczorek. The 2D fractional system described by FM first type was derived. However, the dimensions of input and output vectors increase when the system variable decreases. Tadeusz Kaczorek in [16] showed the result of the asymptotic stability for positive fractional 2D linear system. Refs. [17,18] studied the stabilization issue of 2D fractional systems, they proposed the practical stability of the positive fractional 2D system. Specifically, in [17], Tadeusz Kaczorek introduced a class of fractional 2D system presented by Roesser model. The sufficient criteria of the positivity and stabilization were established. Laila Dami et al. studied the issues of positivity stabilization for the uncertain 2D fractional discrete-time systems in [18]. These papers have well studied the stability and stabilization problem for the positive fractional 2D linear systems. The problem of the stability and stabilization of the general 2D linear discrete systems with fractional order is still an open problem to be solved. Fractional-order 2D systems have attracted increasing interest, due to the fact that many real-world physical systems are well characterized by fractional-order 2D systems.

In this paper, we introduce the issues of stability and stabilization for 2D linear discrete systems with fractional orders. Firstly, based on the existing method, the fractional-order 2D system is transformed into an integral-order system. Secondly, based on the Hurwitz Theorem, we equivalently convert the existing stability condition into a new easily checked condition. Then, we use the discriminant theory of polynomials in [19] to solve the represented condition. We extend the stability results in this paper to the problem of stabilization. The key contributions related to this paper are shown as follows:

(1) We change checking whether the fractional 2D linear discrete system is stable into checking whether the polynomials are positive based on Hurwitz Theorems. Thus, the processing of stability analysis is changed into a mathematical problem whether some polynomials are definitely positive, which can be easily checked. It simplifies the existing methods based on Lyapunov functions in [7–9] and has low complexity.

(2) Based on the results proposed by Kaczorek in [15–18], we give a more general method of stability and stabilization for 2D linear fractional-order discrete systems not only for the positive systems.

(3) For the stabilization, because the condition is necessary and sufficient, we can get a complete solution of gain matrixes called the stable gain matrix region of the considered unstable system. The unstable system with the gain parameters of the stable gain matrix region is stabilizable.

The organization is as follows: Section 2 shows problem formulation and fractional 2D system representation. In Section 3, we give the results of stability analysis and an algorithm for obtaining the stable gain matrix region. In Section 4, an example is given to analyze the stability and get the stable gain matrix region to show the effectiveness of the proposed methods. Section 5 shows the conclusions.

Notations. $\mathbb{C}$ and $\mathbb{R}$ stand for the set of complex numbers and real numbers, respectively. The symbols $Re(x)$ is the real part of x. $\mathbb{C}^+ \triangleq \{x \in \mathbb{C} : Re(x) > 0\}$. I and 0 stand for identity matrix and zero block of appropriate sizes, respectively. j denotes an imaginary unit. $det(\Phi)$ denotes determinant of a matrix Φ. $\mathbb{D} \triangleq \{z \in \mathbb{C} \mid |z| \leq 1\}$,$\mathbb{P} \triangleq \{z \in \mathbb{C} \mid |z| = 1\}$, $\bar{\mathbb{U}} \triangleq \{(z_1, z_2) : z_1, z_2 \in \mathbb{C} \mid |z_1| \leq 1, |z_2| \leq 1\}$. $f(\tau) = \kappa_n \tau^n + \kappa_{n-1}\tau^{n-1} + \ldots + \kappa_0 (\kappa_0 > 0)$ is a real coefficient polynomial, where κ_i is real. The $n \times n$ Hurwitz matrix of $f(\tau)$ denotes

$$M_f = \begin{bmatrix} \kappa_{n-1} & \kappa_{n-3} & \kappa_{n-5} & \cdots & 0 \\ \kappa_n & \kappa_{n-2} & \kappa_{n-4} & \cdots & 0 \\ 0 & \kappa_{n-1} & \kappa_{n-3} & \cdots & 0 \\ 0 & \kappa_n & \kappa_{n-2} & \cdots & 0 \\ \ldots & \ldots & \ldots & \cdots & \ldots \\ 0 & 0 & 0 & \cdots & \kappa_0 \end{bmatrix},$$

$\triangle(f)_k, k = 1, 2 \ldots, n$, represent the k_{th} principal minor determinant of M_f, respectively. $y(\tau)$ is a complex coefficient polynomial and satisfies $y(j\tau) = \varrho_n \tau^n + \varrho_{n-1}\tau^{n-1} + \ldots +$

$\varrho_0 + j(\kappa_n \tau^n + \kappa_{n-1}\tau^{n-1} + \ldots + \kappa_0)$, $\kappa_n \neq 0$, where κ_i and ϱ_i are real. The $2n \times 2n$ Hurwitz matrix of $y(\tau)$ denotes

$$M_y = \begin{bmatrix} \kappa_n & \kappa_{n-1} & \kappa_{n-2} & \cdots & 0 \\ \varrho_n & \varrho_{n-1} & \varrho_{n-2} & \cdots & 0 \\ 0 & \kappa_n & \kappa_{n-1} & \cdots & 0 \\ 0 & \varrho_n & \varrho_{n-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \kappa_0 \\ 0 & 0 & 0 & \cdots & \varrho_0 \end{bmatrix},$$

$\triangle(y)_{2k}, k = 1, 2 \ldots, n$, represent the $2k_{th}$ principal minor determinant of M_y, respectively.

2. Problem Formulation and Fractional-Order 2D System Representation

The aim of this section is to get a fractional 2D linear discrete system represented by integral-order 2D model. We show the process obtained fractional 2D system represented by Roesser model. The transformation from Roesser model to FM second model is introduced for further discussing the stability and stabilization problems. Focus on the following fractional 2D linear system represented as the state-space equations.

$$\begin{bmatrix} \Delta_\alpha^h x_{i+1,j} \\ \Delta_\beta^v x_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}, \tag{1}$$

$$y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du_{ij}, \quad i,j \in \mathbb{Z}_+, \tag{2}$$

where $x_{ij}^h \in \mathbb{R}^{n_1}, x_{ij}^v \in \mathbb{R}^{n_2}, u_{ij} \in \mathbb{R}^m, y_{ij} \in \mathbb{R}^p$ are horizontal state vector, vertical state vector, input vector and output vector at the point (i,j), respectively. And $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}^{n_2 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $B_2 \in \mathbb{R}^{n_2 \times m}$, $C_1 \in \mathbb{R}^{p \times n_1}$, $C_2 \in \mathbb{R}^{p \times n_2}$, $D \in \mathbb{R}^{p \times m}$, $n = n_1 + n_2$.

The boundary conditions are defined by

$$x_{0j}^h, j \in \mathbb{Z}_+ \text{ and } x_{i0}^v, i \in \mathbb{Z}_+ \tag{3}$$

For further getting the Roesser model representing the 2D fractional system, we recall the definitions, horizontal and vertical fractional differences described by the 2D functions, and Lemma 1.

Definition 1 ([17]). *The $\alpha-$order horizontal fractional difference of a 2D function x_{ij}, $i,j \in \mathbb{Z}_+$ is defined by*

$$\Delta_\alpha^h x_{ij} = \sum_{k=0}^{i} c_\alpha(k) x_{i-k,j}, \tag{4}$$

where $\alpha \in \mathbb{R}, n-1 < \alpha < n \in \mathbb{N} = 1, 2, \ldots$ and

$$c_\alpha(k,l) = \begin{cases} 1 & for \quad k = 0 \\ (-1)^k \dfrac{k!}{\alpha(\alpha-1)\ldots(\alpha-k+1)} & \quad k > 0 \end{cases} \tag{5}$$

Definition 2 ([17]). *The $\beta-$order vertical fractional difference of a 2D function x_{ij}, $i,j \in \mathbb{Z}_+$ is defined by*

$$\Delta_\beta^v x_{ij} = \sum_{l=0}^{j} c_\beta(k) x_{i,j-l}, \tag{6}$$

where $\beta \in \mathbb{R}, n-1 < \beta < n \in \mathbb{N} = 1, 2, \ldots$ *and*

$$c_\beta(l) = \begin{cases} 1 & for \quad l = 0 \\ (-1)^l \dfrac{l!}{\beta(\beta-1)\ldots(\beta-l+1)} & \quad l > 0 \end{cases} \tag{7}$$

Remark 1. *We have mentioned that many real-world physical systems are well characterized by a fractional-order model. And the physical systems are generally continuous. We present a definition of fractional derivative and integral by Grünwald-Letnikov as follow:*

$${}^{GL}_{t_0}\mathcal{D}^{\alpha}_{t} f(t) \approx \frac{1}{h^\alpha} \sum_{j=0}^{[(t-t_0)/h]} \omega_j f(t-jh)$$

Forthe purpose of easy calculation, the continuous physical models are usually discretized. Definitions 1 and 2 can be obtained by determining the step size of the fractional order equation. The size of h determines the accuracy of the model. The smaller h is, the closer the model is to the real system. We can select h according to the requirement of precision in practical production. And the fractional order system with different step h can be converted into different integer order system. The system is built as a 2D fractional-order model then convert to a 2D integer-order model instead of directly building a 2D integer-order model, which has a more accurate result.

Lemma 1 ([17])**.** *If* $n-1 < \alpha < n \in \mathbb{N}(n-1 < \beta < n)$, *then*

$$\sum_{k=0}^{\infty} c_\alpha(k) = 0 \quad (resp. \sum_{k=0}^{\infty} c_\beta(k) = 0). \tag{8}$$

According to Definitions 1 and 2, system (1) can be rewritten as follows

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij} \\ &\quad - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) x_{i-k+1,j} \\ \sum_{l=2}^{j+1} c_\beta(k) x_{i,j-l+1} \end{bmatrix}, \end{aligned} \tag{9}$$

where $n = n_1 + n_2$, $\overline{A}_{11} = A_{11} + \alpha I_{n_1}$ and $\overline{A}_{22} = A_{22} + \beta I_{n_2}$. 2D fractional system (1) has been rewritten as the integer-order 2D system with delays.

From Equations (5) and (7), we can get that $c_\alpha(0) = c_\beta(0) = 1$, $c_\alpha(1) = -\alpha$ and $c_\beta(1) = -\beta$. Based on these equations and Lemma 1 we have

$$\sum_{k=2}^{\infty} c_\alpha(k) = \alpha - 1 \; and \; \sum_{k=2}^{\infty} c_\beta(k) = \beta - 1 \tag{10}$$

We firstly analyze the stability of 2D fractional system. Let the input vector $u_{ij} = 0$. We consider the open-loop system

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) x^h_{i-k+1,j} \\ \sum_{k=2}^{j+1} c_\beta(l) x^v_{i,j-l+1} \end{bmatrix}. \tag{11}$$

From [17], the system (11) is asymptotically stable if and only if the following 2D system

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(\begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} - \sum_{k=2}^{\infty} \begin{bmatrix} I_{n_1} c_\alpha(k) & 0 \\ 0 & I_{n_2} c_\beta(k) \end{bmatrix} \right) \cdot \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{12}$$

is asymptotically stable.

According to (10), $\overline{A}_{11} = A_{11} + I_{n_1}\alpha$ and $\overline{A}_{22} = A_{22} + I_{n_2}\beta$, the system (12) is represented as the following form

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} + I_{n_1} & A_{12} \\ A_{21} & A_{22} + I_{n_2} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{13}$$

Remark 2. *For the purposes of analyzing stability of the fractional-order 2D model, we convert the fractional order model into the integer-order model.*

Due to the above discussion, if the 2D system (13) is asymptotically stable, the 2D fractional system (1) with $u_{ij} = 0$ is asymptotically stable. System (13) is a typical Roesser model. The considered stability issue of fractional-order 2D systems is changed into considering the stability of 2D integral-order Roesser model.

Let $x(i,j) = [x^{hT}(i,j)\ x^{vT}(i,j)]^T$ and the matrices

$$A_1 = \begin{bmatrix} A_{11} + I_{n_1} & A_{12} \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} + I_{n_2} \end{bmatrix}$$

System (13) can be represented as the following FM second model:

$$x(i+1,j+1) = A_1 x(i,j+1) + A_2 x(i+1,j), \tag{14}$$

where $x(i+1,j+1)$ denotes the state vector at $(i+1,j+1)$.

Thus, we can analyze the stability of system (14) to know the stability of fractional-order 2D system (1).

For better presenting and understanding the following content in this paper, we give some definitions and recall a lemma and the lemma of Hurwitz stable of the real coefficient polynomial and the complex coefficient polynomial, respectively.

Definition 3. *The gain matrix K is called the stable gain matrix of the closed-loop 2D fractional-order system if the closed-loop 2D fractional-order system with the stable gain matrix is stabilizable.*

Definition 4. *The set of the stable gain matrixes of the closed-loop 2D fractional-order system is called the stable gain matrix region of the closed-loop 2D fractional-order system.*

Lemma 2 ([20])**.** *System (14) is asymptotically stable if and only if*

$$\begin{cases} H(z_1, 0) \neq 0, z_1 \in \mathbb{D} \\ H(z_1, z_2) \neq 0, z_1 \in \mathbb{P}, z_2 \in \mathbb{D} \end{cases} \tag{15}$$

where $H(z_1, z_2) = det(I_n - z_1 A_1 - z_2 A_2)$.

Remark 3. *Condition (15) is not numerically tractable [21]. Next, Condition (15) is transformed into new conditions that can be easily implemented.*

Lemma 3 ([22])**.** *The necessary and sufficient condition for the roots' real part of real coefficient polynomial* $f(\tau)$ *to be negative is* $\triangle(f)_k > 0, k = 1, 2 \ldots, n$.

Lemma 4 ([22])**.** *If* $\triangle(y)_{2k} \neq 0$ *, the necessary and sufficient condition for the roots' real part of complex coefficient polynomial* $y(\tau)$ *to be negative is* $\triangle(y)_{2k} > 0, k = 1, 2 \ldots, n$.

3. Stability and Stabilization Analysis

3.1. Stability Analysis

This subsection is to obtain new tractable conditions based on traditional condition (15). By linear fraction transformation, the stability conditions of 2D system is equivalent to the issue whether the polynomials are Hurwitz stable shown in Theorem 1. We use the criterion of Lemmas 3 and 4 to deal with the new derived conditions of Theorem 1. And we can get the tractable conditions of Theorem 2.

Theorem 1. *System (1) with $u_{ij} = 0$ is asymptotically stable if and only if these criteria are satisfied,*

(1) $H(-1,0) \neq 0, L_1(\gamma,0) \neq 0, \gamma \in \mathbb{C}^+$

(2) $H(-1,-1) \neq 0, L_2(-1,\gamma) \neq 0, \gamma \in \mathbb{C}^+$

(3) $W(s,-1) \neq 0, L_3(s,\gamma) \neq 0, s \in \mathbb{R}, \gamma \in \mathbb{C}^+$

where $W(s,z_2) = (1-js)^m H(\frac{1+js}{1-js}, z_2)$,
$L_1(\gamma,0) = (1+\gamma)^m H(\frac{1-\gamma}{1+\gamma}, 0)$,
$L_2(-1,\gamma) = (1+\gamma)^m H(-1, \frac{1-\gamma}{1+\gamma})$,
$L_3(s,\gamma) = (1+\gamma)^n W(s, \frac{1-\gamma}{1+\gamma})$.

Proof. Substitute $z_1 = \frac{1+js}{1-js}$ to $H(z_1,z_2)$ of condition (15). We can get

$$W(s,z_2) = (1-js)^m H(\frac{1+js}{1-js}, z_2), \tag{16}$$

where m stands for the degree of $H(z_1,z_2)$ in z_1. We can easily obtain that condition (15) of Lemma 2 is equivalent to

$$\begin{cases} H(z_1,0) \neq 0, z_1 \in \mathbb{D} \\ H(-1,z_2) \neq 0, z_2 \in \mathbb{D} \\ W(s,z_2) \neq 0, s \in \mathbb{R}, z_2 \in \mathbb{D} \end{cases} \tag{17}$$

Substitute $z_1 = \frac{1-\gamma}{1+\gamma}$ to $H(z_1,0)$. Substitute $z_2 = \frac{1-\gamma}{1+\gamma}$ to $H(-1,z_2)$ and $W(s,z_2)$, respectively. We can obtain

$$\begin{cases} L_1(\gamma,0) = (1+\gamma)^m H(\frac{1-\gamma}{1+\gamma}, 0), \\ L_2(-1,\gamma) = (1+\gamma)^m H(-1, \frac{1-\gamma}{1+\gamma}), \\ L_3(s,\gamma) = (1+\gamma)^n W(s, \frac{1-\gamma}{1+\gamma}) \end{cases} \tag{18}$$

By the above transformations, condition (17) is equivalent to

$$\begin{cases} H(-1,0) \neq 0, L_1(\gamma,0) \neq 0, \gamma \in \mathbb{C}^+ \\ H(-1,-1) \neq 0, L_2(-1,\gamma) \neq 0, \gamma \in \mathbb{C}^+ \\ H(s,-1) \neq 0, L_3(s,\gamma) \neq 0, s \in \mathbb{R}, \gamma \in \mathbb{C}^+ \end{cases} \tag{19}$$

Condition (15) is equivalently converted into Condition (19). The 2D system (14) is asymptotically stable if and only if the condition (19) is satisfied. The fractional 2D linear discrete system (1) with $u_{ij} = 0$ can be represented by integral-order 2D model (14) in Section 2. So system (1) with $u_{ij} = 0$ is asymptotically stable if and only if the Condition (19) is satisfied. The proof is complete. □

The conditions of Theorem 1 can be represented by the criterion of Lemmas 3 and 4. We show it as follows.

Theorem 2. *System (1) with $u_{ij} = 0$ is asymptotically stable if and only if these conditions are satisfied,*

(1) $H(-1,0) \neq 0, \triangle(L_1)_k > 0, k = 1,2\ldots,n,$

(2) $H(-1,-1) \neq 0, \triangle(L_2)_k > 0, k = 1,2\ldots,n,$

(3) $W(s,-1) \neq 0, \triangle(L_3)_{2k} > 0, k = 1,2\ldots,n, s \in \mathbb{R}.$

Proof. In Theorem 1, $L_1(\gamma, 0)$, $L_2(-1, \gamma)$ are real coefficient polynomial in γ. $L_1(\gamma, 0) \neq 0, \gamma \in \mathbb{C}^+$ and $L_2(-1, \gamma) \neq 0, \gamma \in \mathbb{C}^+$, the roots' real part of L_1 and L_2 are negative, are the criterion of Lemma 3, $\triangle(L_1)_k > 0$ of Hurwitz matrix M_{L_1} and $\triangle(L_2)_k > 0$ of Hurwitz matrix M_{L_2}.

$L_3(s, \gamma) \neq 0, s \in \mathbb{R}, \gamma \in \mathbb{C}^+$, the roots' real part of $L_3(s, \gamma)$ to be negative, is the criterion of Lemma 4, the $2k_{th}$ principal minor determinants $\triangle(L_3)_{2k} > 0$ of the Hurwitz matrix M_{L_3}. According to the above, we rewrite the conditions of Theorem 1 and show them in Theorem 2. The proof is complete. □

In this section, we firstly transform the traditional stability conditions in [20] into Theorem 1 by the linear fraction transformation. Then the obtained conditions of Theorem 1 are the criterions whether the polynomials are Hurwitz stable. These new criterions are represented as Theorem 2 by using the conditions of Lemmas 3 and 4 in [22]. In next subsection, we focus on the stabilization problem applying the similar process of the above proposed method of checking stability.

3.2. Stabilization

This section is to design a state feedback to stabilize the system and get the stable gain matrix region. Consider the system (1) with the following state-feedback

$$u_{ij} = [K_1 K_2] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \tag{20}$$

where $K = [K_1 \ K_2] \in \mathbb{R}^{m \times n}, K_j \in \mathbb{R}^{m \times n_j}, j = 1, 2$ is a gain matrix.

A gain matrix K need to be solved to ensure that the closed-loop system is stabilizable via state feedback. Specifically, K need to be fond to ensure that the following system

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} \overline{A}_{11} + B_1K_1 & A_{12} + B_1K_2 \\ A_{21} + B_2K_1 & \overline{A}_{22} + B_2K_2 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ &\quad - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) x^h_{i-k+1,j} \\ \sum_{k=2}^{j+1} c_\beta(l) x^v_{i,j-l+1} \end{bmatrix} \end{aligned} \tag{21}$$

is asymptotically stable.

Same as the operations of stability analysis, we can easily get the results that the system (21) is asymptotically stable if and only if the following 2D system

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= \Bigg(\begin{bmatrix} \overline{A}_{11} + B_1K_1 & A_{12} + B_1K_2 \\ A_{21} + B_2K_1 & \overline{A}_{22} + B_2K_2 \end{bmatrix} \\ &\quad - \sum_{k=2}^{\infty} \begin{bmatrix} I_{n_1} c_\alpha(k) & 0 \\ 0 & I_{n_2} c_\beta(k) \end{bmatrix} \Bigg) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \end{aligned} \tag{22}$$

is asymptotically stable.

According to the Equation (10), $\overline{A}_{11} = A_{11} + I_{n_1}\alpha$ and $\overline{A}_{22} = A_{22} + I_{n_2}\beta$, system (22) can be represented as follows:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} + I_{n_1} + B_1K_1 & A_{12} + B_1K_2 \\ A_{21} + B_2K_1 & A_{22} + I_{n_2} + B_2K_2 \end{bmatrix} \cdot \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{23}$$

Denote

$$\begin{aligned} \widetilde{A}_1 &= \begin{bmatrix} A_{11} + I_{n_1} + B_1K_1 & A_{12} + B_1K_2 \\ 0 & 0 \end{bmatrix}, \\ \widetilde{A}_2 &= \begin{bmatrix} 0 & 0 \\ A_{21} + B_2K_1 & A_{22} + I_{n_2} + B_2K_2 \end{bmatrix}. \end{aligned}$$

We have a new 2D system in form of FM second model with matrices $\widetilde{A}_1$ and $\widetilde{A}_2$ as follow:

$$x(i+1,j+1) = \widetilde{A}_1 x(i,j+1) + \widetilde{A}_2 x(i+1,j), \tag{24}$$

The proposed method of stability can be applied to consider the stabilization of system (1) with the state-feedback (20). We represent Theorem 2 as follows:

Proposition 1. *The closed-loop 2D fractional-order system (21) is stabilizable if and only if these conditions are satisfied,*

(1) $\widetilde{H}(-1,0) \neq 0, \Delta(\widetilde{L_1}(\gamma))_k > 0, k = 1,2\ldots,n,$

(2) $\widetilde{H}(-1,-1) \neq 0, \Delta(\widetilde{L_2}(\gamma))_k > 0, k = 1,2\ldots,n,$

(3) $\widetilde{W}(s,-1) \neq 0, \Delta(\widetilde{L_3}(j\gamma))_{2k} > 0, k = 1,2\ldots,n, s \in \mathbb{R}.$

where $\widetilde{H}(z_1,z_2) = det(I_n - z_1\widetilde{A_1} - z_2\widetilde{A}_2)$,
$\widetilde{W}(s,z_2) = (1-js)^m \widetilde{H}(\frac{1+js}{1-js}, z_2)$,
$\widetilde{L_1}(\gamma,0) = (1+\gamma)^m \widetilde{H}(\frac{1-\gamma}{1+\gamma}, 0)$,
$\widetilde{L_2}(-1,\gamma) = (1+\gamma)^n \widetilde{H}(-1, \frac{1-\gamma}{1+\gamma})$,
$\widetilde{L_3}(s,\gamma) = (1+\gamma)^n \widetilde{H}(s, \frac{1-\gamma}{1+\gamma}), (z_1,z_2) \in \overline{\mathbb{U}}, s \in \mathbb{R}, z_2 \in \mathbb{D}, \gamma \in \mathbb{C}^+$, *m and n respectively stand for the degree of* $\widetilde{H}(z_1,z_2)$*in* z_1 *and* z_2.

Proof. The proof is same as Theorem 2. □

For obtaining the stable gain matrix region of the closed-loop 2D fractional-order system, we give the following Algorithm 1.

Algorithm 1 2DF Stabilization.

Input: The characteristic equation $\widetilde{H}(z_1,z_2)$ of a closed-loop 2D fractional-order system.
Output: The stable gain matrix region.
Step 1. Let the gain matrix as K.
Step 2. Calculate $\widetilde{H}(z_1,z_2)$ of the closed-loop 2D fractional-order system.
Step 3. Solve the inequalities of Proposition 1 based on the discrimination system of polynomials.
Step 3.1. Get the stable gain matrix region of K by solving $\widetilde{H}(-1,0) \neq 0, \triangle(\widetilde{L}_1)_k > 0, k = 1,2\ldots,n$.
Step 3.2. Get the stable gain matrix region of K by solving $\widetilde{H}(-1,-1) \neq 0, \triangle(\widetilde{L}_2)_k > 0, k = 1,2\ldots,n$.
Step 3.3. Calculate $\widetilde{L}_3(s,\gamma)$, then get $L_3(j\gamma) = \varrho_n\gamma^n + \varrho_{n-1}\gamma^{n-1} + \ldots + \varrho_0 + j(\kappa_n\gamma^n + \kappa_{n-1}\gamma^{n-1} + \ldots + \kappa_0)$, where ϱ_i and $\kappa_i, i = 0,\ldots n$ are real coefficient polynomials in s. Solve $\widetilde{W}(s,-1) \neq 0$, and $\triangle(L_3)_{2k} > 0, k = 1,2\ldots,n, s \in \mathbb{R}$ to the stable gain matrix region of K.
Step 4. From step 3, obtain the final results of the stable gain matrix region of K.

4. Example

In this section, we show a numerical example that the fractional-order 2D system has generality from [17] to show the efficiency of the methods of stability and stabilization in this paper. We focus on the 2D fractional system (1) with $\alpha = 0.4, \beta = 0.5$ and $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} -1.1255 & 0.8 \\ 0.149 & 0.24 \end{pmatrix}$, $B_1 = -19$, $B_2 = -10$

We show the steps of analyzing stability and stabilization processes of fractional 2D systems as follows.

First, we analyze the stability of the considered system.

Step 1. Change the 2D fractional system (1) with $u_{ij} = 0$ into the 2D system (14). Based on the given parameters, we firstly get the polynomial

$$H(z_1, z_2) = 1 - \frac{31}{25}z_2 - \frac{1}{5}z_1 - \frac{49}{125}z_1z_2$$

Step 2. Check whether the inequalities of Theorem 2 hold using Maple.
Step 2.1. We have $H(-1,0) = \frac{6}{5} \neq 0$
Calculate $L_1(\gamma, 0)$, as follows

$$L_1(\gamma, 0) = \frac{6}{5}\gamma + \frac{4}{5}$$

We have $\triangle(L_1)_2 = \frac{24}{25} > 0$. This conditions of Theorem 2 is satisfied from system (1) with the given parameters.
Step 2.2. We have $H(-1,-1) = \frac{256}{125} \neq 0$. Calculate $L_2(-1,\gamma)$, as follows

$$L_2(-1, \gamma) = \frac{256}{125}\gamma + \frac{44}{125}$$

We have $\triangle(L_2)_2 = \frac{11{,}264}{15{,}625} > 0$.
Step 2.3. We have $W(s,-1) = -\frac{256}{125}js + \frac{304}{125} \neq 0$.
Calculate $L_3(s,\gamma)$, as follows

$$L_3(s, \gamma) = \frac{304}{125}\gamma - \frac{104}{125} + j(-\frac{256}{125}s\gamma - \frac{44}{125}s)$$

$$L_3(s, j\gamma) = \frac{256}{125}s\gamma - \frac{104}{125} + j(-\frac{44}{125}s + \frac{304}{125}\gamma)$$

We have $\triangle(L_3)_2 = 88s^2 - 247$. It's easy to know that $\triangle(L_3)_2$ isn't satisfied the condition $\Delta(L_3(j\gamma))_2 > 0, k = 1, 2 \ldots, n.$ of Theorem 2.
Step 3. This fractional-order 2D system is unstable.

Remark 4. *As shown in the example, we know this considered fractional-order 2D system is unstable. The result is in agreement with the literature [17]. While we needn't stabilise the considered system based on the precondition that the system is positive as in the other methods. We extend the existing valuable methods of fractional-order 2D systems in the control theory. The stability condition of the general fractional-order 2D systems instead of the positive fractional-order 2D systems is given.*

Now, we consider the stabilization of the system and obtain the stable gain matrix region according to as follows:
Step 1. Let $K = [k_1 \quad k_2]$.
Step 2.Based on the given parameters of the fractional 2D system, we firstly get the polynomial

$$\begin{aligned}\widetilde{H}(z_1, z_2) =& 1 - \frac{31}{25}z_2 + 10k_2z_2 - \frac{1}{5}z_1 - \frac{49}{125}z_1z_2 \\ &+ \frac{66}{5}k_2z_1z_2 + 19k_1z_1 - \frac{389}{25}z_1k_1z_2\end{aligned}$$

Step 3. Solve the inequalities of Proposition 1 by Maple.
Step 3.1. Calculate the polynomial $\widetilde{L}_1(\gamma, 0)$, as follows

$$\widetilde{L}_1(\gamma, 0) = (\frac{6}{5} - 19k_1)\gamma + 19k_1 + \frac{4}{5} \neq 0, \gamma \in \mathbb{C}^+.$$

Then, from

$$\triangle(\widetilde{L}_1)_2 = (19k_1 - \frac{6}{5})(19k_1 + \frac{4}{5}) < 0,$$

we can get
$$-\frac{4}{95} < k_1 < \frac{6}{95}. \tag{25}$$

From
$$\widetilde{H}(-1,0) = \frac{6}{5} - 19k_1 \neq 0,$$
we can get
$$k_1 \neq \frac{6}{95}. \tag{26}$$

Step 3.2. Calculate the polynomial $\widetilde{L}_2(-1,\gamma)$, as follows
$$\begin{aligned}\widetilde{L}_2(-1,\gamma) =& (-\frac{864}{25}k_1 + \frac{16}{5}k_2 + \frac{256}{125})\gamma - (\frac{86}{25}k_1 - \frac{16}{5}k_2) \\ &+ \frac{44}{125} \neq 0, \gamma \in \mathbb{C}^+.\end{aligned}$$

Then, we can get
$$\triangle(\widetilde{L}_2)_2 = (\frac{864}{25}k_1 - \frac{16}{5}k_2 - \frac{256}{125})(\frac{86}{25}k_1 + \frac{16}{5}k_2 - \frac{44}{125}) > 0, \tag{27}$$
we can get
$$\widetilde{H}(-1,-1) = \frac{256}{125} + \frac{16}{5}k_2 - \frac{864}{25}k_1 \neq 0. \tag{28}$$

Step 3.3. We have
$$\begin{aligned}\widetilde{W}(s,-1) = & (\frac{864}{25}k_1 - \frac{16}{5}k_2 - \frac{256}{125})js + \frac{864}{25}k_1 - \frac{116}{5}k_2 \\ &+ \frac{304}{125} \neq 0, s \in \mathbb{R},\end{aligned}$$

From the above condition, we can get
$$\frac{864}{25}k_1 - \frac{16}{5}k_2 - \frac{256}{125} \neq 0. \tag{29}$$

We have
$$\begin{aligned}\widetilde{L_3}(s,j\gamma) =& -4320k_1s\gamma - 400k_2s\gamma - 256s\gamma - 430k_1 \\ &- 2900k_2 + 104 + j(-430k_1s - 4320k_1\gamma \\ &- 400k_2x + 2900k_2\gamma + 44s - 304\gamma) \\ &\neq 0, s \in \mathbb{R}, \gamma \in \mathbb{C}^+.\end{aligned}$$

Then we can obtain the inequalities $\triangle(\widetilde{L_3}(s,j\gamma))_{2k} > 0, k = 1,2, s \in \mathbb{R}$ to get uncertain parameters k_1 and k_2 of K as follows:
$$\begin{aligned}\triangle(\widetilde{L_3}(s,j\gamma))_2 =& 232{,}200k_1^2s^2 + 194{,}500k_1k_2s^2 - 20{,}000k_2^2s^2 \\ &- 37{,}520k_1s^2 - 10{,}600k_2s^2 + 232{,}200k_1^2 \\ &+ 1{,}410{,}125k_1k2 - 1{,}051{,}250k_2^2 + 1408s^2 \\ &- 39{,}820k_1 + 147{,}900k_2 - 3952 > 0, s \in \mathbb{R}.\end{aligned}$$

In order for $\triangle(\widetilde{L_3}(s,j\gamma))_2 > 0, s \in \mathbb{R}$ to be established, we have
$$\begin{cases}(215k_1 + 200k_2 - 22)(270k_1 - 25k_2 - 16) > 0 \\ (215k_1 + 1450k_2 - 52)(1080k_1 - 725k_2 + 76) > 0\end{cases} \tag{30}$$

Step 4. From (25)–(30), we can find that when

$$\begin{cases} 215k_1 + 200k_2 - 22 < 0 \\ 270k_1 - 25k_2 - 16 < 0 \\ 215k_1 + 1450k_2 - 52 > 0 \\ 1080k_1 - 725k_2 + 76 > 0 \end{cases} \tag{31}$$

all the conditions of Proposition 1 are established. Thus, the closed-loop system (21) with the gain parameters of the stable gain matrix region (31) is stable.

From the above discussion, through solving the conditions of Proposition 1 for getting K, we obtain the stable gain matrix region (31) which is shown as Figure 1:

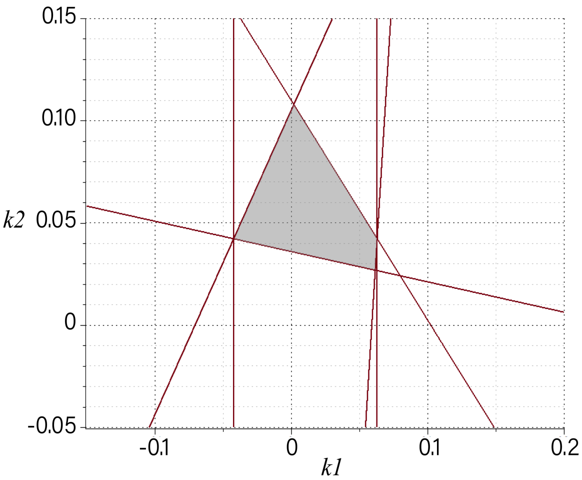

Figure 1. The stable gain matrix region.

For better showing the validity of the method in this paper, we give some simulations. Figures 2 and 3 show the state space responses of system (1). We can find that the state responses of the open-loop system (21) in Figures 2 and 3 is divergent and not stable.

According to the solution (31), let K = [0.01 0.08], and the state responses of the closed-loop system (21) are shown in Figures 4 and 5. As i and j get bigger, the state space response $x_1(i,j)$ and $x_2(i,j)$ go to 0. The system (1) is stable after stabilization. Similarly, let K = [0 0.05] from the region (31), and the state responses of the closed-loop system (21) are shown in Figures 6 and 7. The system (1) is stable after stabilization. It is noted that as long as the gain matrix K is selected in the stable gain matrix region shown in Figure 1, the closed-loop system (21) is stable.

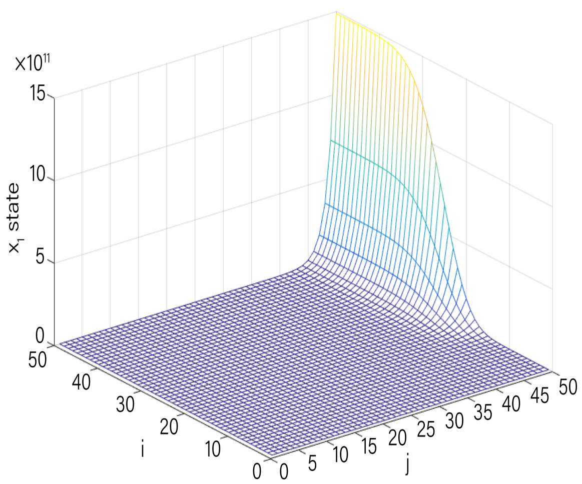

Figure 2. Open-loop state space response of $x_1(i,j)$.

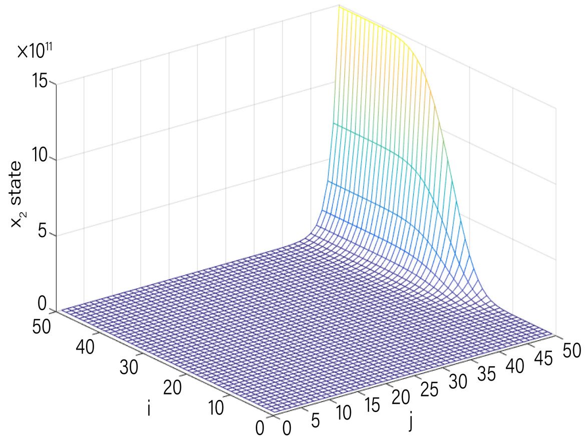

Figure 3. Open-loop state space response of $x_2(i, j)$.

Remark 5. *For better presenting the responses of the closed-loop system, we change the direction of i and j in the Figures 4–7.*

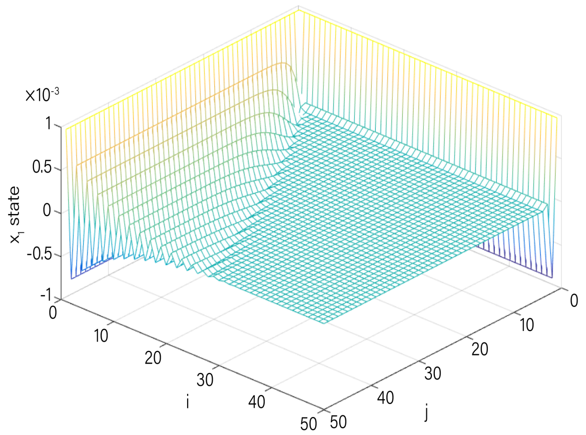

Figure 4. Closed-loop state space response of $x_1(i, j)$ when K = [0.01 0.08].

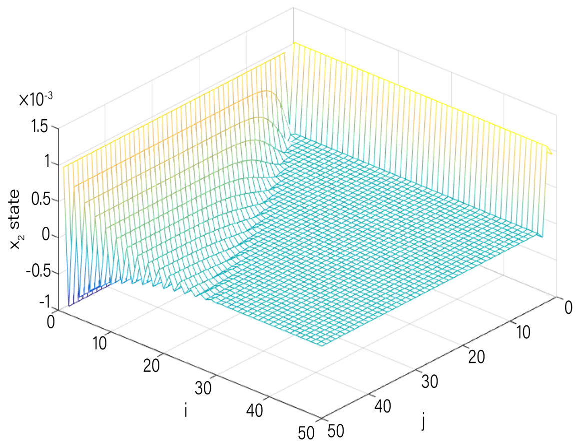

Figure 5. Closed-loop state space response of $x_2(i, j)$ when K = [0.01 0.08].

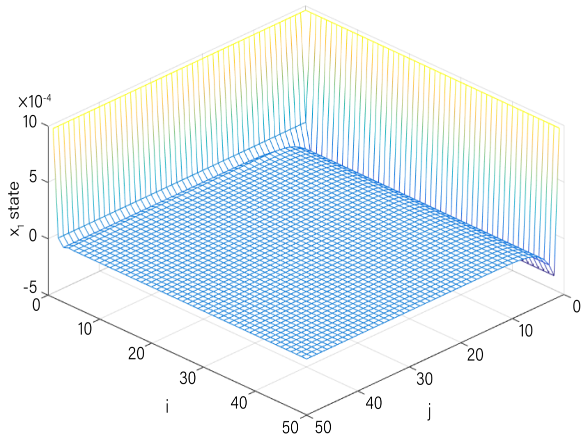

Figure 6. Closed-loop state space response of $x_1(i, j)$ when K = [0 0.05].

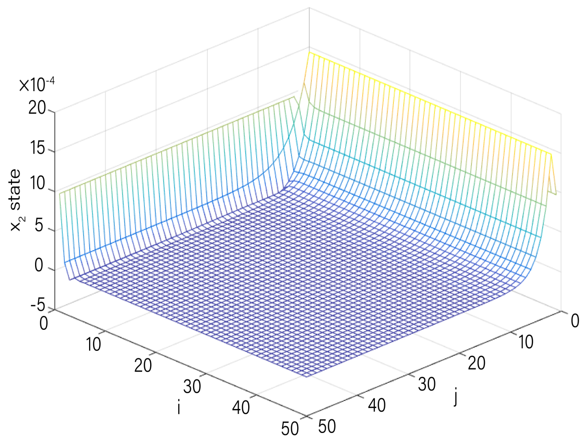

Figure 7. Closed-loop state space response of $x_2(i, j)$ when K = [0 0.05].

Remark 6. *We obtain a stable gain matrix region. All the gain matrixes of the stable gain matrix region can stabilize the system. Further, the closed-loop system (21) with state-feedback can be regarded as a 2D fractional-order system with uncertain parameters. Our method not only extends the existing methods only for positive systems, but can also solve the robust stability problem of fractional 2D systems that has not been solved by other researchers. It can obtain all the parameters that ensure the system is robust and stable, which will be discussed in the future.*

5. Conclusions

This paper has discussed the stability and stabilization problems of fractional-order 2D systems that are common in practice but rarely studied. The stability check process and the algorithm for obtaining the stable gain matrix region have been shown in the example. The method proposed in this paper can be widely used. Compared with the existing method, it is not necessary to stabilise the considered system based on the condition that the system must be positive. And the stabilization method can obtain multiple parameters of control gain to stabilize the fractional-order 2D system. The proposed methods in this paper have low computational complexity, and so are simpler and easier to use.

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