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Exact Solvability Conditions for the Non-Local Initial Value Problem for Systems of Linear Fractional Functional Differential Equations

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Abstract: The exact conditions sufficient for the unique solvability of the initial value problem for a system of linear fractional functional differential equations determined by isotone operators are established. In a sense, the conditions obtained are optimal. The method of the test elements intended for the estimation of the spectral radius of a linear operator is used. The unique solution is presented by the Neumann's series. All theoretical investigations are shown in the examples. A pantograph-type model from electrodynamics is studied.

Keywords: fractional order functional differential equations; unique solvability; Caputo derivative; initial value problem; quasi-interior element; minihedral cone; pantograph-type model

MSC: 26A33; 34A08; 34K08; 34K37; 47H07

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1. Introduction

The theory of fractional functional differential equations (FFDEs) is applied in the modeling of most natural processes. From the point of view of physics, especially of mechanics, the models established by using a fractional differential operator (cap-resistor) were analyzed in [1]. We would also like to highlight [2], where the authors made a complex overview of possible applications of FDEs.

The multiplicity of investigations in the theory of FFDEs covers the variable aspects of the theory of boundary value problems. To investigate the boundary value problem for nonlinear FFDEs (see, for example, [3]), very often, one needs to use the qualitative results for the initial value problem for the linear FFDEs. The main goal of our investigation is to construct the exact conditions sufficient for the unique solvability of linear FFDEs. For establishing these conditions, we use the method of test elements intended for the estimation of the spectral radius of the linear operator. There are many recent results for the Cauchy problem for fractional differential equations (see [2,4–12]) and functional differential equations [13–15]). The authors in [4] investigated the existence and uniqueness of symmetric solutions for fractional differential equations with multi-order fractional integral boundary conditions using fixed-point theorems; the aim of the paper [7] was to propose a new operator named the infinite coefficient-symmetric Caputo–Fabrizio fractional derivative and to study some its properties; the authors investigated the symmetry analysis of the initial and boundary value problem for fractional diffusion and the third order fractional partial differential equation in [10]; by using Banach fixed point theorem in [11] the authors established the existence and uniqueness of the solutions for fractional order functional differential equations involving the Hilfer fractional derivative in the weighted spaces; existence and uniqueness results for a nonlinear Caputo–Riemann–Liouville-type fractional integro-differential boundary value problem with multi-point sub-strip boundary conditions, via Banach and Krasnosel'skii's fixed point theorems were established in [16]. Unlike the general results from [4,7,10,11,16] we obtain here more exact conditions on unique solvability. We use the method characterized by the fact that it allows one to estimate the spectral radius of a linear operator based on knowledge of the value of the operator on a single, suitably chosen element of a space.

We apply the obtained theoretical results for the pantograph-type model from electrodynamics (see Section 6). Another application is an example for a model with a discrete memory effect. We assume that the obtained results can be applied, for example, in the model of the Stieltjes string described in [3,17].

We consider FFDE

$$D_a^q u(t) = (lu)(t) + r(t), \quad t \in [a,b], \tag{1}$$

with initial value problem

$$u(a) = c, \tag{2}$$

where D_a^q is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, operator $l = (l_k)_{k=1}^n : C([a,b],\mathbb{R}^n) \to L([a,b],\mathbb{R}^n)$ is the bounded linear operator, and function r belongs to the $L([a,b],\mathbb{R}^n)$.

The main goal of our investigations is to establish the exact conditions sufficient for the unique solvability of the initial value problem (2) for systems of linear fractional functional differential Equation (1) presented by positive operators. For this aim, we use the method of test elements studied for the estimation of the spectral radius of a linear operator based on knowledge of the value of the operator on a single, suitably chosen, element of the space. Moreover, the unique solution is presented in view of the Neumann's series. The pantograph-type model from electrodynamics is investigated.

The paper is constructed in the following way: we give the notations and all necessary definitions in Section 2. Next, we give all auxiliary statements in Section 3. The main result can be found in Section 4, where it is also presented the unique solution in view of the Neumann's series. The proof of this result is in Section 4.1. The corollary is in Section 4.2. The example that proves the optimality of the condition is in Section 5. All theoretical investigations are shown in the example for the pantograph-type model from electrodynamics in Section 6. The summary of the investigation can be found in Section 7.

2. Notations and Definitions

We use the following notation:

- $q \in (0;1)$ is an order of the Caputo fractional derivative D_a^q;
- $\mathbb{R} := (-\infty,\infty)$, $\mathbb{R}_+ := [0,\infty)$;
- If $u = (u_i)_{i=1}^n \in \mathbb{R}^n$, then
$$|u|_\infty = \max_{1\le i\le n} |u_i|; \tag{3}$$
- $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions $[a,b] \to \mathbb{R}^n$ with the norm
$$C([a,b],\mathbb{R}^n) \ni u \to \max_{s\in[a,b]} |u(s)|_\infty = \max_{s\in[a,b]} \operatorname{ess\,sup} |u(s)|; \tag{4}$$

- For fixed $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset \{-1, 1\}$

$$\vec{\sigma} := \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \cdot \\ \cdot \\ \cdot \\ \sigma_n \end{pmatrix}$$

we set

$$\mathbb{R}^n_+ := \sigma_1\mathbb{R}_+ \times \sigma_2\mathbb{R}_+ \times \cdots \times \sigma_n\mathbb{R}_+.$$

The set $\mathbb{R}^n_{\vec{\sigma}}$ thus defined is obviously a closed solid cone in $\mathbb{R}^n$;
- $C([a,b], \mathbb{R}^n_{\vec{\sigma}})$ is the set of all functions u from the space $C([a,b], \mathbb{R}^n)$ that take values in the cone $\mathbb{R}^n_{\vec{\sigma}}$;
- $C_a([a,b], \mathbb{R}^n_{\vec{\sigma}})$ is the set of all functions u from the space $C([a,b], \mathbb{R}^n)$ that satisfy condition (2) and take values in the cone $\mathbb{R}^n_{\vec{\sigma}}$;
- $B([a,b], \mathbb{R}^n)$ is the Banach space of all bounded functions from $[a,b]$ to $\mathbb{R}^n$ equipped with norm

$$B([a,b], \mathbb{R}^n) \ni u \to \sup_{s\in[a,b]} |u(s)|_\infty;$$

- $L([a,b], \mathbb{R}^n)$ is the Banach space of summable vector functions $[a,b] \to \mathbb{R}^n$ with the norm

$$L([a,b], \mathbb{R}^n) \ni u \to \int_a^b |u(s)|_\infty ds;$$

- $\rho(\Psi)$ is the spectral radius of a bounded linear operator Ψ.

Definition 1. *By a solution of the linear boundary-value problems (1) and (2), we understand continuous vector function $u : [a,b] \to \mathbb{R}^n$ possessing property (2) and satisfying relation (1) for almost all t from the interval $[a,b]$.*

Definition 2 ([8])**.** *For a function u given on the interval $[a,b]$ the Caputo derivative of fractional order q is defined by*

$$D^q_a u(t) = \frac{1}{\Gamma(1-q)}\left(\frac{d}{dt}\right)\int_a^t (t-s)^{-q}(u(s) - u(a))ds, \quad 0 < q < 1,$$

where $\Gamma(q) : [0,\infty) \to \mathbb{R}$ is a Gamma function and

$$\Gamma(q) := \int_0^\infty t^{q-1}e^{-t}dt. \tag{5}$$

Remark 1. *For an, at least, n-times differentiable function u given on the interval $[a,b]$, the Caputo derivative of fractional order q, $n-1 < q < n$, $n = [q]+1$ and $[q]$ denotes the integer part of the real number q defined as*

$$D^q_a u(t) = \frac{1}{\Gamma(n-q)}\int_a^t (t-s)^{n-q-1}u^{(n)}(s)ds.$$

Remark 2 ([2])**.** *The Caputo derivative of order q for a function $u : [a,b] \to \mathbb{R}$ (we mean $D^q_a u(t)$) can be written as*

$$D^q_a u(t) = {}^L D^q_a\Big(u(t) - (t-a)^{-q}u(a)\Big),$$

and it is known, see [2], that the Riemann–Liouville fractional derivative ${}^L D^q_a$ of order q for a function $u : [a,b] \to \mathbb{R}$ does not depend on the initial conditions.

Definition 3. *For a certain given* $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset \{-1, 1\}$ *an operator* $l : C([a,b], \mathbb{R}^n) \to L([a,b], \mathbb{R}^n)$ *is a* $\vec{\sigma}$*-positive operator if the fact that, for all* $t \in [a,b]$*, the relation*

$$\sigma_\nu u_\nu(t) \geq 0, \quad \nu = 1, 2, \ldots, n \tag{6}$$

is true implies that

$$\sigma_\nu (l_\nu u)(t) \geq 0, \quad \nu = 1, 2, \ldots, n$$

for almost every t *from* $[a,b]$*.*

Remark 3. *The inequality sign between vectors and matrices and the other relations is understood component-wise.*

Remark 4. *If* $\sigma_\nu \equiv 1$*,* $\nu = 1, 2, \ldots, n$*, then we have a positive operator in the usual sense.*

Definition 4. *We say that a functional* $g : Y \to \mathbb{R}^n$*, where* Y *denotes one of the spaces* $C_a([a,b], \mathbb{R}^n)$*,* $C([a,b], \mathbb{R}^n)$*, and* $B([a,b], \mathbb{R}^n)$*,* $\vec{\sigma}$*-positive functional if the fact that conditions (6) are satisfied for all components* $(u_k)_{k=1}^n$ *of a vector function,* $u \in Y$ *always yields*

$$g(u) \geq 0.$$

Definition 5. *An element* u *of a cone* $K \subset Y$ *in a Banach space* Y *is called a quasi-interior element if, for arbitrary* $g \in K^* \setminus \{0\}$*, the strict inequality* $g(u) > 0$ *is satisfied. The symbol* K^* *here stands for the set of all functionals* g *from* Y^* *that take non-negative values on the elements of the cone* K*.*

Definition 6. *A cone* $K \subset Y$ *of a Banach space* Y *is called a minihedral cone if a single-valued mapping* $\sup : Y \times Y \to Y$ *is well defined such that, for arbitrary* $\{u_1, u_2, y\} \subset Y$*, we have* $\sup\{u_1, u_2\} - u_1 \in K$ *and* $\sup\{u_1, u_2\} - u_2 \in K$ *and, moreover, it always follows from the relations* $y - u_1 \in K$ *and* $y - u_2 \in K$ *that* $y - \sup\{u_1, u_2\} \in K$*.*

In other words, K *is a minihedral if, in the partial ordering generated by this cone in the space* Y*, every finite set has the least upper bound.*

3. Auxiliary Statements

Lemma 1 ([2] Lemma 2.21)**.** *Let* $0 < q < 1$ *and let* $u(t) \in L([a,b], \mathbb{R}^n)$ *or* $u(t) \in C([a,b], \mathbb{R}^n)$*, then*

$$D_a^q I_a^q u(t) = u(t)$$

and

$$D_b^q I_b^q u(t) = u(t),$$

where

$$I_a^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} u(s) ds, \quad x > a, \tag{7}$$

$$I_b^q u(t) = \frac{1}{\Gamma(q)} \int_t^b (t-s)^{q-1} u(s) ds, \quad x < b \tag{8}$$

and Γ*-function is defined by (5).*

Lemma 2 ([2] Lemma 2.22)**.** *Let* $0 < q < 1$*. If* $u(t) \in C([a,b], \mathbb{R}^n)$*, or* $u(t) \in D([a,b], \mathbb{R}^n)$ *then*

$$I_a^q D_a^q u(t) = u(t) - u(a)$$

and

$$I_b^q D_b^q u(t) = u(t) - u(b),$$

where I_a^q *and* I_b^q *are defined by (7) and (8) correspondingly.*

For the following investigation, we use the well-known result from the general theory of boundary value problems for the functional differential equation.

Lemma 3. *The nonhomogeneous problem* (2) *for linear FFDE* (1) *is uniquely solvable if the corresponding homogeneous problem*

$$u(a) = 0 \tag{9}$$

for linear FFDE

$$D_a^q u(t) = (lu)(t), \quad t \in [a,b], \tag{10}$$

only has a trivial solution.

In view of Definition 1, Lemmas 1 and 2 and relation (5) the next obvious Lemma is true.

Lemma 4. *The problems* (1) *and* (2) *are equivalent to the equation*

$$u(t) = u(t_0) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(lu)(s)ds + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}r(s)ds, \quad t \in [a,b].$$

For establishing the unique solvability conditions for problems (1) and (2), we need the method of the test elements used for the estimation of the spectral radius of a linear operator (see M. Krein [18]). This method is characterized by the fact that, in many cases, it allows one to estimate the spectral radius of a linear operator based on knowledge of the value of the operator on a single, suitably chosen, element of a space.

For the present paper, the following statements are sufficient.

Theorem 1 ([19] Theorem 5.5)**.** *Let Y be a Banach space and let Ψ be a completely continuous linear operator on Y, leaving invariant a certain total cone $K \subset Y$, i.e., $\Psi(K) \subset K$. Suppose that there exists a quasi-interior element z of the cone K such that, for certain positive constant β and integer p, the following relation is true:*

$$\beta z - \Psi^p z \in K. \tag{11}$$

Then the spectral radius $r(\Psi)$ of the operator Ψ admits the estimate

$$\rho(\Psi) \le \beta^{\frac{1}{p}}. \tag{12}$$

We will need the next statement.

Lemma 5. *The set $C_a([a,b],\mathbb{R}^n_{\vec{\sigma}})$ creates a reproducing minihedral cone (see Definition 6) in the Banach space $C_a([a,b],\mathbb{R}^n)$.*

Proof. It is easy to see from the definition of the set $C_a([a,b],\mathbb{R}^n_{\vec{\sigma}})$ (see Notations and Definitions), that set $C_a([a,b],\mathbb{R}^n_{\vec{\sigma}})$ forms nontrivial closed cones in $C_a([a,b],\mathbb{R}^n)$.

The least upper bound $v = \sup\{\tilde{u},\tilde{\tilde{u}}\}$ of the elements $\{\tilde{u},\tilde{\tilde{u}}\} \subset C_a([a,b],\mathbb{R}^n)$ is represented by the formula

$$v_k = \begin{cases} \max\{\tilde{u}_k(t),\tilde{\tilde{u}}_k(t)\}, & \text{if} \quad \sigma_k = 1, \quad t \in [a,b], \quad k = 1,2,\dots,n, \\ \min\{\tilde{u}_k(t),\tilde{\tilde{u}}_k(t)\}, & \text{if} \quad \sigma_k = -1, \quad t \in [a,b], \quad k = 1,2,\dots,n \end{cases}$$

in the partial ordering generated by $C_a([a,b],\mathbb{R}^n_{\vec{\sigma}})$ in $C_a([a,b],\mathbb{R}^n)$. Therefore, the cone $C_a([a,b],\mathbb{R}^n_{\vec{\sigma}})$ is minihedral and reproducing. Lemma 5 is proved. □

We recall the next well-known lemma.

Lemma 6. *An arbitrary bounded linear functional g on the space $C_a([a,b],\mathbb{R}^n)$ is represented as*

$$C_a([a,b],\mathbb{R}^n) \ni u = (u_k)_{k=1}^n \longmapsto g(u(t)) := \sum_{k=1}^n \int_a^b u_k(t)dh_k(t), \tag{13}$$

for every $k = 1,2,\ldots,n$, and some functions $h_k : [a,b] \to \mathbb{R}$ with bounded variations on $[a,b]$ and continuous at t_0.

The same formula (13) allows one to define a bounded functional $\tilde{g}$ coinciding with the natural linear norm-preserving extension of g to $C([a,b],\mathbb{R}^n)$. If, in addition, the original functional g is $\vec{\sigma}$-positive (see Definition 4), then $\tilde{g}$ is also $\vec{\sigma}$-positive.

Lemmas 5 and 6 yield the following statement:

Lemma 7. *A continuous linear functional g of the form (13), where the scalar functions $\{h_k | k = 1,2,\ldots,n\}$ are such that $\sigma_k h_k$ is non-decreasing for any $k = 1,2,\ldots,n$, is $\vec{\sigma}$-positive and, vice versa, every $\vec{\sigma}$-positive bounded linear functional on the space $C([a,b],\mathbb{R}^n)$ can be represented in the form (13) for every $k = 1,2,\ldots,n$, the function $\sigma_k h_k : [a,b] \to \mathbb{R}$ is continuous at the point a and non-decreasing on $[a,b]$.*

Let us fix arbitrary $y_0 \in C_a([a,b],\mathbb{R}^n)$ and introduce the sequence of functions

$$y_k(t) := \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}(ly_{k-1})(s)ds, \quad t \in [a,b], \quad k = 1,2,\ldots. \tag{14}$$

with property

$$\sigma_k y_k(t) > 0, \quad t \in (a,b], \tag{15}$$

$$y_k(a) = 0. \tag{16}$$

The following inclusion is then satisfied obviously:

$$\{y_k \geq 0\} \subset C_a([a,b],\mathbb{R}^n). \tag{17}$$

The next lemma is applied in the sequel.

Lemma 8. *If all components of an arbitrary continuous vector function $(y_\nu)_{\nu=1}^n : [a,b] \to \mathbb{R}^n$ satisfy conditions (15), (16), then $y_\nu(t)$ is a quasi-interior element of the cone $C_a([a,b],\mathbb{R}^n_{\vec{\sigma}})$ in the space $C_a([a,b],\mathbb{R}^n)$.*

Proof of Lemma 8. Suppose that the components $(y_\nu)_{\nu=1}^n$ of a continuous vector function $y = (y_\nu)_{\nu=1}^n : [a,b] \to \mathbb{R}^n$ satisfy conditions (15), (16). Consider an arbitrary nontrivial bounded linear $\vec{\sigma}$-positive functional $g : C_a([a,b],\mathbb{R}^n) \to \mathbb{R}^n$. In view of Lemmas 6 and 7, every functional of this type can be given by (13), where the functions h_ν, $\nu = 1,2,\ldots,n$, are continuous at point a, and the corresponding functions $\sigma_\nu h_\nu$, $\nu = 1,2,\ldots,n$, are non-decreasing. By assumption, $g \neq 0$, and, therefore, for a certain $\nu_0 \in \{1,2,\ldots,n\}$, the linear functional

$$C_a([a,b],\mathbb{R}) \ni y \longmapsto \kappa(y(t)) := \int_a^b y(t)dh_{\nu_0}(t) \tag{18}$$

is not identically equal to zero. In particular, the function h_{ν_0} satisfies the condition

$$\sigma_{\nu_0}\Big(h_{\nu_0}(b) - h_{\nu_0}(a)\Big) > 0. \tag{19}$$

Indeed, if

$$h_{\nu_0}(b) = h_{\nu_0}(a),$$

then the inequality

$$\sigma_{\nu_0}\Big(h_{\nu_0}(s_2) - h_{\nu_0}(s_1)\Big) \geq 0,$$

which is satisfied for all $\{s_1, s_2\} \subset [a, b]$, $s_1 < s_2$, and means that the function $\sigma_{\nu_0} h_{\nu_0}$ is non-decreasing, implies that the function h_{ν_0} is constant, and, hence, functional (18) is trivial.

Let us fix a sufficiently small positive δ and consider the value of functional (18) on the function y_{ν_0}. This functional is $\vec{\sigma}_{\nu_0}$-positive (see Definition 4), whereas the function y_{ν_0} satisfies the condition (15) with $\nu = \nu_0$. Therefore,

$$\int_a^{a+\delta} y_{\nu_0}(t) dh_{\nu_0}(t) \geq 0$$

and, hence,

$$\kappa\big(y_{\nu_0}(t)\big) = \int_a^{a+\delta} y_{\nu_0}(t) dh_{\nu_0}(t) + \int_{a+\delta}^b y_{\nu_0}(t) dh_{\nu_0}(t) \geq \int_{[a+\delta, b]} y_{\nu_0}(t) dh_{\nu_0}(t). \tag{20}$$

We chose certain partitions

$$a + \delta =: \tilde{t}_0 < \tilde{t}_1 < \tilde{t}_2 < \ldots \tilde{t}_{m-1} < \tilde{t}_m := b$$

of the intervals $[a + \delta, b]$. Now we represent the integral on the right-hand side of (20) by the limit of the corresponding integral sums. Then relation (20) ensures the inequality

$$\sum_{i=0}^{m-1} y_{\nu_0}(\tilde{\zeta}_i)\big[h_{\nu_0}(\tilde{t}_{i+1}) - h_{\nu_0}(\tilde{t}_i)\big] \leq \kappa\big(y_{\nu_0}(t)\big) \tag{21}$$

which is fulfilled for arbitrary fixed points $\tilde{\zeta}_i$ from the intervals $(\tilde{t}_i, \tilde{t}_{i+1}]$, whenever the values $\max_{1 \leq i \leq m-1}[\tilde{t}_{i+1} - \tilde{t}_i]$ are sufficiently small.

It is obvious from (15), (16) that, for positive δ, the number

$$\mu_\delta := \min_{t \in [a+\delta, b]} \sigma_{\nu_0} y_{\nu_0}(t)$$

is positive. Taking into account (15) and the non-decreasing $\sigma_{\nu_0} h_{\nu_0}$ we have that

$$\sigma_{\nu_0} y_{\nu_0}(\tilde{\zeta}_i) > 0,$$

and

$$\sigma_{\nu_0} y_{\nu_0}\Big(h_{\nu_0}(\tilde{t}_{i+1}) - h_{\nu_0}(\tilde{t}_i)\Big) \geq 0,$$

which are valid for all $i = 0, 1, \ldots, m-1$, and relation (21) yields

$$\mu_\delta \sigma_{\nu_0} \sum_{i=1}^{m-1} \big(h_{\nu_0}(\tilde{t}_{i+1}) - h_{\nu_0}(\tilde{t}_i)\big) \leq \kappa(y_{\nu_0}(t)),$$

i.e.,

$$\sigma_{\nu_0}\big(h_{\nu_0}(b) - h_{\nu_0}(a+\delta)\big)\mu_\delta \leq \kappa(y_{\nu_0}(t)). \tag{22}$$

The condition (16) and Lemmas 6 and 7 require that the Stieltjes function h_{ν_0} can be assumed to be continuous at the point a. By virtue of inequality (19), such a choice of h_{ν_0} guarantees that

$$\sigma_{\nu_0}\Big(h_{\nu_0}(b) - h_{\nu_0}(a) - \big(h_{\nu_0}(t_0+\delta) - h_{\nu_0}(a)\big)\Big) > 0 \tag{23}$$

whenever δ is sufficiently small. Taking into account the positivity of the value μ_δ, it follows from relations (22) and (23) that, for sufficiently small positive δ, the number $\kappa\big(y_{\nu_0}(t)\big)$ is also positive. The linear functional g under consideration is given by formula (13) (see Lemma 6), in view of (18) we obtain

$$g(y(t)) > 0. \tag{24}$$

Thus, we have shown that if the components of the function y satisfy condition (15), (16), then, for an arbitrary $\vec{\sigma}$-positive functional $g : C_a([a,b],\mathbb{R}^n) \to \mathbb{R}^n$, inequality (24) is fulfilled. Then in view of Definition 5 this means that y is a quasi-interior element of the cone $C_a([a,b],\mathbb{R}^n_{\vec{\sigma}})$. Lemma 8 is proved. □

4. Main Result

We are ready to prove our main achievements.

Theorem 2. *Assume that the linear operator $l = (l_k)_{k=1}^n$ is $\vec{\sigma}$-positive. Suppose that there exist a number $\alpha > 1$, a function $y_0 \in C_a([a,b],\mathbb{R}^n)$, and a certain integer $k \geq 0$ such that the components of the function $(y_{k,\nu})_{\nu=1}^n$ of the respective function y_k are continuous and satisfy conditions* (15) *and* (16)*.*

Additionally, there are fulfilled the succeeding differential inequalities

$$\sigma_\nu\Big(D_a^q y_{k,\nu}(t) - \alpha(l_\nu y_{k+m})(t)\Big) \geq 0 \tag{25}$$

for a certain $m \geq 0$, all $\nu = 1,2,\ldots,n$, and almost all $t \in [a,b]$.

Then the nonhomogeneous non-local boundary-value problem (2) *for FFDE* (1) *has a unique solution $u(\cdot)$ for arbitrary $c \in \mathbb{R}^n$ and $r \in L([a,b],\mathbb{R}^n)$, and this solution is representable in the form*

$$\begin{aligned} u(t) &= r_c(t) + \tfrac{1}{\Gamma(q)}\textstyle\int_a^t (t-s)^{q-1}(lr_c)(s)ds + \\ &+ \tfrac{1}{\Gamma(q)}\textstyle\int_a^t (t-\cdot)^{q-1} l\Big(\tfrac{1}{\Gamma(q)}\int_a^{\cdot}(t-s)^{q-1}(lr_c)(\xi)d\xi\Big)(s)ds + \ldots, \end{aligned} \tag{26}$$

where the functional series converges uniformly on $[a,b]$ and

$$r_c(t) := c + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} r(s)ds, \quad t \in [a,b]. \tag{27}$$

If, furthermore, for all $\nu = 1,2,\ldots,n$ and $t \in [a,b]$, the inequalities

$$\sigma_\nu\Big(c_\nu + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} r_\nu(s)ds\Big) \geq 0 \tag{28}$$

are true, then the components $(u_\nu)_{\nu=1}^n$ of the unique solution $u(\cdot)$ of the initial value problem (2) *for FFDE* (1) *satisfy the conditions* (6)*.*

The homogeneous initial value problem (9) *for FFDE* (10) *has only a trivial solution.*

4.1. Proof

Proof of Theorem 2. Let us put

$$C_a([a,b],\mathbb{R}^n) \ni u \longmapsto I_{a,l}u := \frac{1}{\Gamma(q)}\int_a^{\cdot}(t-s)^{q-1}(lu)(s)ds. \tag{29}$$

It is known from [20,21] that $I_{a,l}$ defined by (29) is a linear operator that transforms the space $C_a([a,b],\mathbb{R}^n)$ into itself. The operator is completely continuous (see Theorem 1.8 from [2]).

Now we establish that the spectrum of operator $I_{a,l}$ contains in the α^{-m-1}-neighborhood of zero under the assumed conditions, or in other words, $\rho(I_{a,l}) < \alpha^{-m-1}$, where $\alpha > 1$ (under the suppositions of the Theorem).

The $\vec{\sigma}$-positivity of the operator l (see Definition 3) guarantees that the corresponding operator $I_{a,l} : C_a([a,b], \mathbb{R}^n_{\vec{\sigma}}) \to C_a([a,b], \mathbb{R}^n_{\vec{\sigma}})$. In view of Lemma 5, the mentioned set is a cone in the Banach space $C_a([a,b], \mathbb{R}^n)$. The function y_k, involved in Theorem 2, is a quasi-interior element of the cone $C_a([a,b], \mathbb{R}^n_{\vec{\sigma}})$ (this fact is proved in Lemma 8). The condition (26) for the function y_k and the corresponding function y_{m+k} ensure the trueness of the following relations:

$$\sigma_\nu \left(D^q_a y_{k,\nu}(t) - \alpha (l_\nu y_{m+k})(t) \right) \geq 0, \quad s \in [a,b], \quad 1 \leq \nu \leq n. \tag{30}$$

After integrating both parts of relations (30) from t_0 to some arbitrary $t \in [a,b]$ and by considering (17), (29) and Lemma 2, we obtain

$$\sigma_\nu \left(y_{k,\nu}(s) - \frac{\alpha}{\Gamma(q)} \int_a^t (t-s)^{q-1} (l_\nu y_{m+k})(s) ds \right) \geq 0, \; t \in [a,b], \quad 1 \leq \nu \leq n.$$

Examination of the relation (14) of the sequence $\{y_k \geq 0\}$ and the relation (29) of the operator $I_{a,l}$, allows to conclude that the recent system of inequalities yields the following inclusion:

$$\alpha^{-1} y_k - I^{m+1}_{a,l} y_k \in C_a([a,b], \mathbb{R}^n_{\vec{\sigma}}), \quad t \in [a,b]. \tag{31}$$

Thus, one can apply Theorem 1 with

$$Y = C_a([a,b], \mathbb{R}^n), \quad K = C_a([a,b], \mathbb{R}^n_{\vec{\sigma}}), \text{ and } \quad p = m+1$$

to the operator $\Psi = I_{a,l}$ (see the relation (29)). Given (31), the conditions (11), (12) of Theorem 2 are satisfied for the element $z = y_k$ with constant $\beta = \alpha^{-1}$. Theorem 1 allows one to claim that the spectral radius $\rho(I_{a,l})$ of the operator $I_{a,l}$ in the space $C_a([a,b], \mathbb{R}^n)$ satisfies the inequality

$$\rho(I_{a,l}) \leq \frac{1}{\alpha^{m+1}}. \tag{32}$$

Consequently, it is strictly smaller than one (the inequality $\alpha > 1$ is supposed in the theorem).

Now we repeat the well-known Neumann's theorem.

Theorem 3 (Theorem 2, p. 69 [22]). *Let $I_{a,l}$ be a bounded linear operator on $C_a([a,b], \mathbb{R}^n)$. Suppose that $\|I_{a,l}\| < 1$. Then $\mathbf{1} - I_{a,l}$ has a unique bounded linear inverse $(\mathbf{1} - I_{a,l})^{-1}$ which is given by Neumann's series*

$$(\mathbf{1} - I_{a,l})^{-1} y_0 = \lim_{n \to \infty} (\mathbf{1} + I_{a,l} + I^2_{a,l} + \ldots + I^n_{a,l}) y_0, \quad y_0 \in C_a([a,b], \mathbb{R}^n),$$

where $\mathbf{1}$ is the identity operator: $\mathbf{1} \cdot y_0 = y_0$.

So, the inequality (32) ensures the uniform convergence on interval $[a,b]$ of the series

$$u_0 = r_0 + I_{a,l} r_0 + I^2_{a,l} r_0 + \ldots \tag{33}$$

to the unique solution u_0 of the equation

$$u_0(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left((l u_0)(s) + r(s) \right) ds, \quad t \in [a,b], \tag{34}$$

or, respectively (see Lemma 4), to the unique solution of the Cauchy problem

$$D^q_a u_0(t) = (l u_0)(t) + r(t), \quad t \in [a,b], \tag{35}$$

$$u_0(t_0) = 0. \tag{36}$$

The function r_0 from (33) is given by (27), where $c = 0$:

$$r_0(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} r(s) ds, \quad t \in [a,b].$$

Note that the above statement concerning problems (35) and (36) is true for arbitrary r from $L([a,b],\mathbb{R}^n)$.

Consider now the nonhomogeneous initial value problems (1) and (2). If u is a solution of problems (1) and (2), then the function

$$u_0(t) := u(t) - c, \quad t \in [a,b],$$

is obviously a solution of the equation

$$u_0(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (lu_0)(s) ds + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \Big(r(s) + (lc)(s) \Big) ds, \tag{37}$$

for $t \in [a,b]$. The equality (37) takes the form (34) if r on its right-hand side is replaced by the function $r - lc$. This corresponds to an analogous transformation in the original functional–differential equation (1). Subsequently, the sum of series (26) represents the solution of problems (1) and (2). Thus, it was shown that the unique solution of the nonlocal initial value problem (2) for FFDE (1) admits the representation (26). The last theorem assertion follows from the definition of the operator (29). The invariance of the set of functions u satisfying condition (6) is given by the problems (1) and (2) solution representation by series (26) and by the $\vec{\sigma}$-positivity of the operator l, the corresponding operator $I_{a,l}$. The condition (6) is guaranteeing by the fact that the function r_c defined by formula (27) is confirmed by inequality (28). The theorem is proved. □

4.2. Corollary

It is easy to see that the following statement follows from Theorem 2.

Theorem 4. *Let the linear operator l in (1) be $\vec{\sigma}$-positive. Also assume that there exist a number $\alpha > 1$ and an absolutely continuous function $x = (x_\nu)_{\nu=1}^n : [a,b] \to \mathbb{R}^n$ such that*

$$x_\nu(a) = 0, \quad \sigma_\nu x_\nu(t) > 0 \quad \textit{for} \quad t \in (a,b], \quad \nu = 1,2,\dots,n \tag{38}$$

is true and, for almost all t from $[a,b]$ and every $\nu = 1,2,\dots,n$, the following differential inequalities are satisfied:

$$\sigma_\nu \Big(D_a^q x_\nu(t) - \alpha (l_\nu x)(t) \Big) \geq 0. \tag{39}$$

Then the non-local initial value problem (1), (2) has a unique solution $u(\cdot)$ for arbitrary $r \in L([a,b],\mathbb{R}^n)$. Moreover, this solution is representable in the form of the uniformly convergent series (26). If, furthermore, for all $\nu = 1,2,\dots,n$ and $t \in [a,b]$, relations (28) are true for the vector c and function r in problems (1) and (2), then the fact that condition (38) is satisfied for a certain solution of the differential inequality (39) implies that the components $(u_\nu)_{\nu=1}^n$ of the unique solution $u(\cdot)$ of problems (1) and (2) are nonnegative in the sense of (6).

Proof. Obviously, (39) is a particular case of the inequality (26) for $k = 0$, $m = 0$, and $x_0 = y_0$. Obviously, the assertion of Theorem 4 is an immediate consequence of Theorem 2. □

5. Optimality of the Exact Conditions on the Unique Solvability for IVP for FFDE

Condition (26) and, hence, condition (39) are unimprovable in the sense that, generally speaking, neither condition can be assumed with $\alpha = 1$. Let us consider the next example.

Example 1. *Consider the initial value problem (2) for the scalar linear FFDE*

$$D_a^{\frac{1}{2}} u(t) = \frac{3(t-a)u(\beta)}{4\Gamma(\frac{1}{2})(\beta-a)^{\frac{3}{2}}}, \quad t \in [a,b], \tag{40}$$

$$u(a) = 0, \tag{41}$$

where β is a given point from $(a,b]$, $q = \frac{1}{2}$, and Γ-function defined by (5).

We fix an arbitrary absolutely continuous function $y_0 : [a,b] \to \mathbb{R}$ satisfying the conditions

$$y_0(t) = \frac{4}{3\Gamma(\frac{1}{2})}(t-a)^{\frac{3}{2}} u(\beta)$$

and construct the corresponding functions $y_1, y_2, \ldots$ determined by formula (14), where the linear operator $l : C_a([a,b],\mathbb{R}) \to L([a,b],\mathbb{R})$ is given by the relation

$$C_a([a,b],R) \ni u(\cdot) \longmapsto (lu)(\cdot) := \frac{3(\cdot - a)u(\beta)}{4\Gamma(\frac{1}{2})(\beta-a)^{\frac{3}{2}}}.$$

This operator is obviously σ-positive in the sense of Definition 3.

Now note that condition (39) is satisfied in the form of an equality with $\alpha = 1$ for

$$u(t) = \lambda \frac{4}{3\Gamma(\frac{1}{2})}(t-a)^{\frac{3}{2}}, \quad t \in [a,b], \quad \lambda \neq 0, \tag{42}$$

where λ is an arbitrary positive constant. Moreover, evidently, (42) is a nontrivial solution of the initial value problems (40) and (41).

So, if $\alpha = 1$ is admitted in (39), then the the assertion of Theorem 4 is not true, A similar conclusion is also true for Theorem 2 because the latter contains Theorem 4 as a particular case.

6. Application

Let us consider a fractional functional differential equation

$$D_a^q u(t) = \sum_{i=1}^{t_i \le t} f_i(t) u(\omega_i(t)) + r(t), \quad t \in [a,b], \tag{43}$$

where $a < t_1 < t_2 < \cdots < t_m < b$ are given, $f_i \in C([a,b], GL_n(\mathbb{R}))$ and $\omega_i \in C([a,b],[a,b])$, $i = 1, \ldots, m$. Setting $\chi_i(t)$ in the Equation (43) as the characteristic function of the interval $[t_i, b]$, we have

$$D_a^q u(t) = \sum_{i=1}^{m} \chi_i(t) f_i(t) u(\omega_i(t)) + r(t), \quad t \in [a,b].$$

So, now we can study a general case

$$D_a^q u(t) = \sum_{i=1}^{m} P_i(t) u(\omega_i(t)) + r(t), \quad t \in [a,b], \tag{44}$$

where P_i, $i = 0, 1, \ldots, m$, are defined by

$$P_i(t) := \begin{pmatrix} p_{11}^i(t) & p_{12}^i(t) & \ldots & p_{1n}^i(t) \\ p_{21}^i(t) & p_{22}^i(t) & \ldots & p_{2n}^i(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}^i(t) & p_{n2}^i(t) & \ldots & p_{nn}^i(t) \end{pmatrix}. \tag{45}$$

Next, taking $\omega_i(t) = t_i$, $i = 1, \ldots, m$, in (43), we obtain

$$D_a^q u(t) = \sum_{i=1}^{t_i \le t} f_i(t) u(t_i) + r(t), \quad t \in [a, b], \tag{46}$$

which is a model with a discrete memory effect.

On the other hand, if $P_i \in L([0,1], GL_n(\mathbb{R}))$, $a = 0$ and $\omega_i(t) = \lambda_i t$ for $\lambda_i \in (0,1)$ in (44), we obtain

$$D_0^q u(t) = \sum_{i=1}^{m} P_i(t) u(\lambda_i t) + r(t), \quad t \in [0, 1], \tag{47}$$

which is a pantograph-type model. Pantograph equations arise in electrodynamics [23].

Now let us establish conditions sufficient for the unique solvability of the initial value problem

$$u(0) = c \tag{48}$$

for FFDE

$$D_0^{\frac{1}{2}} u(t) = \sum_{i=1}^{m} P_i(t) u(\lambda_i t) + r(t), \quad t \in [0, 1] \tag{49}$$

where $\lambda_i \in (0,1)$, $i = 0, 1, \ldots, m$ are constants, function $r : [0,1] \to \mathbb{R}^n$ has summable components, and $c \in \mathbb{R}^n$, $P_i : [0,1] \to GL_n(\mathbb{R})$, $i = 0, 1, \ldots, m$, are defined by (45).

Obviously, Equation (49) is Equation (47) with $q = \frac{1}{2}$.

Let us consider a function

$$x_0(t) = \xi t, \quad \xi \not\equiv 0, \quad t \in [0, 1], \tag{50}$$

where $\xi \in \mathbb{R}^n$.

Theorem 5. *Suppose that*

$$\varsigma P_i(t) \varsigma \ge 0 \quad \textit{for almost all} \quad t \in [0, 1], \quad 1 \le i \le m, \tag{51}$$

where P_i, $i = 1, \ldots, m$ are defined by (45),

$$\varsigma := \begin{pmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n \end{pmatrix}, \tag{52}$$

and assume that there exist a real number $\alpha > 1$ and for a vector $\xi \in \mathbb{R}^n$ such that

$$\sigma_\nu \xi_\nu > 0, \quad 1 \le \nu \le n \tag{53}$$

for almost all t from $[0, 1]$, the following differential inequality is satisfied:

$$\varsigma \left(\frac{4\xi \left(t\right)^{\frac{3}{2}}}{3\Gamma(\frac{1}{2})} - \alpha \sum_{i=1}^{m} P_i(t)\, \xi \lambda_i t \right) \ge 0. \tag{54}$$

Then the initial value problem (49), (48) *has a unique solution $u(\cdot)$ for arbitrary $r \in L([0,1], \mathbb{R}^n)$. Moreover, this solution is representable in the form of the uniformly convergent series*

$$u(t) = \sum_{k=0}^{n} r^{[k]}(t), \quad t \in [0, 1], \tag{55}$$

where

$$r^{[k]}(t) = \frac{1}{\Gamma(\frac{1}{2})} \sum_{i=1}^{m} \int_0^t (t-s)^{-\frac{1}{2}} P_i(s) r^{[k-1]}(\lambda_i s) ds, \quad t \in [0,1]$$

and

$$r^{[0]}(t) = c + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} r(s) ds.$$

Moreover, if c and r satisfy condition (28), then the unique solution (55) of problems (49) and (48) is non-negative in the sense of (6).

For proving Theorem 5, we need the next technical lemma.

Lemma 9. *If each of the functions $P_i : [0,1] \to GL_n(\mathbb{R})$, $i = 1,2,\ldots,m$ satisfies inequality (51), then, for any measurable functions $\lambda_i \in (0,1)$, $i = 1,2,\ldots,m$, the linear operator*

$$C([0,1], \mathbb{R}^n) \ni u \mapsto (lu)(\cdot) := \sum_{i=1}^{m} P_i(\cdot) u(\lambda_i \cdot) \tag{56}$$

is $\vec{\sigma}$-positive.

Proof of Lemma 9. Assume that all components of a vector function u from the space $C_0([0,1], \mathbb{R}^n)$ satisfy condition (6). Since $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset \{-1, 1\}$, relation (56) yields the identity

$$\varsigma(lu)(t) = \varsigma \sum_{i=1}^{m} P_i(t) u(\lambda_i t) = \varsigma \sum_{i=1}^{m} P_i(t) \varsigma \varsigma u(\lambda_i t) \tag{57}$$

for $t \in [0,1]$, ς defined by (52), $P_i(t)$ defined by (45). By virtue of (6), we have $\varsigma u(t) \geq 0$ for all $t \in [0,1]$. Hence, taking assumption (51) into account and using (57), we conclude that, for almost all $t \in [0,1]$, one has

$$\varsigma \sum_{i=1}^{m} P_i(t) u(\lambda_i t) \geq 0 \quad \text{for almost all} \quad t \in [0,1],$$

i.e., the operator l given by relation (56) is $\vec{\sigma}$-positive.

Lemma 9 is proved. □

Proof of Theorem 5. By virtue of Lemma 9, condition (51) guarantees the $\vec{\sigma}$-positivity of operator (56), which defines (49). Inequality (39) is guaranteed by condition (54) if the operator $l : C_0([0,1], \mathbb{R}^n) \to L([0,1], \mathbb{R}^n)$ is defined by relation (56), and the function $x_0 : [0,1] \to \mathbb{R}^n$ is defined by the relation (50), where ξ is the vector appearing in the condition of the theorem. In this case, according to (50), we have

$$D_0^{\frac{1}{2}} x_0(t) = \frac{4\xi \left(t\right)^{\frac{3}{2}}}{3\Gamma(\frac{1}{2})}.$$

By assumption, the vector ξ possesses property (53), and, hence, function (50) satisfies conditions (38). Thus, using Theorem 4, we establish that, for all $r \in L([0,1], \mathbb{R}^n)$ the initial value problem (49), (48) has a unique solution u. Equality (55) obviously follows from (26). Theorem 5 is proved. □

7. Conclusions

Summarizing, we apply a functional–analytical approach for handling certain systems of linear fractional functional differential equations. More concretely, here are established exact conditions sufficient for the unique solvability of the initial-value problem for the system of linear fractional functional differential equations determined by isotone operators.

For the investigation, we use the method of the test elements intended for the estimation of the spectral radius of a linear operator. This method is characterized by the fact that, in many cases, it allows one to estimate the spectral radius of a linear operator based on knowledge of the value of the operator on a single, suitably chosen element of a space. The conditions established are unimprovable in a sense. Moreover, the unique solution is presented in view of Neumann's series. It is necessary to point out that the mentioned method works only for $\vec{\sigma}$-monotone operators. We apply derived abstract results to the pantograph-type model from electrodynamics.

A possible future work would be to study several coupled FFDEs with different fractional derivatives. We also intend to extend the method of this paper to such kinds of systems.

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