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A Sylvester-Type Matrix Equation over the Hamilton Quaternions with an Application

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Abstract: We derive the solvability conditions and a formula of a general solution to a Sylvester-type matrix equation over Hamilton quaternions. As an application, we investigate the necessary and sufficient conditions for the solvability of the quaternion matrix equation, which involves η-Hermicity. We also provide an algorithm with a numerical example to illustrate the main results of this paper.

Keywords: matrix equation; Hamilton quaternion; η-Hermitian matrix; Moore–Penrose inverse; rank

MSC: 15A03; 15A09; 15A24; 15B33; 15B57

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1. Introduction

Let $\mathbb{R}$ stand for the real number field and

$$\mathbb{H} = \{u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ u_0, u_1, u_2, u_3 \in \mathbb{R}\}.$$

$\mathbb{H}$ is called the Hamilton quaternion algebra, which is a non-commutative division ring. Hamilton quaternions and Hermitian quaternion matrices have been utilized in statistics of quaternion random signals [1], quaternion matrix optimization problems [2], signal and color image processing, face recognition [3,4], and so on.

Sylvester and Sylvester-type matrix equations have a large number of applications in different disciplines and fields. For example, the Sylvester matrix equation

$$A_1X + XB_1 = C_1 \tag{1}$$

and the Sylvester-type matrix equation

$$A_1X + YB_1 = C_1 \tag{2}$$

have been applied in singular system control [5], system design [6], perturbation theory [7], sensitivity analysis [8], H_α-optimal control [9], linear descriptor systems [10], and control theory [11]. Roth [12] gave the Sylvester-type matrix Equation (2) for the first time over the polynomial integral domain. Baksalary and Kala [13] established the solvability conditions for Equation (2) and gave an expression of its general solution. In addition, Baksalary and Kala [14] derived the necessary and sufficient conditions for a two-sided Sylvester-type matrix equation

$$A_{11}X_1B_{11} + C_{11}X_2D_{11} = E_{11} \tag{3}$$

to be consistent. Özgüler [15] studied (3) over a principal ideal domain. Wang [16] investigated (3) over an arbitrary regular ring with an identity element.

Due to the wide applications of quaternions, the investigations on Sylvester-type matrix equations have been extended to $\mathbb{H}$ in the last decade (see, e.g., [17–24]). They are applied for signal processing, color-image processing, and maximal invariant semidefinite or neutral subspaces, etc. (see, e.g., [25–28]). For instance, the general solution to Sylvester-type matrix Equation (2) can be used in color-image processing. He [29] derived the matrix Equation (2) as an essential finding. Roman [25] established the necessary and sufficient conditions for Equation (1) to have a solution. Kychei [30] investigated Cramer's rules to drive the necessary and sufficient conditions for Equation (3) to be solvable. As an extension of Equations (2) and (3), Wang and He [31] gave the solvability conditions and the general solution to the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \tag{4}$$

over the complex number field $\mathbb{C}$, which can be generalized to $\mathbb{H}$ and applicable in some Sylvester-type matrix equations over $\mathbb{H}$ (see, e.g., [29,32]).

We know that in system and control theory, the more unknown matrices that a matrix equation has, the wider its application will be. Consequently, for the sake of developing theoretical studies and the applications mentioned above of Sylvester-type matrix equation and their generalizations, in this paper, we aim to establish some necessary and sufficient conditions for the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4 = B \tag{5}$$

to have a solution in terms of the rank equalities and Moore–Penrose inverses of some coefficient quaternion matrices in Equation (5) over $\mathbb{H}$. We derive a formula of its general solution when it is solvable. It is clear that Equation (5) provides a proper generalization of Equation (4), and we carry out an algorithm with a numerical example to calculate the general solution of Equation (5). As a special case of Equation (5), we also obtain the solvability conditions and the general solution for the two-sided Sylvester-type matrix equation

$$A_{11}Y_1B_{11} + A_{22}Y_2B_{22} + A_{33}Y_3B_{33} = T_1. \tag{6}$$

To the best of our knowledge, so far, there has been little information on the solvability conditions and an expression of the general solution to Equation (6) by using generalized inverses.

As usual, we use A^* to denote the conjugate transpose of A. Recall that a quaternion matrix A, for $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, is said to be η-Hermitian if $A = A^{\eta^*}$, where $A^{\eta^*} = -\eta A^* \eta$ [33]. For more properties and information on η^*-quaternion matrices, we refer to [33]. We know that η-Hermitian matrices have some applications in linear modeling and statistics of quaternion random signals [1,33]. As an application of Equation (5), we establish some necessary and sufficient conditions for the quaternion matrix equation

$$A_1X_1 + (A_1X_1)^{\eta^*} + A_2Y_1A_2^{\eta^*} + A_3Y_2A_3^{\eta^*} + A_4Y_3A_4^{\eta^*} = B \tag{7}$$

to be consistent. Moreover, we derive a formula of the general solution to Equation (7) where $B = B^{\eta^*}, Y_i = Y_i^{\eta^*}$ $(i = \overline{1,3})$ over $\mathbb{H}$.

The rest of this paper is organized as follows. In Section 2, we review some definitions and lemmas. In Section 3, we establish some necessary and sufficient conditions for Equation (5) to have a solution. In addition, we give an expression of its general solution to Equation (5) when it is solvable. In Section 4, as an application of Equation (5), we consider some solvability conditions and the general solution to Equation (7), where $Y_i = Y_i^{\eta^*}$ $(i = \overline{1,3})$. Finally, we give a brief conclusion to the paper in Section 5.

2. Preliminaries

Throughout this paper, $\mathbb{H}^{m\times n}$ stands for the space of all $m \times n$ matrices over $\mathbb{H}$. The symbol $r(A)$ denotes the rank of A. I and 0 represent an identity matrix and a zero matrix of appropriate sizes, respectively. In general, $A^{\dagger}$ stands for the Moore–Penrose inverse of $A \in \mathbb{H}^{l\times k}$, which is defined as the solution of $AYA = A$, $YAY = Y$, $(AY)^* = AY$ and $(YA)^* = YA$. Moreover, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ represent two projectors along A.

The following lemma is due to Marsaglia and Styan [34], which can be generalized to $\mathbb{H}$.

Lemma 1 ([34])**.** *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$ and $E \in \mathbb{H}^{l\times i}$ be given. Then, we have the following rank equality:*

$$r\begin{pmatrix} A & BL_D \\ R_EC & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r(D) - r(E).$$

Lemma 2 ([35])**.** *Let $A \in \mathbb{H}^{m\times n}$ be given. Then,*

(1) $(A^\eta)^\dagger = (A^\dagger)^\eta$, $(A^{\eta^*})^\dagger = (A^\dagger)^{\eta^*}$.

(2) $r(A) = r(A^{\eta^*}) = r(A^\eta) = r(A^\eta A^{\eta^*}) = r(A^{\eta^*} A^\eta)$.

(3) $(L_A)^{\eta^*} = -\eta(L_A)\eta = (L_A)^\eta = L_{A^*} = R_{A^{\eta^*}}$.

(4) $(R_A)^{\eta^*} = -\eta(R_A)\eta = (R_A)^\eta = R_{A^*} = L_{A^{\eta^*}}$.

(5) $(AA^\dagger)^{\eta^*} = (A^\dagger)^{\eta^*} A^{\eta^*} = (A^\dagger A)^\eta = A^\eta (A^\dagger)^\eta$.

(6) $(A^\dagger A)^{\eta^*} = A^{\eta^*} (A^\dagger)^{\eta^*} = (AA^\dagger)^\eta = (A^\dagger)^\eta A^\eta$.

Lemma 3 ([16])**.** *Let A_{ii}, B_{ii} and C_i $(i = 1,2)$ be given matrices with suitable sizes over $\mathbb{H}$. $A_1 = A_{22}L_{A_{11}}$, $T = R_{B_{11}}B_{22}$, $F = B_{22}L_T$, $G = R_{A_1}A_{22}$. Then, the following statements are equivalent:*

(1) The system

$$A_{11}X_1B_{11} = C_1,\ A_{22}X_1B_{22} = C_2 \tag{8}$$

has a solution.

(2)

$$A_{ii}A_{ii}^\dagger C_i B_{ii}^\dagger B_{ii} = C_i \ (i = 1,2)$$

and

$$G(A_{22}^\dagger C_2 B_{22}^\dagger - A_{11}^\dagger C_1 B_{11}^\dagger)F = 0.$$

(3)

$$r\begin{pmatrix} A_{ii} & C_i \end{pmatrix} = r(A_{ii}),\ r\begin{pmatrix} C_i \\ B_{ii} \end{pmatrix} = r(B_{ii}) \ (i = 1,2),$$

$$r\begin{pmatrix} A_{11} & C_1 & 0 \\ A_{22} & 0 & -C_2 \\ 0 & B_{11} & B_{22} \end{pmatrix} = r\begin{pmatrix} A_{11} \\ A_{22} \end{pmatrix} + r(B_{11},\ B_{22}).$$

Lemma 4 ([13])**.** *Let A_1, B_1 and C_1 be given matrices with suitable sizes. Then, the Sylvester-type Equation (2) is solvable if and only if*

$$R_{A_1}C_1L_{B_1} = 0.$$

In this case, the general solution to Equation (2) can be expressed as

$$X = A_1^\dagger C_1 - A_1^\dagger U_1 B_1 + L_{A_1}U_2,\ Y = R_{A_1}C_1B_1^\dagger + A_1A_1^\dagger U_1 + U_3R_{B_1},$$

where U_1, U_2, and U_3 are arbitrary matrices with appropriate sizes.

Lemma 5 ([31])**.** *Let $A_1, B_1, C_3, D_3, C_4, D_4$ and E_1 be given matrices over $\mathbb{H}$. Put*

$$A = R_{A_1}C_3,\ B = D_3L_{B_1},\ C = R_{A_1}C_4,\ D = D_4L_{B_1},$$
$$E = R_{A_1}E_1L_{B_1},\ M = R_AC,\ N = DL_B,\ S = CL_M.$$

Then, the following statements are equivalent:

(1) *Equation* (4) *has a solution.*

(2)

$$R_MR_AE = 0,\ EL_BL_N = 0,\ R_AEL_D = 0,\ R_EL_B = 0.$$

(3)

$$r\begin{pmatrix} E_1 & C_4 & C_3 & A_1 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(C_4,\ C_3,\ A_1),$$
$$r\begin{pmatrix} E_1 & A_1 \\ D_3 & 0 \\ D_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} D_3 \\ D_4 \\ B_1 \end{pmatrix} + r(A_1),$$
$$r\begin{pmatrix} E_1 & C_3 & A_1 \\ D_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_1,\ C_3) + r\begin{pmatrix} D_4 \\ B_1 \end{pmatrix},$$
$$r\begin{pmatrix} E_1 & C_4 & A_1 \\ D_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_1 \quad C_4) + r\begin{pmatrix} D_3 \\ B_1 \end{pmatrix}.$$

In this case, the general solution to Equation (4) *can be expressed as*

$$X_1 = A_1^{\dagger}(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^{\dagger}T_7B_1 + L_{A_1}T_6,$$
$$X_2 = R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^{\dagger} + A_1A_1^{\dagger}T_7 + T_8R_{B_1},$$
$$X_3 = A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}ST_2R_NDB^{\dagger} + L_AT_4 + T_5R_B,$$
$$X_4 = M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_D,$$

where $T_1, \ldots, T_8$ are arbitrary matrices with appropriate sizes over $\mathbb{H}$.

3. Some Solvability Conditions and a Formula of the General Solution

In this section, we establish the solvability conditions and a formula of the general solution to Equation (5). We begin with the following lemma, which is used to reach the main results of this paper.

Lemma 6. *Let A_{11}, B_{11}, C_{11}, and D_{11} be given matrices with suitable sizes over $\mathbb{H}$, $A_{11}L_{A_{22}} = 0$ and $R_{B_{11}}B_{22} = 0$. Set*

$$A_1 = A_{22}L_{A_{11}},\ C_{11} = C_2 - A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22}. \tag{9}$$

Then, the following statements are equivalent:

(1) *The system* (8) *is consistent.*

(2)

$$R_{A_{ii}}C_i = 0,\ C_iL_{B_{ii}} = 0\ (i = 1, 2),\ R_{A_1}C_{11} = 0.$$

(3)

$$A_{ii}A_{ii}^{\dagger}C_iB_{ii}^{\dagger}B_{ii} = C_i\ (i = 1, 2),\ C_1B_{11}^{\dagger}B_{22} = A_{11}A_{22}^{\dagger}C_2.$$

(4)

$$r(A_{ii}, C_i) = r(A_{ii}), r\begin{pmatrix} B_{ii} \\ C_i \end{pmatrix} = r(B_{ii}) \ (i = 1,2),$$
$$r\begin{pmatrix} C_1 & 0 & A_{11} \\ 0 & -C_2 & A_{22} \\ B_{11} & B_{22} & 0 \end{pmatrix} = r(A_{22}) + r(B_{11}).$$

In this case, the general solution to system (8) can be expressed as

$$X_1 = A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_{22}^{\dagger}C_2B_{22}^{\dagger} + L_{A_{22}}V_1 + V_2R_{B_{11}} + L_{A_{11}}V_3R_{B_{22}}, \tag{10}$$

where V_1, V_2, and V_3 are arbitrary matrices with appropriate sizes over $\mathbb{H}$.

Proof. (1) $\Leftrightarrow$ (2) It follows from Lemma 3 that

$$\begin{aligned}
&G(A_{22}^{\dagger}C_2B_{22}^{\dagger} - A_{11}^{\dagger}C_1B_{11}^{\dagger})F = 0 \\
\Leftrightarrow &R_{A_1}(A_1 + A_{22}A_{11}^{\dagger}A_{11})A_{22}^{\dagger}C_2B_{22}^{\dagger} - A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} = 0 \\
\Leftrightarrow &R_{A_1}A_{22}A_{11}^{\dagger}A_{11}A_{22}^{\dagger}C_2B_{22}^{\dagger}B_{22} - A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} = 0 \\
\Leftrightarrow &R_{A_1}A_{22}A_{11}^{\dagger}A_{11}A_{22}^{\dagger}A_{22}A_{22}^{\dagger}C_2B_{22}^{\dagger}B_{22} - A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} = 0 \\
\Leftrightarrow &R_{A_1}(A_{22} - A_1)A_{22}^{\dagger}A_{22}A_{22}^{\dagger}C_2B_{22}^{\dagger}B_{22} - A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} = 0 \\
\Leftrightarrow &R_{A_1}C_2 - A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} = 0 \Leftrightarrow R_{A_1}C_{11} = 0,
\end{aligned}$$

where G and F are given in Lemma 3.

(1) $\Rightarrow$ (3) If the system (8) has a solution, then there exists a solution X_0 such that

$$A_{11}X_0B_{11} = C_1, \ A_{22}X_0B_{22} = C_2.$$

It is easy to show that

$$R_{A_{ii}}C_i = 0, \ C_iL_{B_{ii}} = 0 \ (i = 1,2).$$

Thus, $A_{ii}A_{ii}^{\dagger}C_iB_{ii}^{\dagger}B_{ii} = C_i$ $(i = 1,2)$. It follows from $R_{B_{11}}B_{22} = 0$, $A_{11}L_{A_{22}} = 0$ that

$$C_1B_{11}^{\dagger}B_{22} = A_{11}X_0B_{11}B_{11}^{\dagger}B_{22} = A_{11}X_0B_{22} = A_{11}A_{22}^{\dagger}A_{22}X_0B_{22} = A_{11}A_{22}^{\dagger}C_2.$$

(3) $\Rightarrow$ (2) Since $A_{22} - A_1 = A_{22}A_{11}^{\dagger}A_{11}$ and $C_1B_{11}^{\dagger}B_{22} = A_{11}A_{22}^{\dagger}C_2$, we have that

$$\begin{aligned}
R_{A_1}C_{11} &= R_{A_1}C_2 - R_{A_1}A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} = R_{A_1}C_2 - R_{A_1}A_{22}A_{11}^{\dagger}A_{11}A_{22}^{\dagger}C_2 \\
&= R_{A_1}C_2 - R_{A_1}(A_{22} - A_1)A_{22}^{\dagger}C_2 = R_{A_1}C_2 - R_{A_1}A_{22}A_{22}^{\dagger}C_2 = 0.
\end{aligned}$$

(2) $\Leftrightarrow$ (4) It follows from $R_{B_{11}}B_{22} = 0$ and $A_{11}L_{A_{22}} = 0$ that

$$r(B_{22}, \ B_{11}) = r(B_{11}), \ \begin{pmatrix} A_{11} \\ A_{22} \end{pmatrix} = r(A_{22}).$$

By Lemma 1,

$$
\begin{aligned}
&R_{A_{ii}}C_i = 0 \Leftrightarrow r(R_{A_{ii}}C_i) = 0 \Leftrightarrow r(A_{ii},\ C_i) = r(A_{ii})\ (i = 1, 2),\\
&C_i L_{B_{ii}} = 0 \Leftrightarrow r(C_i L_{B_{ii}}) = 0 \Leftrightarrow r\begin{pmatrix} B_{ii} \\ C_i \end{pmatrix} = r(B_{ii})\ (i = 1, 2),\\
&R_{A_1}C_{11} = 0 \Leftrightarrow r(R_{A_1}C_{11}) = 0 \Leftrightarrow r(C_{11},\ A_1) = r(A_1)\\
&\Leftrightarrow r\begin{pmatrix} C_{11} & A_{22}L_{A_{11}} \\ R_{B_{11}}B_{22} & 0 \end{pmatrix} = r(A_{22}L_{A_{11}}) + r(R_{B_{11}}B_{22})\\
&\Leftrightarrow r\begin{pmatrix} C_2 - A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} & A_{22} & 0 \\ B_{22} & 0 & B_{11} \\ 0 & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} A_{11} \\ A_{22} \end{pmatrix} + r(B_{11}, B_{22})\\
&\Leftrightarrow r\begin{pmatrix} C_1 & 0 & A_{11} \\ 0 & -C_2 & A_{22} \\ B_{11} & B_{22} & 0 \end{pmatrix} = r\begin{pmatrix} A_{11} \\ A_{22} \end{pmatrix} + r(B_{11}, B_{22}) = r(A_{22}) + r(B_{11}).
\end{aligned}
$$

We now prove that X_1 in (10) is the general solution of the system (8). We prove it in two steps. We show that X_1 is a solution of system (8) in Step 1. In Step 2, if the system (8) is consistent, then the general solution to system (8) can be expressed as (10).

Step 1. In this step, we show that X_1 is a solution of system (8). Substituting X_1 in (10) into the system (8) yields

$$
A_{11}X_1B_{11} = A_{11}X_0B_{11},\ A_{22}X_1B_{22} = A_{22}X_0B_{22}, \tag{11}
$$

where $X_0 = A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_{22}^{\dagger}C_2B_{22}^{\dagger}$. Since $R_{A_{11}}C_1 = 0$ and $C_1L_{B_{11}} = 0$, we have that

$$
\begin{aligned}
&A_{11}X_0B_{11} = A_{11}A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_1^{\dagger}A_{22}A_{22}^{\dagger}C_{11}B_{22}^{\dagger}B_{11}\\
&= A_{11}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{11} + A_{11}L_{A_{11}}A_1^{\dagger}C_{11} - R_{A_{22}}C_{11}B_{22}^{\dagger}B_{11} = A_{11}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{11}\\
&= -R_{A_{11}}C_1B_{11}^{\dagger}B_{11} - C_1L_{B_{11}} + C_1 = C_1.
\end{aligned}
$$

By

$$
R_{B_{11}}B_{22} = 0,\ R_{A_{22}}C_{22} = 0,\ C_2L_{B_{22}} = 0 \text{ and } C_1B_{11}^{\dagger}B_{22} = A_{11}A_{22}^{\dagger}C_2,
$$

we have that

$$
\begin{aligned}
&A_{22}X_0B_{22} = A_{22}(A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_{22}^{\dagger}C_2B_{22}^{\dagger})B_{22}\\
&= A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} + A_{22}A_{22}^{\dagger}C_2B_{22}^{\dagger}B_{22} - A_{22}A_{11}^{\dagger}A_{11}A_{22}^{\dagger}C_2B_{22}^{\dagger}B_{22}\\
&= C_2 + A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} - A_{22}A_{11}^{\dagger}C_1B_{11}^{\dagger}B_{22} = C_2.
\end{aligned}
$$

Thus, $A_{11}X_1B_{11} = C_1$, $A_{22}X_1B_{22} = C_2$. X_1 is a solution of system (8).

Step 2. In this step, we show that the general solution to the system (8) can be expressed as (10). It is sufficient to show that for an arbitrary solution, say, X_{01} of (8), X_{01} can be expressed in form (10). Put

$$
V_1 = X_{01}B_{22}B_{22}^{\dagger},\ V_2 = X_{01},\ V_3 = X_{01}B_{11}B_{11}^{\dagger}.
$$

It follows from $B_{22} = B_{11}B_{11}^{\dagger}B_{22}$ and $A_{11} = A_{11}A_{22}^{\dagger}A_{22}$ that

$$\begin{aligned}
X_1 &= A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_{22}^{\dagger}C_2B_{22}^{\dagger} + L_{A_{22}}V_1 + V_2R_{B_{11}} + L_{A_{11}}V_3R_{B_{22}} \\
&= A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_{22}^{\dagger}C_2B_{22}^{\dagger} + L_{A_{22}}X_{01}B_{22}B_{22}^{\dagger} + X_{01}R_{B_{11}} + L_{A_{11}}X_{01}B_{11}B_{11}^{\dagger}R_{B_{22}} \\
&= A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_{22}^{\dagger}C_2B_{22}^{\dagger} + X_{01}B_{22}B_{22}^{\dagger} - A_{22}^{\dagger}A_{22}X_{01}B_{22}B_{22}^{\dagger} + X_{01} - X_{01}B_{11}B_{11}^{\dagger} \\
&+ X_{01}B_{11}B_{11}^{\dagger}R_{B_{22}} - A_{11}^{\dagger}A_{11}X_{01}B_{11}B_{11}^{\dagger}R_{B_{22}} \\
&= A_{11}^{\dagger}C_1B_{11}^{\dagger} + L_{A_{11}}A_{22}^{\dagger}C_2B_{22}^{\dagger} - X_{01}R_{B_{11}}B_{22}B_{22}^{\dagger} + X_{01} - A_{22}^{\dagger}A_{22}X_{01}B_{22}B_{22}^{\dagger} \\
&- A_{11}^{\dagger}A_{11}X_{01}B_{11}B_{11}^{\dagger} + A_{11}^{\dagger}A_{11}X_{01}B_{11}B_{11}^{\dagger}B_{22}B_{22}^{\dagger} \\
&= X_{01} + A_{11}^{\dagger}A_{11}X_{01}B_{11}B_{11}^{\dagger}B_{22}B_{22}^{\dagger} - A_{11}^{\dagger}A_{11}A_{22}^{\dagger}A_{22}X_{01}B_{22}B_{22}^{\dagger} \\
&= X_{01} + A_{11}^{\dagger}A_{11}X_{01}B_{22}B_{22}^{\dagger} - A_{11}^{\dagger}A_{11}X_{01}B_{22}B_{22}^{\dagger} = X_{01}.
\end{aligned}$$

Hence, X_{01} can be expressed as (10). To sum up, (10) is the general solution of the system (8). □

Now, we give the fundamental theorem of this paper.

Theorem 1. *Let A_i, B_i, and B $(i = \overline{1,4})$ be given quaternion matrices with appropriate sizes over $\mathbb{H}$. Set*

$$\begin{aligned}
&R_{A_1}A_2 = A_{11},\ R_{A_1}A_3 = A_{22},\ R_{A_1}A_4 = A_{33},\ B_2L_{B_1} = B_{11},\ B_{22}L_{B_{11}} = N_1, \\
&B_3L_{B_1} = B_{22},\ B_4L_{B_1} = B_{33},\ R_{A_{11}}A_{22} = M_1,\ S_1 = A_{22}L_{M_1},\ R_{A_1}BL_{B_1} = T_1,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&C = R_{M_1}R_{A_{11}},\ C_1 = CA_{33},\ C_2 = R_{A_{11}}A_{33},\ C_3 = R_{A_{22}}A_{33},\ C_4 = A_{33}, \\
&D = L_{B_{11}}L_{N_1},\ D_1 = B_{33},\ D_2 = B_{33}L_{B_{22}},\ D_3 = B_{33}L_{B_{11}},\ D_4 = B_{33}D, \\
&E_1 = CT_1,\ E_2 = R_{A_{11}}T_1L_{B_{22}},\ E_3 = R_{A_{22}}T_1L_{B_{11}},\ E_4 = T_1D,
\end{aligned} \tag{13}$$

$$\begin{aligned}
&C_{11} = (L_{C_2},\ L_{C_4}),\ D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix},\ C_{22} = L_{C_1},\ D_{22} = R_{D_2},\ C_{33} = L_{C_3}, \\
&D_{33} = R_{D_4},\ E_{11} = R_{C_{11}}C_{22},\ E_{22} = R_{C_{11}}C_{33},\ E_{33} = D_{22}L_{D_{11}},\ E_{44} = D_{33}L_{D_{11}}, \\
&M = R_{E_{11}}E_{22},\ N = E_{44}L_{E_{33}},\ F = F_2 - F_1,\ E = R_{C_{11}}FL_{D_{11}},\ S = E_{22}L_M,
\end{aligned} \tag{14}$$

$$\begin{aligned}
&F_{11} = C_2L_{C_1},\ G_1 = E_2 - C_2C_1^{\dagger}E_1D_1^{\dagger}D_2,\ F_{22} = C_4L_{C_3},\ G_2 = E_4 - C_4C_3^{\dagger}E_3D_3^{\dagger}D_4, \\
&F_1 = C_1^{\dagger}E_1D_1^{\dagger} + L_{C_1}C_2^{\dagger}E_2D_2^{\dagger},\ F_2 = C_3^{\dagger}E_3D_3^{\dagger} + L_{C_3}C_4^{\dagger}E_4D_4^{\dagger}.
\end{aligned} \tag{15}$$

Then, the following statements are equivalent:

(1) *Equation* (5) *is consistent.*

(2)

$$R_{C_i}E_i = 0,\ E_iL_{D_i} = 0\ (i = \overline{1,4}),\ R_{E_{11}}EL_{E_{44}} = 0. \tag{16}$$

(3)

$$r\begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(A_2,\ A_3,\ A_4,\ A_1), \tag{17}$$

$$r\begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_4,\ A_1) + r\begin{pmatrix} B_3 \\ B_1 \end{pmatrix}, \tag{18}$$

$$r\begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_3,\ A_4,\ A_1) + r\begin{pmatrix} B_2 \\ B_1 \end{pmatrix}, \tag{19}$$

$$r\begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_1 \end{pmatrix} + r(A_4,\ A_1), \tag{20}$$

$$r\begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_3,\ A_1) + r\begin{pmatrix} B_4 \\ B_1 \end{pmatrix}, \tag{21}$$

$$r\begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_2,\ A_1), \tag{22}$$

$$r\begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_4 \\ B_1 \end{pmatrix} + r(A_3,\ A_1), \tag{23}$$

$$r\begin{pmatrix} B & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_1), \tag{24}$$

$$r\begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

$$= r\begin{pmatrix} B_3 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \\ B_4 & B_4 \end{pmatrix} + r\begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix}.$$

Proof. (1) $\Leftrightarrow$ (2) Equation (5) can be written as

$$A_1X_1 + X_2B_1 = B - (A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4). \tag{26}$$

Clearly, Equation (5) is solvable if and only if Equation (26) has a solution. By Lemma 4, Equation (26) is consistent if and only if there exist Y_i $(i = \overline{1,3})$ in Equation (26) such that

$$R_{A_1}[B - (A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4)]L_{B_1} = 0, \tag{27}$$

i.e.,

$$A_{11}Y_1B_{11} + A_{22}Y_2B_{22} + A_{33}Y_3B_{33} = T_1, \tag{28}$$

where A_{ii}, $B_{ii}(i = \overline{1,3})$, and T_1 are defined by (12). In addition, when Equation (26) has a solution, we get the following:

$$X_1 = A_1^{\dagger}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4) - A_1^{\dagger}U_1B_1 + L_{A_1}U_2,$$
$$X_2 = R_{A_1}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4)B_1^{\dagger} + A_1A_1^{\dagger}U_1 + U_3R_{B_1},$$

where U_i $(i = \overline{1,3})$ are any matrices with appropriate dimensions over $\mathbb{H}$. Hence, Equation (26) has a solution if and only if there exist Y_i $(i = \overline{1,3})$ in Equation (26) such that Equation (28) is solvable. According to Equation (28), we have that

$$A_{11}Y_1B_{11} + A_{22}Y_2B_{22} = T_1 - A_{33}Y_3B_{33}. \tag{29}$$

Hence, Equation (28) is consistent if and only if Equation (29) is solvable. It follows from Lemma 5 that Equation (29) has a solution if and only if there exists Y_3 in Equation (29) such that

$$\begin{aligned} R_{M_1}R_{A_{11}}(A_{33}Y_3B_{33}-T_1)=0,\ R_{A_{11}}(T_1-A_{33}Y_3B_{33})L_{B_{22}}=0,\\ R_{A_{22}}(T_1-A_{33}Y_3B_{33})L_{B_{11}}=0,\ (T_1-A_{33}Y_3B_{33})L_{B_{11}}L_{N_1}=0, \end{aligned} \tag{30}$$

i.e.,

$$C_1Y_3D_1=E_1,\ C_2Y_3D_2=E_2,\ C_3Y_3D_3=E_3,\ C_4Y_3D_4=E_4, \tag{31}$$

where C_i, D_i, E_i $(i=\overline{1,4})$ are defined by (13). When Equation (29) is solvable, we have that

$$\begin{aligned} Y_1 &= A_{11}^{\dagger}TB_{11}^{\dagger}-A_{11}^{\dagger}A_{22}M_1^{\dagger}TB_{11}^{\dagger}-A_{11}^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}B_{22}B_{11}^{\dagger}\\ &-A_{11}^{\dagger}S_1U_4R_{N_1}B_{22}B_{11}^{\dagger}+L_{A_{11}}U_5+U_6R_{B_{11}},\\ Y_2 &= M_1^{\dagger}TB_{22}^{\dagger}+S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}+L_{M_1}L_{S_1}U_7+U_8R_{B_{22}}+L_{M_1}U_4R_{N_1}, \end{aligned}$$

where A_{ii}, B_{ii} $(i=\overline{1,3})$, M_1, N_1, S_1, T_1 are defined by (12), $T=T_1-A_{33}Y_3B_{33}$ and U_j $(j=\overline{4,8})$ are any matrices with the appropriate dimensions over $\mathbb{H}$.

It is easy to infer that

$$C_1L_{C_2}=0,\ R_{D_1}D_2=0,\ C_3L_{C_4}=0,\ R_{D_3}D_4=0. \tag{32}$$

Thus, according to Lemma 6, we have that the system (31) is consistent if and only if

$$R_{C_i}E_i=0,\ E_iL_{D_i}=0\ (i=1,2,3,4),\ R_{F_{11}}G_1=0,\ R_{F_{22}}G_2=0. \tag{33}$$

In this case, the general solution to system (31) can be expressed as

$$Y_3=F_1+L_{C_2}V_1+V_2R_{D_1}+L_{C_1}V_3R_{D_2}, \tag{34}$$

$$Y_3=F_2-L_{C_4}W_1-W_2R_{D_3}-L_{C_3}W_3R_{D_4}, \tag{35}$$

where F_1, F_2 are defined by (15) and V_i, W_i $(i=\overline{1,3})$ are any matrices with the appropriate dimensions over $\mathbb{H}$. Thus, system (31) has a solution if and only if (33) holds and there exist V_i, W_i $(i=\overline{1,3})$ such that (34) equals to (35), namely

$$(L_{C_2},\ L_{C_4})\begin{pmatrix}V_1\\W_1\end{pmatrix}+(V_2,\ W_2)\begin{pmatrix}R_{D_1}\\R_{D_3}\end{pmatrix}+L_{C_1}V_3R_{D_2}+L_{C_3}W_3R_{D_4}=F,$$

i.e.,

$$C_{11}\begin{pmatrix}V_1\\W_1\end{pmatrix}+(V_2,\ W_2)D_{11}+C_{22}V_3D_{22}+C_{33}W_3D_{33}=F, \tag{36}$$

where F, C_{ii} and D_{ii} $(i=\overline{1,3})$ are defined by (14). It follows from Lemma 5 that Equation (36) has a solution if and only if

$$R_MR_{E_{11}}E=0,\ EL_{E_{33}}L_N=0,\ R_{E_{11}}EL_{E_{44}}=0,\ R_{E_{22}}EL_{E_{33}}=0. \tag{37}$$

In this case, the general solution to Equation (36) can be expressed as

$$\begin{aligned} V_1 &= (I_m,\ 0)\Big[C_{11}^{\dagger}(F-C_{22}V_3D_{22}-C_{33}W_3D_{33})-C_{11}^{\dagger}U_{11}D_{11}+L_{C_{11}}U_{12}\Big],\\ W_1 &= (0,\ I_m)\Big[C_{11}^{\dagger}(F-C_{22}V_3D_{22}-C_{33}W_3D_{33})-C_{11}^{\dagger}U_{11}D_{11}+L_{C_{11}}U_{12}\Big], \end{aligned}$$

$$W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$

$$V_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}(C_{11})^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$

$$V_3 = E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger}$$
$$- E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},$$
$$W_3 = M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},$$

where $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, and U_{42} are any matrices with the suitable dimensions over $\mathbb{H}$. $M, E, N, S, C_{11}, D_{11}$, and E_{ii} $(i = \overline{1,4})$ are defined by (14), m is the column number of A_4 and n is the row number of B_4. We summarize up that (28) has a solution if and only if (33) and (37) hold. Hence, Equation (5) is solvable if and only if (33) and (37) hold.

In fact, $R_{C_2}E_2 = 0,\ E_1L_{D_1} = 0 \Rightarrow R_{F_{11}}G_1 = 0$; $R_{C_4}E_4 = 0,\ E_3L_{D_3} = 0 \Rightarrow R_{F_{22}}G_2 = 0$; $R_{C_3}E_3 = 0,\ E_1L_{D_1} = 0 \Rightarrow R_MR_{E_{11}}E = 0$; $R_{C_4}E_4 = 0,\ E_1L_{D_1} = 0 \Rightarrow EL_{E_{33}}L_N = 0$; $R_{C_4}E_4 = 0,\ E_2L_{D_2} = 0 \Rightarrow R_{E_{22}}EL_{E_{33}} = 0$. The specific proof is as follows.

Firstly, we prove that $R_{C_2}E_2 = 0,\ E_1L_{D_1} = 0 \Rightarrow R_{F_{11}}G_1 = 0$; $R_{C_4}E_4 = 0,\ E_3L_{D_3} = 0 \Rightarrow R_{F_{22}}G_2 = 0$. It follows from Lemma 1 and elementary transformations that

$$\begin{aligned} &R_{C_1}E_1 = 0 \Leftrightarrow r(E_1,\ C_1) = r(C_1) = r(CT_1,\ CA_{33}) = r(CA_{33}) \Leftrightarrow \\ &r(T_1,\ A_{33},\ A_{11},\ A_{22}) = r(A_{33},\ A_{11},\ A_{22}), \end{aligned} \tag{38}$$

$$R_{C_2}E_2 = 0 \Leftrightarrow r(E_2,\ C_2) = r(C_2) \Leftrightarrow r\begin{pmatrix} T_1 & A_{33} & A_{11} \\ B_{22} & 0 & 0 \end{pmatrix} = r(A_{33},\ A_{11}) + r(B_{22}), \tag{39}$$

$$R_{C_3}E_3 = 0 \Leftrightarrow r(E_3,\ C_3) = r(C_3) \Leftrightarrow r\begin{pmatrix} T_1 & A_{33} & A_{22} \\ B_{11} & 0 & 0 \end{pmatrix} = r(A_{33},\ A_{22}) + r(B_{11}), \tag{40}$$

$$R_{C_4}E_4 = 0 \Leftrightarrow r(E_4,\ C_4) = r(C_4) \Leftrightarrow r\begin{pmatrix} T_1 & A_{33} \\ B_{11} & 0 \\ B_{22} & 0 \end{pmatrix} = r(A_{33}) + r\begin{pmatrix} B_{11} \\ B_{22} \end{pmatrix}, \tag{41}$$

$$E_1L_{D_1} = 0 \Leftrightarrow r\begin{pmatrix} E_1 \\ D_1 \end{pmatrix} \Leftrightarrow r\begin{pmatrix} T_1 & A_{11} & A_{22} \\ B_{33} & 0 & 0 \end{pmatrix} = r(A_{11},\ A_{22}) + r(B_{33}), \tag{42}$$

$$E_2L_{D_2} = 0 \Leftrightarrow r\begin{pmatrix} E_2 \\ D_2 \end{pmatrix} = r(D_2) \Leftrightarrow r\begin{pmatrix} T_1 & A_{11} \\ B_{33} & 0 \\ B_{22} & 0 \end{pmatrix} = r\begin{pmatrix} B_{33} \\ B_{22} \end{pmatrix} + r(A_{11}), \tag{43}$$

$$E_3L_{D_3} = 0 \Leftrightarrow r\begin{pmatrix} E_3 \\ D_3 \end{pmatrix} = r(D_3) \Leftrightarrow r\begin{pmatrix} T_1 & A_{22} \\ B_{33} & 0 \\ B_{11} & 0 \end{pmatrix} = r\begin{pmatrix} B_{33} \\ B_{11} \end{pmatrix} + r(A_{22}), \qquad . \tag{44}$$

$$E_4L_{D_4} = 0 \Leftrightarrow r\begin{pmatrix} E_4 \\ D_4 \end{pmatrix} = r(D_4) \Leftrightarrow r\begin{pmatrix} T_1 \\ B_{33} \\ B_{11} \\ B_{22} \end{pmatrix} = r\begin{pmatrix} B_{33} \\ B_{11} \\ B_{22} \end{pmatrix}. \tag{45}$$

It follows from Lemma 6 and (32) that $R_{F_{11}}G_1 = 0$ and $R_{F_{22}}G_2 = 0$ are equivalent to

$$r\begin{pmatrix} E_1 & 0 & C_1 \\ 0 & -E_2 & C_2 \\ D_1 & D_2 & 0 \end{pmatrix} = r\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + r(D_1,\ D_2), \tag{46}$$

$$r\begin{pmatrix} E_3 & 0 & C_3 \\ 0 & -E_4 & C_4 \\ D_3 & D_4 & 0 \end{pmatrix} = r\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} + r(D_3,\ D_4). \tag{47}$$

According to Lemma 1, we have that

$$
\begin{aligned}
&(46)\\
&\Leftrightarrow r\begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & 0 \\ 0 & -T_1 & A_{33} & 0 & 0 & A_{11} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_{11} & A_{22} & 0 \\ A_{33} & 0 & 0 & A_{11} \end{pmatrix} + r\begin{pmatrix} B_{33} & 0 \\ 0 & B_{22} \end{pmatrix} \\
&\Leftrightarrow r\begin{pmatrix} T_1 & A_{11} & A_{22} & 0 & 0 & 0 \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_1 & A_{33} & A_{11} \\ 0 & 0 & 0 & B_{22} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_{11} & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & A_{11} \end{pmatrix} + r\begin{pmatrix} B_{33} & 0 \\ 0 & B_{22} \end{pmatrix}.
\end{aligned} \tag{48}
$$

Thus, it follows from (48) that (46) holds when (39) and (42) hold. Similarly, if (41) and (44) hold, then (47) holds.

Secondly, we prove that $R_{C_3}E_3 = 0,\ E_1L_{D_1} = 0 \Rightarrow R_M R_{E_{11}} E = 0;\ R_{C_4}E_4 = 0,\ E_1L_{D_1} = 0 \Rightarrow EL_{E_{33}}L_N = 0;\ R_{C_4}E_4 = 0,\ E_2L_{D_2} = 0 \Rightarrow R_{E_{22}}EL_{E_{33}} = 0$. According to Lemma 5 and (32), we have that (37) are equivalent to

$$r\begin{pmatrix} F & L_{C_1} & L_{C_3} \\ R_{D_1} & 0 & 0 \\ R_{D_3} & 0 & 0 \end{pmatrix} = r(L_{C_1},\ L_{C_3}) + r\begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix}, \tag{49}$$

$$r\begin{pmatrix} F & L_{C_2} & L_{C_4} \\ R_{D_2} & 0 & 0 \\ R_{D_4} & 0 & 0 \end{pmatrix} = r(L_{C_2},\ L_{C_4}) + r\begin{pmatrix} R_{D_2} \\ R_{D_4} \end{pmatrix}, \tag{50}$$

$$r\begin{pmatrix} F & L_{C_1} & L_{C_4} \\ R_{D_1} & 0 & 0 \\ R_{D_4} & 0 & 0 \end{pmatrix} = r(L_{C_1},\ L_{C_4}) + r\begin{pmatrix} R_{D_1} \\ R_{D_4} \end{pmatrix}, \tag{51}$$

$$r\begin{pmatrix} F & L_{C_2} & L_{C_3} \\ R_{D_2} & 0 & 0 \\ R_{D_3} & 0 & 0 \end{pmatrix} = r(L_{C_2},\ L_{C_3}) + r\begin{pmatrix} R_{D_2} \\ R_{D_3} \end{pmatrix}, \tag{52}$$

respectively. By Lemma 1, we have that

$$
\begin{aligned}
&(49)\\
&\Leftrightarrow r\begin{pmatrix} F & I & I & 0 & 0 \\ I & 0 & 0 & D_1 & 0 \\ I & 0 & 0 & 0 & D_3 \\ 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} I & D_1 & 0 \\ I & 0 & D_3 \end{pmatrix} + r\begin{pmatrix} I & I \\ C_1 & 0 \\ 0 & C_3 \end{pmatrix} \\
&\Leftrightarrow r\begin{pmatrix} E_1 & 0 & C_1 \\ 0 & -E_3 & C_3 \\ D_1 & D_3 & 0 \end{pmatrix} = r\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + r(D_1,\ D_3).
\end{aligned} \tag{53}
$$

Similarly, we can show that (50)–(52) are equivalent to

$$r\begin{pmatrix} E_1 & 0 & C_1 \\ 0 & -E_4 & C_4 \\ D_1 & D_4 & 0 \end{pmatrix} = r\begin{pmatrix} C_1 \\ C_4 \end{pmatrix} + r(D_1,\ D_4), \tag{54}$$

$$r\begin{pmatrix} E_2 & 0 & C_2 \\ 0 & -E_3 & C_3 \\ D_2 & D_3 & 0 \end{pmatrix} = r\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} + r(D_2,\ D_3), \tag{55}$$

$$r\begin{pmatrix} E_2 & 0 & C_2 \\ 0 & -E_4 & C_4 \\ D_2 & D_4 & 0 \end{pmatrix} = r\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} + r(D_2,\ D_4). \tag{56}$$

Substituting C_i, D_i, and E_i $(i = 1, 3)$ in (13) into the rank equality (53) and by Lemma 1, we have that

$$\begin{aligned}
&(53) \\
&\Leftrightarrow r\begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & 0 \\ 0 & -T_1 & A_{33} & 0 & 0 & A_{22} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_{11} & A_{22} & 0 \\ A_{33} & 0 & 0 & A_{22} \end{pmatrix} + r\begin{pmatrix} B_{33} & 0 \\ 0 & B_{11} \end{pmatrix} \\
&\Leftrightarrow r\begin{pmatrix} T_1 & A_{11} & A_{22} & 0 & 0 & 0 \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_1 & A_{33} & A_{22} \\ 0 & 0 & 0 & B_{11} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_{11} & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & A_{22} \end{pmatrix} + r\begin{pmatrix} B_{33} & 0 \\ 0 & B_{11} \end{pmatrix}.
\end{aligned} \tag{57}$$

Hence, it follows from (40) and (42) that (57) holds. Similarly, we can prove that when (41), (42) hold and (41), (43) hold, we can get that (54) and (56) hold, respectively. Thus, Equation (28) has a solution if and only if (16) holds. That is to say, Equation (5) has a solution if and only if (16) holds.

(2) $\Leftrightarrow$ (3) We prove the equivalence in two parts. In the first part, we want to show that (38) to (45) are equivalent to (17) to (24), respectively. In the second part, we want to show that (55) is equivalent to (25).

Part 1. We want to show that (38) to (45) are equivalent to (17) to (24), respectively. It follows from Lemma 1 and elementary operations to (38) that

$$\begin{aligned}
&(38) \Leftrightarrow r(R_{A_1}BL_{B_{11}},\ R_{A_1}A_4,\ R_{A_1}A_2,\ R_{A_1}A_3) = r(R_{A_1}A_4,\ R_{A_1}A_2,\ R_{A_1}A_3) \\
&\Leftrightarrow r\begin{pmatrix} B & A_4 & A_2 & A_3 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(A_4,\ A_2,\ A_3,\ A_1) + r(B_1) \Leftrightarrow (17).
\end{aligned}$$

Similarly, we can show that (39) to (41) are equivalent to (18) to (20), respectively. Now, we turn to prove that (42) is equivalent to (19). It follows from the Lemma 1 and elementary transformations that

$$\begin{aligned}
&(42) \Leftrightarrow r\begin{pmatrix} R_{A_1}BL_{B_1} & R_{A_1}A_2 & R_{A_1}A_3 \\ B_4L_{B_1} & 0 & 0 \end{pmatrix} = r(R_{A_1}A_2,\ R_{A_1}A_3) + r(B_4L_{B_1}) \\
&\Leftrightarrow r\begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_3,\ A_1) + r\begin{pmatrix} B_4 \\ B_1 \end{pmatrix} \Leftrightarrow (21).
\end{aligned}$$

Similarly, we can show that (43) to (45) are equivalent to (22) to (24). Hence, (38) to (45) are equivalent to (17) to (24), respectively.

Part 2. We want to show that (55) $\Leftrightarrow$ (25). It follows from Lemma 1 and elementary operations to (55) that

$$\begin{aligned}
&(55) \Leftrightarrow \\
&\Leftrightarrow r\begin{pmatrix} R_{A_{11}}T_1L_{B_{22}} & 0 & R_{A_{11}}A_{33} \\ 0 & -R_{A_{22}}T_1L_{B_{11}} & R_{A_{22}}A_{33} \\ B_{33}L_{B_{22}} & B_{33}L_{B_{11}} & 0 \end{pmatrix} = r\begin{pmatrix} R_{A_{11}}A_{33} \\ R_{A_{22}}A_{33} \end{pmatrix} + r(B_{33}L_{B_{22}},\ B_{33}L_{B_{11}}) \\
&\Leftrightarrow r\begin{pmatrix} T_1 & 0 & A_{11} & 0 & A_{33} \\ 0 & -T_1 & 0 & A_{22} & A_{33} \\ B_{22} & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & 0 \\ B_{33} & B_{33} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_{22} & 0 \\ 0 & B_{11} \\ B_{33} & B_{33} \end{pmatrix} + r\begin{pmatrix} A_{11} & 0 & A_{33} \\ 0 & A_{22} & A_{33} \end{pmatrix}
\end{aligned}$$

$$\Leftrightarrow r\begin{pmatrix} B_3 & 0 \\ 0 & B_2 \\ B_4 & B_4 \\ B_1 & 0 \\ 0 & B_1 \end{pmatrix} + r\begin{pmatrix} A_2 & 0 & A_4 & A_1 & 0 \\ 0 & A_3 & A_4 & 0 & A_1 \end{pmatrix}$$

$$= r\begin{pmatrix} B & 0 & A_2 & 0 & A_4 & A_1 & 0 \\ 0 & -B & 0 & A_3 & A_4 & 0 & A_1 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\ B_4 & B_4 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Leftrightarrow (25).$$

Hence, (38) to (45) and (55) are equivalent to (17) to (25), respectively. □

Next, we give the formula of general solution to matrix Equation (5) by using Moore–Penrose. According to Theorem 1, we get the following theorem:

Theorem 2. *Let matrix Equation (5) be solvable. Then, the general solution to matrix Equation (5) can be expressed as*

$$\begin{aligned} X_1 &= A_1^{\dagger}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4) - A_1^{\dagger}U_1B_1 + L_{A_1}U_2, \\ X_2 &= R_{A_1}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4)B_1^{\dagger} + A_1A_1^{\dagger}U_1 + U_3R_{B_1}, \\ Y_1 &= A_{11}^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}A_{22}M_1^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}B_{22}B_{11}^{\dagger} \\ &\quad - A_{11}^{\dagger}S_1U_4R_{N_1}B_{22}B_{11}^{\dagger} + L_{A_{11}}U_5 + U_6R_{B_{11}}, \\ Y_2 &= M_1^{\dagger}TB_{22}^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1}, \\ Y_3 &= F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2}, \text{ or } Y_3 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4}, \end{aligned}$$

where $T = T_1 - A_{33}Y_3B_{33}$, $U_i(i = \overline{1,8})$ are arbitrary matrices with appropriate sizes over $\mathbb{H}$,

$$\begin{aligned} V_1 &= (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\right], \\ W_1 &= (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\right], \\ W_2 &= \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix}, \\ V_2 &= \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix}, \\ V_3 &= E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger} - E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}}, \\ W_3 &= M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}}, \end{aligned}$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, and U_{42} are arbitrary matrices with appropriate sizes over $\mathbb{H}$, m is the column number of A_4 and n is the row number of B_4.

Algorithm with a Numerical Example

In this section, we give Algorithm 1 with a numerical example to illustrate the main results.

Algorithm 1 Algorithm for computing the general solution of Equation (5)

(1) Input the quaternion matrices A_i, B_i $(i = \overline{1,4})$ and B with conformable shapes.
(2) Compute all matrices given by (12)–(15).
(3) Check equalities in (16) or (17)–(25). If not, it returns inconsistent.
(4) Else, compute X_i $Y_j(i = \overline{1,2},\ j = \overline{1,3})$.

Example 1. *Consider the matrix Equation* (5)*. Put*

$$A_1 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & \mathbf{i} \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ \mathbf{i} & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{i} \end{pmatrix}, A_3 = \begin{pmatrix} 1 & \mathbf{i} \\ 0 & 0 \end{pmatrix},$$
$$B_3 = \begin{pmatrix} 1 & \mathbf{j} \\ 0 & 0 \end{pmatrix},\ A_4 = \begin{pmatrix} 1 & \mathbf{k} \\ 0 & 0 \end{pmatrix},\ B_4 = \begin{pmatrix} 0 & 0 \\ \mathbf{k} & \mathbf{i} \end{pmatrix},\ B = \begin{pmatrix} 3\mathbf{i} & \mathbf{i}-1 \\ 0 & \mathbf{j} \end{pmatrix}.$$

Computation directly yields

$$r\begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(A_2,\ A_3,\ A_4,\ A_1) = 3,$$
$$r\begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_4,\ A_1) + r\begin{pmatrix} B_3 \\ B_1 \end{pmatrix} = 4,$$
$$r\begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_3,\ A_4,\ A_1) + r\begin{pmatrix} B_2 \\ B_1 \end{pmatrix} = 4,$$
$$r\begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_1 \end{pmatrix} + r(A_4,\ A_1) = 3,$$
$$r\begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_3,\ A_1) + r\begin{pmatrix} B_4 \\ B_1 \end{pmatrix} = 4,$$
$$r\begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_2,\ A_1) = 3,$$
$$r\begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_4 \\ B_1 \end{pmatrix} + r(A_3,\ A_1) = 3,$$
$$r\begin{pmatrix} B & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_1) = 3,$$
$$r\begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \\ B_4 & B_4 \end{pmatrix} + r\begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix} = 7.$$

All rank equalities in (17) *to* (25) *hold. Hence, according to Theorem 1, Equation* (5) *has a solution. Moreover, by Theorem 2, we have that*

$$X_1 = \begin{pmatrix} 1 & \mathbf{i} \\ 0 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 1 & \mathbf{j} \\ 0 & 0 \end{pmatrix},\ Y_1 = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ 0 & 0 \end{pmatrix},\ Y_2 = \begin{pmatrix} \mathbf{i} & \mathbf{k} \\ 0 & 0 \end{pmatrix},\ Y_3 = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{k} & 0 \end{pmatrix}.$$

Remark 1. *Chu et al. gave potential applications of the maximal and minimal ranks in the discipline of control theory (e.g., [36–38]). We may consider the rank bounds of the general solution of Equation (5).*

4. The General Solution to Equation with η-Hermicity

In this section, as an application of (5), we establish some necessary and sufficient conditions for quaternion matrix Equation (7) to have a solution and derive a formula of its general solution involving η-Hermicity.

Theorem 3. *Let A_i $(i=\overline{1,4})$ and B be given matrices with suitable sizes over $\mathbb{H}$, $B=B^{\eta^*}$. Set*

$$\begin{aligned}
&R_{A_1}A_2=A_{11},\ R_{A_1}A_3=A_{22},\ R_{A_1}A_4=A_{33},\ R_{A_{11}}A_{22}=M_1,\ S_1=A_{22}L_{M_1},\\
&R_{A_1}B(R_{A_1})^{\eta^*}=T_1,\ C=R_{M_1}R_{A_{11}},\ C_1=CA_{33},\ C_2=R_{A_{11}}A_{33},\\
&C_3=R_{A_{22}}A_{33},\ C_4=A_{33},\ E_1=CT_1,\ E_2=R_{A_{11}}T_1(R_{A_{22}})^{\eta^*},\ E_3=R_{A_{22}}T_1(R_{A_{11}})^{\eta^*},\ E_4=T_1C^{\eta^*},\\
&C_{11}=(L_{C_2},\ L_{C_4}),\ C_{22}=L_{C_1},\ C_{33}=L_{C_3},\ E_{11}=R_{C_{11}}C_{22},\ E_{22}=R_{C_{11}}C_{33},\\
&M=R_{E_{11}}E_{22},\ N=(R_{E_{22}}E_{11})^{\eta^*},\ F=F_2-F_1,\ E=R_{C_{11}}F(R_{C_{11}})^{\eta^*},\ S=E_{22}L_M,\\
&F_{11}=C_2L_{C_1},\ G_1=E_2-C_2C_1^{\dagger}E_1(C_4^{\eta^*})^{\dagger}C_3^{\eta^*},\ F_{22}=C_4L_{C_3},\ G_2=E_4-C_4C_3^{\dagger}E_3(C_2^{\eta^*})^{\dagger}C_1^{\eta^*},\\
&F_1=C_1^{\dagger}E_1(C_4^{\eta^*})^{\dagger}+L_{C_1}C_2^{\dagger}E_2(C_3^{\eta^*})^{\dagger},\ F_2=C_3^{\dagger}E_3(C_2^{\eta^*})^{\dagger}+L_{C_3}C_4^{\dagger}E_4(C_1^{\eta^*})^{\dagger}.
\end{aligned}$$

Then, the following statements are equivalent:

(1) *Equation (7) is consistent.*

(2) $R_{C_i}E_i=0$ $(i=\overline{1,4})$, $R_{E_{22}}E(R_{E_{22}})^{\eta^*}=0$.

(3)

$$r\begin{pmatrix} B & A_2 & A_3 & A_4 & A_1\\ A_1^{\eta^*} & 0 & 0 & 0 & 0\end{pmatrix}=r(A_1)+r(A_2,\ A_3,\ A_4,\ A_1),$$

$$r\begin{pmatrix} B & A_2 & A_3 & A_1\\ A_4^{\eta^*} & 0 & 0 & 0\\ A_1^{\eta^*} & 0 & 0 & 0\end{pmatrix}=r(A_2,\ A_3,\ A_1)+r(A_4,\ A_1),$$

$$r\begin{pmatrix} B & A_2 & A_4 & A_1\\ A_3^{\eta^*} & 0 & 0 & 0\\ A_1^{\eta^*} & 0 & 0 & 0\end{pmatrix}=r(A_2,\ A_4,\ A_1)+r(A_3,\ A_1),$$

$$r\begin{pmatrix} B & A_3 & A_4 & A_1\\ A_2^{\eta^*} & 0 & 0 & 0\\ A_1^{\eta^*} & 0 & 0 & 0\end{pmatrix}=r(A_3,\ A_4,\ A_1)+r(A_2,\ A_1),$$

$$r\begin{pmatrix} B & 0 & A_2 & 0 & A_4 & A_1 & 0\\ 0 & -B & 0 & A_3 & A_4 & 0 & A_1\\ A_3^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & A_2^{\eta^*} & 0 & 0 & 0 & 0 & 0\\ A_4^{\eta^*} & A_4^{\eta^*} & 0 & 0 & 0 & 0 & 0\\ A_1^{\eta^*} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & A_1^{\eta^*} & 0 & 0 & 0 & 0 & 0\end{pmatrix}=2r\begin{pmatrix} A_2 & 0 & A_4 & A_1 & 0\\ 0 & A_3 & A_4 & 0 & A_1\end{pmatrix}.$$

In this case, the general solution to Equation (7) can be expressed as

$$
\begin{aligned}
&X_1 = \frac{\widehat{X_1} + (\widehat{X_2})^{\eta^*}}{2},\ Y_1 = \frac{\widehat{Y_1} + (\widehat{Y_1})^{\eta^*}}{2},\ Y_2 = \frac{\widehat{Y_2} + (\widehat{Y_2})^{\eta^*}}{2},\ Y_3 = \frac{\widehat{Y_3} + (\widehat{Y_3})^{\eta^*}}{2},\\
&\widehat{X_1} = A_1^{\dagger}(C_1 - A_2Y_1A_2^{\eta^*} - A_3Y_2A_3^{\eta^*} - A_4Y_3A_4^{\eta^*}) + L_{A_1}U_2,\\
&\widehat{X_2} = R_{A_1}(C_1 - A_2Y_1A_2^{\eta^*} - A_3Y_2A_3^{\eta^*} - A_4Y_3A_4^{\eta^*})(A_1^{\dagger})^{\eta^*} + A_1A_1^{\dagger}U_1 + U_3R_{A_1^{\eta^*}},\\
&\widehat{Y_1} = A_{11}^{\dagger}T(A_{11}^{\dagger})^{\eta^*} - A_{11}^{\dagger}A_{22}M_1^{\dagger}T(A_{11}^{\dagger})^{\eta^*} - A_{11}^{\dagger}U_4A_{22}^{\dagger}T(M_1^{\dagger})^{\eta^*}(A_{22}^{\dagger})^{\eta^*} + L_{A_{11}}U_5 + U_6R_{A_{11}^{\eta^*}},\\
&\widehat{Y_2} = M_1^{\dagger}T(A_{22}^{\dagger})^{\eta^*} + S_1^{\dagger}S_1A_{22}^{\dagger}T(M_1^{\dagger})^{\eta^*} + L_{M_1}L_{S_1}U_7 + U_8R_{A_{22}^{\eta^*}} + L_{M_1}U_4R_{M_1^{\eta^*}},\\
&\widehat{Y_3} = F_1 + L_{C_2}V_1 + V_2R_{C_4^{\eta^*}} + L_{C_1}V_3R_{C_3^{\eta^*}},\ \text{or}\ \widehat{Y_3} = F_2 - L_{C_4}W_1 - W_2R_{C_2^{\eta^*}} - L_{C_3}W_3R_{C_1^{\eta^*}},
\end{aligned}
$$

where $T = T_1 - A_{33}Y_3(A_{33})^{\eta^*}$,

$$
\begin{aligned}
&V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*}) - C_{11}^{\dagger}U_{11}C_{11}^{\eta^*} + L_{C_{11}}U_{12}\right],\\
&W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*}) - C_{11}^{\dagger}U_{11}C_{11}^{\eta^*} + L_{C_{11}}U_{12}\right],\\
&W_2 = \left[R_{C_{11}}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})(C_{11}^{\eta^*})^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_{11}}^{\eta^*}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix},\\
&V_2 = \left[R_{C_{11}}(F - C_{22}V_3C_{33}^{\eta^*} - C_{33}W_3C_{22}^{\eta^*})(C_{11}^{\eta^*})^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_{11}}^{\eta^*}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix},\\
&V_3 = E_{11}^{\dagger}F(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}F(E_{22}^{\eta^*})^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{11}^{\eta^*}(E_{22}^{\eta^*})^{\dagger}\\
&- E_{11}^{\dagger}SU_{31}R_NE_{11}^{\eta^*}(E_{22}^{\eta^*})^{\dagger} + L_{E_{11}}U_{32} + U_{33}L_{E_{22}}^{\eta^*},\\
&W_3 = M^{\dagger}F(E_{11}^{\eta^*})^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}L_{E_{11}}^{\eta^*},
\end{aligned}
$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, *and* U_{42} *are any matrices with suitable dimensions over* $\mathbb{H}$.

Proof. It is easy to show that (7) has a solution if and only if the following matrix equation has a solution:

$$
A_1\widehat{X_1} + \widehat{X_2}A_1^{\eta^*} + A_2\widehat{Y_1}A_2^{\eta^*} + A_3\widehat{Y_2}A_3^{\eta^*} + A_4\widehat{Y_3}A_4^{\eta^*} = B. \tag{58}
$$

If (7) has a solution, say, $(X_1,\ Y_1,\ Y_2,\ Y_3)$, then

$$
(\widehat{X_1},\ \widehat{X_2},\ \widehat{Y_1},\ \widehat{Y_2},\ \widehat{Y_3}) := (X_1,\ X_1^{\eta^*},\ Y_1,\ Y_2,\ Y_3)
$$

is a solution of (58). Conversely, if (58) has a solution, say

$$
(\widehat{X_1},\ \widehat{X_2},\ \widehat{Y_1},\ \widehat{Y_2},\ \widehat{Y_3}).
$$

It is easy to show that (7) has a solution

$$
(X_1,\ Y_1,\ Y_2,\ Y_3) := \left(\frac{\widehat{X_1} + (\widehat{X_2})^{\eta^*}}{2},\ \frac{\widehat{Y_1} + (\widehat{Y_1})^{\eta^*}}{2},\ \frac{\widehat{Y_2} + (\widehat{Y_2})^{\eta^*}}{2},\ \frac{\widehat{Y_3} + (\widehat{Y_3})^{\eta^*}}{2}\right).
$$

□

Letting A_1 and B_1 vanish in Theorem 1, it yields to the following result.

Corollary 1. *Let A_{ii}, B_{ii} $(i=\overline{1,3})$, and T_1 be given matrices with appropriate sizes over $\mathbb{H}$. Set*

$$\begin{aligned}
&M_1 = R_{A_{11}}A_{22},\ N_1 = B_{22}L_{B_{11}},\ S_1 = A_{22}L_{M_1},\\
&C = R_{M_1}R_{A_{11}},\ C_1 = CA_{33},\ C_2 = R_{A_{11}}A_{33},\ C_3 = R_{A_{22}}A_{33},\ C_4 = A_{33},\\
&D = L_{B_{11}}L_{N_1},\ D_1 = B_{33},\ D_2 = B_{33}L_{B_{22}},\ D_3 = B_{33}L_{B_{11}},\ D_4 = B_{33}D,\\
&E_1 = CT_1,\ E_2 = R_{A_{11}}T_1L_{B_{22}},\ E_3 = R_{A_{22}}T_1L_{B_{11}},\ E_4 = T_1D,\\
&C_{11} = (L_{C_2},\ L_{C_4}),\ D_{11} = \begin{pmatrix} R_{D_1}\\ R_{D_3}\end{pmatrix},\ C_{22} = L_{C_1},\ D_{22} = R_{D_2},\ C_{33} = L_{C_3},\\
&D_{33} = R_{D_4},\ E_{11} = R_{C_{11}}C_{22},\ E_{22} = R_{C_{11}}C_{33},\ E_{33} = D_{22}L_{D_{11}},\ E_{44} = D_{33}L_{D_{11}},\\
&M = R_{E_{11}}E_{22},\ N = E_{44}L_{E_{33}},\ F = F_2 - F_1,\ E = R_{C_{11}}FL_{D_{11}},\ S = E_{22}L_M,\\
&F_{11} = C_2L_{C_1},\ G_1 = E_2 - C_2C_1^{\dagger}E_1D_1^{\dagger}D_2,\ F_{22} = C_4L_{C_3},\ G_2 = E_4 - C_4C_3^{\dagger}E_3D_3^{\dagger}D_4,\\
&F_1 = C_1^{\dagger}E_1D_1^{\dagger} + L_{C_1}C_2^{\dagger}E_2D_2^{\dagger},\ F_2 = C_3^{\dagger}E_3D_3^{\dagger} + L_{C_3}C_4^{\dagger}E_4D_4^{\dagger}.
\end{aligned}$$

Then, the following statements are equivalent:

(1) *Equation* (6) *is consistent.*

(2) $R_{C_i}E_i = 0$, $E_iL_{D_i} = 0$ $(i=\overline{1,4})$, $R_{E_{22}}EL_{E_{33}} = 0$.

(3)

$$\begin{aligned}
&r(T_1,\ A_{11},\ A_{22},\ A_{33}) = r(A_{11},\ A_{22},\ A_{33}),\\
&r\begin{pmatrix} T_1\\ B_{11}\\ B_{22}\\ B_{33}\end{pmatrix} = r\begin{pmatrix} B_{11}\\ B_{22}\\ B_{33}\end{pmatrix},\ r\begin{pmatrix} T_1 & A_{11} & A_{22}\\ B_{33} & 0 & 0\end{pmatrix} = r(A_{11},\ A_{22}) + r(B_{33}),\\
&r\begin{pmatrix} T_1 & A_{11} & A_{33}\\ B_{22} & 0 & 0\end{pmatrix} = r(A_{11},\ A_{33}) + r(B_{22}),\\
&r\begin{pmatrix} T_1 & A_{33} & A_{22}\\ B_{11} & 0 & 0\end{pmatrix} = r(A_{33},\ A_{22}) + r(B_{11}),\ r\begin{pmatrix} T_1 & A_{33}\\ B_{11} & 0\\ B_{22} & 0\end{pmatrix} = r\begin{pmatrix} B_{11}\\ B_{22}\end{pmatrix} + r(A_{33}),\\
&r\begin{pmatrix} T_1 & 0 & A_{11} & 0 & A_{33}\\ 0 & -T_1 & 0 & A_{22} & A_{33}\\ B_{22} & 0 & 0 & 0 & 0\\ 0 & B_{11} & 0 & 0 & 0\\ B_{33} & B_{33} & 0 & 0 & 0\end{pmatrix} = r\begin{pmatrix} B_{22} & 0\\ 0 & B_{11}\\ B_{33} & B_{33}\end{pmatrix} + r\begin{pmatrix} A_{11} & 0 & A_{33}\\ 0 & A_{22} & A_{33}\end{pmatrix},\\
&r\begin{pmatrix} T_1 & A_{22}\\ B_{11} & 0\\ B_{33} & 0\end{pmatrix} = r\begin{pmatrix} B_{11}\\ B_{33}\end{pmatrix} + r(A_{22}),\ r\begin{pmatrix} T_1 & A_{11}\\ B_{33} & 0\\ B_{22} & 0\end{pmatrix} = r\begin{pmatrix} B_{33}\\ B_{22}\end{pmatrix} + r(A_{11}).
\end{aligned}$$

In this case, the general solution to Equation (6) *can be expressed as*

$$\begin{aligned}
&Y_1 = A_{11}^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}A_{22}M_1^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}B_{22}B_{11}^{\dagger}\\
&- A_{11}^{\dagger}S_1U_4R_{N_1}B_{22}B_{11}^{\dagger} + L_{A_{11}}U_5 + U_6R_{B_{11}},\\
&Y_2 = M_1^{\dagger}TB_{22}^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},\\
&Y_3 = F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2},\ \text{or}\ Y_3 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4},
\end{aligned}$$

where $T = T_1 - A_{33}Y_3B_{33}$, $U_i (i = 1, \ldots, 8)$ are any matrices with suitable dimensions over $\mathbb{H}$,

$$\begin{aligned}
V_1 &= (I_m,\ 0)\Big[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\Big],\\
W_1 &= (0,\ I_m)\Big[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\Big],
\end{aligned}$$

$$W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$
$$V_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$
$$V_3 = E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger} - E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},$$
$$W_3 = M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$, and U_{42} are any matrices with suitable dimensions over $\mathbb{H}$.

5. Conclusions

We have established the solvability conditions and an exact formula of a general solution to quaternion matrix Equation (5). As an application of Equation (5), we also have established some necessary and sufficient conditions for Equation (7) to have a solution and derived a formula of its general solution involving η-Hermicity. The quaternion matrix Equation (5) plays a key role in studying the solvability conditions and general solutions of other types of matrix equations. For example, we can use the results on Equation (5) to investigate the solvability conditions and the general solution of the following system of quaternion matrix equations

$$\begin{gathered} A_2Y_1 = C_2,\ Y_1B_2 = D_2, \\ A_3Y_2 = C_3,\ Y_2B_3 = D_3, \\ A_4Y_3 = C_4,\ Y_3B_4 = D_4, \\ G_1Y_1H_1 + G_2Y_2H_2 + G_3Y_3H_3 = G \end{gathered}$$

where Y_1, Y_2 and Y_3 are unknown quaternion matrices and the others are given.

It is worth mentioning that the main results of (5) are available over not only $\mathbb{R}$ and $\mathbb{C}$ but also any division ring. Moreover, inspired by [39], we can investigate Equation (5) in tensor form.

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