



# Article Finite-Time Boundedness of Linear Uncertain Switched Positive Time-Varying Delay Systems with Finite-Time Unbounded Subsystems and Exogenous Disturbance

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**Abstract:** The problem of finite-time boundedness for a class of linear switched positive timevarying delay systems with interval uncertainties and exogenous disturbance is addressed. This characteristic research is that the studied systems include the finite-time bounded subsystems and finite-time unbounded subsystems. Both a slow mode-dependent average dwell time and a fast mode-dependent average dwell time switching techniques are utilized reasonably. And by applying a copositive Lyapunov-Krasovskii functional, novel delay-dependent sufficient criteria are derived to guarantee such systems to be finite-time bounded concerning the given parameters and designed switching signal. Furthermore, new finite-time boundedness criteria of the systems without interval uncertainties are also obtained. Finally, the efficiency of the theoretical results is presented in two illustrative examples.

**Keywords:** finite-time boundedness; interval uncertainties; switched positive systems; time-varying delay; exogenous disturbance

## 1. Introduction

Several phenomena can be modeled such as switched systems that compose a family of subsystems and a logical law called a switching signal that determines the switching manner among the multiple subsystems [1–4]. When all subsystems of switched systems are positive under the specific switching rules, the systems are well known as switched positive systems (SPSs). Their applications can encounter in various areas, such as compartmental model [5], water-quality model [6], formation flying [7], congestion control [8], wireless power control [9], and network communication using transmission control protocol [10]. Furthermore, the system's behavior that relies not only on the present state but also on the past state is discovered in many situations, for example, fluid and mechanical transmissions, metallurgical processes, and networked communications. An essential class of dynamical systems with the behavior are referred to as time-delay systems [11–13]. Nevertheless, the existence of the time delay in the systems may cause chaos and instability. Therefore, several beneficial results on SPSs, including time delay as well as time-varying delay, have been published, see [14–19].

Stability analysis of differential equations and dynamical systems has been extensively studied in the literature [20–23]. Remarkably, the concept of classical Lyapunov stability has been utilized to analyze the behavior of the systems over an infinite time interval. Nonetheless, there are some cases where the mentioned stability can not describe the mechanism of studied systems, such as some constraints of operation time and requirements about the transient performance of the systems [24–27]. Hence, a concept of finite-time stability (FTS), which can keep state trajectories of the considered systems within a prescribed bound



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**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). over a fixed time interval under some given constraints of the initial condition [28–30], has been adopted to deal with those cases. In addition, FTS can be extended to the finite-time boundedness (FTB) if exogenous disturbances or the influence of perturbing forces are taken into account together [31]. However, it is well known that the Lyapunov stability and FTS (FTB) are completely independent concepts. Namely, any dynamical system may be the Lyapunov stability but not FTS (FTB), and vice versa [32,33].

Most of the existing researches for studying the Lyapunov stability, FTS, and FTB of switched systems are based on the types of time-dependent switching signals. Examples of time-dependent switching signals are dwell time (DT), average dwell time (ADT), and mode-dependent average dwell time (MDADT), comprising slow mode-dependent average dwell time (SMDADT) and fast mode-dependent average dwell time (FMDADT). The topic of FTS, FTB, and stability over the infinite time interval of SPSs with time delay has been investigated and reviewed in the following. In [34], Liu and Dang analyzed the stability of SPSs with delays. Later, Jian and Weiqun [35] studied the FTS and FTB for the continuous-time and discrete-time SPSs with time-varying delay. Depending on the ADT approach, the issue of FTB and  $L_2$ -gain for SPSs with multiple time delays was discussed in [36]; however, the problem of stability for SPSs with time delay was examined in [37]. As mentioned in [38], Liu et al. used the MDADT method to derive some stability conditions of SPSs with time delay. Reference [39] dealt with the static output-feedback  $L_1$ finite-time control problems for SPSs with time-varying delay by employing the MDADT strategy. The researches mentioned above mainly concentrate on the only stable (bounded) subsystems. Nevertheless, the switched systems composing both stable (bounded) subsystems and unstable (unbounded) subsystems can be implemented widely in practical applications [40–44]. Among them, in [40], Pashaei and Hashemzadeh derived new FTS and FTB conditions for linear switched delayed systems with finite-time unstable and unbounded subsystems by using the ADT tactic. Meanwhile, in [41], Tan et al. investigated FTS and FTB problems of switched systems consisting of both finite-time stable and unstable subsystems by employing the MDADT method. Based on Lyapunov-like functions, FTS and FTB issues of nonlinear switched systems with subsystems that are not finite-time stable or finite-time bounded were discussed by utilizing the ADT strategy in [42]. Still, the time delay phenomena and the positivity of the systems were not considered in the references [41,42]. For SPSs, in [43], Zhang et al. studied the stability problem of linear SPSs with stable and unstable subsystems by adopting a multiple copositive Lyapunov function combined with the ADT approach. Furthermore, the stability of nonlinear switched delayed systems, including stable and unstable subsystems, was analyzed in [44].

For real-world applications, several systems can represent in the form of uncertainties, which indicate the differences or errors between reality and simulation. Nonetheless, the dynamical systems, including the slight uncertainties, may lead to the instability of those systems. Consequently, many researchers have devoted themselves to studying the FTS, FTB, and stability of the systems with uncertainties during the last decades [45–52]. However, to the best of our knowledge, there is no result on the FTB for a class of SPSs, including time-varying delay, interval uncertainties, exogenous disturbance, and finitetime unbounded subsystems in the literature. This practical idea is the motivation of the present paper. The main contributions of this study are highlighted in the following. (i) The FTB problem of the underlying systems with finite-time bounded subsystems and finite-time unbounded subsystems is investigated by applying the SMDADT and FMDADT techniques. (ii) New delay-dependent sufficient criteria (DDSC) for FTB of the systems are derived. (iii) The corresponding result for SPSs, including time-varying delay and exogenous disturbance without interval uncertainties, is also provided. (iv) Unlike the existing results in [36,37,40,42–44], both the SMDADT and FMDADT methods that are less conservative and more applicable in practice than the ADT switching law are employed for studying the FTB of the systems.

The organization of this paper is arranged as follows. The next section, the system descriptions and preliminaries are proposed. Then, in Section 3, the main results are pre-

sented. Next, in Section 4, two numerical examples are shown to support and validate our theoretical results. Lastly, the conclusions are reported in Section 5.

Notations: The following notations are exploited throughout this article. The sets of non-negative integers and positive integers are denoted by  $\mathbb{N}$  and  $\mathbb{N}_+$ , respectively.  $\mathbb{R}^n$  and  $\mathbb{R}^n_+$  refer to the vectors of *n*-tuples of real and positive real numbers, respectively. The set of all  $m \times n$  real matrices is represented by  $\mathbb{R}^{m \times n}$ .  $I_n$  and  $A^T$  are the  $n \times n$  dimensional identity matrix and the transpose of matrix A, respectively. For given vector  $v \in \mathbb{R}^n$ ,  $v_i$   $(1 \le i \le n)$  is the *i*th component of v. The notation  $v \succeq 0$  ( $v \succ 0$ ) stands for non-negative (positive) vector, namely, all components of v are non-negative (positive) for vector  $v \in \mathbb{R}^n$ . Let  $\|v\|_1 = \sum_{i=1}^n |v_i|$  be the 1-norm of  $v \in \mathbb{R}^n$ . The matrix A is called non-negative matrix if all entries are non-negative and defined by  $A \succeq 0$ . In addition,  $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}|$  is the 1-norm of a matrix  $A \in \mathbb{R}^{m \times n}$ .

## 2. System Descriptions and Preliminaries

continuous satisfying the condition:

A class of linear switched time-varying delay system with interval uncertainties and exogenous disturbance can be stated as

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + D_{\sigma(t)}x(t-d(t)) + G_{\sigma(t)}\omega(t), \ t \ge 0, \\ x(t_0+\theta) = \psi(\theta), \ \theta \in [-\hat{d}, 0], \end{cases}$$
(1)

where  $x(t) \in \mathbb{R}^n$ .  $\sigma(t) : [0, +\infty) \to \underline{N} = \{1, 2, ..., N\}$  is the switching signal, which is a piecewise constant function of time t. N is the number of subsystems or modes of the switched system. Without loss of generality, we suppose that  $\sigma(t)$  is continuous from the right everywhere:  $\sigma(t) = \lim_{\chi \to t^+} \sigma(\chi)$ . Let  $0 \le t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} < \cdots < +\infty$  be a sequence of the switching instants, where  $t_0$  is the initial time and  $t_m$  is the *m*th switching instant,  $m \in \mathbb{N}_0$ . We impose that  $\sigma(t) = \sigma(t_m) = i$ ,  $i \in \underline{N}$  and the *i*th subsystem is activated when  $t \in [t_m, t_{m+1})$ . Based on the logical rule of the switching signal  $\sigma(t)$  at the switching instant  $t_m$ , system (1) switches from the *j*th subsystem to the *i*th subsystem, where  $\sigma(t_{m-1}) = j$ ,  $j \in \underline{N}$ . The time-varying delay d(t) satisfies  $0 \le d(t) \le \hat{d}$  and  $\hat{d}(t) \le d < 1$ , where  $\hat{d}$  and d are known constants. For the interval uncertain of system (1),  $A_i$ ,  $D_i$  and  $G_i$  satisfy

and

$$\underline{A}_i \preceq A_i \preceq A_i,$$
$$\underline{D}_i \preceq D_i \preceq \overline{D}_i,$$

where  $\underline{A}_i$ ,  $\underline{D}_i$ ,  $\underline{G}_i$ ,  $\overline{A}_i$ ,  $\overline{D}_i$ ,  $\overline{G}_i$  are the given constant system matrices with appropriate dimensions for all  $i \in \underline{N}$ . As shown in [39], the exogenous disturbance  $\omega(t) \in \mathbb{R}^g$  is

$$\int_0^{T_f} \|\omega(t)\|_1 dt < \rho, \tag{2}$$

where  $T_f \ge 0$  and  $\rho \ge 0$  are a time constant and a known constant, respectively.  $\psi(\cdot)$  is a vector-valued initial state on  $[-\hat{d}, 0]$  with the norm  $\|\psi\|_{\hat{d}} = \sup_{-\hat{d} \le \theta \le 0} \|\psi(\theta)\|_1$ . Furthermore, let  $\Omega$  be the set of switching signal which has only finite number of switching for any finite-time interval.

Next, we will introduce some definitions, lemma, and assumption for studying system (1).

**Definition 1** ([39]). System (1) is said to be positive if for any initial function  $\psi(\theta) \succeq 0$ ,  $\theta \in [-d, 0]$ , for any exogenous disturbance  $\omega(t) \succeq 0$  and for any switching signal  $\sigma(t)$ , the corresponding trajectory  $x(t) \succeq 0$  holds for all  $t \ge t_0$ .

$$\underline{G}_i \preceq G_i \preceq \overline{G}_i,$$

**Definition 2** ([39]). A matrix is said to be a Metzler matrix if all off-diagonal elements are non-negative.

**Lemma 1** ([39]). *System* (1) *is positive if and only if*  $A_i$  *are Metzler matrices,*  $D_i \succeq 0$ *, and*  $G_i \succeq 0$  *for all*  $i \in \underline{N}$ .

In general, several real systems can be modeled by systems in the form of interval uncertainties. Thus, the assumption of the interval uncertainties for studying the FTB of system (1) with exogenous disturbance is stated as follows.

**Assumption 1** ([10,48]). For each  $A_i$ ,  $D_i$  and  $G_i$  in system (1), there are the known Metzler matrices  $\underline{A}_i$  and the matrices  $\underline{D}_i \succeq 0$ ,  $\underline{G}_i \succeq 0$  such that  $A_i \in [\underline{A}_i, \overline{A}_i]$ ,  $D_i \in [\underline{D}_i, \overline{D}_i]$ , and  $G_i \in [\underline{G}_i, \overline{G}_i]$ , where  $\underline{A}_i$ ,  $\underline{D}_i$ ,  $\underline{G}_i$ ,  $\overline{A}_i$ ,  $\overline{D}_i$ ,  $\overline{G}_i$  are the given constant system matrices with appropriate dimensions for all  $i \in \underline{N}$ .

**Definition 3** ([36] (Finite-Time Boundedness)). Given two constants  $c_2 > c_1 > 0$ , a time constant  $T_f$ , two vectors  $l_1 \succ l_2 \succ 0$ , and a switching signal  $\sigma \in \Omega$ . System (1) is said to be finite-time bounded with respect to  $(c_1, c_2, T_f, l_1, l_2, \rho, \sigma)$  if the solution x(t) of the system satisfies the condition:

$$\sup_{\theta \in [-\hat{d},0]} \psi^T(\theta) l_1 \le c_1 \Longrightarrow x^T(t) l_2 < c_2, \quad \forall t \in [0,T_f],$$

where  $\omega(t)$  satisfies inequality (2).

The finite set  $\underline{N}$  is split into  $\underline{B}$  and  $\underline{U}$ ; namely,  $\underline{N} = \underline{B} \cup \underline{U}$  where  $\underline{B} = \{1, 2, ..., B\}$  denotes the set of finite-time bounded subsystems with respect to the required parameters  $(c_1, c_2, T_f, l_1, l_2, \rho, \sigma)$  and  $\underline{U} = \{B + 1, ..., N\}$  represents the set of finite-time unbounded subsystems with respect to the same required parameters  $(c_1, c_2, T_f, l_1, l_2, \rho, \sigma)$ , respectively.

The definitions of both the SMDADT and FMDADT switching laws are stated as follows.

**Definition 4** ([41]). For any  $T \ge t \ge 0$  and a switching signal  $\sigma \in \Omega$ , let  $N_{\sigma p}(T, t)$  be the numbers of the pth subsystem being activated and  $T_p(T, t)$  be the total running time of the pth subsystem,  $p \in \underline{B}$ . We say that  $\sigma$  has the SMDADT  $\tau_{ap}$  if there exist two constants  $N_{0p} \ge 0$  and  $\tau_{ap} > 0$  such that

$$N_{\sigma p}(T,t) \le N_{0p} + \frac{T_p(T,t)}{\tau_{ap}}, \quad \forall T \ge t \ge 0.$$
(3)

**Definition 5** ([41]). For any  $T \ge t \ge 0$  and a switching signal  $\sigma \in \Omega$ , let  $N_{\sigma q}(T, t)$  be the numbers of the qth subsystem being activated and  $T_q(T, t)$  be the total running time of the qth subsystem,  $q \in \underline{U}$ . We say that  $\sigma$  has the FMDADT  $\tau_{aq}$  if there exist two constants  $N_{0q} \ge 0$  and  $\tau_{aq} > 0$  such that

$$N_{\sigma q}(T,t) \ge N_{0q} + \frac{T_q(T,t)}{\tau_{aq}}, \quad \forall T \ge t \ge 0.$$

$$\tag{4}$$

#### 3. Main Results

In this section, we investigate the problem of the FTB for system (1) with exogenous disturbance and interval uncertainties. The subsystems of the studied system are both finite-time bounded and finite-time unbounded. First, a class of quasi-alternative switching signals (QASSs) for system (1) is designed by utilizing a similar approach studied in [33,41].

(a) If  $\sigma(t_m) \in \underline{B}$ , then  $\sigma(t_{m+1}) \in \underline{N}$ ;

(b) If  $\sigma(t_m) \in \underline{U}$ , then  $\sigma(t_{m+1}) \in \underline{B}$ .

This implies that the considered system can switch from a finite-time bounded subsystem to any other subsystems, but cannot switch from a finite-time unbounded subsystem to another finite-time unbounded subsystem.

For convenience, we first define important symbols used in our main theorem as follows: (11)

$$\widetilde{D} = \left(\overline{d}_{kl}\right) \in \mathbb{R}^{n \times n}, \ \overline{d}_{kl} = \max_{i \in \underline{N}} \left\{\overline{D}_i^{(kl)}\right\},\tag{5}$$

where  $\overline{D}_i^{(kl)}$  represents the *k*th row and *l*th column entry of system matrices  $\overline{D}_i$ ,  $i \in \underline{N}$ . And

$$D = (d_{kl}) \in \mathbb{R}^{n \times n}, \ d_{kl} = \max_{i \in \underline{N}} \left\{ D_i^{(kl)} \right\},$$
(6)

where  $D_i^{(kl)}$  denotes the *k*th row and *l*th column entry of system matrices  $D_i$ ,  $i \in \underline{N}$ . Thus, we are ready to derive new DDSC for the FTB of system (1) with finite-time bounded and finite-time unbounded subsystems by designing QASSs above and using the MDADT method in the following theorem. Without loss of generality, we can select the constants in (3) and (4) satisfying  $N_{0p} = 0$  and  $N_{0q} = 0$ ,  $p \in \underline{B}$ ,  $q \in \underline{U}$ .

**Theorem 1.** Consider system (1) with exogenous disturbance satisfying Assumption 1. Let  $\gamma_p > 0$ ,  $\mu_p > 1$ ,  $p \in \underline{B}$ ,  $\gamma_q > 0$ ,  $0 < \mu_q < 1$ ,  $q \in \underline{U}$ , be given constants. For given two constants  $c_2 > c_1 > 0$ , the time constant  $T_f > 0$ , and two vectors  $l_1 \succ l_2 \succ 0$ . Suppose that there exist positive vectors  $v_p \succ 0$ ,  $v_q \succ 0$  and constants  $\xi_p > 0$ ,  $\xi_q > 0$ ,  $\beta_p > 0$ ,  $\beta_q > 0$  such that

$$\left[\overline{A}_{p}^{T} + \left(\frac{1}{1-d}\right)\widetilde{D}^{T} - \gamma_{p}I_{n}\right]\nu_{p} \prec 0,$$
(7)

$$\left[\overline{A}_{q}^{T} + \left(\frac{1}{1-d}\right)\widetilde{D}^{T} - \gamma_{q}I_{n}\right]\nu_{q} \prec 0,$$
(8)

$$\beta_p l_2 \prec \nu_p \prec \xi_p l_1, \tag{9}$$

$$\beta_q l_2 \prec \nu_q \prec \xi_q l_1, \tag{10}$$

$$\nu_n \prec u_n \nu_r, \tag{11}$$

$$\nu_p \preceq \mu_p \nu_r, \tag{11}$$

$$\nu_q \preceq \mu_q \nu_p, \tag{12}$$

$$e^{T_{f}\Gamma} < \frac{(1-d)\beta c_{2}}{\left[(1-d) + \hat{d}\|\widetilde{D}^{T}\|_{1}\right]\xi c_{1} + \rho\|\tilde{\omega}\|_{1}},$$
(13)

hold for every  $p \in \underline{B}$ ,  $q \in \underline{U}$ ,  $r \in \underline{N}$ ,  $p \neq r$ . Then system (1) is positive and finite-time bounded with respect to  $(c_1, c_2, T_f, l_1, l_2, \rho, \sigma)$  under the switching signals with SMDADT satisfying

$$\tau_{ap} \geq \tau_{ap}^* = \frac{T_f \ln \mu_p}{\ln \frac{(1-d)\beta c_2}{\left[(1-d) + \widehat{d} \|\widetilde{D}^T\|_1\right] \xi c_1 + \rho \|\widetilde{\omega}\|_1} - T_f \gamma_p}, \quad \forall p \in \underline{B},$$
(14)

and FMDADT satisfying

$$\tau_{aq} \leq \tau_{aq}^* = -\frac{\ln \mu_q}{\gamma_q}, \quad \forall q \in \underline{U},$$
(15)

where

$$\Gamma = \max_{p \in \underline{B}} \{\gamma_p\}, \ \beta = \min_{p \in \underline{B}, q \in \underline{U}} \{\beta_p, \beta_q\}, \ \xi = \max_{p \in \underline{B}, q \in \underline{U}} \{\xi_p, \xi_q\}, \ \tilde{\omega} = \left[\overline{W}_1 \ \overline{W}_2 \ \dots \ \overline{W}_g\right]^T,$$

 $\overline{W}_{k} = \max_{\sigma(t) \in \Omega} \{\overline{\omega}_{\sigma(t)}^{(k)}\}, \ \overline{\omega}_{\sigma(t)}^{(k)} \text{ is the kth element of the vector } \overline{\omega}_{\sigma(t)} = (1-d)\overline{G}_{\sigma(t)}^{T} \xi l_{1},$ for  $k \in \{1, 2, ..., g\}$ , and  $\widetilde{D}$  is defined as in (5).

**Proof.** We divide the proof process into the following two steps. **Step 1.** We will prove that system (1) is positive.

By Assumption 1, it is immediate that  $\underline{A}_i$  are Metzler matrices and the matrices  $\underline{D}_i \succeq 0$ ,  $\underline{G}_i \succeq 0$  for all  $i \in \underline{N}$ . However, the system matrices for system (1) are supposed to be interval uncertain, namely,  $\underline{A}_i \preceq A_i \preceq \overline{A}_i$ ,  $\underline{D}_i \preceq D_i \preceq \overline{D}_i$ , and  $\underline{G}_i \preceq \overline{G}_i \preceq \overline{G}_i$ , for all  $i \in \underline{N}$ . Thus, it is obvious that  $A_i$  are also Metzler matrices,  $D_i \succeq 0$ , and  $G_i \succeq 0$  for all  $i \in \underline{N}$ . According to Lemma 1, we can conclude that system (1) is positive.

**Step 2.** We will prove the FTB for system (1) under the switching signals with SM-DADT satisfying condition (14) and FMDADT satisfying condition (15).

For any  $T_f > 0$ , let  $t_1, t_2, ..., t_m, t_{m+1}, ..., t_{N_{\sigma}(T_f,0)}$  be the switching time instants over  $[0, T_f]$ where  $t_{N_{\sigma}(T_f,0)} = t_{\sum_{p \in \underline{B}} N_{\sigma p}(T_f,0) + \sum_{q \in \underline{U}} N_{\sigma q}(T_f,0)}$  and  $T_f = \sum_{p \in \underline{B}} T_p(T_f,0) + \sum_{q \in \underline{U}} T_q(T_f,0)$ . For  $t \in [t_m, t_{m+1}), m \in \mathbb{N}$ , we construct the following copositive Lyapunov-Krasovskii functional (CLKF) candidate for system (1):

$$V_{\sigma(t)}(t) \equiv V_{\sigma(t)}(t, x(t)) = (1 - d)x^{T}(t)v_{\sigma(t)} + \int_{t - d(t)}^{t} x^{T}(s)\widetilde{D}^{T}v_{\sigma(t)}ds,$$
(16)

where  $\nu_{\sigma(t)} \succ 0$ ,  $\sigma(t) \in \Omega$ . Along the trajectory of system (1), we have

$$\begin{split} \dot{V}_{\sigma(t_m)}(t) &= (1-d)\dot{x}^T(t)\nu_{\sigma(t_m)} + x^T(t)\widetilde{D}^T\nu_{\sigma(t_m)} - (1-\dot{d}(t))x^T(t-d(t))\widetilde{D}^T\nu_{\sigma(t_m)} \\ &= (1-d)x^T(t)A_{\sigma(t_m)}^T\nu_{\sigma(t_m)} + (1-d)x^T(t-d(t))D_{\sigma(t_m)}^T\nu_{\sigma(t_m)} + (1-d)\omega^T(t)G_{\sigma(t_m)}^T\nu_{\sigma(t_m)} \\ &\quad + x^T(t)\widetilde{D}^T\nu_{\sigma(t_m)} - (1-\dot{d}(t))x^T(t-d(t))\widetilde{D}^T\nu_{\sigma(t_m)} \\ &\leq (1-d)x^T(t)\overline{A}_{\sigma(t_m)}^T\nu_{\sigma(t_m)} + (1-d)x^T(t-d(t))\overline{D}_{\sigma(t_m)}^T\nu_{\sigma(t_m)} + (1-d)\omega^T(t)\overline{G}_{\sigma(t_m)}^T\nu_{\sigma(t_m)} \\ &\quad + x^T(t)\widetilde{D}^T\nu_{\sigma(t_m)} - (1-\dot{d}(t))x^T(t-d(t))\widetilde{D}^T\nu_{\sigma(t_m)}, \end{split}$$

for  $t \in [t_m, t_{m+1})$ ,  $m \in \mathbb{N}$ . We observe that

$$\begin{split} \dot{V}_{\sigma(t_m)}(t) - \gamma_{\sigma(t_m)} V_{\sigma(t_m)}(t) &\leq (1 - d) x^T(t) \overline{A}_{\sigma(t_m)}^T v_{\sigma(t_m)} + (1 - d) x^T(t - d(t)) \overline{D}_{\sigma(t_m)}^T v_{\sigma(t_m)} \\ &+ (1 - d) \omega^T(t) \overline{G}_{\sigma(t_m)}^T v_{\sigma(t_m)} + x^T(t) \widetilde{D}^T v_{\sigma(t_m)} \\ &- (1 - d(t)) x^T(t - d(t)) \widetilde{D}^T v_{\sigma(t_m)} - \gamma_{\sigma(t_m)} (1 - d) x^T(t) v_{\sigma(t_m)} \\ &- \gamma_{\sigma(t_m)} \int_{t - d(t)}^t x^T(s) \widetilde{D}^T v_{\sigma(t_m)} ds, \end{split}$$

for  $t \in [t_m, t_{m+1})$ ,  $m \in \mathbb{N}$ . Together with  $\dot{d}(t) \leq d$ ,  $0 < \gamma_{\sigma(t_m)}$ , and  $\overline{D}_{\sigma(t_m)} \leq D$  for all  $\sigma(t_m) \in \underline{N}$ , one has

$$\begin{split} \dot{V}_{\sigma(t_m)}(t) - \gamma_{\sigma(t_m)} V_{\sigma(t_m)}(t) &= x^T(t) \left[ (1-d) \overline{A}_{\sigma(t_m)}^T \nu_{\sigma(t_m)} + \widetilde{D}^T \nu_{\sigma(t_m)} - \gamma_{\sigma(t_m)} (1-d) \nu_{\sigma(t_m)} \right] \\ &+ (1-d) \omega^T(t) \overline{G}_{\sigma(t_m)}^T \nu_{\sigma(t_m)}, \end{split}$$

for  $t \in [t_m, t_{m+1})$ ,  $m \in \mathbb{N}$ . According to the conditions (7) and (8), we obtain

$$\dot{V}_{\sigma(t_m)}(t) - \gamma_{\sigma(t_m)} V_{\sigma(t_m)}(t) \le (1 - d) \omega^T(t) \overline{G}_{\sigma(t_m)}^T v_{\sigma(t_m)}, \tag{17}$$

for  $t \in [t_m, t_{m+1})$ ,  $m \in \mathbb{N}$ . Substituting the conditions (9) and (10) into the inequality (17), it yields

$$\dot{V}_{\sigma(t_m)}(t) - \gamma_{\sigma(t_m)} V_{\sigma(t_m)}(t) \le (1-d)\omega^T(t)\overline{G}_{\sigma(t_m)}^T \xi l_1,$$
(18)

for  $t \in [t_m, t_{m+1})$ ,  $m \in \mathbb{N}$ , and  $\xi = \max_{p \in \underline{B}, q \in \underline{U}} \{\xi_p, \xi_q\}$ . Setting  $\overline{\omega}_{\sigma(t_m)} = (1 - d)\overline{G}_{\sigma(t_m)}^T \xi_{l_1}$ and integrating both sides of the inequality (18) during the period  $[t_m, t)$  for  $t \in [t_m, t_{m+1})$ ,  $m \in \mathbb{N}$ , it leads to

$$V_{\sigma(t_m)}(t) \le e^{\gamma_{\sigma(t_m)}(t-t_m)} V_{\sigma(t_m)}(t_m) + \int_{t_m}^t e^{\gamma_{\sigma(t_m)}(t-s)} \omega^T(s) \overline{\omega}_{\sigma(t_m)} ds.$$
(19)

Considering the change of the value of the CLKF (16) at the switching time instants and the positivity of x(t) in system (1). According to the condition (11), we get

,

$$V_{p}(t_{m}) = (1-d)x^{T}(t_{m})\nu_{p} + \int_{t_{m}-d(t_{m})}^{t_{m}} x^{T}(s)\widetilde{D}^{T}\nu_{p}ds$$
  

$$\leq (1-d)x^{T}(t_{m}^{-})\mu_{p}\nu_{r} + \int_{t_{m}^{-}-d(t_{m}^{-})}^{t_{m}^{-}} x^{T}(s)\widetilde{D}^{T}\mu_{p}\nu_{r}ds$$
  

$$= \mu_{p}V_{r}(t_{m}^{-}), \qquad (20)$$

for all  $p \in \underline{B}$ ,  $r \in \underline{N}$ , and  $p \neq r$ . Similarly, using the condition (12), we have

$$V_q(t_m) \le \mu_q V_p(t_m^-),\tag{21}$$

for all  $q \in \underline{U}$ ,  $p \in \underline{B}$ . Based on the relationship among the inequalities (19)–(21) for  $t \in [t_m, t_{m+1})$ ,  $m \in \mathbb{N}$ , we can derive

$$\begin{split} V_{\sigma(t_{m})}(t) &\leq \mu_{\sigma(t_{m})} e^{\gamma_{\sigma(t_{m})}(t-t_{m})} V_{\sigma(t_{m-1})}(t_{m}^{-}) + \int_{t_{m}}^{t} e^{\gamma_{\sigma(t_{m})}(t-s)} \omega^{T}(s) \overline{\omega}_{\sigma(t_{m})} ds \\ &\leq \mu_{\sigma(t_{m})} e^{\gamma_{\sigma(t_{m})}(t-t_{m})} \left[ e^{\gamma_{\sigma(t_{m-1})}(t_{m}-t_{m-1})} V_{\sigma(t_{m-1})}(t_{m-1}) + \int_{t_{m-1}}^{t_{m}} e^{\gamma_{\sigma(t_{m-1})}(t_{m}-s)} \omega^{T}(s) \overline{\omega}_{\sigma(t_{m-1})} ds \right] \\ &\quad + \int_{t_{m}}^{t} e^{\gamma_{\sigma(t_{m})}(t-s)} \omega^{T}(s) \overline{\omega}_{\sigma(t_{m})} ds \\ &\leq \cdots \\ &\leq \prod_{p \in \underline{B}} \mu_{p}^{N_{cp}(t,0)} \prod_{q \in \underline{U}} \mu_{q}^{N_{cq}(t,0)} e^{\gamma_{\sigma(t_{m})}(t-t_{m})} e^{\sum_{k=1}^{m} \gamma_{\sigma(t_{k-1})}(t_{k}-t_{k-1})} V_{\sigma(0)}(0) \\ &\quad + \prod_{p \in \underline{B}} \mu_{p}^{N_{cp}(t,0)} \prod_{q \in \underline{U}} \mu_{q}^{N_{cq}(t,0)} e^{\gamma_{\sigma(t_{m})}(t-t_{m})} e^{\sum_{k=2}^{m} \gamma_{\sigma(t_{k-1})}(t_{k}-t_{k-1})} \int_{0}^{t_{1}} e^{\gamma_{\sigma(0)}(t_{1}-s)} \omega^{T}(s) \overline{\omega}_{\sigma(0)} ds \\ &\quad + \dots + \prod_{p \in \underline{B}} \mu_{p}^{N_{cp}(t,0)} \prod_{q \in \underline{U}} \mu_{q}^{N_{cq}(t,0)} e^{\left[\sum_{p \in \underline{B}} \gamma_{p} T_{p}(t,s) + \sum_{q \in \underline{U}} \gamma_{q} T_{q}(t,s)\right]} V_{\sigma(0)}(0) \\ &\quad + \int_{0}^{t_{1}} \prod_{p \in \underline{B}} \mu_{p}^{N_{cp}(t,0)} \prod_{q \in \underline{U}} \mu_{q}^{N_{cq}(t,0)} e^{\left[\sum_{p \in \underline{B}} \gamma_{p} T_{p}(t,s) + \sum_{q \in \underline{U}} \gamma_{q} T_{q}(t,s)\right]} \omega^{T}(s) \widetilde{\omega} ds \\ &\quad + \dots + \int_{t_{m}}^{t} \prod_{p \in \underline{B}} \mu_{p}^{N_{cp}(t,0)} \prod_{q \in \underline{U}} \mu_{q}^{N_{cq}(t,0)} e^{\left[\sum_{p \in \underline{B}} \gamma_{p} T_{p}(t,s) + \sum_{q \in \underline{U}} \gamma_{q} T_{q}(t,s)\right]} \omega^{T}(s) \widetilde{\omega} ds, \end{split}$$

where  $\tilde{\omega} = [\overline{W}_1 \ \overline{W}_2 \ ... \ \overline{W}_g]^T$ ,  $\overline{W}_k = \max_{\sigma(t) \in \Omega} \{\overline{\omega}_{\sigma(t)}^{(k)}\}$ , and  $\overline{\omega}_{\sigma(t)}^{(k)}$  is the *k*th element of the vector  $\overline{\omega}_{\sigma(t)}$ ,  $k \in \{1, 2, ..., g\}$ . It yields that

$$\begin{split} V_{\sigma(t_m)}(t) &\leq \prod_{p \in \underline{B}} \mu_p^{N_{\sigma p}(t,0)} \prod_{q \in \underline{U}} \mu_q^{N_{\sigma q}(t,0)} \mathbf{e}^{\left[\sum_{p \in \underline{B}} \gamma_p T_p(t,0) + \sum_{q \in \underline{U}} \gamma_q T_q(t,0)\right]} V_{\sigma(0)}(0) \\ &+ \prod_{p \in \underline{B}} \mu_p^{N_{\sigma p}(t,0)} \prod_{q \in \underline{U}} \mu_q^{N_{\sigma q}(t,0)} \mathbf{e}^{\left[\sum_{p \in \underline{B}} \gamma_p T_p(t,0) + \sum_{q \in \underline{U}} \gamma_q T_q(t,0)\right]} \int_0^{T_f} \omega^T(s) \tilde{\omega} ds. \end{split}$$

By the property of the exogenous disturbance in (2), it is immediate that

$$V_{\sigma(t_m)}(t) \leq \prod_{p \in \underline{B}} \mu_p^{N_{\sigma p}(t,0)} \prod_{q \in \underline{U}} \mu_q^{N_{\sigma q}(t,0)} \mathbf{e}^{\left[\sum_{p \in \underline{B}} \gamma_p T_p(t,0) + \sum_{q \in \underline{U}} \gamma_q T_q(t,0)\right]} V_{\sigma(0)}(0) + \prod_{p \in \underline{B}} \mu_p^{N_{\sigma p}(t,0)} \prod_{q \in \underline{U}} \mu_q^{N_{\sigma q}(t,0)} \mathbf{e}^{\left[\sum_{p \in \underline{B}} \gamma_p T_p(t,0) + \sum_{q \in \underline{U}} \gamma_q T_q(t,0)\right]} \rho \|\tilde{\omega}\|_1.$$

By Definitions 4 and 5, and employing the inequalities (3) and (4), we obtain

$$V_{\sigma(t_m)}(t) \leq e^{\sum_{p \in \underline{B}} \left(\frac{\ln \mu_p}{\tau_{ap}} + \gamma_p\right) T_p(t,0)} e^{\sum_{q \in \underline{U}} \left(\frac{\ln \mu_q}{\tau_{aq}} + \gamma_q\right) T_q(t,0)} V_{\sigma(0)}(0)$$
$$+ e^{\sum_{p \in \underline{B}} \left(\frac{\ln \mu_p}{\tau_{ap}} + \gamma_p\right) T_p(t,0)} e^{\sum_{q \in \underline{U}} \left(\frac{\ln \mu_q}{\tau_{aq}} + \gamma_q\right) T_q(t,0)} \rho \|\tilde{\omega}\|_1.$$

It can be derived from the inequality (15) that

$$\mathrm{e}^{\sum_{q\in\underline{U}}\left(\frac{\ln\mu_q}{\tau_{aq}}+\gamma_q\right)T_q(t,0)}\leq 1,$$

for all  $q \in \underline{U}$ . Thus, we get

$$V_{\sigma(t_m)}(t) \leq \mathbf{e}^{\sum_{p \in \underline{B}} \left( \frac{\ln \mu_p}{\tau_{ap}} + \gamma_p \right) T_p(t,0)} \left( V_{\sigma(0)}(0) + \rho \| \tilde{\omega} \|_1 \right).$$
(22)

Using the CLKF (16), the conditions (9) and (10), and Definition 3 for the following estimations:

$$\begin{split} V_{\sigma(0)}(0) &= (1-d)x^{T}(0)\nu_{\sigma(0)} + \int_{-d(0)}^{0} x^{T}(s)\widetilde{D}^{T}\nu_{\sigma(0)}ds \\ &\leq (1-d)x^{T}(0)\xi l_{1} + \int_{-d(0)}^{0} x^{T}(s)\widetilde{D}^{T}\xi l_{1}ds \\ &\leq (1-d)x^{T}(0)\xi l_{1} + \int_{-\widehat{d}}^{0} ds\xi \|\widetilde{D}^{T}\|_{1} \sup_{s\in [-\widehat{d},0]} x^{T}(s)l_{1} \\ &\leq \left[ (1-d) + \widehat{d}\|\widetilde{D}^{T}\|_{1} \right]\xi c_{1}, \end{split}$$

and

$$egin{aligned} V_{\sigma(t_m)}(t) &\geq (1-d) x^T(t) 
u_{\sigma(t_m)} \ &\geq (1-d) x^T(t) eta l_2, \end{aligned}$$

where  $\beta = \min_{p \in \underline{B}, q \in \underline{U}} \{\beta_p, \beta_q\}$ . From the inequality (22), we have

$$x^{T}(t)l_{2} \leq e^{\sum_{p \in \underline{B}} \left(\frac{\ln \mu_{p}}{\tau_{ap}} + \gamma_{p}\right) T_{p}(t,0)} \left(\frac{\left[(1-d) + \widehat{d} \| \widetilde{D}^{T} \|_{1}\right] \xi c_{1} + \rho \| \widetilde{\omega} \|_{1}}{(1-d)\beta}\right).$$
(23)

Let  $\epsilon = \max_{p \in \underline{B}} \left( \frac{\ln \mu_p}{\tau_{ap}} + \gamma_p \right)$ . Hence,

$$x^{T}(t)l_{2} \leq e^{\sum_{p \in \underline{B}} \epsilon^{T_{p}(t,0)}} \left( \frac{\left[ (1-d) + \widehat{d} \| \widetilde{D}^{T} \|_{1} \right] \xi c_{1} + \rho \| \widetilde{\omega} \|_{1}}{(1-d)\beta} \right)$$
$$\leq e^{\epsilon^{T_{f}}} \left( \frac{\left[ (1-d) + \widehat{d} \| \widetilde{D}^{T} \|_{1} \right] \xi c_{1} + \rho \| \widetilde{\omega} \|_{1}}{(1-d)\beta} \right), \tag{24}$$

for all  $t \in [0, T_f]$ . It follows from the condition (13) and the inequality (14) that

$$\epsilon T_{f} = \max_{p \in \underline{B}} \left( \frac{T_{f} \ln \mu_{p}}{\tau_{ap}} + T_{f} \gamma_{p} \right)$$

$$\leq \ln \left( \frac{(1-d)\beta c_{2}}{\left[ (1-d) + \widehat{d} \| \widetilde{D}^{T} \|_{1} \right] \xi c_{1} + \rho \| \widetilde{\omega} \|_{1}} \right), \qquad (25)$$

for all  $p \in \underline{B}$ . Utilizing the inequalities (24) and (25), we arrive at

$$x^T(t)l_2 < c_2,$$

for all  $t \in [0, T_f]$ . It can be concluded by Definition 3 that system (1) is finite-time bounded with respect to  $(c_1, c_2, T_f, l_1, l_2, \rho, \sigma)$  for the switching signal  $\sigma(t)$  with SMDADT (14) and FMDADT (15).  $\Box$ 

**Remark 1.** As in [33,41], in our switching scheme, we design slow and fast switching for bounded subsystems and unbounded subsystems, respectively. It gives the lower bounds that bounded subsystems should dwell on and provides the upper bounds for unbounded subsystems. In addition, one can note that if the bounded subsystem is activated, any subsystem can be activated at the next switching time instance. Nevertheless, if an unbounded subsystem is activated, the next activated system must be a bounded subsystem.

Another FTB result of system (1) without its interval uncertainty will be presented as follows:

**Corollary 1.** Consider system (1) with exogenous disturbance. Let  $\gamma_p > 0$ ,  $\mu_p > 1$ ,  $p \in \underline{B}$ ,  $\gamma_q > 0$ ,  $0 < \mu_q < 1$ ,  $q \in \underline{U}$ , be given constants. For given two constants  $c_2 > c_1 > 0$ , the time constant  $T_f > 0$ , and two vectors  $l_1 \succ l_2 \succ 0$ . Suppose that there exist positive vectors  $\nu_p \succ 0$ ,  $\nu_q \succ 0$  and constants  $\xi_p > 0$ ,  $\xi_q > 0$ ,  $\beta_p > 0$ ,  $\beta_q > 0$  such that

$$\begin{bmatrix} A_p^T + \left(\frac{1}{1-d}\right)D^T - \gamma_p I_n \end{bmatrix} v_p \prec 0,$$
$$\begin{bmatrix} A_q^T + \left(\frac{1}{1-d}\right)D^T - \gamma_q I_n \end{bmatrix} v_q \prec 0,$$
$$\beta_p l_2 \prec v_p \prec \xi_p l_1,$$
$$\beta_q l_2 \prec v_q \prec \xi_q l_1,$$
$$v_p \preceq \mu_p v_r,$$
$$v_q \preceq \mu_q v_p,$$
$$e^{T_f \Gamma} < \frac{(1-d)\beta c_2}{\left[(1-d) + \hat{d} \|D^T\|_1\right] \xi c_1 + \rho \|\hat{\omega}\|_1},$$

hold for every  $p \in \underline{B}$ ,  $q \in \underline{U}$ ,  $r \in \underline{N}$ ,  $p \neq r$ . Then system (1) is positive and finite-time bounded with respect to  $(c_1, c_2, T_f, l_1, l_2, \rho, \sigma)$  under the switching signals with SMDADT satisfying

$$\tau_{ap} \geq \tau_{ap}^* = \frac{T_f \ln \mu_p}{\ln \frac{(1-d)\beta c_2}{\left[(1-d) + \widehat{d} \|D^T\|_1\right] \xi c_1 + \rho \|\widehat{\omega}\|_1} - T_f \gamma_p}, \quad \forall p \in \underline{B},$$

and FMDADT satisfying

$$au_{aq} \leq au_{aq}^* = -rac{\ln \mu_q}{\gamma_q}, \ \ orall q \in \underline{U},$$

where

$$\Gamma = \max_{p \in \underline{B}} \{\gamma_p\}, \ \beta = \min_{p \in \underline{B}, q \in \underline{U}} \{\beta_p, \beta_q\}, \ \xi = \max_{p \in \underline{B}, q \in \underline{U}} \{\xi_p, \xi_q\}, \ \widehat{\omega} = \begin{bmatrix} W_1 \ W_2 \ \dots \ W_g \end{bmatrix}^T,$$

 $W_k = \max_{\sigma(t)\in\Omega} \{\omega_{\sigma(t)}^{(k)}\}, \ \omega_{\sigma(t)}^{(k)}$  is the kth element of the vector  $\omega_{\sigma(t)} = (1-d)G_{\sigma(t)}^T \xi l_1$ , for  $k \in \{1, 2, ..., g\}$ , and D is defined as in (6).

**Proof.** With the same symbols in Theorem 1, this corollary can be proved by utilizing the following CLKF candidate for system (1):

$$V_{\sigma(t)}(t) \equiv V_{\sigma(t)}(t, x(t)) = (1 - d)x^{T}(t)v_{\sigma(t)} + \int_{t - d(t)}^{t} x^{T}(s)D^{T}v_{\sigma(t)}ds,$$

where  $\nu_{\sigma(t)} \succ 0$ ,  $\sigma(t) \in \Omega$ . The remainder of the proof is similar to that of Theorem 1. Hence, the detail is omitted.  $\Box$ 

#### 4. Numerical Simulations

In this section, we provide two numerical examples together with the simulation results to demonstrate the correctness and effectiveness of our theoretical analysis presented in the previous section.

**Example 1.** The FTB problem for system (1) comprising of two subsystems is studied in this example. The system data are given as follows:

$$\underline{A}_{1} = \begin{bmatrix} -0.505 & 0.48\\ 0.59 & -0.205 \end{bmatrix}, \ \overline{A}_{1} = \begin{bmatrix} -0.495 & 0.52\\ 0.61 & -0.195 \end{bmatrix}$$
$$\underline{D}_{1} = \begin{bmatrix} 0.09 & 0\\ 0.045 & 0.018 \end{bmatrix}, \ \overline{D}_{1} = \begin{bmatrix} 0.11 & 0\\ 0.055 & 0.022 \end{bmatrix},$$
$$\underline{G}_{1} = \begin{bmatrix} 0.08\\ 0.09 \end{bmatrix}, \ \overline{G}_{1} = \begin{bmatrix} 0.12\\ 0.11 \end{bmatrix},$$
$$\underline{A}_{2} = \begin{bmatrix} -2.7 & 0.49\\ 0.58 & -1.75 \end{bmatrix}, \ \overline{A}_{2} = \begin{bmatrix} -2.3 & 0.51\\ 0.62 & -1.25 \end{bmatrix},$$
$$\underline{D}_{2} = \begin{bmatrix} 0.078 & 0.019\\ 0.008 & 0.045 \end{bmatrix}, \ \overline{D}_{2} = \begin{bmatrix} 0.082 & 0.021\\ 0.012 & 0.055 \end{bmatrix},$$
$$\underline{G}_{2} = \begin{bmatrix} 0.09\\ 0.17 \end{bmatrix}, \ \overline{G}_{2} = \begin{bmatrix} 0.11\\ 0.23 \end{bmatrix},$$

 $d(t) = 0.05 + 0.05 \sin(t)$ , and  $\omega(t) = 0.5e^{-5t}$ .

Under the given time-varying delay above, we select  $\hat{d} = 0.1$  and d = 0.05. According to Definition 2, one can see that  $\underline{A}_1$  and  $\underline{A}_2$  are Metzler matrices. Furthermore, it is obvious that  $\underline{D}_1 \succeq 0$ ,  $\underline{D}_2 \succeq 0$ ,  $\underline{G}_1 \succeq 0$ , and  $\underline{G}_2 \succeq 0$ . By Lemma 1 and Assumption 1, the studied system is positive. For the numerical simulations, we set the initial condition as  $\psi(\theta) = [1 \ 2]^T$ ,  $\theta \in [-\hat{d}, 0]$ , and let the system matrices be

$$A_{1} = \frac{\underline{A}_{1} + \overline{A}_{1}}{2} = \begin{bmatrix} -0.5 & 0.5 \\ 0.6 & -0.2 \end{bmatrix}, \quad D_{1} = \frac{\underline{D}_{1} + \overline{D}_{1}}{2} = \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.02 \end{bmatrix}, \quad G_{1} = \frac{\underline{G}_{1} + \overline{G}_{1}}{2} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$
$$A_{2} = \frac{\underline{A}_{2} + \overline{A}_{2}}{2} = \begin{bmatrix} -2.5 & 0.5 \\ 0.6 & -1.5 \end{bmatrix}, \quad D_{2} = \frac{\underline{D}_{2} + \overline{D}_{2}}{2} = \begin{bmatrix} 0.08 & 0.02 \\ 0.01 & 0.05 \end{bmatrix}, \quad G_{2} = \frac{\underline{G}_{2} + \overline{G}_{2}}{2} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}.$$

Given two positive vectors  $l_1 = [2 \ 2]^T$  and  $l_2 = [1 \ 1]^T$ . Then, we assign the positive constants  $c_1 = 6$ ,  $c_2 = 50$  and the time constant  $T_f = 15$ , which satisfies  $\sup_{\theta \in [-\hat{d},0]} \psi^T(\theta) l_1 \leq c_1$ . From the condition of the exogenous disturbance defined as in (2), it is obvious that  $\rho = 0.1$ . The value of  $x^T(t)l_2$ ,  $t \in [0, T_f]$  for the first subsystem and the second subsystem are shown respectively in Figures 1 and 2. From two subsystems of the simulations, it is verified that the first subsystem is not finite-time bounded and the second one is finite-time bounded.

As defined in (5), it is obviously that

$$\widetilde{D} = \begin{bmatrix} 0.11 & 0.021\\ 0.055 & 0.055 \end{bmatrix}.$$

For given sclars  $\gamma_1 = 0.5$ ,  $\mu_1 = 0.58$ ,  $\gamma_2 = 0.01$  and  $\mu_2 = 1.73$ , we can get a set of feasible solution for Theorem 1:

$$\nu_1 = [0.975 \ 0.975]^T$$
,  $\nu_2 = [1.683 \ 1.683]^T$ ,  $\xi_1 = 0.5$ ,  $\xi_2 = 0.85$ ,  $\beta_1 = 0.95$ ,  $\beta_2 = 1$ .

Thus, system (1) is finite-time bounded with respect to  $(6, 50, 15, [2 2]^T, [1 1]^T, 0.1, \sigma)$  under the switching signal with SMDADT  $\tau_{a2}^* = 3.9976$  and FMDADT  $\tau_{a1}^* = 1.0895$ , which satisfy the conditions specified by (14) and (15), respectively. Let  $\tau_{a1} = 1 < 1.0895$  and  $\tau_{a2} = 4 > 3.9976$ for the first subsystem and the second subsystem, respectively. The value of  $x^T(t)l_2$  of system (1) under the corresponding switching signal is depicted in Figure 3. The plot indicates that the value of  $x^T(t)l_2$  at  $T_f = 15$  does not exceed  $c_2 = 50$ . Consequently, we can conclude that system (1) is finite-time bounded with respect to  $(6, 50, 15, [2 2]^T, [1 1]^T, 0.1, \sigma)$ .



**Figure 1.** The value of  $x^{T}(t)l_{2}$  of the first subsystem in Example 1.



**Figure 2.** The value of  $x^{T}(t)l_{2}$  of the second subsystem in Example 1.



**Figure 3.** The value of  $x^T(t)l_2$  of the system in Example 1 under the corresponding switching signal. **Example 2.** We consider system (1) with two subsystems. The system data are given as follows:

$$\underline{A}_{1} = \begin{bmatrix} -0.305 & 0.48\\ 0.59 & -0.205 \end{bmatrix}, \ \overline{A}_{1} = \begin{bmatrix} -0.295 & 0.52\\ 0.61 & -0.195 \end{bmatrix},$$
$$\underline{D}_{1} = \begin{bmatrix} 0.098 & 0\\ 0.045 & 0.01 \end{bmatrix}, \ \overline{D}_{1} = \begin{bmatrix} 0.102 & 0\\ 0.055 & 0.03 \end{bmatrix},$$
$$\underline{G}_{1} = \begin{bmatrix} 0.05\\ 0.06 \end{bmatrix}, \ \overline{G}_{1} = \begin{bmatrix} 0.15\\ 0.14 \end{bmatrix},$$
$$\underline{A}_{2} = \begin{bmatrix} -0.5 & 0.516\\ 0.426 & -0.5 \end{bmatrix}, \ \overline{A}_{2} = \begin{bmatrix} -0.5 & 0.52\\ 0.43 & -0.5 \end{bmatrix},$$
$$\underline{D}_{2} = \begin{bmatrix} 0.11 & 0.019\\ 0.053 & 0.049 \end{bmatrix}, \ \overline{D}_{2} = \begin{bmatrix} 0.11 & 0.021\\ 0.055 & 0.055 \end{bmatrix},$$
$$\underline{G}_{2} = \begin{bmatrix} 0.09\\ 0.17 \end{bmatrix}, \ \overline{G}_{2} = \begin{bmatrix} 0.11\\ 0.23 \end{bmatrix},$$
$$d(t) = 0.05 + 0.05 \sin(t), \ and \ \omega(t) = 0.5e^{-5t}.$$

The same two constants  $\hat{d}$  and d can be selected as in Example 1. The positivity of the system in this example is also summarized as in Example 1. For the numerical simulations, we set the initial condition as  $\psi(\theta) = [1 \ 2]^T$ ,  $\theta \in [-\hat{d}, 0]$ , and let the system matrices be

$$A_{1} = \frac{\underline{A}_{1} + \overline{A}_{1}}{2} = \begin{bmatrix} -0.3 & 0.5\\ 0.6 & -0.2 \end{bmatrix}, \quad D_{1} = \frac{\underline{D}_{1} + \overline{D}_{1}}{2} = \begin{bmatrix} 0.1 & 0\\ 0.05 & 0.02 \end{bmatrix}, \quad G_{1} = \frac{\underline{G}_{1} + \overline{G}_{1}}{2} = \begin{bmatrix} 0.1\\ 0.1 \end{bmatrix},$$
  
and

$$A_{2} = \frac{\underline{A}_{2} + \overline{A}_{2}}{2} = \begin{bmatrix} -0.5 & 0.518\\ 0.428 & -0.5 \end{bmatrix}, \quad D_{2} = \frac{\underline{D}_{2} + \overline{D}_{2}}{2} = \begin{bmatrix} 0.11 & 0.02\\ 0.054 & 0.052 \end{bmatrix}, \quad G_{2} = \frac{\underline{G}_{2} + \overline{G}_{2}}{2} = \begin{bmatrix} 0.1\\ 0.2 \end{bmatrix}$$

Let  $l_1 = [2 \ 2]^T$  and  $l_2 = [1 \ 1]^T$ . Next, we choose the same positive constants  $c_1 = 6$ ,  $c_2 = 50$ ,  $T_f = 15$ , and  $\rho = 0.1$  which are given in Example 1. The value of  $x^T(t)l_2$ ,  $t \in [0, T_f]$  for the first subsystem and the second subsystem are depicted respectively in Figures 4 and 5. From two subsystems of the simulations, it is verified that the first subsystem is not finite-time bounded and the second one is finite-time bounded.

By (5), we can get the same D proposed in Example 1. For given sclars  $\gamma_1 = 0.5$ ,  $\mu_1 = 0.58$ ,  $\gamma_2 = 0.106$  and  $\mu_2 = 1.728$ , we can get a set of feasible solution for Theorem 1:

$$\nu_1 = [0.975 \ 0.975]^T$$
,  $\nu_2 = [1.684 \ 1.684]^T$ ,  $\xi_1 = 0.6$ ,  $\xi_2 = 0.85$ ,  $\beta_1 = 0.974$ ,  $\beta_2 = 1.2$ .

Hence, system (1) is finite-time bounded with respect to  $(6, 50, 15, [2 \ 2]^T, [1 \ 1]^T, 0.1, \sigma)$ under the switching signal with SMDADT  $\tau_{a2}^* = 12.7870$  and FMDADT  $\tau_{a1}^* = 1.0895$ . Let  $\tau_{a1} = 1 < 1.0895$  and  $\tau_{a2} = 13 > 12.7870$  for the first subsystem and the second subsystem, respectively. The value of  $x^T(t)l_2$  of the system under the corresponding switching signal is presented in Figure 6. It can be seen that the value of  $x^T(t)l_2$  increases significantly, but does not exceed the specified value of  $c_2$ . Therefore, the studied system in this example is finite-time bounded with respect to  $(6, 50, 15, [2 \ 2]^T, [1 \ 1]^T, 0.1, \sigma)$ .



**Figure 4.** The value of  $x^T(t)l_2$  of the first subsystem in Example 2.



**Figure 5.** The value of  $x^T(t)l_2$  of the second subsystem in Example 2.





#### 5. Conclusions

In this paper, the FTB problem for a class of SPSs, including time-varying delay, interval uncertainties, exogenous disturbance, and finite-time unbounded subsystems, has been studied. The design of QASSs for the systems whose subsystems are bounded and unbounded has been addressed. By taking advantage of the positivity of the considered systems combined with the SMDADT and the FMDADT methods, the suitable CLKF has been constructed, and some computable sufficient conditions for FTB have been formulated in the main theorem. Furthermore, novel DDSC of systems without the interval uncertainties have also been acquired in the corollary. Lastly, two numerical examples have been presented to illustrate the validity of the theoretical analysis.

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