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Boundary Value Problem for ψ-Caputo Fractional Differential Equations in Banach Spaces via Densifiability Techniques

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Abstract: A novel fixed-point theorem based on the degree of nondensifiability (DND) is used in this article to examine the existence of solutions to a boundary value problem containing the ψ-Caputo fractional derivative in Banach spaces. Besides that, an example is included to verify our main results. Moreover, the outcomes obtained in this research paper ameliorate and expand some previous findings in this area.

Keywords: fractional differential equations; ψ-Caputo fractional derivative; existence; fixed point; degree of nondensifiability; Banach spaces

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1. Introduction

The study of fractional differential equations (FDEs) has become a hot topic because they can suitably explain the behavior of a wide range of real-world problems more accurately than integer-order derivatives. For the specifics, one can refer to [1–4]. In the same context, many expressions of fractional calculus have been published, but the most prevalent definitions are Riemann–Liouville and Caputo fractional derivatives. The former has an abstraction mathematically, but the latter is mostly used by engineers. At the same time, many attempts have been made in this field to generalize the aforementioned fractional derivatives. The ψ-Caputo fractional derivative suggested by Almeida in [5] is one of these generalized definitions. Furthermore, the authors in [6–23] applied different types of fixed-point techniques in different spaces to tackle the existence, uniqueness, and Ulam-type stability of the solution of ordinary differential equations (ODEs), as well as FDEs.

On the other hand, at the beginning of the 1980s, Cherruault and Guillez [24] proposed the notion of α-dense curves. In addition, Cherruault [25] and Mora [26] were mostly responsible for its development, and in the same direction, the concept of the degree of nondensifiability (DND) was first introduced by Mora and Mira [27], which is based on the so-called α-dense curves. Very recently, García [28] proved a new fixed-point result based on the DND that performs under more generic circumstances than the Darbo fixed-point theorem (DFPT) and its known generalizations. Additionally, for more interesting details about the usefulness of the DND in the study of the existence of solutions to certain differential equations or integral equations in some Banach spaces, we suggest the works [28–32].

Our proposed method is essentially based on the excellent results given by García [28,29] to study the existence of solutions to a boundary value problem (BVP) containing the ψ-Caputo fractional derivative in Banach spaces via densifiability techniques. More specifically, we pose the following fractional BVP:

$$\begin{cases} {}^{c}\mathbb{D}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \mathbb{F}(\mathrm{r},\mathfrak{z}(\mathrm{r})), \ \ \mathrm{r} \in \mathrm{D} := [\mathrm{r}_1, \mathrm{r}_2], \\ \mathfrak{z}(\mathrm{r}_1) = \mathfrak{z}(\mathrm{r}_1) = \theta, \end{cases} \tag{1}$$

where $\mathrm{p} \in (1,2]$, ${}^{c}\mathbb{D}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}$ denotes the ψ-Caputo fractional derivative of order p, $\mathbb{F} : \mathrm{D} \times \mathbb{X} \longrightarrow \mathbb{X}$ is a given function that fulfills certain conditions that will be specified hereafter, $\mathbb{X}$ is a Banach space with norm $\|\cdot\|$, and θ refers to the null vector in the space $\mathbb{X}$.

Comparing the fractional BVP (1) to the previous studies, the highlights of our findings lie in the following features:

- The novelty of the current research work is selecting a more universal fractional derivative that incorporates several traditional fractional derivatives. In other words, the fractional BVP (1) is simplified to the fractional BVP (1) involving Caputo, Caputo–Hadamard, and Caputo–Katugampola fractional derivatives, for $\psi(\mathrm{r}) = \mathrm{r}$, $\psi(\mathrm{r}) = \log \mathrm{r}$, and $\psi(\mathrm{r}) = \frac{\mathrm{r}^{\rho}}{\rho}, \rho > 0$, respectively, in addition to the integer case, by choosing $\psi(\mathrm{r}) = \mathrm{r}$ and $\mathrm{p} = 2$. Therefore, the results obtained in this paper are new;
- Under some suitable assumptions (weak conditions) on the nonlinear part $\mathbb{F}$ and by the application of the new fixed-point theorem combined with densifiability techniques, we obtained the existence of solutions to the considered fractional BVP (1);
- Our obtained results improve and generalize those obtained in [10,15,18].

Our paper is divided into the following sections: In Section 2, we review a few basic definitions and preparation results. Afterward, in Section 3, focused on the aforementioned method, we give our main results, which enabled us to deduce the existence of at least one solution to the BVP described in Equation (1). Further on, in Section 4, an appropriate application is presented to highlight the usefulness of the reported findings. Finally, the paper closes with a brief conclusion and points out some possible future directions of research.

2. Basic Definitions

Assume that $(\mathbb{X}, \|\cdot\|)$ is a Banach space and $\mathfrak{M}_{\mathbb{X}}$ is the class of non-empty and bounded subsets of $\mathbb{X}$. Moreover, we denote by $\big(C(\mathrm{D},\mathbb{X}), \|\cdot\|_\infty\big)$ and $\big(L^1(\mathrm{D},\mathbb{X}), \|\cdot\|_1\big)$ the Banach spaces of continuous and Bochner integrable mappings $\mathfrak{z}$ from D into $\mathbb{X}$, respectively.

We begin with the following definitions that are adapted from [26,33]:

Definition 1. *Assume that $\alpha \geq 0$ and $\mathbb{P} \in \mathfrak{M}_{\mathbb{X}}$; a continuous mapping $\sigma : \Delta := [0,1] \to \mathbb{X}$ is said to be an α-dense curve in $\mathbb{P}$ if the following conditions hold:*

- *$\sigma(\Delta) \subset \mathbb{P}$;*
- *For any $z_1 \in \mathbb{P}$, there is $z_2 \in \sigma(\Delta)$ such that $\|z_1 - z_2\| \leq \alpha$.*

Moreover, if for every $\alpha > 0$, there is an α-dense curve in $\mathbb{P}$, then $\mathbb{P}$ is said to be densifiable.

From the concept of the α-dense curve, we can define the degree of nondensifiability (DND).

Definition 2 ([27,34])**.** *Let $\alpha \geq 0$, and denote by $\Gamma_{\alpha,\mathbb{P}}$ the class of all α-dense curves in $\mathbb{P} \in \mathfrak{M}_{\mathbb{X}}$. The DND is a mapping $\omega\colon \mathfrak{M}_{\mathbb{X}} \to \mathbb{R}_+$ defined as follows:*

$$\omega(\mathbb{P}) = \inf\{\alpha \geq 0 : \Gamma_{\alpha,\mathbb{P}} \neq \emptyset\},$$

for each $\mathbb{P} \in \mathfrak{M}_{\mathbb{X}}$.

Remark 1. *It is worth noting that in [34], a careful study about the DND was performed, and in particular, it was proven that the DND is not a measure of noncompactness (MNC) (see [34] for its* *definition and properties). However, it has characteristics that are extremely similar to it (see [34] Proposition 2.6).*

Certain characteristics of the DND proven in [31,34] are presented the following lemma.

Lemma 1. *Let $\mathbb{P}_1, \mathbb{P}_2 \in \mathfrak{M}_{\mathbb{X}}$. Then, the DND has the following properties listed below:*
(1) $\omega(\mathbb{P}) = 0 \Longleftrightarrow \mathbb{P}$ *is a precompact set, for each nonempty, bounded, and arc-connected subset $\mathbb{P}$ of $\mathbb{X}$;*
(2) $\omega(\overline{\mathbb{P}_1}) = \omega(\mathbb{P}_1)$*, where $\overline{\mathbb{P}_1}$ denotes the closure of $\mathbb{P}_1$;*
(3) $\omega(v\mathbb{P}_1) = |v|\omega(\mathbb{P}_1)$ *for $v \in \mathbb{R}$;*
(4) $\omega(x + \mathbb{P}_1) = \omega(\mathbb{P}_1)$*, for all $x \in \mathbb{X}$;*
(5) $\omega(Conv(\mathbb{P}_1)) \leq \omega(\mathbb{P}_1)$ *and* $\omega(Conv(\mathbb{P}_1 \cup \mathbb{P}_2)) \leq \max\{\omega(Conv(\mathbb{P}_1)), \omega(Conv(\mathbb{P}_2))\}$*, where $Conv(\mathbb{P}_1)$ represent the convex hull of $\mathbb{P}_1$;*
(6) $\omega(\mathbb{P}_1 + \mathbb{P}_2) \leq \omega(\mathbb{P}_1) + \omega(\mathbb{P}_2)$.

On the other hand, we introduce the following class of functions given by:

$$\mathcal{A} = \left\{ \begin{array}{l} \beta \colon \mathbb{R}_+ \to \mathbb{R}_+ : \beta \text{ is monotone increasing} \\ \text{and } \lim_{n\to\infty} \beta^n(t) = 0 \text{ for any } t \in \mathbb{R}_+ \end{array} \right\},$$

where $\beta^n(t)$ denotes the n-th composition of β with itself.

The following version of the Darbo fixed-point theorem for the DND has an important role in this paper.

Theorem 1 ([28], Theorem 3.2)**.** *Let $\mathbb{K}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathbb{X}$, and let $\mathbb{O} : \mathbb{K} \to \mathbb{K}$ be a continuous operator. Assume that there is $\beta \in \mathcal{A}$ such that:*

$$\omega(\mathbb{O}(\mathbb{P})) \leq \beta(\omega(\mathbb{P})), \tag{2}$$

for any non-empty subset $\mathbb{P}$ of $\mathbb{K}$. Then, $\mathbb{O}$ possesses at least one fixed point in $\mathbb{K}$.

Remark 2. *Let us note that the above theorem is in a form very similar to the well-known Darbo fixed-point theorem [11]. However, as shown in [28,31] by several examples, both results are essentially different, as Theorem 1 performs under more generic circumstances than the Darbo fixed-point theorem (DFPT) or its known generalizations.*

Next, we show the following lemma, which we will use later.

Lemma 2 ([28], Lemma 3.2)**.** *Let $\mathbb{P} \subset \mathrm{C}(\mathrm{D}, \mathbb{X})$ be non-empty and bounded. Then:*

$$\sup_{\mathrm{r}\in\mathrm{D}} \omega(\mathbb{P}(\mathrm{r})) \leq \omega(\mathbb{P}).$$

Let $\psi \in \mathrm{C}^1(\mathrm{D}, \mathbb{R})$ be a given function such that $\psi'(\mathrm{r}) > 0$, for all $\mathrm{r} \in \mathrm{D}$.

Definition 3 ([2,5])**.** *Given $\mathrm{p} > 0$, the Riemann–Liouville (R–L) fractional integral of order p for an integrable function $\mathfrak{z} \colon \mathrm{D} \longrightarrow \mathbb{R}$ with respect to ψ is given as follows:*

$$\mathbb{I}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r}) - \psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})}\mathfrak{z}(\varrho)\mathrm{d}\varrho,$$

where $\Gamma(\mathrm{p}) = \int_0^{+\infty} \mathrm{r}^{\mathrm{p}-1}e^{-\mathrm{r}}\mathrm{d}\mathrm{r}, \mathrm{p} > 0$ is called the Gamma function.

Definition 4 ([5])**.** *Given $\psi, \mathfrak{z} \in C^n(\mathrm{D}, \mathbb{R})$, the ψ-Caputo fractional derivative of $\mathfrak{z}$ of order p is expressed by:*

$${}^c\mathbb{D}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \mathbb{I}^{n-\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}^{[n];\psi}(\mathrm{r}),$$

where $n = [\mathrm{p}] + 1$ *for* $\mathrm{p} \notin \mathbb{N}$, $n = \mathrm{p}$ *for* $\mathrm{p} \in \mathbb{N}$, *and* $\mathfrak{z}^{[n];\psi}(\mathrm{r}) = \left(\frac{\frac{d}{dr}}{\psi'(\mathrm{r})}\right)^n \mathfrak{z}(\mathrm{r})$.

Lemmas of the following type are rather standard in the study of fractional differential equations.

Lemma 3 ([5])**.** *Let* $\mathrm{p}, \mathrm{q} > 0$, *and* $\mathfrak{z} \in L^1(\mathrm{D}, \mathbb{R})$. *Then:*

$$\mathbb{I}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathbb{I}^{\mathrm{q};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \mathbb{I}^{\mathrm{p}+\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}), \ a.e. \ \mathrm{r} \in \mathrm{D}.$$

In particular,
if $\mathfrak{z} \in \mathrm{C}(\mathrm{D}, \mathbb{R})$. *Then,* $\mathbb{I}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathbb{I}^{\mathrm{q};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \mathbb{I}^{\mathrm{p}+\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}), \mathrm{r} \in \mathrm{D}$.

Lemma 4 ([5])**.** *For a given* $\mathfrak{z} \in C^n(\mathrm{D}, \mathbb{R})$ *and* $\mathrm{p} > 0$, *then for each* $\mathrm{r} \in \mathrm{D}$, *we have:*

$$\mathbb{I}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}{}^c\mathbb{D}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \mathfrak{z}(\mathrm{r}) - \sum_{j=0}^{n-1}\frac{\mathfrak{z}^{[j];\psi}(\mathrm{r}_1)}{j!}[\psi(\mathrm{r}) - \psi(\mathrm{r}_1)]^j, \quad n-1 < \mathrm{p} \leq n.$$

In particular, if $\mathfrak{z} \in \mathrm{C}(\mathrm{D}, \mathbb{R})$, *then* ${}^c\mathbb{D}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathbb{I}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \mathfrak{z}(\mathrm{r})$, *for all* $\mathrm{r} \in \mathrm{D}$.

The following lemma is essential for the existence of the solutions to the BVP (1)

Lemma 5 ([13])**.** *For* $\mathrm{f} \in L^1(\mathrm{D}, \mathbb{R})$, *the following BVP:*

$$\begin{cases} {}^c\mathbb{D}^{\mathrm{p};\psi}_{\mathrm{r}_1^+}\mathfrak{z}(\mathrm{r}) = \mathrm{f}(\mathrm{r}), \ 1 < \mathrm{p} \leq 2\,, \mathrm{r} \in \mathrm{D}, \\ \mathfrak{z}(\mathrm{r}_1) = \mathfrak{z}(\mathrm{r}_2) = 0, \end{cases} \tag{3}$$

has a unique solution given by:

$$\begin{aligned} \mathfrak{z}(\mathrm{r}) = {} & \int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r}) - \psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})}\mathrm{f}(\varrho)\mathrm{d}\varrho \\ & - \frac{(\psi(\mathrm{r}) - \psi(\mathrm{r}_1))}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2) - \psi(\mathrm{r}_1))}\int_{\mathrm{r}_1}^{\mathrm{r}_2} \psi'(\varrho)(\psi(\mathrm{r}_2) - \psi(\varrho))^{\mathrm{p}-1}\mathrm{f}(\varrho)\mathrm{d}\varrho. \end{aligned} \tag{4}$$

3. Main Results

In this section, we prove the existence outcomes of the suggested system (1). Let us assume the following hypotheses.

Hypothesis 1 (H1). *The function* $\mathbb{F}\colon \mathrm{D} \times \mathbb{X} \to \mathbb{X}$ *satisfies the Carathéodory conditions.*

Hypothesis 2 (H2). *There exist a function* $\mu \in L^\infty(\mathrm{D}, \mathbb{R}_+)$ *and a continuous nondecreasing function* $\Theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that:*

$$\|\mathbb{F}(\mathrm{r}, z)\| \leq \mu(\mathrm{r})\Theta(\|z\|), \quad \textit{for all } z \in \mathbb{X}.$$

Hypothesis 3 (H3). *There exists a number* $v > 0$ *such that:*

$$2\frac{\|\mu\|_{L^\infty}(\psi(\mathrm{r}_2) - \psi(\mathrm{r}_1))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)}\Theta(v) \leq v. \tag{5}$$

Hypothesis 4 (H4). *There are two functions* $g \in L^\infty(\mathrm{D}, \mathbb{R}_+)$ *and* $\beta \in \mathcal{A}$ *such that for any non-empty, bounded, and convex* $\mathbb{P} \subset \mathbb{X}$, *the inequality:*

$$\omega(\mathbb{F}(\mathrm{r}, \mathbb{P})) \leq g(\mathrm{r})\beta(\omega(\mathbb{P})),$$

holds for a.e. $\mathrm{r} \in \mathrm{D}$.

Theorem 2. *Let Conditions (H1)–(H4) be satisfied. Then, BVP (1) admits at least one solution provided that:*

$$2\frac{\|\mu\|_{L^\infty}(\psi(\mathrm{r}_2)-\psi(\mathrm{r}_1))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} \leq 1.$$

Proof. Using Lemma 5, BVP (1) can be switched into an equivalent integral equation defined as follows:

$$\begin{aligned}\mathfrak{z}(\mathrm{r}) = & \int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r})-\psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})}\mathbb{F}(\varrho,\mathfrak{z}(\varrho))\mathrm{d}\varrho \\ & -\frac{\psi(\mathrm{r})-\psi(\mathrm{r}_1)}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2)-\psi(\mathrm{r}_1))}\int_{\mathrm{r}_1}^{\mathrm{r}_2}\psi'(\varrho)(\psi(\mathrm{r}_2)-\psi(\varrho))^{\mathrm{p}-1}\mathbb{F}(\varrho,\mathfrak{z}(\varrho))\mathrm{d}\varrho. \end{aligned} \tag{6}$$

Thus, to investigate the existence of a solution to the BVP (1), we turn it into a fixed-point problem (FPP) for the operator $\mathbb{O} : C(\mathrm{D},\mathbb{X}) \longrightarrow C(\mathrm{D},\mathbb{X})$ defined by the following formula:

$$\begin{aligned}\mathbb{O}\mathfrak{z}(\mathrm{r}) = & \int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r})-\psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})}\mathbb{F}(\varrho,\mathfrak{z}(\varrho))\mathrm{d}\varrho \\ & -\frac{\psi(\mathrm{r})-\psi(\mathrm{r}_1)}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2)-\psi(\mathrm{r}_1))}\int_{\mathrm{r}_1}^{\mathrm{r}_2}\psi'(\varrho)(\psi(\mathrm{r}_2)-\psi(\varrho))^{\mathrm{p}-1}\mathbb{F}(\varrho,\mathfrak{z}(\varrho))\mathrm{d}\varrho. \end{aligned} \tag{7}$$

To demonstrate the intended outcome, we first let:

$$\mathbb{B}_\upsilon = \{\mathfrak{z} \in C(\mathrm{D},\mathbb{X}) : \|\mathfrak{z}\|_\infty \leq \upsilon\},$$

where υ satisfies the inequality (5). We prove that the operator $\mathbb{O}$ fulfills all the hypotheses of Theorem 1.

First, we show $\mathbb{O}\mathbb{B}_\upsilon \subset \mathbb{B}_\upsilon$. Indeed, for any $\mathfrak{z} \in \mathbb{B}_\upsilon$ and under (H2), we obtain:

$$\begin{aligned}\|\mathbb{O}\mathfrak{z}(\mathrm{r})\| \leq & \int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r})-\psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})}\|\mathbb{F}(\varrho,\mathfrak{z}(\varrho))\|\mathrm{d}\varrho \\ & +\frac{\psi(\mathrm{r})-\psi(\mathrm{r}_1)}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2)-\psi(\mathrm{r}_1))}\int_{\mathrm{r}_1}^{\mathrm{r}_2}\psi'(\varrho)(\psi(\mathrm{r}_2)-\psi(\varrho))^{\mathrm{p}-1}\|\mathbb{F}(\varrho,\mathfrak{z}(\varrho))\|\mathrm{d}\varrho \\ \leq & \int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r})-\psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})}\mu(\varrho)\Theta(\|\mathfrak{z}(\varrho)\|)\mathrm{d}\varrho \\ & +\frac{\psi(\mathrm{r})-\psi(\mathrm{r}_1)}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2)-\psi(\mathrm{r}_1))}\int_{\mathrm{r}_1}^{\mathrm{r}_2}\psi'(\varrho)(\psi(\mathrm{r}_2)-\psi(\varrho))^{\mathrm{p}-1}\mu(\varrho)\Theta(\|\mathfrak{z}(\varrho)\|)\mathrm{d}\varrho \\ \leq & \frac{2\|\mu\|_{L^\infty}\Theta(\upsilon)(\psi(\mathrm{r}_2)-\psi(\mathrm{r}_1))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} \\ \leq & \upsilon, \end{aligned}$$

which implies that $\|\mathbb{O}\mathfrak{z}\| \leq \upsilon$, and so, $\mathbb{O}\mathbb{B}_\upsilon \subset \mathbb{B}_\upsilon$. Furthermore, by combining Assumptions (H1) and (H2) and the Lebesgue-dominated convergence theorem, we can obtain easily that the operator $\mathbb{O}$ is continuous on $\mathbb{B}_\upsilon$.

Now, we prove that $\mathbb{O}$ satisfies the contractive condition appearing in Theorem 1. To do this, let V be any non-empty and convex subset of $\mathbb{B}_\upsilon$, and for each $\varrho \in \mathrm{D}$, let $\alpha_\varrho := \omega(V(\varrho))$. Through (H4), there are $g \in L^\infty(\mathrm{D},\mathbb{R}_+)$ and $\beta \in \mathcal{A}$ such that for a.e. $\varrho \in \mathrm{D}$:

$$\omega(\mathbb{F}(\varrho, V(\varrho))) \leq g(\varrho)\beta(\alpha_\varrho).$$

Therefore, given any $\varepsilon > 0$, there is a continuous mapping $\sigma_\varrho \colon \Delta \to \mathbb{X}$, with $\sigma_\varrho(\Delta) \subset \mathbb{F}(\varrho, V(\varrho))$, such that for all $\mathfrak{z} \in V$, there is $\zeta \in \Delta$ with:

$$\|\mathbb{F}(\varrho, \mathfrak{z}(\varrho)) - \sigma_\varrho(\zeta)\| \leq g(\varrho)\beta(\alpha_\varrho) + \varepsilon, \ \textit{for a.e.}\ \varrho \in \mathrm{D}. \tag{8}$$

Construct now the mapping $\tilde{\sigma} \colon \Delta \to (C(\mathrm{D}, \mathbb{X}), \|\cdot\|_\infty)$ as follows:

$$\begin{aligned}\zeta \in \Delta \longmapsto \tilde{\sigma}(\zeta, \mathrm{r}) := &\int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r}) - \psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})} \sigma_\varrho(\zeta) \mathrm{d}\varrho \\ &- \frac{\psi(\mathrm{r}) - \psi(\mathrm{r}_1)}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2) - \psi(\mathrm{r}_1))} \int_{\mathrm{r}_1}^{\mathrm{r}_2} \psi'(\varrho)(\psi(\mathrm{r}_2) - \psi(\varrho))^{\mathrm{p}-1} \sigma_\varrho(\zeta) \mathrm{d}\varrho, \ \ \mathrm{r} \in \mathrm{D}.\end{aligned}$$

It is clear that $\tilde{\sigma}$ is continuous and $\tilde{\sigma}(\Delta) \subset \mathbb{O}(V)$. Additionally, by (8), given $\mathfrak{z} \in V$, we can find $\zeta \in \Delta$ such that:

$$\begin{aligned}\|\mathbb{O}\mathfrak{z}(\mathrm{r}) - \tilde{\sigma}(\zeta, \mathrm{r})\| \leq &\int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r}) - \psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})} \|\mathbb{F}(\varrho, \mathfrak{z}(\varrho)) - \sigma_\varrho(\zeta)\| \mathrm{d}\varrho \\ &+ \frac{(\psi(\mathrm{r}) - \psi(\mathrm{r}_1))}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2) - \psi(\mathrm{r}_1))} \int_{\mathrm{r}_1}^{\mathrm{r}_2} \psi'(\varrho)(\psi(\mathrm{r}_2) - \psi(\varrho))^{\mathrm{p}-1} \|\mathbb{F}(\varrho, \mathfrak{z}(\varrho)) - \sigma_\varrho(\zeta)\| \mathrm{d}\varrho \\ \leq &\int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r}) - \psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})} (g(\varrho)\beta(\alpha_\varrho) + \varepsilon) \mathrm{d}\varrho \\ &+ \frac{(\psi(\mathrm{r}) - \psi(\mathrm{r}_1))}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2) - \psi(\mathrm{r}_1))} \int_{\mathrm{r}_1}^{\mathrm{r}_2} \psi'(\varrho)(\psi(\mathrm{r}_2) - \psi(\varrho))^{\mathrm{p}-1} (g(\varrho)\beta(\alpha_\varrho) + \varepsilon) \mathrm{d}\varrho.\end{aligned}$$

Applying Lemma 2 and the features of β, setting $\alpha := \omega(V)$, we can deduce that $\beta(\alpha_\varrho) \leq \beta(\alpha)$ for a.e. $\varrho \in \mathrm{D}$, and from the last estimate, we obtain:

$$\begin{aligned}\|\mathbb{O}\mathfrak{z}(\mathrm{r}) - \tilde{\sigma}(\zeta, \mathrm{r})\| \leq &\int_{\mathrm{r}_1}^{\mathrm{r}} \frac{\psi'(\varrho)(\psi(\mathrm{r}) - \psi(\varrho))^{\mathrm{p}-1}}{\Gamma(\mathrm{p})} (g(\varrho)\beta(\alpha) + \varepsilon) \mathrm{d}\varrho \\ &+ \frac{\psi(\mathrm{r}) - \psi(\mathrm{r}_1)}{\Gamma(\mathrm{p})(\psi(\mathrm{r}_2) - \psi(\mathrm{r}_1))} \int_{\mathrm{r}_1}^{\mathrm{r}_2} \psi'(\varrho)(\psi(\mathrm{r}_2) - \psi(\varrho))^{\mathrm{p}-1} (g(\varrho)\beta(\alpha) + \varepsilon) \mathrm{d}\varrho.\end{aligned}$$

Since the previous inequality is valid, for every $\varepsilon > 0$, we conclude:

$$\begin{aligned}\|\mathbb{O}\mathfrak{z}(\mathrm{r}) - \tilde{\sigma}(\zeta, \mathrm{r})\| &\leq 2 \frac{\|g\|_{L^\infty} (\psi(\mathrm{r}_2) - \psi(\mathrm{r}_1))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} \beta(\alpha) \\ &\leq \beta(\alpha),\end{aligned}$$

which means, from the arbitrariness of $\mathrm{r} \in \mathrm{D}$, that $\omega(\mathbb{O}V) \leq \beta(\alpha)$.

Since the assumptions in Theorem 1 are fulfilled, the intended result follows. □

4. Application

Take $\mathbb{X} = c_0 = \{\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_i, \cdots) : \mathfrak{z}_i \to 0\ (i \to \infty)\}$, the Banach space of real sequences converging to zero, with the standard norm:

$$\|\mathfrak{z}\|_\infty = \sup_i |\mathfrak{z}_i|.$$

Let $\mathrm{r}_1 = 1, \psi(\mathrm{r}) = \ln \mathrm{r}$; fix $1 < \mathrm{p} < \frac{\ln 0.25}{\ln \ln(\sqrt{\ln 4 - 1} + 1)} \simeq 1.9$; let $1 < \mathrm{r}_2 < \exp 0.25^{\frac{1}{\mathrm{p}}}$.

Example 1. *Consider the BVP:*

$$\begin{cases} {}^{c}_{H}\mathbb{D}^{\mathrm{p};\psi}_{1^+} \mathfrak{z}(\mathrm{r}) = \mathbb{F}(\mathrm{r}, \mathfrak{z}(\mathrm{r})), \quad \mathrm{r} \in \mathrm{D} := [1, \mathrm{r}_2], \\ \mathfrak{z}(1) = \mathfrak{z}(\mathrm{r}_2) = \theta. \end{cases} \tag{9}$$

System (9) *is a special case of BVP* (1)*, with* $\mathbb{F}\colon \mathrm{D}\times \mathrm{c}_0 \to \mathrm{c}_0$ *defined as:*

$$\mathbb{F}(\mathrm{r},\mathfrak{z}) = \left\{ \frac{2i}{(i+1)((\mathrm{r}-1)^2+1)} \left(\frac{1}{i} + \ln(1+|\mathfrak{z}_i|) \right) \right\}_{i\geq 1},$$

for $\mathrm{r}\in \mathrm{D}, \mathfrak{z} = \{\mathfrak{z}_i\}_{i\geq 1} \in c_0$. *Obviously, Assumption (H1) of Theorem* 2 *is satisfied. Moreover, for each* $\mathrm{r}\in\mathrm{D}$ *and* $\mathfrak{z}\in \mathrm{c}_0$*, we obtain:*

$$\begin{aligned}\|\mathbb{F}(\mathrm{r},\mathfrak{z})\|_\infty &\leq \left\| \frac{2i}{(i+1)((\mathrm{r}-1)^2+1)} \left(\frac{1}{i} + |\mathfrak{z}_i| \right) \right\|_\infty \\ &\leq \frac{2}{(\mathrm{r}-1)^2+1}(\|\mathfrak{z}\|_\infty + 1) \\ &= \mu(\mathrm{r})\Theta(\|\mathfrak{z}\|).\end{aligned}$$

Hence, (H2) holds for $\mu(\mathrm{r}) = \frac{2}{(\mathrm{r}-1)^2+1}, \mathrm{r}\in\mathrm{D}$ *and* $\Theta(\ell) = 1+\ell,\ \ell\in[0,\infty)$. *The inequality appearing in (H3) has the following expression:*

$$\frac{4(\log \mathrm{r}_2)^{\mathrm{p}}(v+1)}{\Gamma(\mathrm{p}+1)} \leq v,$$

Then, v can be chosen as:

$$v \geq \frac{1}{\Gamma(\mathrm{p}+1)-1},$$

so (H3) is satisfied. On the other hand, for any non-empty, bounded, and convex subset V of $C(\mathrm{D},\mathrm{c}_0)$ *and* $\mathrm{r}\in\mathrm{D}$ *fixed, let* σ *be an* α_{r}*-dense curve in* $V(\mathrm{r})$ *for some* $\alpha_{\mathrm{r}} > 0$. *Then, for* $\mathfrak{z}\in V$, *there is* $\zeta\in\Delta$ *satisfying:*

$$\|\mathfrak{z}(\mathrm{r}) - \sigma(\zeta,\mathrm{r})\|_\infty \leq \alpha_{\mathrm{r}}.$$

Therefore, we have:

$$\begin{aligned}\|\mathbb{F}(\mathrm{r},\mathfrak{z}(\mathrm{r})) - \mathbb{F}(\mathrm{r},\sigma(\zeta,\mathrm{r}))\|_\infty &\leq \frac{2}{(\mathrm{r}-1)^2+1}\left\|\ln\left(1+\frac{\|\mathfrak{z}(\mathrm{r})-\sigma(\zeta,\mathrm{r})\|}{1+\|\sigma(\zeta,\mathrm{r})\|}\right)\right\| \\ &\leq \frac{2}{(\mathrm{r}-1)^2+1}\ln(1+\|\mathfrak{z}(\mathrm{r})-\sigma(\zeta,\mathrm{r})\|) \\ &\leq \frac{2}{(\mathrm{r}-1)^2+1}\ln(1+\alpha_{\mathrm{r}}),\end{aligned}$$

and $\beta(\mathrm{r}) = \ln(1+\mathrm{r})$. *This function is continuous, and it is easily seen that* $\beta\in\mathcal{A}$*, so Condition (H4) is satisfied taking* $g(\mathrm{r}) := \frac{2}{(\mathrm{r}-1)^2+1}$. *Hence, all the conditions of Theorem* 2 *are satisfied, and consequently, Problem* (9) *has at least one solution* $\mathfrak{z}\in C(\mathrm{D},\mathrm{c}_0)$.

Remark 3. *We would like to point out that in the aforementioned example, Darbo's fixed-point theorem (DFPT) for the Hausdorff MNC* χ *cannot be implemented. To begin, remember that the Hausdorff MNC* χ *in the space* c_0 *may be obtained by the following expression:*

$$\chi(V) = \lim_{i\to\infty}\left\{\sup_{\mathfrak{z}\in V}\left(\sup_{k\geq i}|\mathfrak{z}_k|\right)\right\}. \tag{10}$$

where $V \in \mathfrak{M}_{c_0}$ *(cf. [12]). Next, by taking the standard basis of* c_0, $V = \{e_i : i \in \mathbb{N}\}$, *given* $\mathrm{r} \in \mathrm{D}$ *from* (10), *we have:*

$$\begin{aligned}
&\chi(\{\mathbb{F}(\mathrm{r}, e_i) : e_i \in V\}) \\
&= \lim_{i \to \infty} \left\{ \sup_{e_i \in V} \left(\sup_{k \geq i} \frac{2k}{(k+1)((\mathrm{r}-1)^2+1)} \left(\frac{1}{k} + \ln(1+|e_k|) \right) \right) \right\} \\
&= \frac{2}{(\mathrm{r}-1)^2+1} \lim_{i \to \infty} \left\{ \sup_{k \geq i} \left(\frac{1}{k} + \ln 2 \right) \right\} \geq \frac{\ln 4}{(\mathrm{r}_2-1)^2+1},
\end{aligned}$$

and:

$$\frac{\ln 4}{(\mathrm{r}_2-1)^2+1} \geq 1 \iff \mathrm{r}_2 \leq \sqrt{\ln 4 - 1} + 1 \simeq 1.6215.$$

This shows that $\chi(\{\mathbb{F}(\mathrm{r}, e_i) : e_i \in V\})$ *is strictly greater than* $\chi(V) = 1$.

5. Conclusions

Through this study, we focused our attention on the problem of the existence of solutions to a boundary value problem (BVP) containing the ψ-Caputo fractional derivative in Banach spaces. The proofs of our results were based on a new fixed point combined with the technique of the degree of nondensifiability (DND), which seems to perform under more general conditions than Darbo's fixed point theorem (DFPT) along with the concept of measures of noncompactness (MNC), and it was shown through some research papers that this novel approach is a useful tool for seeking solutions of differential and integral equations. We also provided an example to make our results clear. Finally, it would be interesting to apply the aforementioned technique to more generalized fractional operators in the future. Another area of investigation is the development of numerical methods for approximating the solutions suggested by our Theorem 2.

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