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Additive Noise Effects on the Stabilization of Fractional-Space Diffusion Equation Solutions

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Abstract: This paper considers a class of stochastic fractional-space diffusion equations with polynomials. We establish a limiting equation that specifies the critical dynamics in a rigorous way. After this, we use the limiting equation, which is an ordinary differential equation, to approximate the solution of the stochastic fractional-space diffusion equation. This equation has never been studied before using a combination of additive noise and fractional-space, therefore we generalize some previously obtained results as special cases. Furthermore, we use Fisher's and Ginzburg–Landau equations to illustrate our results. Finally, we look at how additive noise affects the stabilization of the solutions.

Keywords: approximate solutions; fractional Laplacian; limiting equation; stochastic fractional-space

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1. Introduction

Stochastic partial differential equations (SPDEs) are crucial in understanding the dynamics of many fascinating phenomena. In recent years, the significance of taking random influences into consideration in modeling, analyzing, simulating, and predicting complex phenomena has become widely realized in physics, chemistry, biology, materials science, and climate dynamics, as well as in geophysical and other areas; see [1,2].

Furthermore, fractional derivatives have drawn tremendous interest mainly because of their possible implementations in different areas, such as, for example, in physics [3–6], biology [7], finance [8–10], biochemistry and chemistry [11], and hydrology [12,13]. These fractional-order equations are better suited than equations with integer-orders because derivatives of the fractional order are allowed the memory and hereditary properties of various substances to be represented [14].

It seems that examining fractional equations with some random force is more significant. Therefore, we are concerned, here, with the fractional space-diffusion equation perturbed by additive noise on a bounded domain $G \subset \mathbb{R}$:

$$d\varphi = [-\varepsilon^{-2} D(-\Delta)^{\frac{r}{2}}\varphi + \mathcal{P}(\varphi)]dt + \varepsilon^{-1}dW, \quad t \geq 0, \quad x \in G, \tag{1}$$

where $\varepsilon \ll 1$ is a small parameter, D is the diffusion coefficient, $(-\Delta)^{\frac{r}{2}}$ is the fractional Laplacian with $r \in (1,2]$, $\mathcal{P}$ is a polynomial with the degree m and represents reaction kinetics, and W is a finite dimensional Wiener process.

In normal diffusion with time, the mean square displacement of an equation particle linearly increases, i.e., $\langle x^2(t)\rangle \ltimes t$. In contrast, anomalous diffusion is a diffusion process not following this linear relation. In some cases, they have a power-law scaling relation, namely $\langle x^2(t)\rangle \ltimes t^r$, that is present in various types of equations. r is defined as the anomalous exponent of diffusion in the case of $r = 1$ of normal diffusion, whereas $r = 2$, $r \in (0,1)$, and $r \in (1,2)$ correspond to a ballistic diffusion, a sub-diffusion, and a Levy super-diffusion, respectively [4]. By the transformation of a Fourier, the anomalously diffusive operator $(-\Delta)^{\frac{r}{2}}$ is defined [8,15–17] as:

$$\mathcal{L}\{(-\Delta)^{\frac{r}{2}}\varphi\}(\eta) = |\eta|^r \mathcal{L}\{\varphi\}(\eta),$$

where $\mathcal{L}\{\varphi\}$ is the Fourier transform of φ.

It is interesting to note that if we input $\mathcal{P}(\varphi) = \varphi(1-\varphi)(\varphi - a)$, Equation (1) becomes the stochastic fractional space Fitzhugh–Nagumo equation, which is used in the field of biology and population genetics, and also is used to model nerve impulse transmission [18,19]. We derive the stochastic space-fractional heat equation if we set $\mathcal{P}(\varphi) = \varphi - \varphi^3$, which is used in physics and describes the heat distribution within a given time interval in a given region [20]. Furthermore, if $\mathcal{P}(\varphi) = \varphi(1 - \frac{\varphi}{N})$, then (1) gives rise to the stochastic fractional-space Fisher equation, which is used as the spatial and temporal propagation model in an infinite medium of a virile gene [21]. Additionally, it is used in chemical kinetics [22], auto catalytic chemical reactions [23], flame propagation [24], neurophysiology [25], and nuclear reactor theory [26].

Recently, Equation (1) with $r = 2$ was addressed in the stochastic case by [27–30]. This equation with $r = 2$ was studied analytically by [31,32] in the deterministic case, i.e without noise. Recently, this Equation (1) was discussed by [33,34] with multiplicative noise. Furthermore, many analytical and numerical methods have been proposed to find the solution of the fractional-space partial differential Equation (1) without noise, such as in [35–41]. In this paper, we analytically approximate the solution of Equation (1) by using the perturbation method. This equation has never been addressed before using a combination of additive noise and fractional-space, therefore we generalize some previously obtained results as a special case.

The first aim of this paper is to show that the approximate solution of (1) is given by

$$\varphi(t,x) = \xi(t) + \chi(t,x) + \text{error}\ , \tag{2}$$

where ξ solves

$$d\xi = [\mathcal{P}(\xi) + G(\xi)]dt. \tag{3}$$

The polynomial $G(\xi)$, is defined later in (22) and has a degree of $m-2$. The term $\chi(t,x)$ in Equation (2) is referred to as a fast Ornstein–Uhlenbeck process (FOUP for short) and it will be defined later in (10). We note that the ordinary differential Equation (3) contains the same polynomial $\mathcal{P}$ as in Equation (1), plus an additional polynomial G that occurs as a result of the interaction between the non-linear term and additive noise. The second aim of this paper is to discuss the impact of additive noise on the solutions of Equation (1).

As an applications of how our results can be applied, we provide theoretical examples from physics (the real Ginzburg–Landau equation) and biology (the Fisher's equation). To clarify our results, let us consider the very simple real-valued Ginzburg–Landau equation with Neumann boundary conditions on $[0,\pi]$ as follows

$$d\varphi = [-\varepsilon^{-2}(-\Delta)^{\frac{r}{2}}\varphi + \varphi - \varphi^3]dt + \varepsilon^{-1}dW. \tag{4}$$

The approximation Theorem 2 shows us that the solution of the Ginzburg–Landau Equation (4) shall be of the kind (2), where ξ is the solution of

$$d\xi = [\Big(1 - \sum_{k=1}^{N} \frac{3\alpha_k^2}{2k^r}\Big)\xi - \xi^3]dt,$$

where $\alpha_k$for $k = 1,2,\cdots,N$ are real numbers and where there is noise intensity. If we input $r = 2$, we have the previous result that was obtained by [28].

In this paper, one great innovation of our approach is the explicit estimation of error in terms of arbitrarily high moments of error, as usually only weak convergence is handled against approximation. Moreover, this paper is the first paper, to the best of our knowledge, to analytically find the approximate solution of stochastic fractional-space partial differential equations.

The rest of this article is set out as follows. In the next section, we present some notations, assumptions, and preliminaries that we need in this paper. We also estimate an equation representing the high modes and give bounds on it in Section 3. We will state a general case of the averaging-over OU-process in Section 4. After that, we deduce the limiting equation and prove the main result of Theorem 2 in Section 5. In Section 6, there are two examples to clarify our results, including the Ginzburg–Landau and Fisher's equations. Finally, the conclusion of this paper is given.

2. Preliminaries

Let $\mathcal{H}$=$\mathcal{L}^2([0,\pi])$ be a separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$.

Since the operator $-\Delta$ is self-adjoint, there exists a complete orthonormal system $\{e_j\}_{j=0}^{\infty}$ and a sequence $\{\lambda_j\}_{j=0}^{\infty}$ such that

$$-\Delta e_j = \lambda_j e_j\ ,$$

with

$$0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_j \le \cdots$$

Here, we consider $-\Delta$ with the Neumann boundary condition on $[0,\pi]$. Therefore,

$$e_j(x) = \begin{cases} 1 & if \quad j = 0, \\ \sqrt{\frac{2}{\pi}}\cos(jx) & if \quad j \neq 0, \end{cases}$$

and

$$\lambda_j = j^2.$$

Define $\mathcal{H}^c$ and $\mathcal{H}^s$ as

$$\mathcal{H}^c := \text{kernel}\{\Delta\} = span\{1\} \text{ and } \mathcal{H}^s = (\mathcal{H}^c)^{\perp}.$$

Define the projections

$$P_c := \mathcal{H} \to \mathcal{H}^c \text{ and } P_s := \mathcal{I} - P_c,$$

where $\mathcal{I}$ is the identity operator on $\mathcal{H}$.

Consider the fractional space $\mathcal{H}^r$ to be the domain of $\mathcal{A}^{\frac{r}{2}}$ for $r > 0$, which is defined as

$$(-\Delta)^{\frac{r}{2}}\varphi = \sum_{j=0}^{\infty} \lambda_j^{\frac{r}{2}} \langle \varphi, e_j\rangle e_j,$$

$$\mathcal{H}^r = \mathcal{D}((-\Delta)^{\frac{r}{2}}) = \left\{\varphi \in \mathcal{H} : \sum_{j=0}^{\infty} \lambda_j^r \varphi_j^2 < \infty\right\},$$

with norm

$$\|\varphi\|_r^2 = \left\|(-\Delta)^{\frac{r}{2}}\varphi\right\|^2 = \sum_{k=0}^{\infty} \lambda_j^r \varphi_j^2.$$

Furthermore, let $\mathbb{T}_r(t) = \exp(-t(-\Delta)^{\frac{r}{2}})$ for $t \ge 0$ be the analytic semigroup generated by the fractional Laplacian $-(-\Delta)^{\frac{r}{2}}$ and satisfy

$$\|\mathbb{T}_r(t)\varphi\|_r \le e^{-\omega t}\|\varphi\|_r \text{ for all } \varphi \in \mathcal{H}^r, \tag{5}$$

where a constant $\omega > 0$ exists.

For the non-linear $\mathcal{P}$ in Equation (1), we assume:

Assumption 1. *Let $\mathcal{P} : \mathcal{H}^r \to \mathcal{H}^r$ satisfy for all $\varphi \in \mathcal{H}^r$ such that*

$$\|\mathcal{P}(\varphi)\|_r \leq C(1 + \|\varphi\|_r^m), \tag{6}$$

where m is the degree of $\mathcal{P}$.

Put shortly, we are using $\mathcal{P}_c(\varphi) = P_c\mathcal{P}(\varphi)$ and $\mathcal{P}_s(\varphi) = P_s\mathcal{P}(\varphi)$.

Assumption 2. *Let $\mathbb{F}(u) = \mathcal{P}_c(u) + G(u)$, where $G(u)$ is defined in (22). Assume for $u, w, v \in \mathcal{H}^c$ that*

$$\langle \mathbb{F}(u + v + w) - \mathbb{F}(v), u\rangle \leq C(|u|^2 + |w|^2 + |w|^m)\,. \tag{7}$$

We note that from Assumption 2, if we input $v = w = 0$,, we derive

$$\langle \mathbb{F}(u), u\rangle \leq C|u|^2 \text{ for } u \in \mathcal{H}^c \tag{8}$$

For the noise in Equation (1), see the following.

Assumption 3. *Suppose that the Wiener process $W(t)$ for $t \geq 0$, is finite dimensional and acts only on $\mathcal{H}^s$. Corresponding to [42], one can write it as*

$$W(t, x) = \sum_{j=1}^{N} \alpha_j \beta_j(t) e_j(x),$$

where $\alpha_j \in \mathbb{R}$ for all $j \in \{1, 2, \cdots, N\}$ and $(\beta_j)_{j\in\{1,\,2,\cdots,\,N\}}$ are mutually independent real-valued Brownian motions.

Definition 1. *Define the FOUP χ as*

$$\chi(t) = \sum_{j=1}^{N} \chi_j(t) e_j, \tag{9}$$

where

$$\chi_j(t) = \alpha_j \varepsilon^{-1} \int_0^t e^{-\varepsilon^{-2} D \geq_j^{\frac{r}{2}} (t-s)} d\beta_j(s). \tag{10}$$

In the following definition, we assume that the solution of Equation (1) is not too large.

Definition 2. *Stopping time: define the stopping time τ^* as*

$$\tau^* := \inf\{t > 0 : \|\varphi(t)\|_r > \varepsilon^{-\kappa}\} \wedge T_0, \tag{11}$$

for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{2m})$.

3. High Modes and Its Bounds

In this section, we deduce an equation representing high modes and bound it. We start by splitting the solution φ of (1) into

$$\varphi(t, x) = \varphi_c(t) + \varphi_s(t, x), \tag{12}$$

where $\varphi_c \in \mathcal{H}^c$ and $\varphi_s \in \mathcal{H}^s$. By substituting (12) into (1), we have

$$d(\varphi_c + \varphi_s) = [-\varepsilon^{-2} D(-\Delta)^{\frac{r}{2}} \varphi_s + \mathcal{P}(\varphi_c + \varphi_s)]dt + \varepsilon^{-1} dW. \tag{13}$$

By projecting to $\mathcal{H}^s$, we obtain

$$d\varphi_s = [-\varepsilon^{-2}D(-\Delta)^{\frac{r}{2}}\varphi_s + \mathcal{P}_s(\varphi_c + \varphi_s)]dt + \varepsilon^{-1}\alpha dW. \tag{14}$$

This equation can be expressed in the integral form as

$$\varphi_s(t) = \mathbb{T}_r(\varepsilon^{-2}Dt)\varphi_s(0) + \int_0^t \mathbb{T}_r(\varepsilon^{-2}D(t-\tau))(\varphi_c + \varphi_s)d\tau + \chi(t), \tag{15}$$

where χ is defined in Definition 1.

In the following lemma, we will show how $\varphi_s(t)$ equals to $\chi(t)$, plus a small term.

Lemma 1. *Assume that Assumption 1 is satisfied. Then, there is $C > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} \left\| \varphi_s(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Dt)\varphi_s(0) \right\|_r^p \leq C\varepsilon^{p-mp\kappa}, \tag{16}$$

for $p \geq 1$ and $\kappa > 0$ from the definition of τ^*.

Proof. Using the triangle inequality for (15), the equation yields

$$\begin{aligned} \left\| \varphi_s(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Dt)\varphi_s(0) \right\|_r &\leq \left\| \int_0^t \mathbb{T}_r(\varepsilon^{-2}D(t-s))\mathcal{P}_s(\varphi_c + \varphi_s)ds \right\|_r \\ &\leq C\|\mathcal{P}_s(\varphi_c + \varphi_s)\|_r \int_0^t e^{-\varepsilon^{-2}\omega D(t-s)}ds \\ &\leq C\varepsilon^2(1 + \|\varphi_c + \varphi_s\|_r^m). \end{aligned}$$

By taking $\mathbb{E}\sup_{t\in[0,\tau^*]}$ on both sides, we find that

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,\tau^*]} \left\| \varphi_s(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Dt)\varphi_s(0) \right\|_r &\leq C\varepsilon^2(1 + \mathbb{E} \sup_{t\in[0,\tau^*]} \|\varphi_c + \varphi_s\|_r^m) \\ &\leq C\varepsilon^2, \end{aligned}$$

where we used (5), representing Assumption 1 and the definition of τ^*, respectively. □

Now, let us, without proof, declare the uniform bounds on $\chi(t)$. For the proof, see Lemma 4.2 in [28].

Lemma 2. *Let $\chi(t)$ be defined in Definition 1. Then, there is $C > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\chi(t)\|_r^p \leq C\varepsilon^{-\kappa_0}, \tag{17}$$

for every $p \geq 1$ and $\kappa_0 > 0$.

The next corollary declares that $\varphi_s(t)$ is much smaller than $\varepsilon^{-\kappa}$, as stated in the definition of stopping time τ^*.

Corollary 1. *Assume that the assumptions of Lemmas 1 and 2 are satisfied. Let $\varphi_s(0) = \mathcal{O}(1)$. Then, for $p \geq 1$, $\kappa < \frac{1}{m}$ and $C > 0$,*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} \|\varphi_s(t)\|_r^p \leq C\varepsilon^{-\kappa_0} \quad \textit{for } p \geq 1. \tag{18}$$

Proof. Using Equation (16) and triangle inequality, we find that

$$\begin{aligned}\mathbb{E}\sup_{[0,\tau^*]}\|\varphi_s\|_r^p &\leq c\mathbb{E}\sup_{[0,\tau^*]}\left\|\varphi_s-\chi-\mathbb{T}_r(\varepsilon^{-2}Dt)\varphi_s(0)\right\|_r^p \\ &\quad +c\mathbb{E}\sup_{[0,\tau^*]}\left\|\mathbb{T}_r(\varepsilon^{-2}Dt)\varphi_s(0)\right\|_r^p+c\mathbb{E}\sup_{[0,\tau^*]}\|\chi\|_r^p.\end{aligned}$$

Apply Lemmas 1 and 2 to finish the proof. □

Lemma 3. *Let $\varphi_s(0)=\mathcal{O}(1)$, then*

$$\int_0^t\left\|\mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\right\|^n ds\leq C\varepsilon^2 \ \textit{for}\ \ n\geq 1.$$

Proof.

$$\int_0^t\left\|\mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\right\|^n ds\leq\int_0^t e^{-\varepsilon^{-2}\omega nD\tau}\|\varphi_s(0)\|^n ds=C\varepsilon^2.$$

□

4. Averaging over FOUP

Here, we use a comprehensive version of Lemma 5.1 from [28] over the FOUP χ_j. This lemma declares that odd powers of χ_j are small powers of the order $\mathcal{O}(\varepsilon^{1-\kappa_0})$, while the power of χ_j averages to a constant.

Lemma 4. *Assume that ϕ is a stochastic process with real values and $\phi(0)=\mathcal{O}(\varepsilon^{-\gamma})$ for some $\gamma\geq 0$. If $d\phi=Fdt$ together with $F=\mathcal{O}(\varepsilon^{-\gamma})$, then*

$$\int_0^t\phi\prod_{i=1}^N\chi_i^{n_i}d\tau=\begin{cases}\sum_{i=1}^N\frac{n_i(n_i-1)\alpha_i^2}{2\sum_{j=1}^N n_jD\lambda_j^{\frac{r}{2}}}\int_0^t\phi\prod_{j=1}^N\chi_j^{n_j}\chi_i^{-2}d\tau & \\ +\mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}) & \textit{if all } n_i \textit{ are even and} \\ \mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}) & \textit{if one of the } n_i \textit{ is odd.}\end{cases} \tag{19}$$

In the next lemma, we utilize Lemma 4 repeatedly and display the outcome that we require afterwards in our application.

Lemma 5. *Assume that ϕ is as in Lemma 4. Then, there is a constant C_{2k} for $k\in\mathbb{N}$ such that*

$$\int_0^t\phi\chi^{2k}d\tau=C_{2k}\int_0^t\phi d\tau+\mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}), \tag{20}$$

where χ is defined in Definition 1.

Proof. We address three cases as follows.

First case when $k=1$:

$$\begin{aligned}\int_0^t\phi\chi^2d\tau &= P_c\int_0^t\phi\left(\sum_{j=1}^N\chi_je_j\right)^2d\tau \\ &= \sum_{j=1}^N(e_j^2)\int_0^t\phi\chi_j^2d\tau+2\sum_{j\neq i}^N(e_je_i)\int_0^t\phi\chi_j\chi_id\tau.\end{aligned}$$

From Lemma 4, we have

$$\int_0^t \phi\chi^2 d\tau = \sum_{j=1}^{N} \frac{e_j^2\alpha_j^2}{2D\lambda_j^{\frac{r}{2}}} \int_0^t \phi d\tau + \mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}).$$

Thus,

$$C_{2k} \overset{k=1}{=} C_2 = \sum_{j=1}^{N} \frac{e_j^2\alpha_j^2}{2D\lambda_j^{\frac{r}{2}}}.$$

Second case when $k = 2$:

$$\begin{aligned}\int_0^t \phi\chi^4 d\tau &= \int_0^t \phi\left(\sum_{j=1}^{N} \chi_j e_j\right)^4 d\tau \\ &= \sum_{j=1}^{N} e_j^4 \int_0^t \phi\chi_j^4 d\tau + 6\sum_{j\neq i}^{N} e_j^2 e_i^2 \int_0^t \phi\chi_j^2\chi_i^2 d\tau + 4\sum_{k\neq i}^{N} e_j e_i^3 \int_0^t \phi\chi_j\chi_i^3 d\tau.\end{aligned}$$

Again, from Lemma 4, we derive

$$\int_0^t \phi\chi^4 d\tau = [\sum_{j=1}^{N} \frac{3e_j^4\alpha_j^4}{4D^2\lambda_j^r} + \sum_{j\neq i}^{N} \frac{3e_j^2 e_i^2\alpha_j^2\alpha_i^2}{2\lambda_j^{\frac{r}{2}}\lambda_i^{\frac{r}{2}}}] \int_0^t \phi d\tau + \mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}).$$

Hence,

$$C_{2k} \overset{k=2}{=} C_4 = \sum_{j=1}^{N} \frac{3e_j^4\alpha_j^4}{4D^2\lambda_j^r} + \sum_{j\neq i}^{N} \frac{3e_j^2 e_i^2\alpha_j^2\alpha_i^2}{2D^2\lambda_j^{\frac{r}{2}}\lambda_i^{\frac{r}{2}}}.$$

Third case when $k > 2$: We can follow the previous cases by expanding

$$\left(\sum_{j=1}^{N} \chi_j e_j\right)^{2k}.$$

□

5. Limiting Equation and Main Theorem

Here, the limiting equation is derived for Equation (1). Additionally, the main theorem of this paper is stated and proved.

Lemma 6. *Assume that Assumptions 1, 2, and 3 are satisfied. If $\varphi_s(0) = \mathcal{O}(1)$, then*

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c) d\tau + \int_0^t G(\varphi_c) d\tau + \mathcal{R}(t), \tag{21}$$

where

$$G(\varphi_c) = \sum_{k\geq 1} \frac{C_{2k}}{(2k)!}[\mathcal{P}_c(\varphi_c)]^{(2k)}, \tag{22}$$

and

$$\mathcal{R} = \mathcal{O}(\varepsilon^{1-(2m-1)\kappa}). \tag{23}$$

Proof. By recalling (13) and projecting to $\mathcal{H}^c$, we have

$$d\varphi_c = \mathcal{P}_c(\varphi_c + \varphi_s) dt. \tag{24}$$

W rewrite the above equation in the integral form as

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c + \varphi_s) d\tau. \tag{25}$$

Recall Lemma 1 which states

$$\varphi_s(t) = \chi(t) + \psi(t) + z(t), \tag{26}$$

with

$$\psi(t) = \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0) \text{ and } z(t) = \mathcal{O}(\varepsilon^{1-m\kappa}) .$$

By substituting (26) into (25), we have

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c + \chi + \psi + z)(\tau) d\tau. \tag{27}$$

Now, by applying Taylor's expansion to $\mathcal{P}_c$, we derive

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c + \chi)(\tau) d\tau + \tilde{z}(t),$$

where

$$\tilde{z}(t) = \sum_{k=1}^{m} P_c \int_0^t \frac{[\mathcal{P}_c(\varphi_c + \chi)]^{(k)}}{k!} (\psi + z)^k d\tau. \tag{28}$$

We next apply Taylor's expansion again to polynomial $\mathcal{P}_c$

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c)(\tau) d\tau + \sum_{k=1}^{m} \int_0^t \frac{[\mathcal{P}_c(\varphi_c)]^{(k)}}{k!} \chi^k d\tau + \tilde{z}(t),$$

where m is the degree of $\mathcal{P}$. Using (20), we derive

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c) d\tau + \int_0^t G(\varphi_c) d\tau + \mathcal{R}(t),$$

where

$$\mathcal{R}(t) = \tilde{z}(t) + \mathcal{O}(\varepsilon^{1-\kappa_0}). \tag{29}$$

To bound the error $\mathcal{R}$, we take the $\mathbb{E}\sup_{t\in[0,\tau^*]} \|\cdot\|_r^p$ on both sides of (28).

$$\mathbb{E} \sup_{[0,\tau^*]} \|\tilde{z}\|_r^p \le C \sum_{k=1}^{m} \mathbb{E} \sup_{[0,\tau^*]} \left\| \int_0^t \frac{[\mathcal{P}_c(\varphi_c + \chi)]^{(k)}}{k!} (\psi + z)^k d\tau \right\|_r^p.$$

Using Lemmas 1 and 2, Assumption 1, and the theorem of Burkholder–Davis–Gundy (cf. Theorem 1.2.4 in [43]), the equation yields

$$\mathbb{E} \sup_{[0,\tau^*]} \|\tilde{z}\|_r^p \le C\varepsilon^{1-(2m-1)\kappa}. \tag{30}$$

Substituting (30) into (29) yields (23). □

Lemma 7. *Assume that Assumption 1 is satisfied. Define $\xi(t)$ in $\mathcal{H}^c$ as a solution of (3) with $\mathbb{E}|\xi(0)|^p \le C$. Then, for all $T_0 > 0$ there is $C > 0$ such that*

$$\mathbb{E} \sup_{[0,T_0]} |\xi(t)|^p \le C. \tag{31}$$

Proof. By taking the scalar product $\langle \cdot, \xi \rangle$ on both sides of (3), we derive

$$\frac{1}{2} d_t |\xi|^2 = \langle \mathcal{P}_c(\xi) + G(\xi), \xi \rangle.$$

Using Equation (8) yields

$$\frac{1}{2} d_t |\xi|^2 \leq C|\xi|^2.$$

By integrating from 0 to t, we have

$$|\xi(t)|^2 \leq |\xi(0)|^2 + 2C \int_0^t |\xi(\tau)|^2 d\tau.$$

By applying Gronwall's lemma, we attain for $t \in [0, T_0]$

$$|\xi(t)| \leq |\xi(0)| e^{CT_0}. \tag{32}$$

We take the expectation on both sides after the supremum for Equation (32) to obtain (31). □

In fact, we cannot control the error terms that are defined in terms of φ_s or φ_c. Therefore, we are limited to a sufficiently large subset of Ω where all our estimates of errors are true.

Definition 3. *Define the set $\Omega^* \subset \Omega$ so that all of these estimations are included:*

$$\sup_{[0,\tau^*]} \|\varphi_s\|_r < \varepsilon^{-\kappa_0 - \frac{1}{2}\kappa}, \tag{33}$$

$$\sup_{[0,\tau^*]} \|\varphi_s - \chi - \mathbb{T}_r(\varepsilon^{-2} D s) \varphi_s(0)\|_r < \varepsilon^{1 - m\kappa - \kappa}, \tag{34}$$

$$\sup_{[0,\tau^*]} |\sim| < \varepsilon^{-\frac{1}{2}\kappa}, \tag{35}$$

and

$$\sup_{[0,\tau^*]} \|\mathcal{R}\|_r < \varepsilon^{1 - 2m\kappa}, \tag{36}$$

are valid on Ω^*.

As shown below, the set Ω^* has a probability of nearly to one.

Proposition 1. *Assume that Assumptions 1 and 2 are satisfied. Then, Ω^* has probability*

$$\mathbb{P}(\Omega^*) \geq 1 - C\varepsilon^p. \tag{37}$$

Proof. We notice that

$$\mathbb{P}(\Omega^*) \geq 1 - \mathbb{P}(\sup_{[0,\tau^*]} \|\varphi_s\|_r \geq \varepsilon^{-\kappa_0 - \frac{1}{2}\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} \|\varphi_s - \chi - \mathbb{T}_r(\varepsilon^{-2} D s) \varphi_s(0)\|_r \geq \varepsilon^{1 - \kappa(m+1)})$$

$$-\mathbb{P}(\sup_{[0,\tau^*]} |\xi| > \varepsilon^{-\frac{1}{2}\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} \|\mathcal{R}\|_r \geq \varepsilon^{1 - 2m\kappa}).$$

By first using the Chebychev inequality and afterwards using Lemmas 1, 6, and 7, and Corollary 1, we derive

$$\mathbb{P}(\Omega^*) \geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{q\kappa} + \varepsilon^{q\kappa}] \geq 1 - C\varepsilon^{q\kappa} \geq 1 - C\varepsilon^p.$$

□

Theorem 1. *Let Assumption 2 hold. Assume that ξ and φ_c are solutions of (3) and (21), respectively, with $\xi(0) = \varphi_c(0) = \mathcal{O}(1)$. Then,*

$$\sup_{t\in[0,\tau^*]} \|\varphi_c(t) - \xi(t)\|_r \le C\varepsilon^{1-2m\kappa}\,, \tag{38}$$

and

$$\sup_{t\in[0,\tau^*]} \|\varphi_c(t)\|_r \le C\varepsilon^{-\frac{1}{2}\kappa}\,. \tag{39}$$

Proof. By subtracting (3) from (21), we derive

$$\varphi_c(t) - \xi(t) = \int_0^t \mathcal{P}_c(\varphi_c) - \mathcal{P}_c(\xi)d\tau + \int_0^t G(\varphi_c) - G(\xi)d\tau + \mathcal{R}(t).$$

Let $\mathbb{F}(u) = \mathcal{P}_c(u) + G(u)$ to have

$$\varphi_c(t) - \xi(t) = \int_0^t [\mathbb{F}(\varphi_c) - \mathbb{F}(\xi)]d\tau + \mathcal{R}(t).$$

Now, define $\Theta = \varphi_c - \xi - \mathcal{R}$ to obtain

$$\Theta(t) = \int_0^t [\mathbb{F}(\Theta + \xi + \mathcal{R}) - \mathbb{F}(\xi)]d\tau.$$

Thus,

$$d_t\Theta(t) = \mathbb{F}(\Theta + \xi + \mathcal{R}) - \mathbb{F}(\xi).$$

By taking the scalar product $\langle\cdot,\Theta\rangle$ on both sides and using Assumption 2, we have

$$\begin{aligned} \frac{1}{2}d_t|\Theta(t)|^2 &= \langle\mathbb{F}(\Theta + \xi + \mathcal{R}) - \mathbb{F}(\xi), \Theta\rangle \\ &\le |\Theta(t)|^2 + |\mathcal{R}(t)|^2 + |\mathcal{R}(t)|^m \\ &\le |\Theta(t)|^2 + C\varepsilon^{2-4m\kappa} \quad \text{on } \Omega^*. \end{aligned}$$

By using Gronwall's lemma, we obtain

$$\sup_{[0,\tau^*]}|\Theta| \le C\varepsilon^{1-2m\kappa} \quad \text{on } \Omega^*.$$

We finish the first part by using

$$\sup_{[0,\tau^*]}|\varphi_c - \xi| \le \sup_{[0,\tau^*]}|\Theta| + \sup_{[0,\tau^*]}|\mathcal{R}| \le C\varepsilon^{1-2m\kappa}.$$

For the second part, consider

$$\begin{aligned} \sup_{[0,\tau^*]}|\varphi_c| &\le \sup_{[0,\tau^*]}|\varphi_c - \xi| + \sup_{[0,\tau^*]}|\xi| \\ &\le C\varepsilon^{1-2m\kappa} + C\varepsilon^{-\frac{1}{2}\kappa} \\ &\le C\varepsilon^{-\frac{1}{2}\kappa}, \end{aligned}$$

for $\kappa < \frac{1}{2m}$, where we used the first part and (35). □

Theorem 2. *(Approximation: suppose that Assumptions 1–3 are satisfied. Let φ be a solution of (1) with splitting $\varphi = \varphi_c + \varphi_s$ as defined in (12). Additionally, let ξ be a solution of (3) with $\xi(0) = \varphi_c(0)$. Then, for all $\kappa \in (0, \frac{1}{2m})$ and $T_0 > 0$, there is $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{t\in[0,T_0]} \Big\|\varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\Big\|_r > \varepsilon^{1-2m\kappa}\Big) \leq C\varepsilon^p, \tag{40}$$

for all $p > 0$.

Proof. We notice that for τ^*:

$$\Omega^* \subseteq \{\sup_{[0,\tau^*]} \|\varphi_c\|_r < \varepsilon^{-\kappa}, \sup_{[0,\tau^*]} \|\varphi_s\|_r < \varepsilon^{-\kappa}\} \subseteq \{\tau^* = T_0\} \subset \Omega.$$

By using Equation (12) and the triangle inequality, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T_0]} \|\varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r \\
&\quad= \sup_{t\in[0,\tau^*]} \|\varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r \\
&\quad\leq \sup_{[0,\tau^*]} \|\varphi_c - \xi\|_r + \sup_{[0,\tau^*]} \|\varphi_s - \chi - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r \\
&\quad\leq C\varepsilon^{1-2m\kappa} \quad \text{on } \Omega^*.
\end{aligned}$$

where we used (34) and (38). Hence,

$$\mathbb{P}\Big(\sup_{t\in[0,T_0]} \|\varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r > C\varepsilon^{1-2m\kappa}\Big) \leq 1 - \mathbb{P}(\Omega^*)\,.$$

By using (37), the equation yields (40). □

6. Application

Throughout chemistry, physics, biology, and other fields of reaction-diffusion equations with non-linearities of polynomials, there are many models in which the main theory of approximation is applied; for example, consider Fisher's and Fitzhugh–Nagumo equations in biology and the real-valued Ginzburg–Landau equation in physics. Here, we are looking at two models, namely one from physics and the other from biology, as follows.

6.1. Physical Example

The first example is the Ginzburg–Landau equation [20]. The Ginzburg–Landau equation is used for modeling a wide variety of physical systems. Additionally, it was first formulated in the sense of pattern formation as a long-wave amplitude equation in the case of convection in binary mixtures close to the onset of instability. The fractional space Ginzburg–Landau equation with additive noise is

$$d\varphi = [-\varepsilon^{-2}(-\Delta)^{\frac{r}{2}}\varphi + \varphi - \varphi^3]dt + \varepsilon^{-1}dW \quad \text{for } t \geq 0, \tag{41}$$

where the variable $\varphi(t,x)$ is a real-valued function of t and x.

To check Assumption 1, we note that $\mathcal{P}(\varphi) = \varphi - \varphi^3$ and then for $r > \frac{1}{2}$

$$\begin{aligned} \|\mathcal{P}(\varphi)\|_r &= \left\|\varphi - \varphi^3\right\|_r \leq \|\varphi\|_r + \left\|\varphi^3\right\|_r \\ &\leq \frac{2}{3} + \frac{1}{3}\|\varphi\|_r^3 + \|\varphi\|_r^3 \\ &\leq \frac{4}{3}(1 + \|\varphi\|_r^3), \end{aligned}$$

where we used the Young inequality.

Moreover, we use (22) with $k = 1$ to obtain

$$G(\xi) = \sum_{j=1}^{N} \frac{3\alpha_j^2}{2j^r}\xi,$$

Hence, the limiting equation is

$$d\xi = [\Big(1 - \sum_{j=1}^{N} \frac{3\alpha_j^2}{2j^r}\Big)\xi - \xi^3]dt. \tag{42}$$

Now, the solution of (41) by our main theorem is approximated by

$$\varphi(t,x) \simeq \xi(t) + \chi(t,x),$$

where ξ is a solution of (42) and χ is defined in (1). If we suppose that the noise acts only in one mode, i.e., $W(t) = \alpha_j \beta_j \cos(jx)$, then Equation (42) takes the form

$$d\xi = [\Big(1 - \frac{3\alpha_j^2}{2j^r}\Big)\xi - \xi^3]dt. \tag{43}$$

If we choose α_j such that $\alpha_j^2 < \frac{2j^r}{3}$ for $r \in (1,2]$, then the term $(1 - \frac{3\alpha_j^2}{2j^r})$ is negative. We may say, in this case, that the dynamics of the dominant modes were stabilized by the degenerated additive noise.

6.2. Biological Example

The second example is Fisher's equation [21]. Fisher's equation becomes one of the most important types of non-linear equations due to its existence in many chemical and biological processes. Fisher's equation with fractional space and, by being forced by additive noise, takes the form

$$d\varphi = [-\varepsilon^{-2}D(-\Delta)^{\frac{r}{2}}\varphi + A\varphi(1 - \frac{\varphi}{K})]dt + \varepsilon^{-1}dW, \tag{44}$$

where A and K are positive constants. Here, $\varphi(t,x)$ describes the evolution of the state over the spatial–temporal domain defined by the coordinates t and x, respectively.

Our main theory shows that the approximate solution of Fisher's Equation (44) is

$$\varphi(t,x) = \xi(t) + \chi(t,x) + error,$$

where ξ is the solution of

$$d\xi = [A\xi(1 - \frac{\xi}{K}) - \frac{A}{2DK}\sum_{j=1}^{N}\frac{\alpha_j^2}{j^r}]dt.$$

7. Conclusions

In this article, we obtained the approximation solutions of stochastic fractional-space diffusion equations via the solutions of ordinary differential equations, which are called limiting equations. This equation has never been studied before using a combination of additive noise and fractional-space. We applied our results to many example such as Fisher's equation and the Ginzburg–Landau models. Additionally, we discussed the influence of degenerate additive noise on the stabilization of the approximate solutions. These solutions are of considerable importance in understanding many important complex physical phenomena as fractional diffusion equations arise in the modeling of turbulent flow, contaminant transport in groundwater flow, and chaotic dynamics of classical conservative systems.

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