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Article

Does a Least-Preferred Candidate Win a Seat? A Comparison of Three Electoral Systems

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Abstract: In this paper, the differences between two variations of proportional representation (PR), open-list PR and closed-list PR, are analyzed in terms of their ability to accurately reflect voter preference. The single nontransferable vote (SNTV) is also included in the comparison as a benchmark. We construct a model of voting equilibria with a candidate who is least preferred by voters in the sense that replacing the least-preferred candidate in the set of winners with any loser is Pareto improving, and our focus is on whether the least-preferred candidate wins under each electoral system. We demonstrate that the least-preferred candidate never wins under the SNTV, but can win under open-list PR, although this is less likely than winning under closed-list PR.

Keywords: open list; closed list; proportional representation; single nontransferable vote; voting equilibria

JEL classifications: D72

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1. Introduction

Proportional representation (PR) is used in national elections in many countries. Under *open-list* PR, voters can affect which candidates fill the seats allocated to each party list, whereas this is not the case under *closed-list* PR. As reported by Kunicova and Rose-Ackerman [1], as of 1997, 53 out of 99 democracies use PR to elect at least half of their lower house members. Of the 53 countries, 37 use closed-list PR, while 16 use open-list PR.[1] However, there has been little formal analysis of the distinction between open-list PR and closed-list PR, because the focus of the literature has been on the differences between the majoritarian and the PR systems. Moreover, it is relatively difficult to deal with the order of candidates in party lists.[2] The goal of this paper is to construct a formal model that enables us to deal with the order of candidates in party lists and clarify the differences between open-list PR and closed-list PR in terms of their ability to accurately reflect voter preference.

Elections under closed-list PR consist of two stages. First, parties determine the order of candidates in their party lists. Second, given the order of candidates in each party list, voters cast their ballots for one of the parties, which determines how many seats each party obtains. These seats are then allocated to candidates of each party from the top of the list. Under open-list PR, on the other hand, parties do not have perfect control over the order of candidates in their lists, but the determination of winners in each party list is affected by voters. Open-list PR has some variations with respect to the extent to which voters can affect the winners in each party list. To contrast the differences between open-list PR and closed-list PR, we consider an extreme variation of open-list PR, under which parties are not allowed to choose the order of their candidates. That is, given the lists of candidates without ranking, voters cast their ballots for one of the parties, which determines how many seats each party obtains. Among the candidates on the list of the party they choose, voters can also cast their personal ballots. These personal ballots determine which of the candidates on the list fill the seats allocated to their party.

As a benchmark, we also include the *single nontransferable vote* (SNTV) in our comparison of electoral systems. As PR does, the SNTV also elects more than one candidate in each electoral district, but the rules of SNTV are much simpler than those of the two variations of PR because neither the parties nor the transfer of votes affects the election results. Under the SNTV, candidates run independently of parties. Each voter has only one vote, and voters cast their ballots for one of the candidates. As many candidates as the number of seats win under the plurality rule. In summary, given the set of candidates, both the SNTV and open-list PR are modeled as one-shot games played among voters, whereas closed-list PR consists of two stages, in which parties rank their candidates in their lists, and then voters vote for a party list given the order of candidates on each list.

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1 For the classification, Kunicova and Rose-Ackerman [1] used the World Bank's Database on Political Institutions and the index provided by Freedom House Annual Surveys. Here, "democracies" are defined as countries that achieved average scores below 5.5 in the Freedom House Annual Surveys taken during the years 1992/93 to 2000/01. This score is a measure of political freedom that takes values between 1 (most free) and 7 (least free).

2 See Persson and Tabellini [2] for a survey of the formal analysis of why electoral systems are important (Ch. 8). They consider the effect of the ranks of candidates in a closed list, which are exogenously given, on their policy choices through their winning probabilities (Ch. 9). Bawn and Thies [3] examine a similar effect using a decision theoretic model. Magar, Rosenblum, and Samuels [4] constructed a Downsian model of open-list PR with two parties and two candidates per party to analyze which policy platform each candidate chooses. Studies on "personal vote" provide a way of distinguishing the variations of PR. Namely, electoral systems are classified into two groups in terms of whether parties have control over which of their candidates win seats (e.g., single-member district systems and closed-list PR) or not (e.g., majoritarian systems with primary elections, the single nontransferable vote, and open-list PR). Candidates' incentives to cultivate their personal reputation among their constituencies (Cain, Ferejohn, and Fiorina [5]) and the degree of internal disunity of parties (Katz [6]) under each category of electoral system are analyzed empirically in the literature.

Does allowing voters to write in a candidate's name as well as a party's name improve the reflection of voter preference? The answer is not obvious for the following two reasons. First, a larger choice set for voters under open-list PR (*i.e.*, party names and candidate names) than closed-list PR (*i.e.*, only party names) does not necessarily imply the superiority of open-list PR to closed-list PR in reflecting voter preference. If this were merely a decision problem in which one person could affect the outcome as he/she liked, a larger choice set would not adversely affect the reflection of his/her preference. However, since an election is a collective decision-making event involving multiple voters, when the choice set is larger, some "strange" equilibria may be supported by some equally "strange" but self-enforcing expectations. Second, in determining the order of candidates under closed-list PR, parties are competing for seats with each other. Hence, if parties could win more seats by ranking more-preferred candidates higher in their lists, closed-list PR might yield electoral outcomes that would better reflect voter preference than would open-list PR. Formal analyses are helpful in resolving this ambiguity.

In order to clearly measure how well voter preference is reflected, we introduce a least-preferred candidate in our model, from whom every voter receives the lowest utility, so that replacing this candidate in the set of winners with any loser is Pareto improving. We focus on whether this candidate wins under each electoral system. As an equilibrium concept in the voting stage, we define voting equilibrium, which is based on, but different from, the definition given by Myerson and Weber [7]. As explained in detail in Section 2, voters perceive how likely a close race is to occur between each pair of candidates and cast their ballots to maximize their expected utilities achieved by the electoral outcome.

The following results are obtained from our analysis. The least-preferred candidate never wins under the SNTV, wins with high probability under open-list PR only when a portion of voters use weakly dominated strategies, and wins with high probability under closed-list PR even though no one uses weakly dominated strategies.[3] Therefore, in our specification of voter preference toward candidates, electoral outcomes realized under the SNTV are never Pareto inferior to those realized under open-list or closed-list PR. Although the Pareto relationship of outcomes between the two variations of PR depends on which of multiple equilibria is realized, the least-preferred candidate is regarded to be less likely to win under open-list PR than under closed-list PR if we compare them with respect to the nonuse of weakly dominated strategies.

The reasoning behind our results is as follows. The difference in rules between the SNTV and PR lies in whether seats are allocated based on individual candidates or based on parties. Since the least-preferred candidate cannot attract voters by himself/herself, he/she never wins under the SNTV. Under PR, on the other hand, popular candidates on the same party list as the least-preferred candidate attract voters to their party, which enables the least-preferred candidate to win a seat. The difference in rules between open-list PR and closed-list PR is whether the order of candidates in each party list is determined by votes or by parties. In the electoral outcomes under open-list PR, popular candidates

[3] In Myerson's [8] voting model with corruption, the presence of another policy dimension as well as the corruption level enables corrupt parties (or candidates) to win seats under the plurality rule and the Borda rule. In our model, on the other hand, the presence of a popular candidate in the same party enables the least-preferred candidate to win a seat under PR. Myerson [9] also shows that a least-preferred candidate can win a seat under negative voting in a large Poisson game.

can be, by votes, either ranked high and on the borderline between winning and losing or ranked low and on the borderline. Since candidates above the borderline win seats while candidates below the borderline lose seats, parties win more seats when their popular candidates are, by votes, ranked lower and on the borderline. Under closed-list PR, therefore, parties rank their popular candidates lower. As a result, the least-preferred candidate is ranked higher and wins a seat with high probability. Note that such a strategy to rank popular candidates lower is successful in attracting more voters to the party only if those candidates are sufficiently popular. In other words, the least-preferred candidate wins with high probability under closed-list PR, but requires sufficient popularity of other candidates on the same party list.

Our analysis is related to the following two streams of strategic-voting models. First, with respect to the electoral systems considered herein, that is, a multimember district of the SNTV and PR, our study is related to studies by Cox [10] and Cox and Shugart [11]. They focused on how many candidates attract votes, whereas we focus on what types of candidates win. Cox [10] considered a multimember district and formally derived the $M+1$ *rule* (Reed [12]), whereby in an M-member district, exactly $M+1$ candidates obtain votes under the SNTV. Moreover, he assumed additive utility so that the payoff of each voter from each set of M winning candidates is calculated as the sum of utilities that he/she receives from each winning candidate, which simplifies the analysis of multimember districts. Cox and Shugart [11] extended Cox's [10] model to the largest-remainders PR, which includes the plurality rule and the SNTV as special cases. They showed that the number of lists which attract votes under the largest-remainders PR is also bounded above by $M+1$ as a result of the strategic behavior of voters. Their results are in contrast to the view of Duverger, who suggested that strategic voting would not appear under PR.

Second, the model-building strategy proposed herein is based on the research of Myerson and Weber [7] and Gerber, Morton and Rietz [13]. Myerson and Weber [7] introduced voter perception of the probability of close races between each pair of candidates and required consistency between the perception and the electoral outcome. Their method simplifies the procedure for solving voting models by enabling us to avoid calculating the complicated forms of exact *pivot probabilities* (*i.e.*, probabilities of each vote affecting the outcome). They compared electoral outcomes under the plurality rule, approval voting, and the Borda rule, and showed that no minority candidate wins a seat with certainty only under approval voting. As Cox [10] did, Gerber, Morton and Rietz [13] assumed additive utility and extended Myerson and Weber's [7] model to a two-member district with two majority candidates and one minority candidate. They focused on whether the minority candidate wins a seat under straight voting and cumulative voting.[4] They showed that under straight voting there exists an equilibrium in which the minority candidate never wins a seat, while under cumulative voting he/she always wins with positive probability but is less likely to win when voters supporting the minority candidate prefer one majority candidate over another.

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[4] Under cumulative voting, each voter has two votes and is allowed to cast two votes for one candidate. Under straight voting, each voter is required to split the two votes between two candidates. Under both rules, casting only one vote is also allowed. Gerber, Morton and Rietz [13] also conducted laboratory experiments to test the theoretical predictions.

We construct a model of a two-member district with four candidates in order to compare the SNTV, open-list PR, and closed-list PR. Under PR, each two-candidate set constitutes a party. Under closed-list PR, a stage of party decision making regarding the order of their candidates is introduced before voting. Our model is related to the models proposed by Myerson and Weber [7] and Gerber, Morton and Rietz [13], but we use a different equilibrium criterion, as explained in Section 2. We focus on whether the least-preferred candidate wins a seat.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 obtains three propositions to compare the outcomes under the three electoral systems. The conclusion is presented in Section 4. Finally, proofs of the propositions are provided in the Appendix.

2. The Model

2.1. Preferences

We consider a two-member district in which four candidates, referred to as candidates 1, 2, 3, and 4, compete. Candidates 1 and 2 constitute the left-wing party, while candidates 3 and 4 constitute the right-wing party. There are three types of voters, a, b, and c, into which a finite number of voters are equally divided. The voters have preferences regarding the four candidates specified in Table 1. The rows represent candidate names, whereas the columns represent the types of voters. For example, the first row describes the type-a voter preference. Type-a voters receive utilities of 1, a_2, a_3, and 0 if candidates 1, 2, 3, and 4, respectively, win. We normalize voter utility by measuring it from 0 to 1, and so the parameter values of a_2, a_3, b_1, b_3, c_1, and c_2 are strictly between 0 and 1. Later, we examine what types of equilibria exist for each set of parameter values. Following Cox [10] and Gerber, Morton and Rietz [13], we assume additivity of utility. For example, if candidates 1 and 2 win seats, type-c voters receive a utility of $c_1 + c_2$.

Table 1. Voters' Preferences Regarding Candidates.

		Candidates			
		1	2	3	4
Voters' Types	a	1	a_2	a_3	0
	b	b_1	1	b_3	0
	c	c_1	c_2	1	0

In Table 1, we impose the following features on the preferences. Each type of voter receives zero utility from candidate 4, while he/she receives positive utility from the other three candidates. In addition, candidates 1, 2, and 3 are similarly preferred by voters in the sense that each candidate has one type of voter who receives the maximal utility by his/her winning. These features differentiate candidate 4 from others, so that replacing him/her in the set of winners with any loser is Pareto improving.[5] We call him/her the least-preferred candidate among the four candidates and focus on whether he/she wins under each electoral system despite such preferences.

2.2. Voting Equilibria

Following Myerson and Weber [7] and Gerber, Morton and Rietz [13], we specify voter decision making as follows. At the beginning, for each pair of candidates, say i and j, where $i \neq j$, each voter perceives the probability that the pair of candidates are in a close race for the second seat, denoted by p_{ij}. Note that close races for the first seat do not matter for voters because the second-place candidate also wins a seat. As mentioned by Myerson and Weber [7], the perceived probabilities of close races for the second seat between each pair of candidates are dealt with like a price vector in the general equilibrium model of competitive markets in microeconomic theory. That is, each voter decides for which candidate he/she will vote or abstains, given a perception of close-race probabilities, and voters' behaviors in fact realize an electoral outcome which is consistent with the perception.[6]

Mathematically, given a vector of close-race probabilities $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$, each voter chooses a vote vector (v_1, v_2, v_3, v_4) to maximize the following objective function:

$$\sum_{i=1}^{4} v_i \Big(\sum_{j \neq i} p_{ij}(u_i - u_j) + \theta \sum_{l \neq i,k} p_{kl}(u_k - u_l) \Big)$$

where v_i represents the number of votes the voter casts for candidate i, u_i represents the utility he/she receives from candidate i's winning, and $k = i + 1$ if $i = 1, 3$ while $k = i - 1$ if $i = 2, 4$.[7] The term containing $\theta \in \{0, 1\}$ expresses the transfer of votes from candidate i to candidate k in a party. As expressed by the subtraction $u_i - u_j$, the objective function approximately describes the increment of a voter's expected utility from voting for a candidate or a party.[8] Each electoral system is characterized by

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5 Suppose that one of two winners is candidate 4. All types of voters receive zero utility from the winning of candidate 4. If candidates 4 is replaced with candidate 1 in the set of winners, for example, then the utilities of types a, b, and c increase by 1, b_1, and c_1, respectively. Since the utilities of all types of voters increase without decreasing the utility of any type of voter, this replacement is Pareto improving. A similar calculation applies to the replacement of candidate 4 with candidates 2 and 3. Note that every outcome in which candidate 4 does not win is Pareto efficient because each of candidates 1, 2, and 3 is most preferred by one of the three types of voters.

6 In the general equilibrium model, we first derive how many units of each commodity each consumer (firm, respectively) buys (sells) according to the price vector of commodities. We then find a price vector for which the payoff-maximization behaviors of consumers and firms equate demand and supply in each market. Here, we first examine how each type of voter votes according to the vector of close-race probabilities. We then find a vector of close-race probabilities for which the payoff-maximization behaviors of voters result in an electoral outcome that is consistent with the close-race probabilities.

7 Here, we omit the index of each voter to simplify the expressions.

8 If close-race probabilities, which are only perceived by voters in their decisions, are replaced with pivot probabilities (*i.e.*, actual probabilities of each vote affecting the outcome, which are calculated based on the strategies other voters choose), the objective function then becomes the exact, rather than approximate, expression of the increment of a voter's expected utility. The pivot probability of a vote for candidate i changing the winner from candidate j to candidate i is different from the pivot probability of a vote for candidate j changing the winner from candidate i to candidate j: the former (latter,

the set of vote vectors each voter can choose, the value of θ in his/her objective function, the manner in which votes are converted into seats, and the role of parties.

A voting equilibrium consists of (1) a profile of voter strategies that maximize the above objective function for each type of voter, given a vector of close-race probabilities; and (2) a vector of close-race probabilities that satisfies the following two conditions. First, the sum of the close-race probabilities must be one ($\sum p_{ij} = 1$). This condition is imposed only to avoid a trivial equilibrium in which voters vote arbitrarily perceiving $p_{ij} = 0$ for all i and j.[9] Second, the perception of close-race probabilities must be common to all voters[10] and must be consistent with the electoral outcome induced by voters' payoff-maximization behaviors based on the close-race probabilities.[11] In order to maintain the consistency between the perception and the outcome, we impose the following three requirements: (i) if a tie for the second seat occurs with positive probability between two distinct candidates i and j, then voters must perceive $p_{ij} > 0$; (ii) if a tie for the second seat is more likely to occur between two distinct candidates i and j than between two other distinct candidates k and l, then voters must perceive $p_{ij} > p_{kl} > 0$; and (iii) if and only if ties never occur, voters perceive arbitrary close-race probabilities, although voters must perceive $p_{ij} = 0$ when candidates i and j can never be in a tie for the final seat under the electoral rules irrespective of the behaviors of voters.

We distinguish "close races" and "ties" as follows. Close races are perceived by voters in their decisions, whereas ties are created by votes in the election. Whether or not ties actually occur, voters must perceive the positive probabilities of close races in their making decisions (*i.e.*, the sum of close-race probabilities must be one). Even when ties actually occur, the probabilities of the ties occurring between each pair of candidates are not necessarily the same as the close-race probabilities voters perceive between each pair of candidates. In fact, requirement (i) determines only whether each close-race probability is positive or zero. Requirement (ii) also determines only the relative values of close-race probabilities.

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respectively) is the sum of the probability that candidate i is one vote behind (ahead) candidate j and the probability that candidates i and j obtain the same number of votes, multiplied by $1/2$. Such a distinction is absent if candidates' competitions are described with close-race probabilities.

[9] This condition requires voters to perceive that at least one close race occurs. Myerson and Weber [7] interpret such close-race probabilities as probabilities of each vote affecting the outcome conditional on the occurrence of close races for the second seat. In other words, when voters make voting decisions, they care about how their votes would affect the outcome if their votes affected the outcome. Note that only the large/small relationship between close-race probabilities matters in voting decisions.

[10] We may allow voters to perceive different close-race probabilities from each other to some extent as long as the consistency between the close-race probabilities and the electoral outcome is maintained. However, allowing differences in perception of close-race probabilities between voters enlarges the set of close-race probabilities that are consistent with each electoral outcome. It also requires us to specify, with a reasonable criterion, the extent to which the perception of close-race probabilities is allowed to differ between voters. In order to avoid such problems and simplify our model by reducing the number of possible profiles of close-race probabilities, we require the close-race probabilities to be common to all voters.

[11] The consistency between perceptions (or beliefs) and outcomes is required in most theories that assume the rationality of decision makers. In perfect Bayesian equilibria of dynamic games with incomplete information, for example, to each decision node in each information set, players assign (as a belief) a probability of themselves being on that decision node, and these probabilities must be consistent with the probabilities calculated by Bayes' rule based on all players' strategies that maximize their respective payoffs based on the belief.

We refer to voting equilibria in which ties occur as *tie equilibria*, whereas voting equilibria in which ties do not occur are referred to as *no-tie equilibria*. When ties occur, voting equilibria require the close-race probabilities voters perceive to be consistent with the ties, as specified by requirements (i) and (ii). On the other hand, when ties do not occur, it is difficult to specify the consistency between close-race probabilities and ties. In such cases, voting equilibria allow voters to perceive close-race probabilities arbitrarily, as mentioned in requirement (iii), while the only consistency imposed on such cases is that voters' behaviors based on the arbitrary perception of close-race probabilities create no ties.[12]

The possibility of ties between candidates provides an opportunity for each vote to affect the outcome, and hence it is the key for rational voters, who try to maximize the increment of their expected utilities by affecting the electoral outcome, to make voting decisions seriously. In the literature on game-theoretic analysis of voting, particularly that by Palfrey and Rosenthal [14], formal models have been constructed so that ties between candidates occur with positive probability.[13] Although our voting equilibrium allows no ties to occur, as indicated by requirement (iii), we mention no-tie equilibria only briefly and focus on tie equilibria in our analysis because tie equilibria are in line with the previous literature due to their consistency between perceptions and outcomes. In no-tie equilibria, various types of electoral outcomes are obtained due to the arbitrariness of perception, but the basic properties of no-tie equilibria with respect to the winning probability of the least-preferred candidate under the three electoral systems are revealed to be similar to those of tie equilibria.

Myerson and Weber's [7] *ordering condition* imposed on their voting equilibrium is defined as follows if it is applied to a two-member district of the SNTV. Given an election result (*i.e.*, an aggregate behavior of voters) and any $\epsilon \in [0, 1)$, close-race probabilities satisfy the ordering condition for ϵ if, for every three distinct candidates i, j, and h, inequality $p_{ih} \leq \epsilon p_{jh}$ holds if the expected number of votes for candidate i is smaller than that for candidate j, where j is a candidate other than the candidate who obtains the largest number of votes. The requirement for the consistency between perceptions and outcomes in our

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[12] The idea of arbitrary perception of close-race probabilities in no-tie equilibria comes from the belief players hold off equilibrium paths in dynamic games with incomplete information. That is, if an information set is not reached in a perfect Bayesian equilibrium, the player who makes decisions based on this information set is allowed to assign arbitrary probabilities to each decision node in the information set as long as the sum of the probabilities is one.

[13] Ties between candidates are created in several ways. In Palfrey and Rosenthal's [14] model with fixed voting costs, voters randomize between going to the polls and abstaining (*i.e.*, they choose mixed strategies) as their optimal behaviors, which creates the possibility of a tie between two candidates. In Palfrey and Rosenthal's [15] model with incomplete information about voting costs, in which voting costs are randomly determined for each voter, each voter goes to the poll if and only if his/her voting cost is sufficiently small. As a result, each voter goes to the poll with a probability between zero and one. In Schram and Sonnemans' [16] voting environment for laboratory experiments, two candidates have the same number of supporters, and voting costs are sufficiently small. Thus, under the plurality rule, all of the voters go to the polls, and the two candidates will be in a tie. In some models of referendums (e.g., Aguiar-Conraria and Magalhães [17]; Hizen and Shinmyo [18]), a fixed number of voters are randomly assigned one of two groups (*i.e.*, voters who want to keep the status quo and voters who want to change the current situation). In Myerson's [19] Poisson game, the total number of voters is also a random variable. Such uncertainty about the number of voters also creates a tie between two alternatives with positive probability.

model is different from theirs with respect to whether it is based on the possibility of ties or on the expected number of votes. There is no inclusion relationship between the two requirements.

To clarify this difference, suppose that, in a two-member district under the SNTV, candidates 1, 2, 3, and 4 are strictly first, second, third, and fourth, respectively, in their expected number of votes. The ordering condition requires $p_{23} = 1$. Under our requirement on tie equilibria, close-race probabilities depend on whether there is the possibility of a tie for the second seat between each pair of candidates. Hence, if some voters cast their ballots randomly among several candidates and if candidate 4 can be in a tie for the second seat with candidate 1 with positive probability, then voters perceive $p_{14} > 0$, even though the expected numbers of votes for candidates 1 and 4 do not approach the number of votes required to come in second place and third place.

We use our specification of the requirement for the following two reasons. First, as Myerson [9] points out, the ordering condition can generate equilibrium outcomes that are inconsistent with those generated by models that calculate exact pivot probabilities. Second, and more importantly, replacing our specification of the requirement with the ordering condition excludes some tie-equilibrium outcomes that are obtained under our specification of the requirement, but yields many other outcomes. Although this multiplicity of electoral outcomes under the ordering condition makes the difference between the two variations of PR less clear, it does not affect the primary results of our analysis.

2.3. Electoral Systems

Now, we describe each electoral system. We use the *d'Hondt method* to convert votes into seats under PR. Any ties are broken randomly.

Single Nontransferable Vote: Under the SNTV, parties play no role, and the game consists of only one stage of voting. Voters have one vote each and cast it simultaneously for one of the four candidates or abstain. Mathematically, the strategy set of each voter is a probability distribution $[0, 1]^5$ in which the support is $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (0, 0, 0, 0)\}$. The top two candidates win seats. No votes are transferred between candidates, even when a candidate obtains more votes than required to win a seat. Therefore, the objective function of each voter has $\theta = 0$.

Open-List Proportional Representation: Under open-list PR, parties are not players but are used when votes are converted into seats. As in the SNTV, the game consists of one stage of voting. Voters have one vote each and cast it simultaneously for one of the four candidates or one of the two parties, or abstain. Since casting a vote for a party is equivalent to casting one half of a vote for each of the two candidates of that party, the strategy set of each voter is described as a probability distribution $[0, 1]^7$ in which the support is $\{(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(0.5, 0.5, 0, 0)$, $(0, 0, 0.5, 0.5)$, $(0, 0, 0, 0)\}$. The number of votes that a party obtains is calculated as the sum of the ballots for the party and the ballots for either candidate of the party. The expression used in this case is the sum of v_1s and v_2s for the left-wing party and the sum of v_3s and v_4s for the right-wing party. Votes for parties are converted into the allocation of seats to parties by the d'Hondt method. In our model, the number of seats a party wins is 0 if it obtains less than $1/3$ of all valid votes, 1 if it obtains between $1/3$ and $2/3$ of all valid votes, and 2 if it obtains more than $2/3$ of all valid votes. In each party, the candidate who obtains more personal votes (with his/her name) is assigned a seat first, if available. Then, if another seat is available for that party, the second candidate wins a seat.[14]

Under this rule, casting a vote for candidate 1, for example, also helps candidate 2 win a seat because such a vote helps the left-wing party win more seats, although it lowers the rank of candidate 2 in the left-wing party. That is, a vote for candidate 1 favors candidate 1 over candidates 2, 3, and 4, but favors candidate 2 over candidates 3 and 4. Therefore, the objective function of each voter has $\theta = 1$.

Closed-List Proportional Representation: Under closed-list PR, parties are also players. The game consists of two stages. In the first stage, the leaders of the two parties, who may or may not be candidates, determine the order of their candidates simultaneously for the purpose of maximizing the expected number of seats that each party will win. Mathematically, the strategy set is a probability distribution $[0, 1]^2$ in which the support is $\{(1, 2), (2, 1)\}$ for the left-wing party and $\{(3, 4), (4, 3)\}$ for the right-wing party, where (x, y) means that the party ranks candidate x first and candidate y second. Therefore, we have four subgames according to the combination of the order of candidates in two parties.

In the second stage, given the order of candidates determined in the first stage, voters simultaneously vote for either the left-wing party or the right-wing party, or abstain. Casting a vote for a party is equivalent to allowing each voter to cast one vote for each of the two candidates of that party under $\theta = 0$. Therefore, the strategy of each voter is described as a function $\{(1, 2), (2, 1)\} \times \{(3, 4), (4, 3)\} \to [0, 1]^3$, where $[0, 1]^3$ is a probability distribution in which the support is $\{(1, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 0)\}$.[15] The number of votes that each party obtains, which, in our expression, is the sum of v_1s and v_2s for the left-wing party and the sum of v_3s and v_4s for the right-wing party, is converted by the d'Hondt method into the allocation of seats to parties. In each party, the candidate ranked first is assigned a seat first, if available, and the second candidate wins a seat if there is another seat available for that party. The equilibrium concept applied to the entire game is subgame perfection. That is, in the first stage, given a voting equilibrium in each subgame, the order of candidates is determined as a Nash equilibrium between the two parties.

3. Results

In this section, we analyze what types of equilibrium outcomes are realized under each electoral system, and, in particular, whether the least-preferred candidate wins a seat. We first examine the one-shot games of SNTV and open-list PR, and then proceed to the two-stage game of closed-list PR. The present analysis focuses on tie equilibria. No-tie equilibria are mentioned briefly later herein, where the basic properties of electoral outcomes are shown to be similar between the two types of equilibria.

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[14] For rational voters in our model, this rule is equivalent to the following rule. Each voter writes in the name of a party on his/her ballot (party vote) and he/she *can also* choose one candidate from the list of the party that he/she has chosen (personal vote). Party votes are converted into the allocation of seats to parties, and personal votes are used only to determine which candidates fill the seats in each party list.

[15] Another way of formulating closed-list PR is to replace $\theta = 0$ and $\{(1, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 0)\}$ with $\theta = 1$ and $\{(0.5, 0.5, 0, 0), (0, 0, 0.5, 0.5), (0, 0, 0, 0)\}$. We use the former because the expression is simpler.

3.1. Single Nontransferable Vote

We begin with the following proposition on the electoral outcome under the SNTV:

Proposition 1. *Under the single nontransferable vote, there exists a unique tie equilibrium outcome. Three candidates* 1*,* 2*, and* 3*, two of whom win seats, compete for the second seat, and voters perceive that* $p_{12}, p_{13}, p_{23} > 0$ *and* $p_{14} = p_{24} = p_{34} = 0$*. Candidate* 4*, who is the least-preferred candidate, never wins.*

Since the least-preferred candidate is not preferred by any voter, he/she needs help to win a seat. Under the SNTV, however, no votes are transferred, and parties play no role. Hence, he/she never wins. Here, we derive a tie equilibrium that generates the electoral outcome described in the proposition. The proof of the nonexistence of other types of tie equilibrium outcomes is given in the Appendix.

Suppose that candidates 1, 2, and 3 compete for the second seat. Then, the close-race probabilities must be $p_{12}, p_{13}, p_{23} > 0$ and $p_{14} = p_{24} = p_{34} = 0$. We have consistency between these close-race probabilities and the electoral outcome if the three candidates equally share votes with positive probability, possibly one. Now, let us consider the case in which each type of voter votes for the candidate he/she prefers most; that is, type-a, type-b, and type-c voters vote for candidates 1, 2, and 3, respectively.

The payoff for type-a voters is

$$\begin{gathered} p_{12}(1-a_2) + p_{13}(1-a_3) \text{ if type-}a\text{ voters vote for candidate } 1; \\ -p_{12}(1-a_2) + p_{23}(a_2-a_3) \text{ if type-}a\text{ voters vote for candidate } 2; \\ -p_{13}(1-a_3) + p_{23}(a_3-a_2) \text{ if type-}a\text{ voters vote for candidate } 3; \\ 0 \text{ if type-}a\text{ voters vote for candidate } 4 \text{ or abstain.} \end{gathered}$$

Since voting for candidate 1 gives a strictly positive payoff for any parameter values, type-a voters neither vote for candidate 4 nor abstain. Either voting for candidate 2 or voting for candidate 3 (or both) results in a strictly negative payoff according to which of a_2 and a_3 is larger. Therefore, the necessary condition for type-a voters to vote for candidate 1 is

$$\begin{gathered} p_{12}(1-a_2) + p_{13}(1-a_3) \geq -p_{12}(1-a_2) + p_{23}(a_2-a_3) \text{ if } a_2 > a_3; \\ p_{12}(1-a_2) + p_{13}(1-a_3) \geq -p_{13}(1-a_3) + p_{23}(a_3-a_2) \text{ if } a_2 < a_3, \end{gathered}$$

and non-binding if $a_2 = a_3$.

Similarly, we can write the necessary conditions for type-b and type-c voters to vote for candidates 2 and 3, respectively, as follows. For type-b voters, it is

$$\begin{gathered} p_{12}(1-b_1) + p_{23}(1-b_3) \geq p_{13}(b_3-b_1) - p_{23}(1-b_3) \text{ if } b_3 > b_1; \\ p_{12}(1-b_1) + p_{23}(1-b_3) \geq -p_{12}(1-b_1) + p_{13}(b_1-b_3) \text{ if } b_3 < b_1, \end{gathered}$$

and non-binding if $b_1 = b_3$. For type-c voters, it is

$$\begin{gathered} p_{13}(1-c_1) + p_{23}(1-c_2) \geq p_{12}(c_1-c_2) - p_{13}(1-c_1) \text{ if } c_1 > c_2; \\ p_{13}(1-c_1) + p_{23}(1-c_2) \geq p_{12}(c_2-c_1) - p_{23}(1-c_2) \text{ if } c_1 < c_2, \end{gathered}$$

and non-binding if $c_1 = c_2$.

The gray hexagon $defghi$ in Figure 1 describes the range of close-race probabilities under which type-a, type-b, and type-c voters vote for candidates 1, 2, and 3, respectively, where $a_2 > a_3$, $b_3 > b_1$, and $c_1 > c_2$ are assumed. The height of the equilateral triangle is one, and each point in the triangle represents a vector of (p_{12}, p_{13}, p_{23}). If we take a point j, for example, then p_{12}, p_{13}, and p_{23} are measured by the distances from point j to each side, the sum of which is equal to one. As the values of a_2, b_3, and c_1 converge to one, which makes each necessary condition more difficult to satisfy, the three segments de, fg, and hi move monotonically toward the center. At the limit, the gray area shrinks, and we have $p_{12} = p_{13} = p_{23}$. The relationship between parameter values and the limit of the range of close-race probabilities that satisfy the necessary conditions for type-a, type-b, and type-c voters to vote for candidates 1, 2, and 3, respectively, is summarized in Table 2.

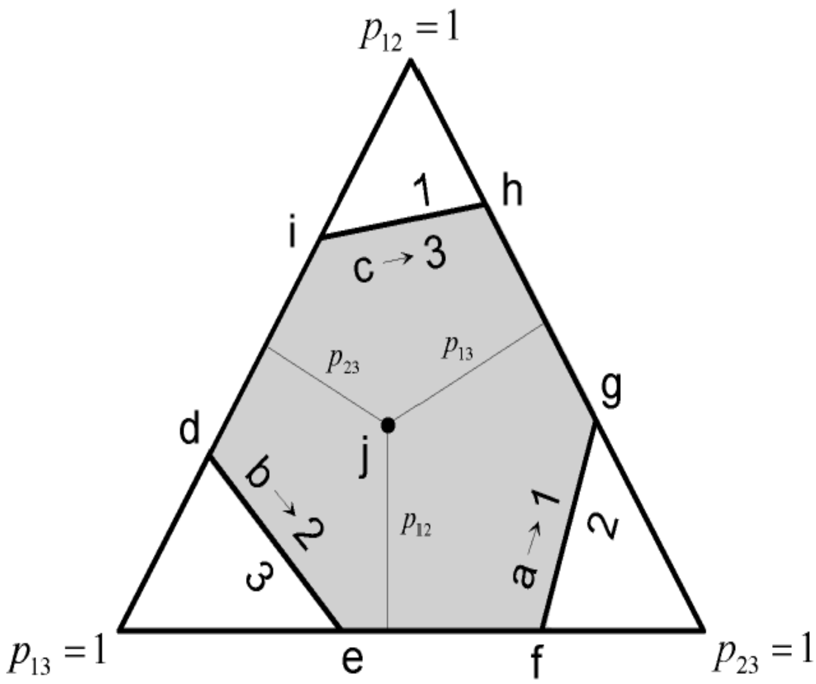

Figure 1. Equilibrium Close-Race Probabilities under the SNTV: Case of $a_2 > a_3$, $b_3 > b_1$, and $c_1 > c_2$.

In Table 2, the limit is taken by letting the larger parameter of each pair, a_2 and a_3, b_3 and b_1, and c_1 and c_2, converge to one in each necessary condition, so that each condition becomes more difficult to satisfy. Combined with the continuity of each necessary condition with respect to the parameters, this summary of the limit of close-race probabilities implies that, for any parameter values, there exists a vector of close-race probabilities that induces the above-described voting behaviors, for example, $p_{12} = p_{13} = p_{23} = 1/3$.[16] In other words, for any preferences regarding candidates, if voters perceive that close races are equally likely to occur between each pair from candidates 1, 2, and 3, then type-a,

[16] For parameter values other than the first and the last rows in Table 2, one of p_{12}, p_{13}, and p_{23} can be smaller than the other two at the limit. This indicates that the three-candidate equilibrium outcome described in Proposition 1 is relatively easy

type-b, and type-c voters vote for candidates 1, 2, and 3, respectively, as their optimal strategies, and such voting behaviors in fact create ties between each pair from candidates 1, 2, and 3. Therefore, this case is a tie equilibrium outcome for any parameter values.

Table 2. Close-Race Probabilities at the Limit of Parameter Values under the SNTV.

Parameter values	At the limit
$a_2 > a_3, b_3 > b_1, c_1 > c_2$	$p_{12} = p_{13} = p_{23}$
$a_2 > a_3, b_3 > b_1, c_1 < c_2$	$p_{12} = p_{13} \geq p_{23}$
$a_2 > a_3, b_3 < b_1, c_1 > c_2$	$p_{13} = p_{23} \geq p_{12}$
$a_2 > a_3, b_3 < b_1, c_1 < c_2$	$p_{13} = p_{23} \geq p_{12}$
$a_2 < a_3, b_3 > b_1, c_1 > c_2$	$p_{12} = p_{23} \geq p_{13}$
$a_2 < a_3, b_3 > b_1, c_1 < c_2$	$p_{12} = p_{13} \geq p_{23}$
$a_2 < a_3, b_3 < b_1, c_1 > c_2$	$p_{12} = p_{23} \geq p_{13}$
$a_2 < a_3, b_3 < b_1, c_1 < c_2$	$p_{12} = p_{13} = p_{23}$

3.2. Open-List Proportional Representation

The transfer of votes in a party under open-list PR generates other types of tie equilibria in addition to that realized under the SNTV. The following proposition summarizes the tie equilibrium outcomes under open-list PR:

Proposition 2. *Under open-list proportional representation, there exist three types of tie equilibrium outcomes:*

*(i) Three candidates, 1, 2, and 3, compete for the second seat (*i.e.*, $p_{12}, p_{13}, p_{23} > 0$, while $p_{14} = p_{24} = p_{34} = 0$). Candidate 4, who is the least-preferred candidate, can win a seat with certainty only if $(a_3 - a_2)(1 - b_1) > (1 - a_2)(1 - b_3)$, but never wins otherwise.*

(ii) Two candidates, 1 and 3 (2 and 3, respectively), are in a tie for the second seat (i.e., $p_{13} = 1$ ($p_{23} = 1$), while other probabilities are zero). Candidate 4, who is the least-preferred candidate, wins a seat with certainty only if $b_1 \leq b_3$ ($a_2 \leq a_3$), but never wins otherwise.

(iii) Assume $b_1 < b_3$ ($a_2 < a_3$, respectively). Then, two pairs of candidates, 1 and 3, and 2 and 4, (1 and 4, and 2 and 3) are separately in a tie for the second seat (i.e., $p_{13}, p_{24} > 0$ ($p_{14}, p_{23} > 0$), while other probabilities are zero). Candidate 4, who is the least-preferred candidate, wins a seat with positive probability.

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to realize in the sense that such voting behaviors are induced not only by $p_{12} = p_{13} = p_{23} = 1/3$ but also by some other vectors of close-race probabilities even for extreme parameter values.

If $p_{34} > 0$, every voter who prefers the right-wing party votes for candidate 3, which results in inconsistency between the close-race probability and the outcome. Therefore, voters must perceive no close race between candidates 3 and 4 in tie equilibria.[17] Under open-list PR with $p_{34} = 0$, voters are indifferent between voting for candidate 3 and voting for candidate 4 because either vote merely helps the right-wing party win more seats. This enables voters who prefer candidate 3 to vote for candidate 4 so that candidate 4 wins with high probability in some equilibria, although voting for candidate 4 is a weakly dominated strategy. To clarify how such perception is supported by electoral outcomes, we derive a three-candidate equilibrium in which candidate 4 wins with certainty (part (i) of the proposition). After that, we derive two-candidate equilibria (part (ii) of the proposition) and two-pair equilibria (part (iii) of the proposition) for the purpose of future comparison with closed-list PR. The proofs of the existence of three-candidate equilibria in which candidate 4 never wins and the nonexistence of other types of tie equilibria are given in the Appendix.

3.2.1. Three-Candidate Equilibria

Suppose that candidates 1, 2, and 3 compete for the second seat. Then, close-race probabilities must be $p_{12}, p_{13}, p_{23} > 0$, and $p_{14} = p_{24} = p_{34} = 0$. Suppose also that candidate 4 wins a seat with certainty. Then, the following three conditions on voting behaviors must be satisfied in order to be consistent with such close-race probabilities and to generate such an outcome: (i) candidate 4 obtains more votes than candidate 3; (ii) candidates 1 and 2 obtain an equal share of votes with positive probability; and (iii) the right-wing (left-wing) party obtains exactly $2/3$ $(1/3)$ of valid votes with positive probability. According to the d'Hondt method, when the right-wing party obtains exactly $2/3$ of valid votes (*i.e.*, condition (iii)), the first seat is allocated to the right-wing party, which is filled by candidate 4 who is ranked first by votes in the right-wing party (*i.e.*, condition (i)). The second seat is randomly won by either the second candidate of the right-wing party, who is candidate 3, or the first candidate of the left-wing party. The first rank of the left-wing party is competed for between candidates 1 and 2 (*i.e.*, condition (ii)), so that voters perceive $p_{12} > 0$. If the first rank is won by candidate 1 (2, respectively), then the second seat is competed for between candidate 3 and candidate 1 (2) so that voters perceive $p_{13} > 0$ $(p_{23} > 0)$.

As shown in the Appendix, we only have to consider the following tie-equilibrium voting behavior which satisfies the above three conditions: half of type-a (b, respectively) voters vote for candidate 1 (2) and the other half vote for candidate 4, while all type-c voters vote for candidate 4.[18] For this behavior to be incentive compatible, type-a (b) voters must be indifferent between voting for candidate 1 (2) and voting for candidate 4, *i.e.*,

$$p_{12}(1 - a_2) + p_{13}(1 - a_3) + p_{23}(a_2 - a_3) = -p_{13}(1 - a_3) + p_{23}(a_3 - a_2) \tag{1}$$

$$p_{12}(1 - b_1) + p_{23}(1 - b_3) + p_{13}(b_1 - b_3) = p_{13}(b_3 - b_1) - p_{23}(1 - b_3) \tag{2}$$

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[17] If all voters abstained, then $p_{34} > 0$ could hold. However, at least type-c voters receive a positive payoff by voting for candidate 3 and so go to the polls.

[18] As long as the three conditions are satisfied, some of the voters may vote in a different way. For example, a part of type-a (b, respectively) voters may vote for candidates 1 (2) and 4 randomly.

These two equations hold only if $a_2 < a_3$ and $b_1 < b_3$. Although type-a (b) voters prefer candidate 1 (2) most and candidate 4 least, they can be indifferent between voting for candidate 1 (2) and voting for candidate 4 because of the rules of open-list PR: voting for candidate 1 (2) helps candidate 2 (1) win against candidate 3 while voting for candidate 4 helps candidate 3 win against candidate 2 (1). Thus, if type-a (b) voters prefer candidate 3 to candidate 2 (1), and if they perceive a close race between candidates 2 (1) and 3 to be highly likely to occur (*i.e.*, p_{23} (p_{13}) is sufficiently large), then voting for candidate 4 can help type-a (b) voters increase their expected utilities as much as voting for candidate 1 (2).

Substituting $p_{12} = 1 - p_{13} - p_{23}$ into Equations (1) and (2) determines the values of p_{13} and p_{23} as the functions of parameters a_2, a_3, b_1, and b_3, which we write as p^*_{13} and p^*_{23}. First of all, p^*_{13} and p^*_{23} must be consistent as close-race probabilities, *i.e.*,

$$p^*_{13}, p^*_{23} \in (0, 1) \tag{3}$$

$$p^*_{13} + p^*_{23} < 1 \tag{4}$$

Second, for type-a and b voters not to abstain, p^*_{13} and p^*_{23} must lead to nonnegative payoffs from voting:

$$-p^*_{13}(1 - a_3) + p^*_{23}(a_3 - a_2) \geq 0 \tag{5}$$

$$p^*_{13}(b_3 - b_1) - p^*_{23}(1 - b_3) \geq 0 \tag{6}$$

Finally, under p^*_{13} and p^*_{23}, type-c voters must prefer voting for candidate 4 to voting for candidates 1 and 2:

$$\begin{aligned} p^*_{13}(1 - c_1) + p^*_{23}(1 - c_2) \geq (1 - p^*_{13} - p^*_{23}) \mid c_1 - c_2 \mid \\ -p^*_{13}(1 - c_1) - p^*_{23}(1 - c_2) \end{aligned} \tag{7}$$

Inequalities (3) through (7), some of which may be non-binding, give us conditions that must be imposed on parameter values for the existence of the equilibrium that is currently under consideration. After calculations (see Appendix), we obtain only one condition to satisfy, which is

$$(a_3 - a_2)(1 - b_1) > (1 - a_2)(1 - b_3) \tag{8}$$

This inequality implies that a_3 and b_3 must be sufficiently larger than a_2 and b_1, respectively, so that voters are willing to vote for candidate 4. That is, when candidate 3, who belongs to the same party as candidate 4, is sufficiently popular, candidate 4, who is the least-preferred candidate, can win a seat with certainty in a three-candidate equilibrium under open-list PR.

3.2.2. Two-Candidate Equilibria

A close race for the second seat between only two candidates can also be perceived in tie equilibria under open-list PR. In this case, the close race must be between candidate 3 and either candidate 1 or 2 because nobody votes for candidate 4 if one of the two candidates is candidate 4 and also because too many voters vote for the left-wing candidates to support the close-race probability if a close race is perceived between candidates 1 and 2.

Suppose that the two candidates are 2 and 3. Then, the close-race probability must be $p_{23} = 1$. The payoff to type-a voters is

$$a_2 - a_3 \text{ if type-}a\text{ voters vote for } 1, 2, \text{ or the left-wing party;}$$
$$a_3 - a_2 \text{ if type-}a\text{ voters vote for } 3, 4, \text{ or the right-wing party.}$$

Type-b voters obtain

$$1 - b_3 \text{ if type-}b\text{ voters vote for } 1, 2, \text{ or the left-wing party;}$$
$$-(1 - b_3) \text{ if type-}b\text{ voters vote for } 3, 4, \text{ or the right-wing party.}$$

Type-c voters obtain

$$-(1 - c_2) \text{ if type-}c\text{ voters vote for } 1, 2, \text{ or the left-wing party;}$$
$$1 - c_2 \text{ if type-}c\text{ voters vote for } 3, 4, \text{ or the right-wing party.}$$

These payoffs imply that type-b (c, respectively) voters vote for the left-wing (right-wing) party and/or its candidates for any parameter values and that the behavior of type-a voters depends on the values of a_2 and a_3. Hence, we find the following two voting behaviors which satisfy the consistency between the close-race probability and the outcome. (i) For $a_2 \geq a_3$, type-a and type-b voters vote for 1, 2, and the left-wing party, so that candidate 1 receives more votes than candidate 2. Type-c voters vote for 3, 4, and the right-wing party, so that candidate 3 receives more votes than candidate 4. (ii) For $a_2 \leq a_3$, type-a and c voters vote for 3, 4, and the right-wing party, so that candidate 4 receives more votes than candidate 3. Type-b voters vote for 1, 2, and the left-wing party, so that candidate 2 receives more votes than candidate 1.

Under the first (second, respectively) voting behavior, the left-wing (right-wing) party obtains $2/3$ of votes and wins one or two seats. The right-wing (left-wing) party obtains $1/3$ of votes and wins one or zero seats. Therefore, the second candidate of the left-wing (right-wing) party and the first candidate of the right-wing (left-wing) party, who are candidates 2 and 3, are in a tie for the second seat. As a result, candidate 1 (4) wins the first seat with certainty, and either candidate 2 or 3 wins the second seat. In this way, this case is an equilibrium outcome.

By the symmetry of candidates 1 and 2, we also have an equilibrium outcome, in which candidates 1 and 3 are in a tie for the second seat with $p_{13} = 1$. In this case, candidate 2 wins a seat with certainty only if $b_1 \geq b_3$, and candidate 4 wins with certainty only if $b_1 \leq b_3$.

3.2.3. Two-Pair Equilibria

Two candidates who are not in a tie in two-candidate equilibria, who are candidate 4 and either candidate 1 or 2, can also be in a tie for the second seat separately from the other pair. Suppose that ties for the second seat occur between candidates 1 and 4 and separately between candidates 2 and 3. This outcome requires the following two conditions to be satisfied: (i) candidates 1 and 3 obtain more votes than candidates 2 and 4, respectively, or candidates 2 and 4 obtain more votes than candidates 1 and 3, respectively; and (ii) each party obtains exactly $1/3$ and $2/3$ of valid votes with positive probabilities and can win any number of seats, 0, 1, and 2. Condition (i) leads to $p_{12} = p_{34} = p_{13} = p_{24} = 0$ because ties do not occur within each party and because ties for the second seat can occur only between the first-ranked candidate of one party and the second-ranked candidate of the other party. Thus, only p_{14} and p_{23} can be strictly positive, and in fact condition (ii) leads to $p_{14}, p_{23} > 0$.

Under such close-race probabilities, type-b voters strictly prefer voting for 1, 2, and the left-wing party because they prefer 1 to 4, and 2 to 3. Hence, condition (ii) requires that half or fewer of the remaining voters (*i.e.*, type-a and type-c voters) vote for 3, 4, and the right-wing party, and that the remaining voters cast their ballots randomly between the two parties. The payoff for type-a voters is

$$p_{14} + p_{23}(a_2 - a_3) \text{ if type-}a \text{ voters vote for } 1, 2, \text{ or the left-wing party;}$$
$$-p_{14} + p_{23}(a_3 - a_2) \text{ if type-}a \text{ voters vote for } 3, 4, \text{ or the right-wing party.}$$

The payoff for type-c voters is

$$p_{14}c_1 - p_{23}(1 - c_2) \text{ if type-}c \text{ voters vote for } 1, 2, \text{ or the left-wing party;}$$
$$-p_{14}c_1 + p_{23}(1 - c_2) \text{ if type-}c \text{ voters vote for } 3, 4, \text{ or the right-wing party.}$$

Therefore, we need either $p_{14} = (a_3 - a_2)p_{23}$ and $p_{14} \leq ((1 - c_2)/c_1)p_{23}$ (*i.e.*, type-a voters randomly vote for the two parties and their candidates, while type-c voters vote for the right-wing party and its candidates) or $p_{14} \leq (a_3 - a_2)p_{23}$ and $p_{14} = ((1 - c_2)/c_1)p_{23}$ (*i.e.*, type-a voters vote for the right-wing party and its candidates, while type-c voters randomly vote for the two parties and their candidates). Note that a condition $a_2 < a_3$ is required for the existence of p_{14} and p_{23} which satisfy either pair of the two incentive constraints. In other words, candidate 3 must be somewhat popular.

Figure 2 illustrates the behavior of type-a and type-c voters when $a_3 - a_2 < (1 - c_2)/c_1$. In this case, the first pair of constraints is satisfied at the circled point, where $p_{23} = 1/(1 + a_3 - a_2)$ and $p_{14} = (a_3 - a_2)/(1 + a_3 - a_2)$. If $a_3 - a_2 > (1 - c_2)/c_1$, the second pair is satisfied with $p_{23} = c_1/(1 - c_2 + c_1)$ and $p_{14} = (1 - c_2)/(1 - c_2 + c_1)$. If $a_3 - a_2 = (1 - c_2)/c_1$, both pairs are satisfied with equality. Note that since $a_3 - a_2 < 1$, the line $p_{14} = (a_3 - a_2)p_{23}$ is less steep than the 45-degree line, so that $p_{23} > p_{14}$ holds whether $a_3 - a_2$ is greater or smaller than $(1 - c_2)/c_1$. If close races are perceived to be more likely to occur between candidates 2 and 3 than between candidates 1 and 4, then voters who prefer candidate 3 can vote for the right-wing party and its candidates, despite the presence of candidate 4 who is the least-preferred candidate. For consistency with the electoral outcome, these close-race probabilities require voters who randomly cast their ballots between the two parties to behave so that a tie for the second seat is more likely to occur between candidates 2 and 3 than between candidates 1 and 4.

Since candidate 3 is more likely to be in a tie for the second seat than candidate 4, the expected number of seats is greater for the right-wing party when candidate 4 receives more votes than candidate 3, and vice versa. Note that if the second-ranked candidate in a party is more likely to be in a tie for the second seat, then the first-ranked candidate of the same party is highly expected to win the first seat. Hence, candidate 4 wins a seat with high (low, respectively) probability when candidates 2 and 4 receive more (fewer) votes than candidates 1 and 3, respectively. In this way, both high and low winning probabilities of candidate 4 can be realized in equilibrium.

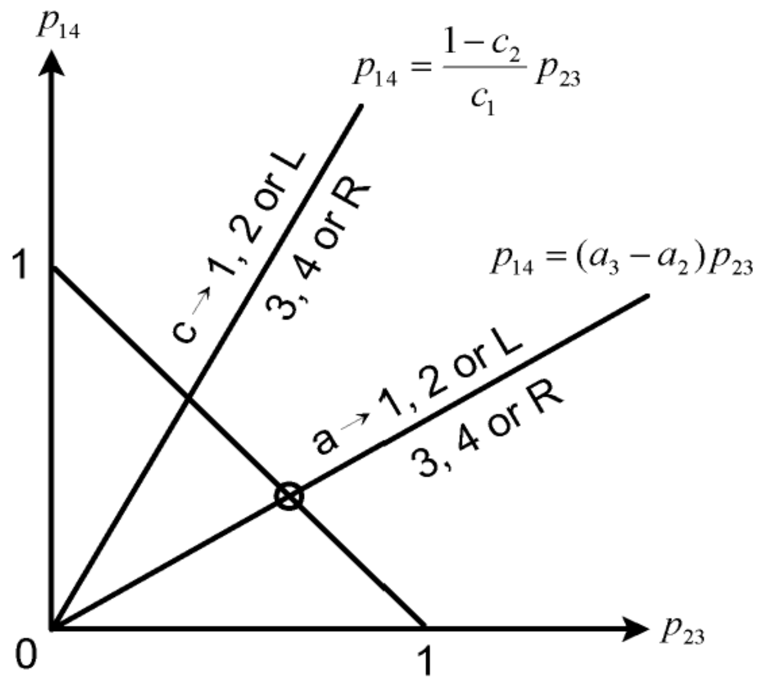

Figure 2. Two-Pair Equilibrium Close-Race Probabilities with Ties Between Candidates 1 and 4 and Between Candidates 2 and 3.

Similarly, ties can occur between candidates 1 and 3 and separately between candidates 2 and 4. In this case, since type-a voters strictly prefer voting for 1, 2, and the left-wing party, half or fewer of type-b and type-c voters must vote for 3, 4, and the right-wing party, and the remaining type-b and type-c voters must randomize their votes between the two parties. For such voting behaviors to be optimal, voters must perceive either $p_{24} = p_{13}(b_3 - b_1)$ and $p_{24} \leq ((1 - c_1)/c_2)p_{13}$ or $p_{24} \leq (b_3 - b_1)p_{13}$ and $p_{24} = ((1 - c_1)/c_2)p_{13}$. A condition $b_1 < b_3$ is required for the existence of p_{13} and p_{24} which satisfy either of the two pairs of incentive constraints.

In summary, under open-list PR, the least-preferred candidate can win a seat only if another candidate on the same party list is sufficiently popular. In addition, voters who prefer the popular candidate may vote for the least-preferred candidate because they perceive no close races within the party. If they do so, although it is a weakly dominated strategy, then the least-preferred candidate wins with high probability.

3.3. Closed-List Proportional Representation

In each subgame under closed-list PR, two-candidate equilibria and two-pair equilibria are realized, just as they are under open-list PR. The only difference in these equilibria between the two variations of PR is whether intraparty competition is eliminated endogenously or exogenously. Under open-list PR, it is eliminated by equilibrium behaviors of voters, whereas, under closed-list PR, it is eliminated by the inherent rule that the order of candidates is determined by parties in advance of voting.

Under closed-list PR, each party can choose one of the two types of equilibrium order of candidates realized under open-list PR for the purpose of maximizing the expected number of seats the party will win. Tie equilibria require candidate 3 to be perceived to be in a tie for the second seat with a sufficiently high probability, so that at least $1/3$ of the voters vote for the right-wing party, thereby creating a tie. Since the expected number of seats is greater for a party when its second-ranked candidate is more likely to be in a tie for the second seat than when its first-ranked candidate is more likely, the right-wing party ranks candidate 3 second under closed-list PR. Then, the remaining rank for candidate 4 is the first rank. As a result, candidate 4, who is the least-preferred candidate, always wins a seat with high probability, possibly one, under closed-list PR. The following proposition summarizes the tie equilibrium outcomes under closed-list PR:

Proposition 3. *Assume that $a_2 \leq a_3$ and $b_1 \leq b_3$. In each subgame under closed-list proportional representation, there exist two types of tie equilibrium outcomes:*

*(i) Two candidates 1 and 3 (2 and 3, respectively) are in a tie for the second seat (*i.e.*, $p_{13} = 1$ ($p_{23} = 1$), while other probabilities are zero);*

*(ii) Two pairs of candidates, 1 and 3, and 2 and 4, (1 and 4, and 2 and 3, respectively) are separately in a tie for the second seat (*i.e.*, $p_{13}, p_{24} > 0$ ($p_{14}, p_{23} > 0$), while other probabilities are zero).*

In the entire game with ties, candidate 4, who is the least-preferred candidate, is always ranked first and wins a seat with high probability or with certainty.

In the following, we present the derivation of the proposition following the backward induction procedure.

3.3.1. The Voting Stage

First, we derive tie equilibria in each subgame. Note that since the order of candidates in each party list is determined in advance, candidates of the same party cannot be in a tie under closed-list PR. Therefore, voters must perceive $p_{12} = p_{34} = 0$ in every subgame.

Let us examine the subgame after $((2, 1), (4, 3))$. Under this order of candidates, ties for the second seat can occur only between candidates 1 and 4 and between candidates 2 and 3. Hence, voters must perceive $p_{13} = p_{24} = 0$, and possible cases are (i) $p_{14} = 1$ and $p_{23} = 0$, (ii) $p_{14} = 0$ and $p_{23} = 1$, and (iii) $p_{14} > 0$ and $p_{23} > 0$. In case (i), since every type of voter prefers candidate 1 to candidate 4, the left-wing party receives all votes, and so candidates 1 and 2 win seats with certainty, which is inconsistent with $p_{14} = 1$. Hence, case (i) is ruled out. To the remaining two cases, we can apply the analyses of two-candidate equilibria and two-pair equilibria conducted in the subsection of open-list PR. The only difference from open-list PR is that, here, the order of candidates is exogenously fixed and voters write in only the names of parties.

Given the order of candidates $((2, 1), (4, 3))$, $p_{23} = 1$ implies that the first-ranked candidate of the left-wing party and the second-ranked candidate of the right-wing party are perceived to be in a close race for the second seat. For this perception to be consistent with the outcome, the left-wing party needs to win one or zero seats by receiving exactly $1/3$ of all valid votes with positive probability, possibly one, whereas the right-wing party needs to win one or two seats. Since type-b and type-c voters with the perception $p_{23} = 1$ strictly prefer voting for the left-wing and right-wing parties, respectively, tie equilibria require type-a voters to vote for the right-wing party, some of whom may use mixed strategies among the two parties and abstention. They do so only if $a_2 \leq a_3$.[19]

The case of $p_{14} > 0$ and $p_{23} > 0$ implies that each party is perceived to win any number of seats, 0, 1, and 2. Type-b voters strictly prefer voting for the left-wing party for any positive values of p_{14} and p_{23}, and the behaviors of type-a and type-c voters are similar to those described in Figure 2, where we need only replace "$1, 2$ *or* L" and "$3, 4$ *or* R" with "L" and "R", respectively. Therefore, the tie equilibrium outcomes in the subgame after $((2,1),(4,3))$ are summarized as follows:

Case 1 ($a_2 > a_3$): Tie equilibria do not exist.

Case 2 ($a_2 = a_3$): A tie occurs between candidates 2 and 3 ($p_{23} = 1$). The expected number of seats is less than 1 for the left-wing party and more than 1 for the right-wing party.

Case 3 ($a_2 < a_3$): Two types of equilibrium outcomes exist. One is the same as Case 2. The other is separate ties between candidates 1 and 4, and between candidates 2 and 3 ($p_{14} > 0$ and $p_{23} > 0$). The expected number of seats is less than 1 for the left-wing party and more than 1 for the right-wing party.

Since tie equilibria do not exist in this subgame if $a_2 > a_3$, our analysis of tie equilibria under closed-list PR hereinafter restricts the range of parameter values to $a_2 \leq a_3$.

The subgame after $((1,2),(3,4))$ is approximately the same as that after $((2,1),(4,3))$ because ties for the second seat can occur only between candidates 1 and 4 and between candidates 2 and 3 in these two subgames. The only difference is that when $p_{23} = 1$, tie equilibria after $((1,2),(3,4))$ require type-a voters to vote for the left-wing party, some of whom may use mixed strategies among the two parties and abstention, so that the left-wing party receives exactly $2/3$ of all valid votes with positive probability, possibly one, which results in a tie between candidates 2 and 3 for the second seat. Hence, we need $a_2 \geq a_3$. However, the case of $a_2 > a_3$ has already been excluded, as described above. The tie equilibrium outcomes in the subgame after $((1,2),(3,4))$ are summarized as follows:

Case 1 ($a_2 = a_3$): A tie occurs between candidates 2 and 3 ($p_{23} = 1$). The expected number of seats is more than 1 for the left-wing party and less than 1 for the right-wing party.

Case 2 ($a_2 < a_3$): Ties occur between candidates 1 and 4, and separately between candidates 2 and 3 ($p_{14} > 0$ and $p_{23} > 0$). The expected number of seats is more than 1 for the left-wing party and less than 1 for the right-wing party.

In the subgames after $((1,2),(4,3))$ and $((2,1),(3,4))$, type-b voters play the same role as type-a voters play in the subgames examined above. Here, we have $p_{14} = p_{23} = 0$. In the case of $p_{13} = 1$, type-a and type-c voters strictly prefer voting for the left-wing and right-wing parties, respectively. Hence, for this close-race probability to constitute two-candidate equilibria after $((1,2),(4,3))$, type-b voters need to vote for the right-wing party, some of whom may use mixed strategies among the two parties and abstention. Type-b voters behave in such a manner only if $b_1 \leq b_3$. In two-candidate equilibria after $((2,1),(3,4))$, on the other hand, type-b voters need to vote for the left-wing party, some of whom may

[19] Only if $a_2 = a_3$, some type-a voters may use mixed strategies among the two parties and abstention as their payoff-maximization behaviors because both voting for either party and abstaining result in zero payoffs for them (*i.e.*, $p_{23}(a_2 - a_3) = p_{23}(a_3 - a_2) = 0$ if $a_2 = a_3$).

use mixed strategies. Type-b voters behave in such a manner only if $b_1 \geq b_3$. In the case of $p_{13} > 0$ and $p_{24} > 0$, type-a voters strictly prefer voting for the left-wing party. Hence, in both of the above two subgames, two-pair equilibria require type-b and type-c voters not to vote for the left-wing party with certainty, which is realized only when $b_1 < b_3$.[20] Therefore, the tie equilibrium outcomes in the subgame after $((1, 2), (4, 3))$ are summarized as follows:

Case 1 ($b_1 > b_3$): Tie equilibria do not exist.

Case 2 ($b_1 = b_3$): A tie occurs between candidates 1 and 3 ($p_{13} = 1$). The expected number of seats is less than 1 for the left-wing party and more than 1 for the right-wing party.

Case 3 ($b_1 < b_3$): Two types of equilibrium outcomes exist. One is the same as Case 2. The other is separate ties between candidates 1 and 3, and between candidates 2 and 4 ($p_{13} > 0$ and $p_{24} > 0$). The expected number of seats is less than 1 for the left-wing party and more than 1 for the right-wing party.

Since tie equilibria do not exist in this subgame if $b_1 > b_3$, our analysis of tie equilibria under closed-list PR also restricts the range of parameter values to $b_1 \leq b_3$. The tie equilibrium outcomes in the subgame after $((2, 1), (3, 4))$ are summarized as follows:

Case 1 ($b_1 = b_3$): A tie occurs between candidates 1 and 3 ($p_{13} = 1$). The expected number of seats is more than 1 for the left-wing party and less than 1 for the right-wing party.

Case 2 ($b_1 < b_3$): Ties occur between candidates 1 and 3, and separately between candidates 2 and 4 ($p_{13} > 0$ and $p_{24} > 0$). The expected number of seats is more than 1 for the left-wing party and less than 1 for the right-wing party.

3.3.2. The Ranking Stage

The analysis of the voting stage has determined the expected number of seats that each party will win according to the order of their candidates. In the analysis, we have found that, under closed-list PR, the conditions on the parameter values, $a_2 \leq a_3$ and $b_1 \leq b_3$, are required for the existence of tie equilibria in every subgame. Assuming these conditions, we now examine the decision making of the parties with respect to the order of their candidates in the first stage. The expected number of seats for each party according to the order of candidates is summarized in Table 3.

In the table, the rows represent the order of candidates of the left-wing party, and the columns represent the order of candidates of the right-wing party. "$More$" indicates that the expected number of seats for that party is greater than one, whereas "$Less$" indicates that the expected number of seats for that party is smaller than one. Note that the exact value of the expected number of seats for each party in each cell depends on what type of tie equilibrium is realized in each subgame.

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[20] If some type-b and type-c voters vote for the left-wing party with certainty, the left-wing party obtains more than $1/3$ of valid votes with certainty, and hence wins at least one seat. Then, the first-ranked candidate of the left-wing party, who is either candidate 1 or 2, wins a seat with certainty, which is inconsistent with either $p_{13} > 0$ or $p_{24} > 0$.

Table 3. Strategic Form of the Ranking Stage under Closed-List PR.

		Party *R*	
		(3, 4)	(4, 3)
Party *L*	(1, 2)	*More, Less*	*Less, More*
	(2, 1)	*More, Less*	*Less, More*

Let $((p, 1-p), (q, 1-q))$ denote the strategy profile of the two parties, which means that the left-wing party chooses $(1, 2)$ with probability p and $(2, 1)$ with probability $1 - p$, whereas the right-wing party chooses $(3, 4)$ with probability q and $(4, 3)$ with probability $1 - q$. Then, Nash equilibria at the first stage are $((p, 1 - p), (0, 1))$, where $p = 1$ ($p = 0$, respectively) if the $Less$ in the cell of $((1, 2), (4, 3))$ is greater (smaller) than the $Less$ in the cell of $((2, 1), (4, 3))$. If they are equal, we have any $p \in [0, 1]$.

As we can see in Table 3, at the first stage, ranking its popular candidate second (*i.e.*, $q = 0$) is the strongly dominant strategy for the right-wing party. In fact, the popular candidate is the only candidate in the right-wing party who can compete with left-wing candidates for the second seat, and the right-wing party wins more seats if its second-ranked candidate competes for the second seat than if its first-ranked candidate does so. As a result, the least-preferred candidate is ranked at the top of the list and wins with high probability.

3.4. Comparing the Three Electoral Systems

From the three propositions regarding tie equilibria, we know that the least-preferred candidate never wins a seat under the SNTV, wins under open-list PR with various probabilities (zero, low, high, and one), and wins with high probability or with certainty under closed-list PR. The difference in rules between the SNTV and PR is whether seats are allocated based on individual candidates or based on parties. Since the least-preferred candidate cannot attract votes by himself/herself, he/she never wins under the SNTV but can win under PR. The difference in rules between open-list PR and closed-list PR is whether the order of candidates is determined by votes or by parties. The expected number of seats is greater for the right-wing party, whose list includes the least-preferred candidate, if its popular candidate finishes second in its list, but open-list PR does not allow parties to choose which candidate finishes second in their lists. Under closed-list PR, on the other hand, parties can choose the order of candidates by themselves before a voting equilibrium is realized. Therefore, the right-wing party ranks its popular candidate second, which must be accompanied by the least-preferred candidate being ranked first and winning a seat with high probability.

To prevent the discrepancy in interest between parties and voters under closed-list PR, for example, the pre-2001 Public Officers Election Act of Japan required parties to submit two documents to the head of the election administration commission: (i) a statement that reported how parties selected their candidates and determined the order of these candidates in their PR lists, and who was responsible for the selection decision; and (ii) a written oath in which the person who was responsible for the selection decision promised that he/she selected the candidates and determined the order of these candidates appropriately (Article 86-II-6). In addition, each party was required to pay four million yen per candidate as deposit money. It seemed difficult for the election administration commission to determine whether the selection decision of each party was appropriate because the word "appropriately" can be interpreted in various ways. However, Article 86-II-6 was evidence that, in the pre-2001 Public Officers Election Act of Japan, there was an attempt to induce political parties to rank their candidates in accordance with voter preference.

In real politics, parties are concerned with not only the number of seats that they acquire, but also which candidates win. Moreover, some candidates have an influence on the decision of the party regarding candidate ranking. These considerations may change the prediction of our model of closed-list PR. However, politicians to whom the party wants to give seats and/or who have an influence on the decisions of the party are not necessarily popular among voters. In our model, such candidates can be represented by either candidate 3 or candidate 4. In this sense, these effects are neutral in determining the order of candidates based on their popularity, and so they are not critical for our analysis.

In actual politics, it is also possible that ranking popular candidates at the top of the list under closed-list PR will improve the voter impression of the party and attract additional votes to the party. This advertisement effect of popular candidates at the top of the list is related to the popularity of individual candidates, which is the focus herein, and relatively preferred candidates, such as candidate 3 of the right-wing party in our model, should be more likely to be ranked higher. If we attempt to analyze such an advertisement effect, we need to introduce into the model some type of informational mechanism which works between parties and voters through the order of candidates in party lists, or we may simply introduce nonstrategic voters who respond mechanically to highly ranked popular candidates. Our model should indicate what would happen if other such effects were eliminated.

Even though we increase the number of seats and add some candidates to the right-wing party, candidate 3, who is a relatively preferred candidate, is still ranked at the bottom of the list under closed-list PR if newly added candidates are unpopular among voters in the same way as candidate 4 is. In this case, only candidate 3 can compete for the final seat with the candidates of the left-wing party. Therefore, ranking candidate 3 at the bottom of the list gives the largest expected number of seats to the right-wing party. On the other hand, if some of the newly added candidates are somewhat preferred by voters, the least-preferred candidate, instead of candidate 3, can be ranked at the bottom of the list. This is because we can construct a tie equilibrium in the subgame after candidate 3 is ranked lowest so that some preferred candidates ranked around the top of the right-wing party list compete for the final seat, which eliminates the incentive for the right-wing party to rank candidate 3 lowest. However, this is the case for only one of multiple equilibria, and there still exist equilibria in which candidate 3 competes for the final seat with high probability in every subgame, and hence candidate 3 is ranked lowest, while the least-preferred candidate is ranked near the top of the list. In the case of open-list PR, voters must write in the name of the least-preferred candidate on their ballots if he/she wins a seat with high probability, which is a weakly dominated strategy. Therefore, the statement that the least-preferred candidate is less likely to win under open-list PR than under closed-list PR holds true even when there are more than two seats in the district and more than two candidates in each party list.

3.5. No-Tie Equilibria

Finally, we summarize the electoral outcomes in no-tie equilibria. Under the SNTV, candidate 4, who is the least-preferred candidate, never wins even in no-tie equilibria for the following reason. Whatever close-race probabilities voters perceive, they never vote for candidate 4 because candidates run individually under the SNTV. Hence, candidate 4 would have a chance to win a seat only if one of the other three candidates obtained all of the valid votes, while the remaining candidates, including candidate 4, obtained no votes and were in a tie for the second seat. However, this would contradict the requirement of no-tie equilibrium, whereby ties for the second seat do not occur.

In tie equilibria under open-list PR, as shown in Subsection 3.2, candidate 4 wins with high probability only when voters use weakly dominated strategies. In principle, this property is also true in no-tie equilibria, but there is one exception, in which voters perceive a sufficiently high probability of a close race within the right-wing party (*i.e.*, p_{34} is sufficiently large), where all voters vote for candidate 3, but they finally find that their votes for candidate 3 have also given a seat to candidate 4 as well as candidate 3, which has created no ties.[21] This type of inconsistency between perception and outcomes is allowed in no-tie equilibria. In this no-tie equilibrium, candidate 4 wins a seat with certainty even though no voter writes in the name of candidate 4 on his/her ballot (*i.e.*, no voter uses weakly dominated strategies).

Under closed-list PR, if the necessary conditions for the existence of tie equilibria are satisfied (*i.e.*, $a_2 \leq a_3$ and $b_1 \leq b_3$), then two types of no-tie equilibria exist in each subgame: (1) the left-wing party wins two seats; and (2) each party wins one seat. In the first type of no-tie equilibrium, voters perceive a sufficiently high probability of a close race between candidate 4 and one of the left-wing candidates (*i.e.*, $p_{i4} > 0$, $i \in \{1, 2\}$) that all voters vote for the left-wing candidate. In the second type of no-tie equilibrium, on the other hand, candidate 3 is perceived to be in a close race with a high probability with one of the left-wing candidates. According to which type of no-tie equilibrium is realized in each subgame, there exist the following three types of Nash equilibria in the first stage: (i) the left-wing party wins two seats; (ii) each party wins one seat; and (iii) the left-wing (right-wing, respectively) party chooses $(1, 2)$ ($(3, 4)$) with probability $1/2$ and $(2, 1)$ ($(4, 3)$) with probability $1/2$. In case (i), the order of the right-wing candidates does not matter because they win no seats. In case (ii), in some equilibria, candidate 4 is ranked first and wins a seat with certainty, whereas in the other equilibria, candidate 4 is ranked second and loses. In case (iii), candidate 4 wins with probability $1/4$.

If candidate 3 is less popular than the left-wing candidates (*i.e.*, $a_2 > a_3$ and $b_1 > b_3$), then the right-wing party wins no seats in any subgame and hence in the entire game. This type of no-tie equilibrium implies that the place in which the least-preferred candidate will be ranked should be seriously watched only when there is a popular candidate in the same party list, whose popularity provides the least-preferred candidate an opportunity to win.

If candidate 3 is popular to some extent in the sense that $a_2 > a_3$ and $b_1 \leq b_3$ ($a_2 \leq a_3$ and $b_1 > b_3$, respectively) hold, then the left-wing party wins two seats with certainty in subgames after $((1, 2), (3, 4))$ and $((2, 1), (4, 3))$ ($((1, 2), (4, 3))$ and $((2, 1), (3, 4))$), while there exist the following two types of no-tie

[21] By the same logic, we also have no-tie equilibria under open-list PR in which voters perceive a sufficiently high probability of a close race between candidates 1 and 2, where the left-wing party wins two seats with certainty.

equilibria in subgames after $((1,2),(4,3))$ and $((2,1),(3,4))$ ($((1,2),(3,4))$ and $((2,1),(4,3))$): (i) the left-wing party wins two seats; and (ii) each party wins one seat. In the first stage, therefore, there exist two types of Nash equilibria: (i) the left-wing party wins two seats with certainty; and (ii) each party chooses the two types of order of candidates randomly. Therefore, candidate 4 either loses with certainty or wins a seat with probability $1/4$.

The difference in electoral outcomes between the two variations of PR is less clear in no-tie equilibria than in tie equilibria due to the variety of electoral outcomes. However, the following three properties of the electoral outcomes in no-tie equilibria show that the basic properties of electoral outcomes that we observe in tie equilibria survive in no-tie equilibria. First, the least-preferred candidate never wins under the SNTV. Second, in no-tie equilibria under open-list PR, the least-preferred candidate can win with certainty even without the use of weakly dominated strategies, which is different from tie equilibria. However, this occurs only when voters do not care about the competition between parties, but rather are concerned only with the race within the right-wing party. Such perception seems extreme because the PR system is intended to enhance the competition between parties rather than between individual candidates. Finally, under closed-list PR, if candidate 3 is not very popular, the order of the candidates of the right-wing party does not matter because all voters vote for the left-wing party. If candidate 3 is popular, on the other hand, various types of electoral outcomes are realized, but the least-preferred candidate can still be ranked first and win a seat with certainty.

4. Conclusions

We have compared three electoral systems, the SNTV, open-list PR, and closed-list PR, in terms of whether a least-preferred candidate wins a seat. We focused on tie equilibria, in which ties for seats occur between candidates. Under the SNTV, seats are allocated based on individual candidates. Thus, the least-preferred candidate, who cannot attract voters by himself/herself, has no chance to win. Under PR, on the other hand, votes are cast for parties, and hence the least-preferred candidate can win if popular candidates are included in the same party list. Under open-list PR, the order of candidates in each party list is determined by votes, and hence the least-preferred candidate can win a seat with various probabilities. However, his/her winning with high probability requires voters to write in his/her name on their ballots although this is no better than writing in a popular candidate's name on the same party list. Only when voters perceive no close race within the party are they indifferent between voting for the least-preferred candidate and voting for other candidates of the same party. Under closed-list PR, parties can choose the order of their candidates in advance of voting. Since strategic voters attempt to elect popular candidates, ranking popular candidates lower attracts more voters to the party. Therefore, the party whose list includes the least-preferred candidate ranks popular candidates lower, and, as a result, the least-preferred candidate is ranked higher and wins with high probability.

In no-tie equilibria, where no ties occur, the least-preferred candidate still never wins under the SNTV. Under PR, on the other hand, electoral outcomes that are different from those realized in tie equilibria are also obtained. The presence of other such outcomes makes the difference between the two variations of PR less clear, but the comparison between the two variations of PR reveals that the least-preferred candidate is still less likely to win under open-list PR than under closed-list PR.

In trying to determine which electoral system should be used, however, we also need to take into account other aspects related to each electoral system. In this sense, our analysis, which deals with only the reflection of voter preference, is positive rather than normative. For example, the high campaign cost for each candidate was one of the main reasons why the Japanese House of Councilors replaced the SNTV with closed-list PR in 1983 in its national constituency. Shiratori [20] reported that "it is generally said that the conservative candidates each spend between 300 m. yen and 500 m. yen" under the SNTV (p. 154). The severe corruption in postwar Italy is explained in part by its open-list PR (Golden and Chang [21]). In addition to these costs, the type of party politics that each country aspires to also affects the choice of electoral systems. For example, intra-party electoral competition tends to result in the internal disunity of parties (Katz [6]).

In our model, candidates are dealt with as non-players, but their behaviors can also change between the two variations of PR. Under closed-list PR, they compete for higher ranks in advance of elections, whereas under open-list PR they compete for personal votes in elections. Comparing the two variations of PR in terms of candidates' incentives is a topic for future research.

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Appendix

Proof of Proposition 1: We examine the cases that were not dealt with in Section 3.

Case 1: Four candidates compete for the second seat.

In this case, the close-race probability between any pair of candidates must be strictly positive. Then, voting for candidates 1, 2, and 3, respectively, gives strictly positive payoffs to type-a, type-b, and type-c voters. On the other hand, voting for candidate 4 results in a strictly negative payoff for any type of voter. Therefore, candidate 4 obtains no votes, while some of the other candidates obtain a positive number of votes, which is inconsistent with the close-race probabilities.

Case 2-1: Three candidates, one of whom is candidate 4, compete for the second seat.

Suppose that the three candidates are candidates 2, 3, and 4. Then, the close-race probabilities must be $p_{23}, p_{24}, p_{34} > 0$ and $p_{12} = p_{13} = p_{14} = 0$, under which the payoff for type-a voters is

$$\begin{gathered}0 \text{ if type-}a\text{ voters vote for candidate } 1 \text{ or abstain;}\\ p_{23}(a_2 - a_3) + p_{24}a_2 \text{ if type-}a\text{ voters vote for candidate } 2;\\ p_{23}(a_3 - a_2) + p_{34}a_3 \text{ if type-}a\text{ voters vote for candidate } 3;\\ -p_{24}a_2 - p_{34}a_3 \text{ if type-}a\text{ voters vote for candidate } 4.\end{gathered}$$

If $a_2 > a_3$ ($a_2 < a_3$, respectively), then voting for candidate 2 (3) gives a strictly positive payoff. If $a_2 = a_3$, then voting for either candidate 2 or 3 gives a strictly positive payoff. Similarly, we can ensure that voting for either candidate 2 or 3 (or both candidates) gives a strictly positive payoff to type-b and type-c voters for any values of b_1, b_3, c_1, and c_2. On the other hand, voting for candidate 1 and voting for candidate 4 give zero and strictly negative payoffs, respectively, to any type of voter. Therefore, no types of voters vote for candidate 4, whereas candidates 2 and/or 3 obtain votes, which is inconsistent with the close-race probabilities. By the symmetry of candidates 1, 2, and 3, we face such an inconsistency whenever candidate 4 is included in the three candidates.

Case 2-2: Candidates 1*, 2, and* 3 *compete for the second seat, and candidate* 4 *wins with positive probability.*

As mentioned in Section 3, voters neither vote for candidate 4 nor abstain under $p_{12}, p_{13}, p_{23} > 0$ and $p_{14} = p_{24} = p_{34} = 0$. Hence, if candidate 4 wins with positive probability, it must be that, with positive probability, one of candidates 1, 2, and 3 obtains all of the valid votes and the other three candidates obtain no votes. Then, candidate 4 is included in the competition for the second seat, which is inconsistent with the close-race probabilities.

Case 3: Two candidates are in a tie for the second seat.

Suppose that candidates 1 and 2 are in a tie for the second seat. Then, the close-race probability must be $p_{12} = 1$. In this case, type-a and type-b voters vote for candidates 1 and 2, respectively, for any parameter values. For the tie between candidates 1 and 2 to be for the second seat, either candidate 3 or 4 must obtain more votes than candidates 1 and 2. However, this is impossible because the remaining voters are only type-c voters, who make up $1/3$ of the population. Similarly, a tie for the second seat occurs neither between candidates 1 and 3 nor between candidates 2 and 3.

Now suppose that the two candidates are $i \in \{1, 2, 3\}$ and 4. Then, $p_{i4} = 1$. In this case, to any type of voter, voting for candidate i gives a strictly positive payoff, voting for candidate 4 gives a strictly negative payoff, and voting for other candidates or abstaining gives zero payoffs. Therefore, all voters vote for candidate i, which is inconsistent with the close-race probability.

Case 4: Two pairs of candidates are separately in a tie for the second seat.

Suppose that candidates 1 and 2 are in a tie, while, separately, candidates 3 and 4 are also in a tie. Since voting for candidate 4 gives a strictly negative payoff to any type of voter, candidate 3 must obtain no votes with positive probability so that $p_{34} > 0$ holds. When candidates 3 and 4 are in a tie with no votes, (i) it will be for third place if both candidates 1 and 2 obtain positive numbers of votes; (ii) it will be a competition among three candidates if one of candidates 1 and 2 obtains a positive number of votes while the other obtains no votes; and (iii) it will be a competition among four candidates if neither candidate 1 nor candidate 2 obtains votes. Therefore, this case never occurs. *Q.E.D.*

Proof of Proposition 2: We examine the cases that were not dealt with in Section 3.

Case 1: Four candidates compete for the second seat.

In this case, the close-race probability between any pair of candidates must be strictly positive. Then, to any type of voter, voting for candidate 3 gives a greater payoff than voting for candidate 4, and so candidate 4 obtains no votes. Therefore, for $p_{34} > 0$ to hold, it must occur with positive probability that not only candidate 3 but also candidates 1 and 2 obtain no votes. Note that if candidates 1 and 2 obtain a positive number of votes while candidate 3 obtains no votes, then the tie between candidates 3 and 4 becomes a tie for third place. Hence, all voters must abstain simultaneously with positive probability. For type-a voters to abstain, for example, voting for candidate 1 and voting for candidate 3 must give nonpositive payoffs. However, if voting for candidate 3 gives a nonpositive payoff, voting for candidate 1 must give a strictly positive payoff. Mathematically, if

$$-p_{13}(1-a_3) + p_{23}(a_3 - a_2) + p_{34}a_3 - p_{14} - p_{24}a_2 \leq 0$$

then we have

$$\begin{aligned} p_{12}(1-a_2) + p_{13}(1-a_3) + p_{14} + p_{23}(a_2 - a_3) + p_{24}a_2 \\ \geq p_{12}(1-a_2) + p_{34}a_3 > 0 \end{aligned}$$

Therefore, type-a voters never abstain, and so this case never occurs in equilibrium.

Case 2-1: Three candidates, one of whom is candidate 4*, compete for the second seat.*

If the three candidates include 3 and 4, the same logic as in Case 1 applies. If the three candidates are 1, 2, and 4, then any type of voter is indifferent with regard to voting for candidates 3 and 4 and the right-wing party, each of which gives a strictly negative payoff. On the other hand, voting for the left-wing party assures a strictly positive payoff. Hence, candidates 1 and 2 win seats with certainty, which excludes candidate 4 from the competition.

Case 2-2: Candidates 1*,* 2*, and* 3 *compete for the second seat, and candidate* 4 *never wins.*

If the three candidates are 1, 2, and 3, then close-race probabilities must be $p_{12}, p_{13}, p_{23} > 0$ and $p_{14} = p_{24} = p_{34} = 0$. Suppose that two of the three candidates are winners. Then, the following three conditions are required in order for the outcome to be consistent with the close-race probabilities: (i) with positive probability, the left-wing party obtains $2/3$ of valid votes, while the right-wing party obtains $1/3$; (ii) candidates 1 and 2 obtain an equal share of votes with positive probability; and (iii) candidate 3 obtains more votes than candidate 4. As a voting behavior that satisfies these three conditions, let us suppose that voters of types a, b, and c vote for candidates 1, 2, and 3, respectively. Since type-a voters strictly prefer voting for candidate 1 to voting for candidate 2, and voting for candidate 3 to voting for candidate 4 under the current close-race probabilities, they vote for candidate 1 only if

$$p_{12}(1-a_2) + p_{13}(1-a_3) + p_{23}(a_2 - a_3) \geq max\{-p_{13}(1-a_3) + p_{23}(a_3 - a_2), 0\}$$

Similarly, type-b voters vote for candidate 2 only if

$$p_{12}(1-b_1) + p_{23}(1-b_3) + p_{13}(b_1 - b_3) \geq max\{p_{13}(b_3 - b_1) - p_{23}(1-a_3), 0\}$$

Whether type-c voters prefer voting for candidate 1 to voting for candidate 2 depends on which of c_1 and c_2 is larger. Type-c voters vote for candidate 3 only if

$$p_{13}(1-c_1)+p_{23}(1-c_2) \geq p_{12} \mid c_1-c_2 \mid -p_{13}(1-c_1)-p_{23}(1-c_2)$$

If $a_2 \geq a_3$, $b_1 \geq b_3$, and $c_1 = c_2$ hold, then the incentive constraints for types a, b, and c, respectively, are satisfied by any $p_{12}, p_{13}, p_{23} > 0$. Now suppose that $a_2 < a_3$, $b_1 < b_3$, and $c_1 \neq c_2$. The incentive constraints for type-a, type-b, and type-c voters become more difficult to satisfy as a_3, b_3, and c_1 for $c_1 > c_2$ or c_2 for $c_1 < c_2$ approach 1. At the limit, the incentive constraints for type-a and type-b voters converge, respectively, to $(1/2)p_{12} \geq p_{23}$ and $(1/2)p_{12} \geq p_{13}$. The incentive constraints regarding type-c voters for $c_1 > c_2$ and $c_1 < c_2$ converge to $p_{23} \geq (1/2)p_{12}$ and $p_{13} \geq (1/2)p_{12}$, respectively. Combining them, we have $(1/2)p_{12} = p_{23} \geq p_{13}$ for $c_1 > c_2$, and $(1/2)p_{12} = p_{13} \geq p_{23}$ for $c_1 < c_2$. Since these inequalities are satisfied with $p_{12} = 1/2$ and $p_{13} = p_{23} = 1/4$, for example, we conclude that there exists a set of close-race probabilities that induce the voting behavior we have supposed above for any parameter values. That is, this case is realized in equilibrium.

Case 2-3: Candidates 1*,* 2*, and* 3 *compete for the second seat, and candidate* 4 *wins with certainty.*
Derivation of Condition (8):

First, let us derive condition (8) from Equations (1) and (2) and inequalities (3) to (7).

Since Equations (1) and (2) require at least that $a_2 < a_3$ and $b_1 < b_3$, these inequalities are assumed throughout this derivation. Then, we obtain

$$p_{13}^* = A/C$$
$$p_{23}^* = B/C$$

where

$$\begin{aligned} A &= (a_3-a_2)(1-b_1)+(1-a_2)(1-b_3) \\ B &= (1-a_2)(b_3-b_1)+(1-a_3)(1-b_1) \\ C &= 2\left[(1-a_2)(b_3-b_1)+(a_3-a_2)(1-b_1)\right] \end{aligned}$$

A, B, and C are strictly positive for $a_2 < a_3$ and $b_1 < b_3$, which ensures that $p_{13}^* > 0$ and $p_{23}^* > 0$.

Before confirming $p_{13}^* < 1$ and $p_{23}^* < 1$, let us examine inequality (4). Inequality (4) is equivalent to $C > A + B$ and is rewritten as condition (8). Note that condition (8) implies $a_2 < a_3$ and $b_1 < b_3$. Note also that, for any values of a_2, b_1, and a_3 (b_3, respectively), there exists a value of b_3 (a_3) which satisfies condition (8) because the convergence of b_3 (a_3) to 1 enables condition (8) to hold given the other parameter values.

Let us go back to confirming $p_{13}^* < 1$ and $p_{23}^* < 1$. Inequality $p_{13}^* < 1$ is equivalent to $C > A$, which is rewritten as

$$2(1-a_2)(b_3-b_1)+(a_3-a_2)(1-b_1) > (1-a_2)(1-b_3)$$

This condition is weaker than (8).

Inequality $p_{23}^* < 1$ is equivalent to $C > B$, which is rewritten as

$$3(a_3-a_2)(1-b_1) > (1-a_2)(1-b_3)$$

This condition is also weaker than (8), and hence inequality (3) is satisfied.

Substituting p^*_{13} and p^*_{23} into either inequality (5) or (6) yields

$$(a_3 - a_2)(1 - b_1) \geq (1 - a_2)(1 - b_3)$$

which is also weaker than (8).

The remaining condition to be examined is inequality (7). Inequality (7) is clearly satisfied if $c_1 = c_2$. Suppose $c_1 > c_2$. Then, inequality (7) is rewritten as

$$(1 - a_2)(1 - c_1) + (1 - a_3)(c_1 - c_2) \geq 0$$

which always holds. In the case of $c_1 < c_2$, inequality (7) is rewritten as

$$(1 - b_1)(1 - c_2) + (1 - b_3)(c_2 - c_1) \geq 0$$

which always holds. Therefore, the above equilibrium requires only condition (8).

Nonexistence of Other Three-Candidate Equilibria in Which Candidate 4 *Wins with Certainty*:

Next, we show that no voting behaviors other than that described in Subsection 3.2.1 can constitute three-candidate equilibria (*i.e.*, $p_{12}, p_{13}, p_{23} > 0$ and $p_{14} = p_{24} = p_{34} = 0$) in which candidate 4 wins with certainty. In this proof, we consider the case of $c_1 < c_2$. By symmetry, the case of $c_1 > c_2$ is dealt with in a similar manner. Of the following arguments about seven profiles of voting behaviors, profiles from (i) to (iii) are not dependent on the values of c_1 and c_2, while profiles from (iv) to (vii) need to be examined only if $c_1 < c_2$ holds. Therefore, in the case of $c_1 = c_2$, the arguments about profiles from (i) to (iii) still hold, and we do not need to discuss profiles (iv) to (vii).

The following five findings restrict the voting behaviors that we need to consider in this proof. First, for the purpose of looking for voting behaviors that enable candidate 4 to win a seat with certainty, we do not have to let voters vote for candidate 3 nor for the right-wing party. Second, type-c voters never abstain because they obtain a strictly positive payoff if they vote for candidate 4 (*i.e.*, $p_{13}(1 - c_1) + p_{23}(1 - c_2) > 0$). Third, type-$a$ voters prefer voting for candidate 1 to voting for candidate 2 and to voting for the left-wing party, whereas type-b and type-c voters prefer voting for candidate 2 to voting for candidate 1 and to voting for the left-wing party. Fourth, the share of valid votes for the left-wing candidates must be less than $2/3$. In addition, with positive probability, the share of valid votes for the left-wing candidates must be $1/3$, and candidates 1 and 2 must be in a tie. Finally, the incentive constraint for voters of a type to vote randomly for candidates i and j is the same as the incentive constraint for a part of voters of a type to vote for candidate i and for the remaining voters of that type to vote for candidate j. In either case, voters of that type must be indifferent between voting for candidate i and voting for candidate j.

If we exclude the cases in which all types of voters use mixed strategies, then we have only the following seven profiles of voting behaviors that are consistent with the above five findings. We show now that none of them constitute three-candidate equilibria. This implies that profiles in which all types of voters use mixed strategies also do not constitute equilibria because the use of mixed strategies requires more conditions on parameter values to be satisfied. In the following, "$t \to i$", "$t \to abstain$", and "$t \to i\&j$", respectively, indicate that type-$t \in \{a, b, c\}$ voters vote for candidate $i \in \{1, 2, 3, 4\}$, that those voters abstain, and that those voters use mixed strategies by voting for candidates i with some probability and j with the remaining probability.

(i) $a \rightarrow 1\&4$*,* $b \rightarrow 2\&abstain$*, and* $c \rightarrow 4$

In this case, we only have to replace Equation (2) with

$$p_{12}(1-b_1)+p_{23}(1-b_3)+p_{13}(b_1-b_3)=0 \tag{A1}$$

From (1) and ($A1$), we obtain

$$p^*_{13} = A_i/C_i$$
$$p^*_{23} = B_i/C_i$$

where

$$\begin{aligned} A_i &= (1-a_2)(1-b_3)+2(a_3-a_2)(1-b_1) \\ B_i &= (1-a_2)(b_3-b_1)+2(1-a_3)(1-b_1) \\ C_i &= (1-a_2)(1-b_1)+2(1-a_2)(b_3-b_1)+2(a_3-a_2)(1-b_1) \end{aligned}$$

Inequality (4) is equivalent to $C_i > A_i + B_i$ and is rewritten as (8). On the other hand, the incentive constraint for type-b voters to prefer abstention to voting for candidate 4 (*i.e.*, $p^*_{13}(b_3-b_1)-p^*_{23}(1-b_3) \leq 0$) is rewritten as

$$(a_3-a_2)(1-b_1) \leq (1-a_2)(1-b_3) \tag{A2}$$

which contradicts (8). Hence, this case does not constitute an equilibrium.

(ii) $a \rightarrow 1\&abstain$*,* $b \rightarrow 2\&4$*, and* $c \rightarrow 4$

In this case, we only have to replace Equation (1) with

$$p_{12}(1-a_2)+p_{13}(1-a_3)+p_{23}(a_2-a_3)=0 \tag{A3}$$

From (2) and ($A3$), we obtain

$$p^*_{13} = A_{ii}/C_i$$
$$p^*_{23} = B_{ii}/C_i$$

where

$$A_{ii} = 2(1-a_2)(1-b_3)+(a_3-a_2)(1-b_1)$$
$$B_{ii} = 2(1-a_2)(b_3-b_1)+(1-a_3)(1-b_1)$$

Inequality (4) is equivalent to $C_i > A_{ii} + B_{ii}$ and is rewritten as (8). On the other hand, the incentive constraint for type-a voters to prefer abstention to voting for candidate 4 (*i.e.*, $-p^*_{13}(1-a_3)+p^*_{23}(a_3-a_2) \leq 0$) is rewritten as ($A2$), which contradicts (8). Hence, this case does not constitute an equilibrium.

(iii) $a \rightarrow 1\&abstain$*,* $b \rightarrow 2\&abstain$*, and* $c \rightarrow 4$

In this case, we only have to replace Equations (1) and (2) with Equations ($A3$) and ($A1$), respectively. From ($A1$) and ($A3$), we obtain

$$p_{13}^* = A/C_{iii}$$
$$p_{23}^* = B/C_{iii}$$

where

$$C_{iii} = (1-a_2)(1-b_1) + (1-a_2)(b_3-b_1) + (a_3-a_2)(1-b_1)$$

Inequality (4) is equivalent to $C_{iii} > A + B$ and is rewritten as (8). On the other hand, the incentive constraint for type-a voters to prefer abstention to voting for candidate 4 (*i.e.*, $-p_{13}^*(1-a_3) + p_{23}^*(a_3-a_2) \le 0$) is rewritten as ($A2$), which contradicts (8). Hence, this case does not constitute an equilibrium.

(iv) $a \to 1\&4$, $b \to 4$, *and* $c \to 2\&4$

In this case, inequality (7) must hold with equality. From (1) and (7) with equality, we obtain

$$p_{13}^* = A_{iv}/C_{iv}$$
$$p_{23}^* = B_{iv}/C_{iv}$$

where

$$\begin{aligned} A_{iv} &= (a_3-a_2)(c_2-c_1) - (1-a_2)(1-c_2) \\ B_{iv} &= (1-a_2)(1-c_1) + (1-a_3)(c_2-c_1) \\ C_{iv} &= 2\left[(1-a_2)(1-c_1) + (a_3-a_2)(c_2-c_1)\right] \end{aligned}$$

The incentive constraint for type-b voters to prefer voting for candidate 4 to voting for candidate 2 is written as

$$p_{13}^*(b_3-b_1) - p_{23}^*(1-b_3) \ge p_{12}^*(1-b_1) + p_{23}^*(1-b_3) + p_{13}^*(b_1-b_3) \tag{A4}$$

which is rewritten as

$$(1-b_1)(1-c_2) + (1-b_3)(c_2-c_1) \le 0 \tag{A5}$$

Since the left-hand side of this inequality is strictly positive, this inequality does not hold. Hence, this case does not constitute an equilibrium.

(v) $a \to 1\&4$, $b \to abstain$, *and* $c \to 2\&4$

From (1) and (7) with equality, we obtain $p_{13}^* = A_{iv}/C_{iv}$ and $p_{23}^* = B_{iv}/C_{iv}$. Note that, for Equation (1) to hold, we must have $a_2 < a_3$. The incentive constraint for type-b voters to prefer abstention to voting for candidate 2 is written as

$$p_{12}^*(1-b_1) + p_{23}^*(1-b_3) + p_{13}^*(b_1-b_3) \le 0 \tag{A6}$$

For this constraint to hold, we must have $b_1 < b_3$. ($A6$) is rewritten as

$$\begin{gathered} 2(1-a_2)(1-b_3)(1-c_1) + 2(1-a_2)(b_3-b_1)(1-c_2) \\ +(a_3-a_2)(1-b_1)(1-c_1) + (1-a_3)(1-b_1)(1-c_2) \le 0 \end{gathered}$$

Since the left-hand side of this inequality is strictly positive, this inequality does not hold. Hence, this case does not constitute an equilibrium.

(vi) $a \to 1\&abstain$, $b \to 4$, *and* $c \to 2\&4$

From $(A3)$ and (7) with equality, we obtain

$$p_{13}^* = A_{vi}/C_{vi}$$
$$p_{23}^* = B_{vi}/C_{vi}$$

where

$$\begin{aligned} A_{vi} &= (a_3 - a_2)(c_2 - c_1) - 2(1 - a_2)(1 - c_2) \\ B_{vi} &= 2(1 - a_2)(1 - c_1) + (1 - a_3)(c_2 - c_1) \\ C_{vi} &= 3(1 - a_2)(c_2 - c_1) + 2(a_3 - a_2)(1 - c_1) + 2(1 - a_3)(1 - c_2) \end{aligned}$$

The incentive constraint for type-b voters to prefer voting for candidate 4 to voting for candidate 2 is written as $(A4)$, which is rewritten as $(A5)$. Since the left-hand side of $(A5)$ is strictly positive, $(A5)$ does not hold. Hence, this case does not constitute an equilibrium.

(vii) $a \to 1\&abstain$, $b \to abstain$, *and* $c \to 2\&4$

From $(A3)$ and (7) with equality, we obtain $p_{13}^* = A_{vi}/C_{vi}$ and $p_{23}^* = B_{vi}/C_{vi}$. Note that, for $(A3)$ to hold, we must have $a_2 < a_3$. The incentive constraint for type-b voters to prefer abstention to voting for candidate 2 is written as $(A6)$, and hence we must have $b_1 < b_3$. $(A6)$ is rewritten as

$$3(1 - a_2)(1 - b_3)(1 - c_1) + 3(1 - a_2)(b_3 - b_1)(1 - c_2)$$
$$+(a_3 - a_2)(1 - b_1)(1 - c_1) + (1 - a_3)(1 - b_1)(1 - c_2) \leq 0$$

Since the left-hand side of this inequality is strictly positive, this inequality does not hold. Hence, this case does not constitute an equilibrium.

Case 3-1: Two candidates, one of whom is candidate 4*, are in a tie for the second seat.*

If the two candidates are $i \in \{1, 2, 3\}$ and 4, then the close-race probability must be $p_{i4} = 1$. Then, all types of voters receive strictly positive payoffs by voting for candidate i, whereas voting for other candidates or abstaining gives at most a zero payoff. This results in all voters voting for candidate i, which is inconsistent with the close-race probability.

Case 3-2: Candidates 1 *and* 2 *are in a tie for the second seat.*

Suppose that the two candidates are 1 and 2. Then, we have $p_{12} = 1$, and type-a and type-b voters vote for candidates 1 and 2, respectively. If $c_1 > c_2$ ($c_1 < c_2$, respectively), then type-c voters vote for candidate 1 (2), and candidates 1 and 2 win seats with certainty. Hence, it is necessary that $c_1 = c_2$. Unless all type-c voters vote for the right-wing party and its candidates, the tie between candidates 1 and 2 becomes a tie for the first seat. If all of the type-c voters vote for the right-wing party and its candidates, then the right-wing party obtains $1/3$ of all votes, and either candidate 3 or 4 (or both candidates) joins the competition for the second seat, which is inconsistent with the close-race probability.

Case 4: Two pairs of candidates, 1 *and* 2*, and* 3 *and* 4*, are separately in a tie for the second seat.*

Suppose that ties for the second seat occur between candidates 1 and 2 and separately between 3 and 4. Then, close-race probabilities must be $p_{12}, p_{34} > 0$. Under $p_{34} > 0$, every type of voter prefers voting for candidate 3 to voting for candidate 4, so that candidate 4 obtains no votes. Therefore, in order for $p_{34} > 0$ to hold, candidates 1, 2, and 3 must obtain no votes at the same time with positive probability. However, voting for 1 (2) gives type-a (type-b) voters a strictly positive payoff for any parameter values. Hence, this case can never occur in equilibrium. *Q.E.D.*

Conflicts of Interest

The author declares no conflict of interest.

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