Supplementary 1: Time as relative dimension in space

This supplemental attempts to explain by analogy section 3.2 to 4.2 of the main paper. Section numbers are non-sequential as their numbers relate to the equivalent sections of the main paper. References are largely left to the main text except where other sources are used for illustration here and the main paper should be referenced in preference to this supplement.

3.1 Is space made of dimensions or locations?

Intuitively we tend to ignore the air around us. We might refer to a jar as empty even though we know it is in fact filled with air. Whilst today wind or breath seem self-evidently to be movement of air the idea was not always obvious, as this passage from Pneumatics (1stC AD), by the Greek philosopher Hero of Alexandria observes "Vessels which seem empty to the common man are not empty as they suppose, but full of air" [1][1]. The distinction is perhaps mute unless one wishes to know the properties of air i.e. what it *is* and what it *does*. Indeed it was not until 1644 that Torricelli demonstrated the idea of a vacuum [2] and not until 1996 that the Casimir effect was measured [3] showing that not only is a vacuum not actually empty but this has macroscopic effects. These are of course physical properties with which a conceptual space, waiting to be filled with theoretical objects such as geometric shapes, need not be endowed. The concept of space seems similarly intuitive until we wish to know the properties of that space. In particular how consistently and precisely objects might be positioned therein i.e. until we wish the space to contain locations. But *can* such a space be filled with locations?

3.2.1 Imprecise locations on uncertain dimensions

Section 1 of the main paper focused on the question of what address system to use to locate objects precisely. Here we take this discussion into two further questions (a) what does it really mean to say we have information about where something is in space i.e. what *is* an information-space and (b) how does information about *when* something is relate to *where* it is?

Generally, one starts with a space in which information is to be mapped and assign some of it addresses via a set of coordinates. The problem is that one cannot do so perfectly precisely, there is always some measurement error and some limit to coordinate precision. For example when someone reads a sign stating "you are here" it really means "you are close enough to here to read this sign" i.e. within some radius of it. If there were two such signs near to one another one might use that radius to decide which one applies. But if the radii overlap, are both signs correct? When there was only one such sign we knew where we were, now we are not so sure. The signs are more useful if their area of relevance is unique.

That threshold of uniqueness becomes a sphere in 3D representing where a measured position might really be which inevitably leaves a substantial portion of the space unaddressed because

[1] Hero was not the first to realise this, Anaximander of Miletus (610 – c. 546 BC) appears to have understood it half a millennia earlier [4].

these sphere's don't completely pack together, just as one cannot stack oranges without leaving space in between. But if the oranges were small as a speck of dust or a fleck of mud, then perhaps it is not so intuitive whether the result is a solid mass or still has spaces in between those specks? Can we address all of space with these infinitesimal specks and if not what does it mean to have a spatial continuum of locations?

The following thought experiment reverses the endeavor. It starts with a sufficient set of addresses for everywhere and asks how the underlying space might be reconstructed from them. These addresses are not coordinates or names, but simply a notion of ordinal relations (a sequence of waypoints). From this it creates a graph across which there is a measure of relative travel time and that is then converted to spatial distances between origin and destination. These distances need to be "puzzled" together to find a spatial arrangement in common. The final step is to then appreciate that depending on what routes are taken the arrangement looks different but by taking lots of combinations, over time, all routes are eventually mapped.

3.2.2 Locations as relations

The simplest logical building blocks are those which we cannot break down into combinations of other ideas. For example a spatial dimension can be broken down into an origin and a direction (which doesn't mean much unless there are other independent directions for comparison). A length along that direction must be defined by some scale consisting of a unit, e.g. 1 meter, and a position relative to the origin e.g. the 9th meter away. Position can be broken down into an ordinal sequence of relations; steps along the way e.g. the place between the 8th meter and the 10th meter. Relations between objects can be broken down into the idea that two things are discernably different but connected, if only because they are next to one another in the sequence.

This will be our basic building block, called a natural number, it is simply the idea that objects may be conceived which can be arranged in a sequence such that to get from A to C one must pass B. This is illustrated in Figure S1, each object is a circle and the lines define the sequence. From now on we will call the circles 'nodes', the lines 'edges' and the sequence a 'graph'.

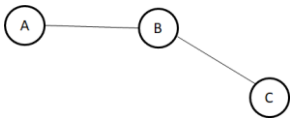


Figure S1 A "triad" of three nodes.

3.2.3 Ordered paths can emerge from random relations

We will start with some amount of these nodes and edges, and will connect them together entirely at random over and over again until the result is a single sequence such that every node has only two neighbours connected by edges. We can arrange them geometrically how we want, in a curvy or straight line. The sequence remains the same. We have a graph (Figure S2):

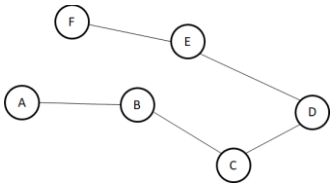


Figure S2 Triads are "chained" to build larger networks (graphs)

3.2.4 Paths become circles

If the last node has an edge that connects back to the first node then anyone stepping along the sequence will find themselves back where they started, going around in circles as it were even if the path is not arranged as a literal circle. So, if a node starts swapping its position in the same sequence it will achieve the same effect as were the whole graph rotated back to its original position. In a sense, the same effect occurs even if the order of the swapping of nodes is not in the original sequence but the same number of steps leads to the same result. In that sense we have system which can revolve by rearranging into a symmetric version of itself (technically an automorphism) (Figure S3).

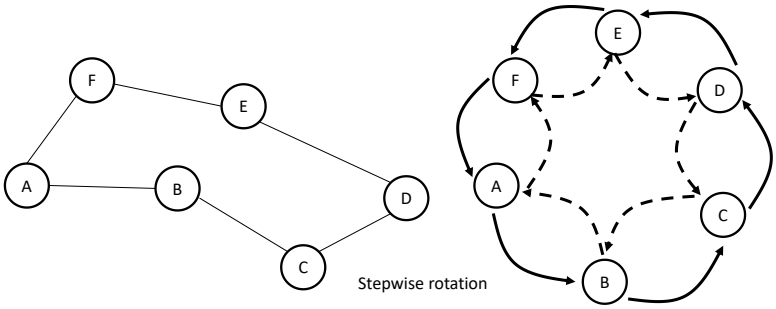


Figure S3 By swapping position with neighbours a node can make a complete circuit of the graph (even if it is not arranged in a circle) when all nodes have done so this defines a "revolution" of the entire graph.

Rotating like clockwork

Analogously, a clock measures time in revolutions, be they an hour of the minute hand or 12 hours of the hour hand. Clock time is simply a subdivision of the daily revolution of the earth, and its annual revolution around the sun. So, by identifying one "revolution" (technically an 'orbit') of the graph we implicitly identify a unit of time which can be subdivided in a similar way since a fraction of the nodes could rearrange representing a fraction of a full revolution. Unlike our usual conception of time however, we do not need to reference some real event such as the ticking of a clock or the daily rotation of the earth. One revolution of this system does not occur 'over time'; it defines one unit of its own time. To avoid confusion we will call this Topological Time.

3.2.5 Topological gears

Two processes of topological rearrangement can happen at different relative "rates" depending on how big a fraction of one revolution they represent, like gears in a clock. One can think of the revolution of the whole graph as a "fly wheel" which determines the maximum change possible, but this change can be expended differentially with parts of the graph changing at different rates (Figure S4):

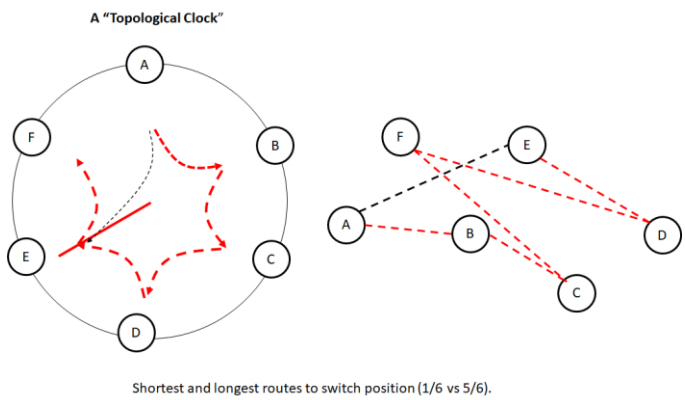


Figure S4: A complete revolution of the graph can be considered a unit of "time" independent of any external clock; all topological changes less than a complete revolution are thus a fraction of this unit.

Each flip in position is like one "tick" of the second hand on our topological clock. So for an independent observer of that second hand, who had their own normal clock ticking along, it would appear the second hand in the topological clock was varying in speed depending on how many node position swaps occur. This fixes the maximum amount of change that can occur at any one 'time', an effect analogous to conservation of momentum as each set of changes must first "borrow momentum" from the common pool of one complete revolution. If process (a) takes allot of moves, the topological clock ticks onward quickly, making process (b) appear slower. What this does is produce clusters of change (Figure S5):

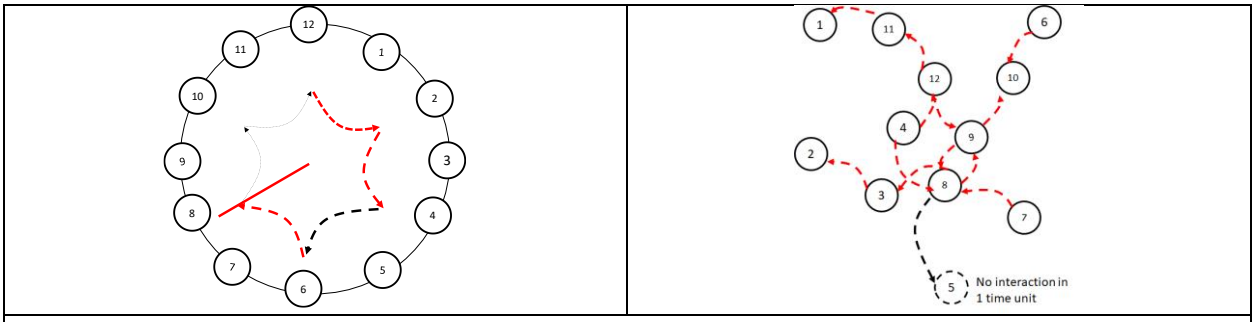


Figure S5 Left: The topological clock ticks once for each change in connection. Right: Various paths are created but most paths are via nodes 4, 9 and 8 creating a cluster.

As the edges randomly flip between nodes some regions in the graph have, through sheer chance, more changes happening than others. So these areas take up more of the available "momentum" and because the nodes in this region are reconnecting relatively quicker, it is more likely that they will connect with others in that region than "colder" or "slower" parts of the graph, effectively forming a subgraph with higher connectivity. Again, not a physical process but simply an emergent cluster over topological-time (iterations). It is this process we shall use to try and build a continuum. To make the logic easier to grasp, we will replace the nodes with road junctions and the edges with roads, access to which can be flipped on and off by traffic lights each change of which represents one "tick" of the topological clock.

3.2.6 Topological clocks as a measure of distance

Manhattan has a famously gridded street plan with a numbered naming convention, with streets running in one direction and avenues cutting perpendicular across these. In a sense the street numbers are coordinates and it is common practice to refer to positions by these intersections, e.g. "Let's meet at 5th and 42nd" meaning the intersection of 5th Avenue and 42nd Street. They are not scalar coordinates, but they do combine to define a location.

Figure S6 A Manhattan street sign showing the intersection between an Avenue and a Street. (Wikicommons). Right: An early plan for the Manhattan street grid (Wikicommons)

In our thought experiment a visitor knowing nothing of Manhattan is driven, blindfold, around the city at random. The driver announces the names of the intersections as they pass but does not use the official numbers, replacing them with letters in the order in which it is reached, e.g. 5th and 42nd might be junction "C" if it happens to be the third junction reached. The sneaky driver also gently changes speed so that all adjacent junctions are exactly the same travel time apart regardless of distance and turns corners as gently as possible so the passenger loses all sense of direction.

The passenger must then draw a map of the city based on what they know, which is only the junction letters reached repeatedly via various routes. By considering the sequence in which different junctions were passed, the passenger can figure out two things (a) which junctions are next to which and (b) that the junctions are not all in one straight line because some can be reached via two different sequences of junctions with only one junction letter in common (Figure S7).

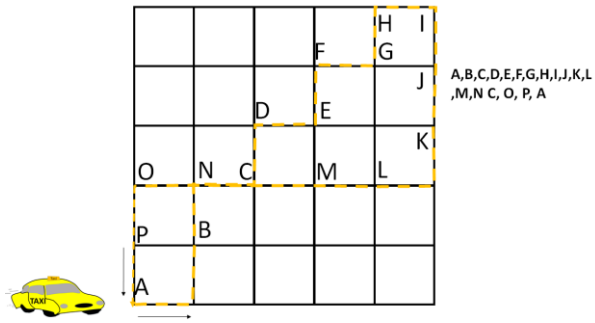
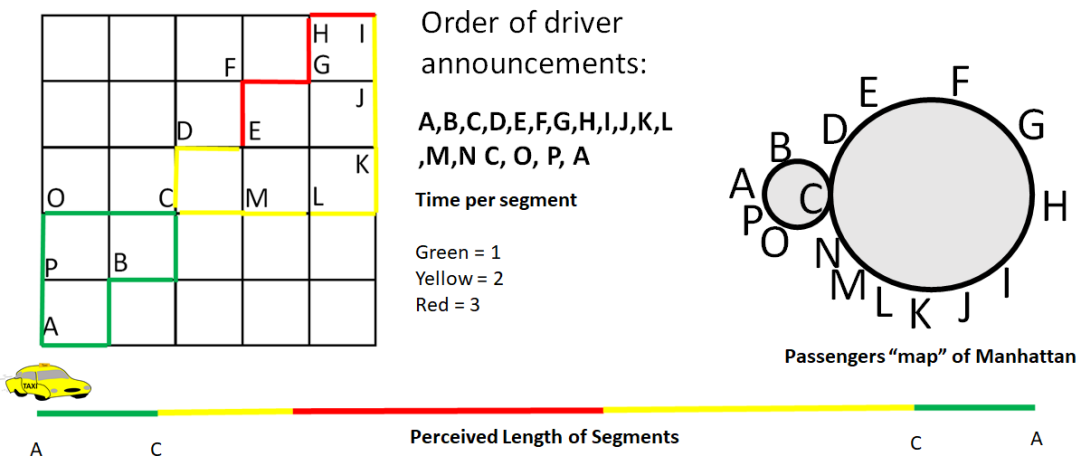


Figure S7 : Mapping an approximate route from junction order. Note, it is possible to establish two loops as C and A are both reached from different places on the outward and homeward journeys.

With enough thought the passenger could figure out the pattern of all the junctions over a 2D map (like a classic London Underground map), but could not give it a scale (there is ordinal position, some places are more stops away than others). Now let us add a complicating factor. The driver still maintains the same time interval between junctions, but each junction has traffic lights. So because the driver has to wait at the lights the passenger, unaware of what speed they are going, gets the impression some junctions are further apart. We can add the conservation of momentum idea in by stating that only a certain number of lights in the city can be on green at any given time. Each change in the lights is a "tick" of the topological clock because it changes the graph[2]. But in some parts of town the lights change to green often and other parts of town only occasionally. Our New York cab driver only needs a tiny window of green to drive through, so the areas where the lights change often afford more opportunities and the taxi moves more quickly over all. To the passenger, unaware of the stops and starts, it seems like some areas of the city are smaller because the junctions are passed more quickly. The passenger can now draw a map such as Figure s8.



[2] When all lights have been lit in all possible combinations that is a full revolution of the graph.

Figure S8 Mapping the apparent relative positions of waypoints based on travel time.

3.2.7 Breaking away from the grid

Adding dimensions

Of course, the 2D spatial layout is still underpinning the information reaching the passenger to some extent. So let us make the taxi fly. Now it can take any route across town, it does not need to visit the nodes sequentially by road, but it does need to observe the traffic lights at those junctions it passes over. The passenger still manages to correctly position all the different nodes with the correct travel times between each pair using just 2D, in effect the passenger is *projecting* the 3D route onto 2D in the same way hills are projected onto a 2D walking map.

But the taxi no longer needs to take the street network; it can go directly between two distant nodes. So what does the passenger make of the arrangement of the city when any node can be connected to any other node in the same amount of time? Our passenger can map out the graph of these lines, but to position in them in 2D space it seems that the city is a sort of circle (Figure S9a). With only the routes via "A" completed it seems to have a diameter of 2 junctions with A the center but as each node becomes the point of departure, they also seem to be the center (Figure S9b).

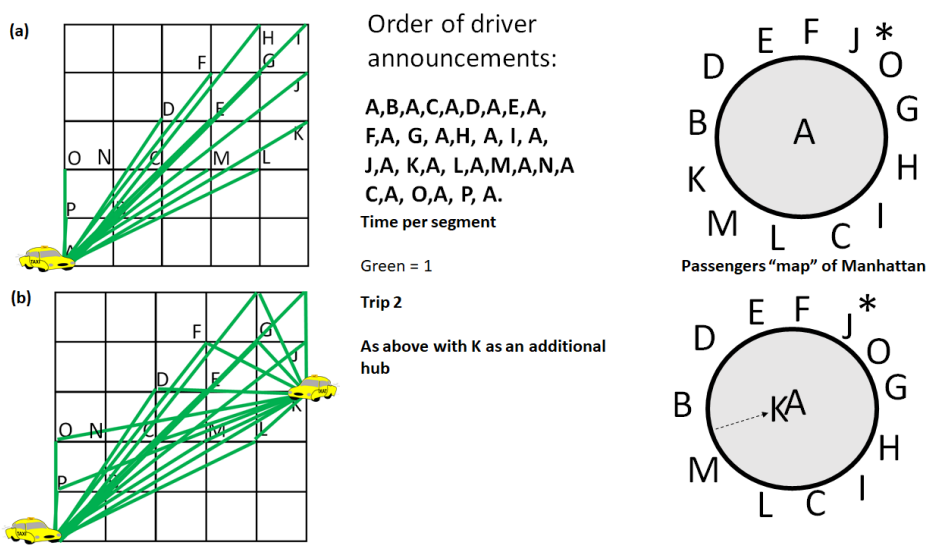


Figure S9 Mapping the apparent relative positions of way points when travel time between them all is equal (a) starting at intersection A (b) Adding intersection K... each letter moves to the center of its own set of paths.

With all routes completed it appears as if all points are equidistant from each other. Our confused passenger can only map it as a zero-dimensional point (Fig S10 a). However when the taxi must wait for a green light it takes longer to travel over certain parts of the city. To make the travel consistent with a (perceived) constant speed of movement certain sets of nodes which have slow traffic lights on many of the routes between them need to be further away from each other. An area "balloons" out of the dot on our passenger's map as each junction becomes differentiated in its distance to all the others (Figure S10b).

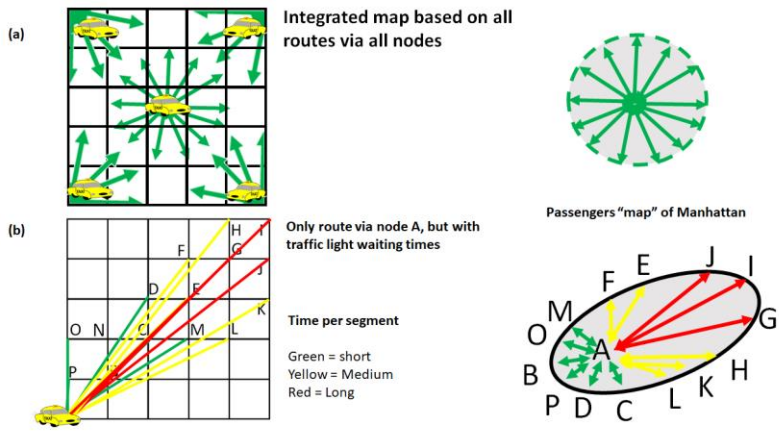


Figure S10 (a) When all distances are equal the projected space collapses to a point (b) differentiation in travel time between intersections results in projected space expanding.

Depending on what route the taxi takes the passenger may be able to map it as 1D (the taxi takes a single snaking route over all the junctions) or in 2D (the taxi loops back through the same point) or perhaps may realize they need 3D to make all the distances work as one big puzzle. The important point is that a (relative) scalar field has emerged for our passenger to map. It is not at all obvious to our passenger that the original junction layout is a grid. Nor does it actually matter, the nodes could simply be in a graph with no spatial arrangement at all but the passenger travelling around that graph will still map them to the same emergent space based on the travel times, which are determined by the traffic lights not the distance (Figure S11):

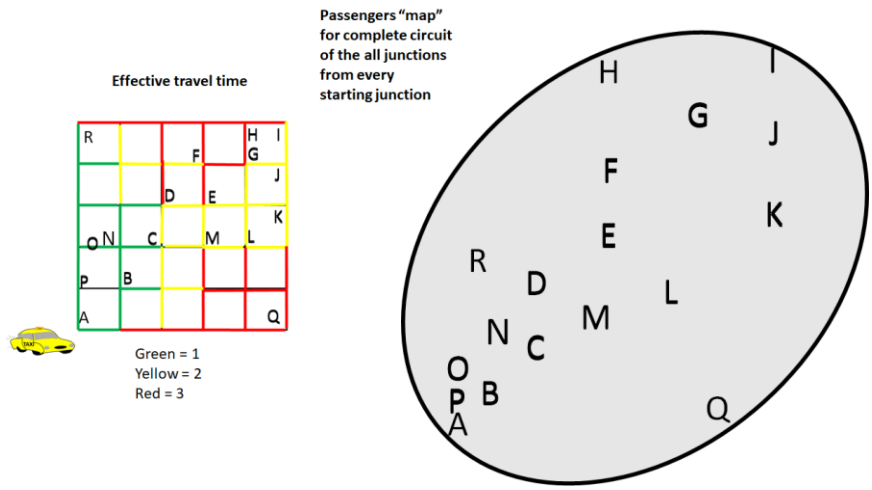


Figure S11: Mapping a discreet set of relationship as a continuous set of relative distances based on relative travel time assuming a constant speed.

Key Observation 1: A smooth space emerged from entirely discreet concepts. Time is discretized by the changes of traffic light colour. But spatial position is not based on a single such journey, it is based on the travel time ratios between two or more such journeys. Ratios of integers can be infinitely precise, thus relative distance based on this may also be infinitely precise.

3.2.8 Mapping relative position to a universal precision

Just because *any* arrangement of relative position can be encoded into the travel times of the taxi, does not mean they *all* could be mapped precisely in 2D or 3D. The measurement uncertainty our passenger faces is from their own estimation of the relative time taken between stops. Once it is projected into 3D space that uncertainty becomes a sphere as to potential error in relative distance between nodes. If two nodes have overlapping spheres it would be uncertain they were not in fact in the same place. Thus the spheres must be hard, which limits the density with which they can be packed and how many nodes the passenger can locate in one frame of reference. This point might well require some *un*packing!

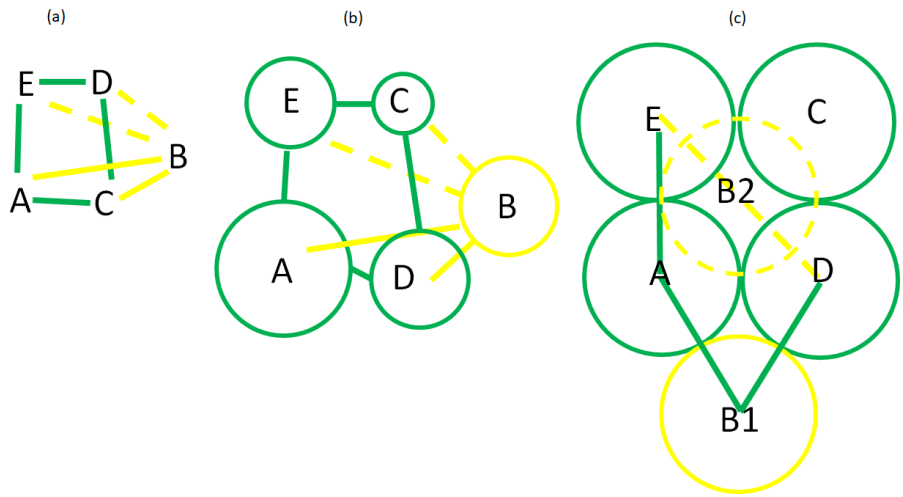


Figure S12 (a) A graph of relations with no metric. (b) A graph of relations where the circles represent uncertainty as to time of departure/arrival but edges have yet to be given a defined metric for travel time (c) A graph of relations where edges represent one time unit and circles a constant uncertainty radius of half a time unit.

Figure S12a shows the graph for routes between five neigbouring nodes (no longer on a grid). In S12a these are in no particular spatial arrangement, each edge has the same time value "1 unit" but the lines are not drawn to that scale. In S12b the circles are to scale for the measurement error. Not all nodes have the same (perhaps the car moved at varying speed at the start and end of its journey). The circles on the graph represent this easily without overlap because the edges are not to the same scale. But maps define precision to a certain constant scale, and to achieve this must assume measurement error is symmetric and equal for all locations. By averaging out the car's speed, the different uncertainties are averaged to a constant radius (r^2) around each node (Figure S12c).

From the route AECDA it is established that all journeys are the same distance, exactly 1 time unit, making the uncertainty in position bisect each exactly half way. So the map can be projected to the green nodes in S12c. Adding in the journey ABD, it is established B is also one unit from both A and D. This can also be projected as B1. Then the journey EBC is made and it is established that B is also 1 unit from E and C. The most likely projection is now a circle with B at the center (B2). The graph tells us they are all exactly 1 unit +/- half a unit from each other, so all are definitely in distinct locations. But mapped to a grid, the error margins overlap and B is no longer 1 unit from the other nodes. The problem is the inability to represent this arrangement in a square lattice. One needs a 3rd dimension making a pyramid, but then the unit is no longer a grid in 3D, rather it has become a triangular lattice[3].

[3] Lattice: A regular mesh of points and edges. In this context the points are spheres, like beads on a string.

Key Observation 2: The emergent geometry cannot be guaranteed to be perfectly represented in a lattice, matrix or other system measured by independent dimensions at a universal precision.

4.2 Addressing the in-between

Instead of considering the nodes as points in a path with no direction of time, consider point B as a "stepping stone" in a path between the other nodes over time. This provides the rationale for an additional dimension into which the (now) spheres can all pack. It makes point B not a place the taxi stops at and restarts from but a place it passed *through* half way between origin and destination, an inferred point offset in the vertical dimension (time) and interstitial to the other nodes in space. By this reasoning B can fit "around" a vector of travel, representing the uncertainty as to actual route left by the packing of the lattice (Figure S13a). Over many paths, these offset points build up inter-leaven layers of spatial coordinates as time progresses (Figure S13b).

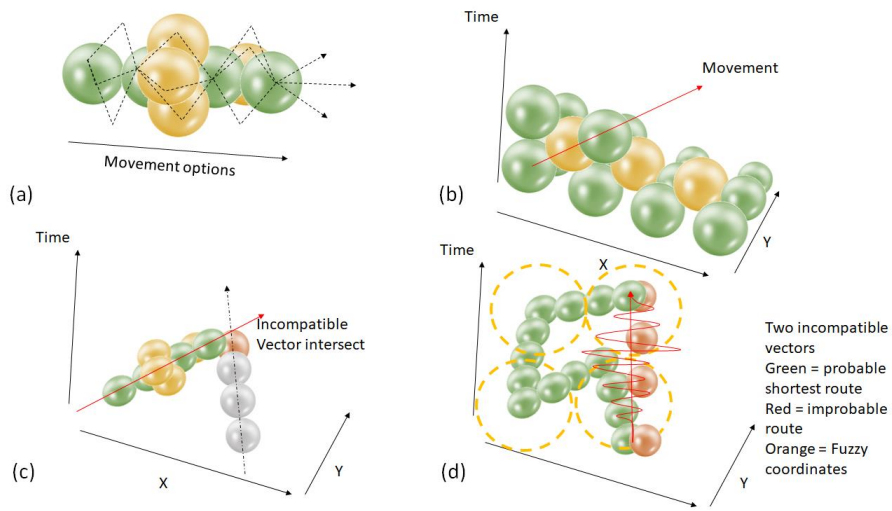


Figure S13 (a) A 'fuzzy' vector whereby the uncertainty of the path in between time points (green) is represented by a ring of points (yellow). (b) A packing which positions in-between "time slices" by alternately offsetting layers overtime. (c) Illustrating how one way of packing space with addresses may still result in multiple vectors having intersections that do not fall exactly onto any address in common (d) Fuzzy coordinates represent at a larger spatial scale two vectors- one which is a probable path (but meanders) the other which involves fewer nodes (but the probability of their connecting is low) such that both map to a common fuzzy vector.

Discussion up to here shows that a field emerges based on relative travel time between nodes but this cannot all be mapped onto a suitable lattice. However, what is needed is only a relative addressing of positions between the graph nodes, not from them to every other point in space regardless of whether anything is located there or not. The best lattice to project a specific set of

nodes and vectors onto depends on where those nodes are relative to one another. So for different subsets of the graph there are different ideal packings (Fig S13c).

One can intuitively imagine that routes with faster traffic lights will represent the most important routes through the network (the ones most likely to consistently deliver a minimum travel time) which thus define the minimum relative distance between all the nodes. These key routes will roughly map out the extent of the latent[4] field. Of course it is entirely possible that a line of slow traffic lights turn green in just the right sequence making another route much shorter – but only occasionally. If one includes the effect of these less likely routes, the ideal lattice changes a little. There is a tradeoff between addressing key vectors onto a lattice and addressing as many vectors as possible.

As the underlying process changes (i.e. the traffic lights change frequency) the key routes change and so too does the best lattice packing. So we see that there is a latent space of relative distances which is in principle genuinely continuous (*any* set of travel times is possible). This latent travel time space can be partially projected into multiple different mappings, so in 3D space the coordinate system becomes fuzzy from overlaying competing best projections (Fig S13d). Multiple views (such as across scales or over several time steps) can place a lattice address at any and every location in a continuous space. But a single view/instance at single precision radius cannot.

As the crow (probably) flies

Having established that is not possible to address all locations simultaneously, one is then faced with the question of what objects, tracks, edges and points look like in a fuzzy space-time? The shortest route is no longer the geometric straight line (which may not exist at any one instant) but the most probable shortest path. Fortunately we already have a way to describe this mathematically via a 'Feynman path integral' (invented by the physicist Richard Feynman to describe statistically how subatomic particles move). In this context integral might be considered to mean "when all taken together" or "net result of", so when all possible paths are weighed together given their respectively likely hoods, what is the most likely path? Equation 9 in the main paper just relates the probability that a topological path exists on the graph to that of a particle taking a straight line between the same two points. Anecdotally an object moving along a path integral vector might look something like Figure S13a if the highest packing density is hexagonal.

[4] Latent: the underlying or concealed phenomenon, in this case travel time underlies how "close" places are.

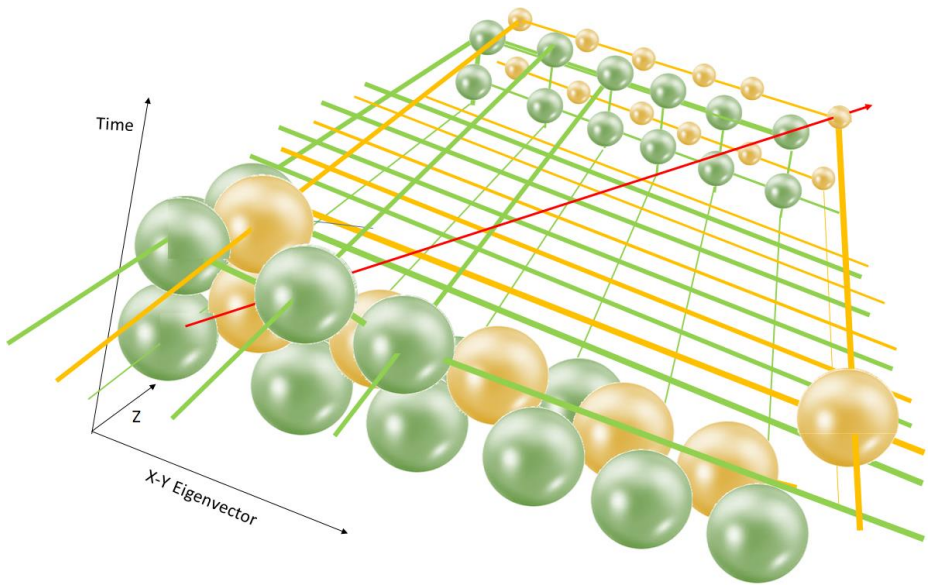


Figure S14 Face Centered Cubic Packing in '4'D (to illustrate a vector is chosen over X and Y). The red arrow represents a path such as from an airplane taking off, moving straight down the runway and gradually rising (in Z) over time. Green spheres represent position at each time step, yellow spheres its position in between each time step.

The price paid for a higher packing density than the Cartesian lattice, is that one cannot rotate the orientation of the packing so that a vector becomes a perfectly packed straight line along one axis. To move in one direction it is necessary to also weave to and fro in the perpendicular direction. This is why in section 4 of the main paper it is considered that, by using the 4th Dimension as time in a 4D hexagonal packing, time and space become "conjugate" i.e. one cannot move in one dimension without also moving in the other. Timeless paths are conceivable but arguably they are not meaningful. If nothing changes time is not measurable, if change takes no time distance is not measurable. Consequently, between any two points there is always "halfway" in both space *and* time. Fuzzy coordinates can represent that, but are inconvenient when one needs a single, crisp, map. The Face Centered Cubic Packing approximates this principle in a regular lattice by defining "halfway" as the alternate layers in two interleaved and offset cubic lattices creating one triangular lattice (S14)[5]. Although still not entirely space filling, FCCP at least provides a means to explicitly represent the incompleteness inherent to measurements of spatial change over time, representing the possible "half way" points. The key difference in using FCCP rather than simply doubling the resolution in Cartesian coordinates (cubic packing) is that higher resolution only redistributes the uncertainty of "in-between" at a smaller scale while FCCP explicitly represents it.

[5] In the supermarket trays of eggs are often stacked this way, each layer offset atop the one below. It is almost as efficient as a hexagonal packing but each layer is a convenient grid.

2.1.8 Summary

This section showed how a set of discrete relations (measured in time) can be represented as a truly continuous set of spatial relations with well defined (im)precision. It has also shown why, despite this being so, when one seeks to map (project) these spatial relations as absolute locations in Euclidean dimensions, relations between distinct positions in relative terms can become indistinct. In particular it can become unclear exactly where "half way" points along a relation actually are, either because the uncertainty in their true position overlaps the uncertainties of both start and end point or because there is no coordinate available to address the halfway point. One solution is to explicitly recognize that start and end points are at different positions in time, this opens up packing space for halfway points, offset in both space *and* time, creating a spatio-temporal Face Centered Cubic Packing.

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(Please see the main paper for previous work on the issues discussed above)