

Review

Phenomenologies in Hypersphere Soliton and Stringy Photon Models

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Abstract: We consider the Dirac quantization in the first-class formalism to investigate the hypersphere soliton model (HSM) defined on the S^3 hypersphere. To do this, we construct the first-class Hamiltonian possessing the Weyl ordering correction. In the HSM, we evaluate the baryon physical quantities such as the baryon masses, magnetic moments, axial coupling constant and charge radii, most predicted values of which are in good agreement with the corresponding experimental data. Moreover, shuffling the baryon and transition magnetic moments, we find the model independent sum rules. In the HSM we also evaluate the baryon intrinsic frequencies such as $\omega_N = 0.87 \times 10^{23} \text{ s}^{-1}$ and $\omega_\Delta = 1.74 \times 10^{23} \text{ s}^{-1}$ of the nucleon and delta baryon, respectively, to yield the identity $\omega_\Delta = 2\omega_N$. Next, making use of the Nambu-Goto string action and its extended rotating bosonic string theory, we formulate the stringy photon model to obtain the energy of the string configuration, which consists of the rotational and vibrational energies of the open string. Exploiting this total string energy, we evaluate the photon intrinsic frequency $\omega_\gamma = 9.00 \times 10^{23} \text{ s}^{-1}$, which is comparable to the corresponding baryon intrinsic frequencies. We also predict the photon size $\langle r^2 \rangle^{1/2}(\text{photon}) = 0.17 \text{ fm}$, which is approximately 21% of the proton magnetic charge radius.

Keywords: hypersphere soliton model; baryon physical quantities; intrinsic frequencies; stringy photon model; photon size

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1. Introduction

In this work, we will consider the extended objects instead of the point particles. It is well known that, as the extended objects, we have the solitons [1–9] and strings [10–13]. In particular, in this review we will mainly use the hypersphere soliton model (HSM) [4,6,9] and stringy photon model (SPM) [13]. To be more specific, in the soliton models, we have the standard Skyrmion, which describes baryon static properties in R^3 manifold [1,3,5,8]. This model was proposed by Skyrme in 1961 [1]. In this paper we will consider the paper by Adkins, Nappi and Witten (ANW) [3,5], to compare with the HSM. Next, we will investigate the HSM, which is formulated on the hypersphere S^3 instead of R^3 [4,6,9]. Exploiting the HSM, we will evaluate the baryon physical quantities, most of which are in good agreement with the corresponding experimental data.

In 1962, the electron was proposed as a charged conducting surface by Dirac [14]. According to his proposal, the electron shape and size should pulsate. Here, the surface tension of the electron was supposed to prevent the electron from flying apart under the repulsive forces of the charge. Motivated by his idea, we will investigate pulsating baryons in the first-class formalism in the HSM [4,6], to evaluate the intrinsic frequencies of the baryons, baryon masses with the Weyl ordering correction (WOC) and axial coupling constant [9].

On the other hand, as regards string theories, we have the critical higher-dimensional string theory [10–12], and the recently proposed SPM defined in four-dimensional spacetime, which predicts the photon radius, and the photon intrinsic frequency comparable to

the corresponding baryon intrinsic frequencies [13]. In the SPM, we have exploited the open string, which performs both rotational and vibrational motions [13]. Note that the rotational degrees of freedom of the photon have been investigated in the early universe [15,16].

In this review, we will exploit the HSM in the first-class Dirac Hamiltonian formalism to evaluate the physical quantities such as the baryon masses, magnetic moments, axial coupling constant, charge radii and baryon intrinsic frequencies. Next, in the SPM, we will predict the photon intrinsic frequency, which is shown to be comparable to the baryon intrinsic ones. To do this, we will exploit the Nambu–Goto string theory [17,18]. In the SPM, we will next introduce an open string action associated with the photon [19]. Making use of the rotational and vibrational energies of the string, we will evaluate explicitly the photon intrinsic frequency with which, assuming that the photon size is given by the string radius in the SPM, we will predict the photon size.

In Section 2, we will predict the baryon properties in the HSM. In Section 3, we will evaluate the intrinsic frequencies of the baryons in the HSM. In Section 4, we will exploit the SPM to predict the photon intrinsic frequency and photon size. Section 5 includes conclusions.

2. Baryon Predictions in HSM

Now, we consider the baryon predictions in the first-class Hamiltonian formalism in the HSM. To do this, we introduce the Skyrminion Lagrangian density given by

$$\mathcal{L} = \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{tr}[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 \tag{1}$$

where U is an SU(2) chiral field, and f_π and e are a pion decay constant and a dimensionless Skyrme parameter, respectively. In this work, we will treat f_π and e as the model parameters. Here, the quartic term is necessary to stabilize the soliton in the baryon sector.

Next, we introduce the hyperspherical three metric on S^3 of the form

$$ds^2 = \lambda^2 d\mu^2 + \lambda^2 \sin^2 \mu (d\theta^2 + \sin^2 \theta d\phi^2), \tag{2}$$

where the ranges of the three angles are defined as $0 \leq \mu \leq \pi, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, and λ ($0 \leq \lambda < \infty$) is a radius parameter of S^3 . In the HSM, using the Skyrminion Lagrangian density in (1), we obtain the soliton energy E of the form

$$E = \frac{f_\pi}{e} \left[2\pi L \int_0^\pi d\mu \sin^2 \mu \left(\left(\frac{df}{d\mu} + \frac{1}{L} \frac{\sin^2 f}{\sin^2 \mu} \right)^2 + 2 \left(\frac{1}{L} \frac{df}{d\mu} + 1 \right)^2 \frac{\sin^2 f}{\sin^2 \mu} \right) + 6\pi^2 B \right], \tag{3}$$

where $L = ef_\pi\lambda$ ($0 \leq L < \infty$) is a dimensionless radius parameter and B is topological baryon number, which is unity for a single soliton. Here, $f(\mu)$ is a profile function for hypersphere soliton, and satisfies $f(0) = \pi$ and $f(\pi) = 0$ for unit topological baryon number. Note that the the profile function f in the soliton energy E in (3) satisfies the first-order differential equations

$$\frac{df}{d\mu} + \frac{1}{L} \frac{\sin^2 f}{\sin^2 \mu} = 0, \quad \frac{1}{L} \frac{df}{d\mu} + 1 = 0, \tag{4}$$

to attain the BPS topological lower bound in the soliton energy [2,4–7] given by

$$E_B = \frac{6\pi^2 f_\pi}{e}. \tag{5}$$

Moreover, in this case, we find the equation of motion for the hypersphere soliton [4,6]

$$\left(L \sin^2 \mu + \frac{2}{L} \sin^2 f \right) \frac{d^2 f}{d\mu^2} + \left(L \sin 2\mu + \frac{1}{L} \frac{df}{d\mu} \sin 2f \right) \frac{df}{d\mu} - \left(L + \frac{1}{L} \frac{\sin^2 f}{\sin^2 \mu} \right) \sin 2f = 0. \tag{6}$$

One of the simplest solutions of (6) is the identity map

$$f(\mu) = \pi - \mu, \tag{7}$$

in which case the soliton energy in (3) can be rewritten as [4,6]

$$E = \frac{3\pi^2 f_\pi}{e} \left(L + \frac{1}{L} \right). \tag{8}$$

Note that, in order to obtain the BPS topological lower bound E_B in (5) by exploiting the soliton energy E in (8) associated with the identity map $f(\mu) = \pi - \mu$ in (7), we use the fixed value $L = L_B$ where

$$L_B \equiv e f_\pi \lambda_B = 1. \tag{9}$$

Note also that the identity map in (7) is a minimum energy solution for $L < \sqrt{2}$, while for $L > \sqrt{2}$, it is a saddle point [20,21].

Now, we briefly discuss the Dirac quantization of constrained systems [1,3–9,22]. In the HSM, we have the second class constraints for the collective coordinates a^μ ($\mu = 0, 1, 2, 3$) and the corresponding canonical momenta π^μ conjugate to a^μ of the form

$$\Omega_1 = a^\mu a^\mu - 1 \approx 0, \quad \Omega_2 = a^\mu \pi^\mu \approx 0. \tag{10}$$

Exploiting the Poisson bracket for a^μ and π^μ ,

$$\{a^\mu, \pi^\nu\} = \delta^{\mu\nu}, \tag{11}$$

we obtain the Poisson algebra for the commutator of Ω_1 and Ω_2 [8,22]

$$\{\Omega_1, \Omega_2\} = 2a^\mu a^\mu \neq 0. \tag{12}$$

Since this Poisson algebra does not vanish, we call the constraints Ω_1 and Ω_2 the second class.

In the HSM, spin and isospin states can be treated by collective coordinates $a^\mu = (a^0, \vec{a})$ ($\mu = 0, 1, 2, 3$), corresponding to the spin and isospin collective rotation $A(t) \in \text{SU}(2)$ given by $A(t) = a^0 + i\vec{a} \cdot \vec{\tau}$. Exploiting the coordinates a^μ , we obtain the Hamiltonian of the form

$$H = E_B + \frac{1}{8\mathcal{I}} \pi^\mu \pi^\mu, \tag{13}$$

where π^μ are canonical momenta conjugate to the collective coordinates a^μ . Here, the soliton energy lower bound E_B is given by (5) and moment of inertia \mathcal{I} is given by

$$\mathcal{I} = \frac{3\pi^2}{e^3 f_\pi}. \tag{14}$$

Note that the identity map $f(\mu) = \pi - \mu$ in (7) with condition $L = L_B$, where L_B is given by (9), is used to predict the physical quantities such as the moment of inertia \mathcal{I} in (14), baryon masses, charge radii, magnetic moments, axial coupling constant g_A and intrinsic pulsation frequencies ω_I in the HSM. Note also that the hypersphere coordinates (μ, θ, ϕ) are integrated out in (3), and E_B in (5) is a function of $\lambda_B = \frac{1}{e f_\pi}$ or equivalently f_π and e only. Similarly, after integrating out the hypersphere coordinates (μ, θ, ϕ) , the physical quantities in (14), (16) and (26)–(30) and (44) are formulated in terms of f_π and e only.

After performing the canonical quantization in the second class formalism in the HSM, we now construct the Hamiltonian spectrum

$$\langle H \rangle = E_B + \frac{1}{2\mathcal{I}} I(I + 1), \tag{15}$$

where $I (=1/2, 3/2, \dots)$ are baryon isospin quantum numbers. Exploiting (15), we find the nucleon mass M_N for $I = 1/2$ and delta baryon mass M_Δ for $I = 3/2$, respectively [6,9]

$$M_N = ef_\pi \left(\frac{6\pi^2}{e^2} + \frac{e^2}{8\pi^2} \right), \quad M_\Delta = ef_\pi \left(\frac{6\pi^2}{e^2} + \frac{5e^2}{8\pi^2} \right). \tag{16}$$

Next, we formulate the first-class constraints $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ by adding the terms related with the Stückelberg fields θ and π_θ

$$\tilde{\Omega}_1 = a^\mu a^\mu - 1 + 2\theta = 0, \quad \tilde{\Omega}_2 = a^\mu \pi^\mu - a^\mu a^\mu \pi_\theta = 0. \tag{17}$$

Here, θ and π_θ satisfy the following Poisson bracket

$$\{\theta, \pi_\theta\} = 1, \tag{18}$$

to produce the first-class Poisson algebra for the first-class constraints $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$

$$\{\tilde{\Omega}_1, \tilde{\Omega}_2\} = 0. \tag{19}$$

Now, we investigate the operator ordering problem in the first-class Hamiltonian formalism. To do this, we construct the first-class Hamiltonian [9]

$$\tilde{H} = E_B + \frac{1}{8\mathcal{I}} (\pi^\mu - a^\mu \pi_\theta) (\pi^\mu - a^\mu \pi_\theta) \frac{a^\nu a^\nu}{a^\nu a^\nu + 2\theta}. \tag{20}$$

Applying the first-class constrains in (17) to (20), we find

$$\tilde{H} = E_B + \frac{1}{8\mathcal{I}} (a^\mu a^\mu \pi^\nu \pi^\nu - \pi^\mu a^\mu a^\nu \pi^\nu). \tag{21}$$

Next, we introduce the Weyl ordering procedure [23] to obtain the Weyl-ordered operators

$$\begin{aligned} (a^\mu a^\mu \pi^\nu \pi^\nu)_W^{op} &= \frac{1}{4} [a^\mu (a^\mu \pi^\nu + \pi^\nu a^\mu) \pi^\nu + \pi^\nu (a^\mu \pi^\nu + \pi^\nu a^\mu) a^\mu] = -\frac{1}{4} (4a^\mu a^\mu \partial_\nu^2 + 8a^\mu \partial_\mu + 3\delta_{\mu\mu}), \\ (\pi^\mu a^\mu a^\nu \pi^\nu)_W^{op} &= \frac{1}{4} (\pi^\mu a^\mu + a^\mu \pi^\mu) (a^\nu \pi^\nu + \pi^\nu a^\nu) = -\frac{1}{4} (4a^\mu a^\nu \partial_\mu \partial_\nu + 20a^\mu \partial_\mu + \delta_{\mu\mu} \delta_{\nu\nu}), \end{aligned} \tag{22}$$

where we have used the quantum operator $\pi^\mu = -i \frac{\partial}{\partial a^\mu} \equiv -i\partial_\mu$. Inserting (22) into (21), we arrive at the Weyl-ordered first-class Hamiltonian operator

$$\tilde{H}_W^{op} = H^{op} + \frac{1}{32\mathcal{I}} \delta_{\mu\mu} (\delta_{\nu\nu} - 3), \tag{23}$$

where H^{op} is the second class Hamiltonian operator given by

$$H^{op} = E_B + \frac{1}{8\mathcal{I}} (-a^\mu a^\mu \partial_\nu^2 + 3a^\mu \partial_\mu + a^\mu a^\nu \partial_\mu \partial_\nu). \tag{24}$$

Here, the last three terms are the three-sphere Laplacian, given in terms of the collective coordinates and their derivatives to yield the eigenvalues $4I(I + 1)$ [24]. Inserting the relation $\langle H^{op} \rangle = E_B + \frac{1}{2\mathcal{I}} I(I + 1)$, which is also given in (15), and the identity $\delta_{\mu\mu} = 4$ into (23), we construct the Hamiltonian spectrum with the WOC in the first-class formalism [9]

$$\langle \tilde{H} \rangle = E_B + \frac{1}{2\mathcal{I}} \left[I(I + 1) + \frac{1}{4} \right], \tag{25}$$

where E_B is the soliton energy BPS lower bound in (5) and \mathcal{I} is the moment of inertia in (14). Comparing the canonical quantization spectrum result $\langle H \rangle$ in (15) with $\langle \tilde{H} \rangle$ obtained via Dirac quantization with the WOC, the latter has the additional term $\frac{1}{8\mathcal{I}}$ in (25). This

additional contribution originates from the first-class constraints in (17). The nucleon mass M_N ($I = 1/2$) and delta baryon mass M_Δ ($I = 3/2$) are then given as follows [9]

$$M_N = ef_\pi \left(\frac{6\pi^2}{e^2} + \frac{e^2}{6\pi^2} \right), \quad M_\Delta = ef_\pi \left(\frac{6\pi^2}{e^2} + \frac{2e^2}{3\pi^2} \right). \tag{26}$$

Next, we formulate the magnetic moments of the form [9]

$$\begin{aligned} \mu_p &= \frac{2M_N}{ef_\pi} \left(\frac{e^2}{48\pi^2} + \frac{\pi^2}{2e^2} \right), & \mu_n &= \frac{2M_N}{ef_\pi} \left(\frac{e^2}{48\pi^2} - \frac{\pi^2}{2e^2} \right), \\ \mu_{\Delta^{++}} &= \frac{2M_N}{ef_\pi} \left(\frac{e^2}{16\pi^2} + \frac{9\pi^2}{10e^2} \right), & \mu_{N\Delta} &= \frac{2M_N}{ef_\pi} \cdot \frac{\sqrt{2}\pi^2}{2e^2}, \end{aligned} \tag{27}$$

where M_N is now given by the nucleon mass with the WOC in (26), given in the first-class formalism. Next, we similarly obtain the axial coupling constant [9]

$$g_A = \frac{4\pi}{e^2} \int_0^\pi d\mu \sin^2 \mu (1 + \cos \mu) = \frac{2\pi^2}{e^2}. \tag{28}$$

Now, we consider the charge radii. The electric and magnetic isovector mean square charge radii are given in the HSM, respectively [6,9]

$$\langle r^2 \rangle_{E,I=1} = \langle r^2 \rangle_{M,I=1} = \frac{2}{3e^2\mathcal{I}} \int_{S^3} dV_B \sin^2 \mu \sin^2 f \left(1 + \left(\frac{df}{d\mu} \right)^2 + \frac{\sin^2 f}{\sin^2 \mu} \right) = \frac{5}{6e^2 f_\pi^2}, \tag{29}$$

where the subscripts E and M denote electric and magnetic charge radii, respectively, and $dV_B = \lambda_B^3 \sin^2 \mu \sin \theta d\mu d\theta d\phi$ on the hypersphere S^3 . Note that dV_B is given by the product of three arc lengths: $\lambda_B d\mu$, $\lambda_B \sin \mu d\theta$ and $\lambda_B \sin \mu \sin \theta d\phi$ and λ_B is radius of hypersphere soliton. Moreover, we find the charge radii in terms of ef_π [6,9]

$$\begin{aligned} \langle r^2 \rangle_{M,I=0}^{1/2} &= \langle r^2 \rangle_{M,I=1}^{1/2} = \langle r^2 \rangle_{M,p}^{1/2} = \langle r^2 \rangle_{M,n}^{1/2} = \langle r^2 \rangle_{E,I=1}^{1/2} = \sqrt{\frac{5}{6}} \frac{1}{ef_\pi}, \\ \langle r^2 \rangle_{E,I=0}^{1/2} &= \frac{\sqrt{3}}{2} \frac{1}{ef_\pi}, \quad \langle r^2 \rangle_p = \frac{19}{24} \frac{1}{(ef_\pi)^2}, \quad \langle r^2 \rangle_n = -\frac{1}{24} \frac{1}{(ef_\pi)^2}. \end{aligned} \tag{30}$$

Shuffling the above baryon and transition magnetic moments, we obtain the model independent sum rules in the HSM [6]

$$\begin{aligned} \mu_{\Delta^{++}} &= \frac{3}{5}(4\mu_p + \mu_n) \\ \mu_{N\Delta} &= \frac{\sqrt{2}}{2}(\mu_p - \mu_n) \\ \mu_p + \mu_n &= \frac{2}{9}M_N(M_\Delta - M_N)\langle r^2 \rangle_{E,I=0} \\ \mu_p - \mu_n &= \frac{M_N}{M_\Delta - M_N}. \end{aligned} \tag{31}$$

Next, we choose $\langle r^2 \rangle_{M,p}^{1/2} = 0.80$ fm as an input parameter. We then have

$$ef_\pi = 225.23 \text{ MeV} = (0.876 \text{ fm})^{-1}, \tag{32}$$

and exploiting this fixed value of ef_π and the phenomenological formulas in (26)–(30), we can proceed to calculate the physical quantities, as shown in Table 1.

Table 1. In Predictions I and II (Hong) [9], we use the hypersphere soliton model. In Prediction III (ANW), we exploit the standard Skyrmion model. The input parameters are indicated by *.

Quantity	Prediction I (Hong)	Prediction II (Hong)	Prediction III (ANW)	Experiment
$\langle r^2 \rangle_{M,I=0}^{1/2}$	0.80 fm	0.63 fm	0.92 fm	0.81 fm
$\langle r^2 \rangle_{M,I=1}^{1/2}$	0.80 fm	0.63 fm	∞	0.80 fm
$\langle r^2 \rangle_{M,p}^{1/2}$	0.80 fm *	0.63 fm	∞	0.80 fm
$\langle r^2 \rangle_{M,n}^{1/2}$	0.80 fm	0.63 fm	∞	0.79 fm
$\langle r^2 \rangle_{E,I=0}^{1/2}$	0.76 fm	0.60 fm	0.59 fm	0.72 fm
$\langle r^2 \rangle_{E,I=1}^{1/2}$	0.80 fm	0.63 fm	∞	0.88 fm
$\langle r^2 \rangle_p$	$(0.780 \text{ fm})^2$	$(0.61 \text{ fm})^2$	∞	$(0.805 \text{ fm})^2$
$\langle r^2 \rangle_n$	$-(0.179 \text{ fm})^2$	$-(0.14 \text{ fm})^2$	$-\infty$	$-(0.341 \text{ fm})^2$
μ_p	2.98	1.88	1.87	2.79
μ_n	-2.45	-1.32	-1.31	-1.91
$\mu_{\Delta^{++}}$	5.69	3.72	3.72	4.7 – 6.7
$\mu_{N\Delta}$	3.84	2.27	2.27	3.29
M_N	939 MeV *	939 MeV *	939 MeV *	939 MeV
M_Δ	1112 MeV	1232 MeV *	1232 MeV *	1232 MeV
g_A	1.30	0.98	0.61	1.23

Now, we discuss the predictions in the soliton models. In Table 1, Prediction I and II are given by Hong [9] using the HSM, while Prediction III is given by ANW [3], exploiting the standard Skyrmion model defined in R^3 manifold. Here, the input parameters are indicated by *. In Prediction I, the two experimental values for $\langle r^2 \rangle_{M,p}^{1/2}$ and M_N are used as input parameters. In Predictions II and III, we have exploited the same input parameters associated with M_N and M_Δ to compare their predictions effectively. Note that in Prediction II we have finite charge radii, while in Prediction III we have infinite charge radii.

Next, we discuss the evaluations of Prediction I. First, the six predicted values for $\mu_{\Delta^{++}}$, $\langle r^2 \rangle_{M,I=0}^{1/2}$, $\langle r^2 \rangle_{M,I=1}^{1/2}$, $\langle r^2 \rangle_{M,n}^{1/2}$ (in addition to the input parameters M_N and $\langle r^2 \rangle_{M,p}^{1/2}$) are within about 1% of the corresponding experimental data. Second, the three predictions for g_A , $\langle r^2 \rangle_{E,I=0}^{1/2}$ and $\langle r^2 \rangle_p$ are within about 6% of the experimental values. Third, the three predictions for M_Δ , μ_p and $\langle r^2 \rangle_{E,I=1}^{1/2}$ are within about 10% of the experimental values.

Now we comment on the hypersurface A_3 of the hypersphere S^3 of radius parameter λ_B , and the charge radius $\langle r^2 \rangle_{E,I=1}^{1/2}$ in (29). Exploiting the hyperspherical three metric in (2), we find that A_3 can be analyzed in terms of three arc length elements $\lambda_B d\mu$, $\lambda_B \sin \mu d\theta$ and $\lambda_B \sin \mu \sin \theta d\phi$, from which we find the three-dimensional hypersurface manifold with $A_3 = 2\pi^2 \lambda_B^3$. Note that λ_B is the radial distance from the center of S^3 to the hypersphere manifold S^3 in R^4 . In fact, inserting the value $ef_\pi = (0.876 \text{ fm})^{-1}$ in (32) into the condition $L_B = 1$ in (9), in the HSM we obtain the fixed radius parameter given by $\lambda_B = \frac{1}{ef_\pi} = 0.876 \text{ fm}$. On the other hand, the charge radius $\langle r^2 \rangle_{E,I=1}^{1/2}$ is the physical quantity expressed in (29). Integrating over a relevant density on S^3 corresponding to the integrand in (29), we evaluate $\langle r^2 \rangle_{E,I=1}$, which is now independent of μ , to yield a specific value of the electric isovector root mean square charge radius. The calculated charge radius then can be defined as the fixed radial distance to the point on a hypersurface manifold, which does not need to be located only on the compact manifold S^3 of radius parameter λ_B . This hypersurface manifold is now a submanifold in R^4 , which is located at $r = 0.80 \text{ fm}$, far from the center of S^3 . Note that $\langle r^2 \rangle_{E,I=1}^{1/2}$ denotes the radial distance, which is a geometrical invariant giving the same value both in R^3 (for instance, in volume R^3 , which contains the center of S^3 and is described in terms of (r, θ, ϕ) at $\mu = \frac{\pi}{2}$) and in R^4 . Next, the physical quantity $\langle r^2 \rangle_{E,I=1}^{1/2}$ calculated in R^3 (and in R^4) then can be compared with the corresponding experimental value, similar to the other physical quantities such as M_N and M_Δ .

Note that, as a toy model of soliton embedded in R^3 , we consider a uniformly charged manifold S^2 described in terms of (θ, ϕ) and a fixed radius parameter λ_B where we have $A_2 = 4\pi\lambda_B^2$. By integrating over a surface charge density residing on S^2 , one can calculate the physical quantity such as the electric potential, at an arbitrary observation point, which does not need to be located only on the compact manifold S^2 of radius parameter λ_B . Next, since the S^2 soliton of fixed radius parameter λ_B is embedded in R^3 , we manifestly define an arbitrary radial distance from the center of the compact manifold to an observation point, which is located in $R^3 = S^2 \times R$. Here, S^2 denotes foliation leaves [25] of the spherical shell of radius parameter λ ($0 \leq \lambda < \infty$) and R is a manifold associated with radial distance. Note that the radial distance itself is a fixed geometrical invariant producing the same value both in R^2 (for instance, on equatorial plane R^2 , which contains the center of S^2 and is delineated by (r, ϕ) at $\theta = \frac{\pi}{2}$) and in R^3 . The same mathematical logic can be applied to S^3 soliton of fixed radius parameter λ_B embedded in $R^4 = S^3 \times R$, where S^3 stands for the foliation leaves of the hyperspherical shell of radius parameter λ ($0 \leq \lambda < \infty$) and R is a manifold related with radial distance.

Finally, we have some comments on the Betti numbers associated with the manifold S^3 in the HSM. First of all, the p -th Betti number $b_p(M)$ is defined as the maximal number of p -cycles on M :

$$b_p(M) = \dim H_p(M), \tag{33}$$

where $H_p(M)$ is the homology group of the manifold M [26–28]. For the case of S^3 , we obtain

$$\begin{aligned} H_0(S^3) &= H_3(S^3) = \mathbf{Z}, \\ H_p(S^3) &= 0, \text{ otherwise.} \end{aligned} \tag{34}$$

The non-vanishing Betti numbers related with S^3 are thus given by $b_0(S^3) = b_3(S^3) = 1$.

3. Intrinsic Frequencies of Baryons

Now, we investigate the intrinsic frequencies of baryons in the first-class Hamiltonian formalism in the HSM. To do this, in the HSM we construct the equivalent first-class Hamiltonian \tilde{H}' as follows

$$\tilde{H}' = \tilde{H} + \frac{1}{4\mathcal{I}} \pi_\theta \tilde{\Omega}_2, \tag{35}$$

to yield the corresponding Gauss law constraint algebra

$$\{\tilde{\Omega}_1, \tilde{H}'\} = \frac{1}{2\mathcal{I}} \tilde{\Omega}_2, \quad \{\tilde{\Omega}_2, \tilde{H}'\} = 0. \tag{36}$$

Note that $\{\tilde{\Omega}_1, \tilde{H}\} = 0$ and $\{\tilde{\Omega}_2, \tilde{H}\} = 0$. We then find the Hamiltonian spectrum for \tilde{H}'

$$\langle \tilde{H}' \rangle = E_B + \frac{1}{2\mathcal{I}} \left[I(I+1) + \frac{1}{4} \right], \tag{37}$$

which is equal to that for \tilde{H} in (25), as expected. Next, we consider the equation of motion in Poisson bracket form

$$\dot{\tilde{W}} = \{\tilde{W}, \tilde{H}'\}, \quad \text{for the first class variable } \tilde{W}, \tag{38}$$

where the over-dot denotes the time derivative. Making use of the equation of motion in (38), we obtain these two equations

$$\dot{\tilde{a}}^\mu = \{\tilde{a}^\mu, \tilde{H}'\} = \frac{1}{4\mathcal{I}} \tilde{\pi}^\mu, \quad \dot{\tilde{\pi}}^\mu = \{\tilde{\pi}^\mu, \tilde{H}'\} = -\frac{1}{4\mathcal{I}} \tilde{\pi}^\nu \tilde{\pi}^\nu \tilde{a}^\mu, \tag{39}$$

where the first-class fields \tilde{a}^μ and $\tilde{\pi}^\mu$ are given as follows

$$\tilde{a}^\mu = a^\mu \left(\frac{a^\nu a^\nu + 2\theta}{a^\nu a^\nu} \right)^{1/2}, \quad \tilde{\pi}^\mu = (\pi^\mu - a^\mu \pi_\theta) \left(\frac{a^\nu a^\nu}{a^\nu a^\nu + 2\theta} \right)^{1/2}. \tag{40}$$

In order to formulate the equations in (39), we have used the following identities among the physical fields

$$\begin{aligned} \{\tilde{a}^\mu, \tilde{\pi}^\nu\} &= \delta^{\mu\nu} - \tilde{a}^\mu \tilde{a}^\nu, & \{\tilde{\pi}^\mu, \tilde{\pi}^\nu\} &= \tilde{\pi}^\mu \tilde{a}^\nu - \tilde{a}^\mu \tilde{\pi}^\nu, \\ \{\tilde{a}^\mu, \tilde{H}\} &= \frac{1}{4\mathcal{I}} (\tilde{\pi}^\mu - \tilde{a}^\mu \tilde{a}^\nu \tilde{\pi}^\nu), & \{\tilde{\pi}^\mu, \tilde{H}\} &= \frac{1}{4\mathcal{I}} (\tilde{\pi}^\mu \tilde{a}^\nu \tilde{\pi}^\nu - \tilde{a}^\mu \tilde{\pi}^\nu \tilde{\pi}^\nu), \\ \{\tilde{a}^\mu, \pi_\theta\} &= \tilde{a}^\mu, & \{\tilde{\pi}^\mu, \pi_\theta\} &= -\tilde{\pi}^\mu. \end{aligned} \tag{41}$$

Applying the equation of motion algorithm in (38) to \tilde{a}^μ , we find

$$\ddot{\tilde{a}}^\mu = \{\dot{\tilde{a}}^\mu, \tilde{H}'\} = \frac{1}{4\mathcal{I}} \dot{\tilde{\pi}}^\mu = -\frac{1}{4\mathcal{I}^2} \left[I(I+1) + \frac{1}{4} \right] \tilde{a}^\mu, \tag{42}$$

to yield the equation of motion for a simple harmonic oscillator

$$\ddot{\tilde{a}}^\mu = -\omega_I^2 \tilde{a}^\mu, \tag{43}$$

where ω_I is the intrinsic frequency of pulsating baryon with isospin quantum number I given by

$$\omega_I = \frac{1}{2\mathcal{I}} \left[I(I+1) + \frac{1}{4} \right]^{1/2}. \tag{44}$$

Making use of the formula for ω_I in (44) for the nucleon N ($I = \frac{1}{2}$) and the delta baryon Δ ($I = \frac{3}{2}$), we obtain predictions of intrinsic frequencies ω_N and ω_Δ of the baryons given in Table 2. Note that we find the identity $\omega_\Delta = 2\omega_N$.

Table 2. The intrinsic frequencies of particles [9,13].

Particle Type	Notation	Intrinsic Frequency
Nucleon	ω_N	$0.87 \times 10^{23} \text{ s}^{-1}$
Delta baryon	ω_Δ	$1.74 \times 10^{23} \text{ s}^{-1}$
Photon	ω_γ	$9.00 \times 10^{23} \text{ s}^{-1}$

Finally, it seems appropriate to comment on the gauge fixing problem within the first-class constraints of the Dirac Hamiltonian formalism. In order to investigate the gauge fixing of the first-class Hamiltonian \tilde{H}' in (35), we introduce two canonical sets of ghost and anti-ghost fields together with auxiliary fields $(\mathcal{C}^i, \bar{\mathcal{P}}_i), (\mathcal{P}^i, \bar{\mathcal{C}}_i), (\mathcal{N}^i, \mathcal{B}_i), (i = 1, 2)$, which satisfy the super-Poisson algebra, $\{\mathcal{C}^i, \bar{\mathcal{P}}_j\} = \{\mathcal{P}^i, \bar{\mathcal{C}}_j\} = \{\mathcal{N}^i, \mathcal{B}_j\} = \delta_j^i$. Here, the super-Poisson bracket is defined as $\{A, B\} = \frac{\delta A}{\delta q} |_r \frac{\delta B}{\delta p} |_l - (-1)^{\eta_A \eta_B} \frac{\delta B}{\delta q} |_r \frac{\delta A}{\delta p} |_l$, where η_A denotes the number of fermions, called the ghost number, in A and the subscript r and l denotes right and left derivatives, respectively. The BRST charge for the first-class constraint algebra related with \tilde{H}' is then given by

$$Q = \mathcal{C}^i \tilde{\Omega}_i + \mathcal{P}^i \mathcal{B}_i. \tag{45}$$

We choose the unitary gauge

$$\chi^1 = \Omega_1, \quad \chi^2 = \Omega_2, \tag{46}$$

by selecting the fermionic gauge fixing function $\Psi = \bar{C}_i \chi^i + \bar{P}_i \mathcal{N}^i$. Exploiting the BRST charge Q in (45), we find the BRST transformation rule defined as $\delta_Q F = \{Q, F\}$ for a physical field F

$$\begin{aligned} \delta_Q a^\mu &= -C^2 a^\mu, & \delta_Q \pi^\mu &= 2C^1 a^\mu + C^2(\pi^\mu - 2a^\mu \pi_\theta), & \delta_Q \theta &= C^2 a^\mu a^\mu, & \delta_Q \pi_\theta &= 2C^1, & \delta_Q C^i &= 0, \\ \delta_Q \bar{P}_i &= \bar{\Omega}_i, & \delta_Q \mathcal{P}^i &= 0, & \delta_Q \bar{C}_i &= \mathcal{B}_i, & \delta_Q \mathcal{N}^i &= -\mathcal{P}^i, & \delta_Q \mathcal{B}_i &= 0. \end{aligned} \tag{47}$$

Note that \tilde{H}' is not BRST invariant, which implies that $\delta_Q \tilde{H}' \neq 0$. Next, we obtain the gauge-fixed Hamiltonian

$$\tilde{H}'' = \tilde{H}' - \frac{1}{2\mathcal{L}} C^1 \bar{P}_2, \tag{48}$$

which is now invariant under the BRST transformation rule in (47), namely $\delta_Q \tilde{H}'' = 0$. Note that the BRST charge Q in (45) is nilpotent so that we can have $\delta_{Q^2} F = \{Q, \{Q, F\}\} = 0$ for a physical field F . Note also that $H_{eff} \equiv \tilde{H}'' - \{Q, \Psi\}$ is the BRST invariant Hamiltonian including the fermionic gauge-fixing function Ψ .

4. SPM Predictions

In this section, we will predict the physical quantities such as the photon intrinsic frequency and photon size in the SPM [13]. To do this, we will exploit the Nambu–Goto string action [17,18] and its extended rotating bosonic string model in $D = 3 + 1$ dimensions of spacetime [29]. Note that in the $D = 26$ dimension open string theory, which will be briefly discussed below, it is well known that there exists the vector boson with 24 independent polarizations [10,11], corresponding to the photon in the stringy photon model defined in the $D = 3 + 1$ dimensions of spacetime [13] considered in this paper.

Before we construct the SPM, we pedagogically summarize a mathematical formalism for the Nambu–Goto open string, which is related with a photon. In order to define the action on a curved manifold, we introduce (M, g_{ab}) , which is a D dimensional spacetime manifold M associated with the metric g_{ab} . Given g_{ab} , we can have a unique covariant derivative ∇_a satisfying [30]

$$\begin{aligned} \nabla_a g_{bc} &= 0, \\ \nabla_a \omega^b &= \partial_a \omega^b + \Gamma^b_{ac} \omega^c, \\ (\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c &= R_{abc}{}^d \omega_d. \end{aligned} \tag{49}$$

We parameterize an open string by two world sheet coordinates τ and σ , and then we have the corresponding vector fields $\xi^a = (\partial/\partial\tau)^a$ and $\zeta^a = (\partial/\partial\sigma)^a$. The Nambu–Goto string action is now given by [17,18]

$$S = -\kappa \int \int d\tau d\sigma f(\tau, \sigma), \tag{50}$$

where the coordinates τ and σ have ranges $\tau_1 \leq \tau \leq \tau_2$ and $0 \leq \sigma \leq \pi$, respectively, and

$$f(\tau, \sigma) = [(\xi \cdot \zeta)^2 - (\xi \cdot \xi)(\zeta \cdot \zeta)]^{1/2}. \tag{51}$$

Here, the string tension κ is defined by $\kappa = \frac{1}{2\pi\alpha'}$, with α' being the universal slope of the linear Regge trajectories [31].

We perform an infinitesimal variation of the world sheets $\gamma_\alpha(\tau, \sigma)$ traced by the open string during its evolution in order to find the string geodesic equation from the least action principle. Here, we impose the restriction that the length of the string is τ independent. We introduce the deviation vector $\eta^a = (\partial/\partial\alpha)^a$, which represents the displacement to an infinitesimally nearby world sheet, and we consider Σ , which denotes the three-dimensional

submanifold spanned by the world sheets $\gamma_\alpha(\tau, \sigma)$. We then may choose τ, σ and α as coordinates of Σ to yield the commutator relations

$$\begin{aligned} \mathcal{L}_{\tilde{\zeta}}\eta^a &= \tilde{\zeta}^b \nabla_b \eta^a - \eta^b \nabla_b \tilde{\zeta}^a = 0, \\ \mathcal{L}_{\zeta}\eta^a &= \zeta^b \nabla_b \eta^a - \eta^b \nabla_b \zeta^a = 0, \\ \mathcal{L}_{\tilde{\zeta}}\zeta^a &= \tilde{\zeta}^b \nabla_b \zeta^a - \zeta^b \nabla_b \tilde{\zeta}^a = 0. \end{aligned} \tag{52}$$

Now, we find the first variation as follows

$$\frac{dS}{d\alpha} = \int \int d\tau d\sigma \eta_b (\tilde{\zeta}^a \nabla_a p^b + \zeta^a \nabla_a \pi^b) - \int d\sigma p^b \eta_b |_{\tau=\tau_1}^{\tau=\tau_2} - \int d\tau \pi^b \eta_b |_{\sigma=0}^{\sigma=\pi}, \tag{53}$$

where the world sheet currents associated with τ and σ directions are respectively given by [31],

$$\begin{aligned} p^a &= \frac{\kappa}{f} [(\tilde{\zeta} \cdot \zeta) \zeta^a - (\zeta \cdot \zeta) \tilde{\zeta}^a], \\ \pi^a &= \frac{\kappa}{f} [(\tilde{\zeta} \cdot \zeta) \tilde{\zeta}^a - (\tilde{\zeta} \cdot \tilde{\zeta}) \zeta^a]. \end{aligned} \tag{54}$$

Using the endpoint conditions

$$\eta^a(\tau = \tau_1; \sigma) = \eta^a(\tau = \tau_2; \sigma) = 0, \tag{55}$$

and

$$\pi^a(\tau; \sigma = 0) = \pi^a(\tau; \sigma = \pi) = 0, \tag{56}$$

we have the string geodesic equation

$$\tilde{\zeta}^a \nabla_a p^b + \zeta^a \nabla_a \pi^b = 0, \tag{57}$$

and constraint identities [31]

$$\begin{aligned} p \cdot \zeta &= 0, \quad p \cdot p + \kappa^2 \zeta \cdot \zeta = 0, \\ \pi \cdot \tilde{\zeta} &= 0, \quad \pi \cdot \pi + \kappa^2 \tilde{\zeta} \cdot \tilde{\zeta} = 0. \end{aligned} \tag{58}$$

For more details of the string theory and deviation vector on the curved manifold, see the references [15,16,30].

Next, we consider the open rotating string in the (3 + 1) dimensional flat spacetime and delineate the string in terms of the coordinates

$$x_\mu = (x_0, x_i) = (\tau, x_i(\tau; \sigma)), \quad (i = 1, 2, 3). \tag{59}$$

The Nambu–Goto string action in (50) is then described in terms of $f(\tau, \sigma)$ given by

$$f(\tau, \sigma) = [(\dot{x}_\mu x'_\mu)^2 - (\dot{x}_\mu \dot{x}_\mu)(x'_\mu x'_\mu)]^{1/2}, \tag{60}$$

where the overdot and prime denote derivatives with respect to τ and σ , respectively. In this paper, we use the metric signature (+, −, −, −).

Inserting (59) into (60), we find

$$f(\tau, \sigma) = [(\dot{x}_i x'_i)^2 + (1 - \dot{x}_i \dot{x}_i) x'_j x'_j]^{1/2}. \tag{61}$$

Moreover, we proceed to construct the world sheet currents

$$\begin{aligned}
 p_0 &= \frac{\kappa}{f} \dot{x}_i x'_i, & p_i &= -\frac{\kappa}{f} [(\dot{x}_j x'_j) x'_i - (x'_j x'_j) \dot{x}_i], \\
 \pi_0 &= -\frac{\kappa}{f} \dot{x}_i x'_i, & \pi_i &= -\frac{\kappa}{f} [(\dot{x}_j x'_j) \dot{x}_i + (1 - \dot{x}_j \dot{x}_j) x'_i].
 \end{aligned}
 \tag{62}$$

Now, exploiting (57), we obtain the string equation of motion

$$\frac{\partial p_\mu}{\partial \tau} + \frac{\partial \pi_\mu}{\partial \sigma} = 0,
 \tag{63}$$

and the string boundary condition

$$\pi_\mu(\tau; \sigma = 0) = \pi_\mu(\tau; \sigma = \pi) = 0.
 \tag{64}$$

Inserting p_μ and π_μ in (62) into the string equation of motion in (63), we find

$$\begin{aligned}
 \frac{\partial}{\partial \tau} \left(\frac{x'_i x'_i}{f} \right) - \frac{\partial}{\partial \sigma} \left(\frac{\dot{x}_i x'_i}{f} \right) &= 0, \\
 \frac{\partial}{\partial \tau} \left[\frac{(\dot{x}_j x'_j) x'_i - (x'_j x'_j) \dot{x}_i}{f} \right] + \frac{\partial}{\partial \sigma} \left[\frac{(\dot{x}_j x'_j) \dot{x}_i + (1 - \dot{x}_j \dot{x}_j) x'_i}{f} \right] &= 0.
 \end{aligned}
 \tag{65}$$

Exploiting the boundary conditions in (64), we also obtain at $\sigma = 0$ and $\sigma = \pi$

$$\dot{x}_i x'_i = 0, \quad (1 - \dot{x}_j \dot{x}_j) x'_i = 0.
 \tag{66}$$

Next, in order to describe an open string, which is rotating in the (x_1, x_2) plane and residing on the string center of mass, we take an ansatz [29]

$$x_i^{rot} = (r(\sigma) \cos \omega \tau, r(\sigma) \sin \omega \tau, 0).
 \tag{67}$$

Here, we propose that $r(\sigma)$ and ω represent respectively the diameter and angular velocity of the photon with solid spherical shape, which is delineated by the open string. Note that $r(\sigma = \pi/2)$ denotes the center of the diameter of the string. More specifically, $r(\sigma = \pi/2)$ is located in the center of the solid sphere, which describes the photon. The first boundary condition in (66) is trivially satisfied and the second one yields

$$r'(\sigma = 0, \pi) = 0.
 \tag{68}$$

We then obtain $r(\sigma)$, which fulfills the above condition in (68)

$$r(\sigma) = \frac{1}{\omega} \cos \sigma.
 \tag{69}$$

Note that the photon has a finite size, which is filled with mass. Using the photon configuration in (67) and (69) together with (62), we find the rotational energy of the photon

$$E^{rot} = \int_0^\pi d\sigma p_0^{rot} = \frac{1}{2\alpha' \hbar \omega},
 \tag{70}$$

where we have included \hbar factor explicitly, and the value of α' is given by $\alpha' = 0.95 \text{ GeV}^{-2}$ [31].

Next, we evaluate the photon intrinsic frequency and size in the SPM. To do this, we calculate the vibrational energy of the photon by introducing the string coordinate configurations

$$x_i = x_i^{rot} + y_i, \quad i = 1, 2, 3.
 \tag{71}$$

Exploiting the coordinates in (71), we expand the string Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial \dot{x}_j} |_{0} \dot{y}_i \dot{y}_j + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial x'_j} |_{0} \dot{y}_i y'_j + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial x'_i \partial x'_j} |_{0} y'_i y'_j + \dots, \tag{72}$$

where the subscript 0 denotes that the terms in (72) are evaluated by using the coordinates in (67). The ellipsis stands for the higher derivative terms. Here, the first term is a constant given by $\mathcal{L}_0 = \mathcal{L}(x_i^{rot})$. The first derivative terms vanish after exploiting the string equation of motion in (63). Next, in order to obtain the vibration energy of the photon, we define coordinates z_i , which co-rotates with the string itself

$$\begin{aligned} z_1 &= y_1 \cos \omega \tau + y_2 \sin \omega \tau, \\ z_2 &= -y_1 \sin \omega \tau + y_2 \cos \omega \tau, \\ z_3 &= y_3. \end{aligned} \tag{73}$$

After some algebra, we obtain the Lagrangian density associated with the coordinates z_i

$$\begin{aligned} \mathcal{L}(z_i) &= \frac{\kappa}{2 \sin^2 \sigma} \left[\frac{1}{\omega} (\dot{z}_2 + \omega z_1)^2 + 2 \sin \sigma \cos \sigma ((\dot{z}_1 - \omega z_2) z'_2 \right. \\ &\quad \left. - (\dot{z}_2 + \omega z_1) z'_1) - \omega z_2'^2 \right] + \frac{\kappa}{2\omega} (\dot{z}_3^2 - \omega^2 z_3'^2). \end{aligned} \tag{74}$$

The equations of motion for the directions z_2 and z_3 are then given by

$$\begin{aligned} \ddot{z}_2 + \omega^2 z_2 + 2\omega^2 \cot \sigma z'_2 - \omega^2 z_2'' &= 0, \\ \ddot{z}_3 - \omega^2 z_3'' &= 0. \end{aligned} \tag{75}$$

Now, the photon is assumed to be in the ground state of the string energy spectrum. From (75), we find the eigenfunctions for the ground states

$$z_2 = c_2 \sin(\omega \tau + \phi_2), \tag{76}$$

$$z_3 = c_3 \cos \sigma \sin(\omega \tau + \phi_3). \tag{77}$$

Here, ϕ_2 and ϕ_3 are arbitrary phase constants, which are irrelevant to the physics arguments of interest.

It seems appropriate to address comments on the photon vibration modes. The transverse mode z_2 in (76) is independent of the string coordinate σ , so that the photon can tremble back and forth with a constant amplitude, while the longitudinal mode z_3 in (77) possesses sinusoidal dependence on σ . Here, note that z_3 does not move at the center of the string, namely at $\sigma = \pi/2$, independent of τ and the other parts of the string oscillate with the frequency ω . As for the transverse mode z_1 , we can find that any value for z_1 satisfies the Euler–Lagrange equation for z_1 obtained from the Lagrangian density in (74). Up to now, we have considered a single massive photon with the solid sphere shape, whose diameter is delineated in terms of the length of the open string. The photon thus has a disk-like cross section on which the coordinates z_1 and z_2 reside. Note that, similar to the phonon associated with massive particle lattice vibrations, the photon is massive so that we can have three polarization directions: two transverse directions, as in the massless photon case, and an additional longitudinal one. Keeping this argument in mind, we find that there exist two transverse modes z_1 and z_2 associated with the photon vibrations on z_1 - z_2 cross sectional disk, in addition to one longitudinal mode z_3 . We thus have the transverse mode in z_1 direction to yield the eigenfunction for the ground state, with an arbitrary phase constant ϕ_1 similar to ϕ_2 and ϕ_3 discussed above,

$$z_1 = c_1 \sin(\omega \tau + \phi_1). \tag{78}$$

Note that, as in the case of massless photons, z_1 mode oscillates with the same frequency ω that z_2 mode does. Note also that the above solutions z_i satisfy their endpoint conditions at $\sigma = 0$ and $\sigma = \pi$

$$z'_i = 0, \quad i = 1, 2, 3. \tag{79}$$

The energy eigenvalues in the ground states in (76)–(78) are then given by

$$E_i^{vib} = \frac{1}{2} \hbar \omega, \quad i = 1, 2, 3. \tag{80}$$

Exploiting the energies in (80), we arrive at the vibrational energy of the open string ground state

$$E^{vib} = \sum_{i=1}^3 E_i^{vib} = \frac{3}{2} \hbar \omega. \tag{81}$$

In the SPM, the classical energy of the string is given by E^{rot} in (70), while the corresponding quantum mechanical energy is given by E^{vib} in (81). Now, we define the total energy of the string configuration as a function of ω

$$E = E^{rot} + E^{vib} = \frac{1}{2\alpha' \hbar \omega} + \frac{3}{2} \hbar \omega. \tag{82}$$

Note that we have already removed the translational degree of freedom, by considering the string observer residing on the photon center of mass in the SPM. Varying the energy E with respect to ω , we find the minimum value condition for E at $\omega = \omega_\gamma$

$$\frac{dE}{d\omega}(\omega = \omega_\gamma) = 0, \tag{83}$$

to yield the intrinsic frequency ω_γ for the photon [13]

$$\omega_\gamma = \frac{1}{\hbar(3\alpha')^{1/2}} = 9.00 \times 10^{23} \text{ s}^{-1}, \tag{84}$$

which is greater than the baryon intrinsic frequencies, as shown in Table 2. Note that we have related the HSM and SPM, in terms of the intrinsic frequencies of the baryons and photon, both of which are the extended objects. Next, we consider the photon radius as a half of the open string length. Exploiting the photon intrinsic frequency ω_γ in (84), we obtain the photon radius [13]

$$\langle r^2 \rangle^{1/2}(\text{photon}) = \frac{c}{2\omega_\gamma} = 0.17 \text{ fm}, \tag{85}$$

which is 21% of the proton magnetic charge radius $\langle r^2 \rangle_M^{1/2}(\text{proton}) = 0.80 \text{ fm}$ as shown in Table 1.

Next, even though up to now we have investigated the stringy photon model in $D = 3 + 1$ dimensional spacetime [13], we parenthetically discuss the bosonic string theory in the critical dimension $D = 26$. In the light-cone gauge quantization in bosonic string theory, the so-called anomaly associated with the commutator of Lorentz generators has been canceled in $D = 26$ critical dimensions and with the condition $a = -\frac{D-2}{2}R = 1$ [10,11], where R is the Ramanujan evaluation [32] for the infinite sum

$$R \equiv 1 + 2 + 3 + \dots = -\frac{1}{12}. \tag{86}$$

Now, we investigate the Ramanujan evaluation procedure for R . To accomplish this, we manipulate the difference $R - 4R$ to yield $R - 4R = -3R = 1 + (2 - 4) + 3 + (4 - 8) + 5 + \dots = 1 - 2 + 3 - 4 + 5 + \dots = \frac{1}{4}$. Here, we have used the identity $1 + 2x + 3x^2 + 4x^3 + \dots = \frac{1}{(1-x)^2}$, which yields $1 - 2 + 3 - 4 + \dots = \frac{1}{4}$ at $x = -1$. Following the above relation

$R - 4R = -3R = \frac{1}{4}$, we finally obtain the Ramanujan evaluation $R = -\frac{1}{12}$ in (86). Next, exploiting the Ramanujan evaluation in (86), we can obtain the relation

$$\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \cdots = \frac{1}{2}R - R = \frac{1}{24}, \quad (87)$$

which has been used in the $D = 10$ superstring theory [11,13]. Note that the stringy photon model has been described in $D = 3 + 1$ dimensional spacetime, without resorting to (86) and (87).

5. Conclusions

In summary, we have formulated the HSM to discuss the Dirac quantization in the first-class formalism and the predictions of baryon physical quantities in HSM. To be specific, we have evaluated the intrinsic pulsating frequencies of the baryons. To accomplish this, we have exploited the first-class Hamiltonian possessing the WOC and quantized the hypersphere soliton in the HSM. Here, we have noticed that the profile function in the soliton energy of the hypersphere soliton satisfies the two first-order differential equations to attain the BPS topological lower bound in the soliton energy. Next, we have evaluated the baryon physical quantities such as baryon masses, magnetic moments, charge radii and axial coupling constant. Shuffling the baryon and transition magnetic moments, we have constructed the model independent sum rules. The intrinsic frequency for more massive particles also has been shown to be greater than that for less massive ones. Explicitly, we have evaluated the intrinsic frequencies $\omega_N = 0.87 \times 10^{23} \text{ s}^{-1}$ and $\omega_\Delta = 1.74 \times 10^{23} \text{ s}^{-1}$ of the nucleon and delta baryon, respectively, to yield the identity $\omega_\Delta = 2\omega_N$.

Next, making use of the Nambu–Goto string action and its extended rotating bosonic string theory, we have formulated the SPM to find the energy of the string configuration, which consists of the rotational and vibrational energies of the open string. We have then investigated a photon intrinsic frequency by exploiting the string, which performs both rotational and pulsating motions. Here, we have interpreted the string as a diameter of a solid spherical shape photon. Explicitly, we have found that the intrinsic frequency of the photon is comparable to those of the baryons such as nucleon and delta baryon. In the SPM, we also have evaluated the photon size given by the string radius, which is approximately 21% of the proton magnetic charge radius. It will be interesting to search for strong experimental evidence for the photon size, which could be associated with the manifest photon phenomenology such as the photoelectric effect, Compton scattering and Raman scattering, for instance. Assuming that the SPM exploited in this paper could be a precise description for the photon, the photon intrinsic frequency $\omega_\gamma = 9.00 \times 10^{23} \text{ s}^{-1}$ and photon size $\langle r^2 \rangle^{1/2}(\text{photon}) = 0.17 \text{ fm}$ could be fundamental predictions in the extended object phenomenology, similar to ω_N , ω_Δ and the charge radii in the hypersphere soliton model.

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References

1. Skyrme, T.H.R. A nonlinear field theory. *Proc. R. Soc. Lond. A* **1961**, *260*, 127.
2. Faddeev, L.D. Some comments on the many dimensional solitons. *Lett. Math. Phys.* **1976**, *1*, 289. [[CrossRef](#)]
3. Adkins, G.S.; Nappi, C.R.; Witten, E. Static properties of nucleons in the Skyrme model. *Nucl. Phys. B* **1983**, *228*, 552. [[CrossRef](#)]
4. Manton, N.S.; Ruback, P.J. Skyrmions in flat space and curved space. *Phys. Lett. B* **1986**, *181*, 137. [[CrossRef](#)]
5. Adkins, G.S. *Chiral Solitons*; Liu, K.F., Ed.; World Scientific: Singapore, 1987.
6. Hong, S.T. The static properties of Skyrmions on a hypersphere. *Phys. Lett. B* **1998**, *417*, 211; Corrigendum in *Phys. Lett. B* **2020**, *803*, 135179. [[CrossRef](#)]

7. Manton, N.S.; Sutcliffe, P. *Topological Solitons*; Cambridge University Press: Cambridge, UK, 2004.
8. Hong, S.T. *BRST Symmetry and de Rham Cohomology*; Springer: Heidelberg, Germany, 2015.
9. Hong, S.T. Dirac quantization and baryon intrinsic frequencies in hypersphere soliton model. *Nucl. Phys. B* **2021**, *973*, 115611. [[CrossRef](#)]
10. Green, M.B.; Schwarz, J.H.; Witten, E. *Superstring Theory*; Cambridge University Press: Cambridge, UK, 1987.
11. Govaerts, J. The geometrical principles of string theory constructions. In *Particle Physics Cargèse 1989*; Levy, M., Basdevant, J.L., Jacob, M., Speiser, D., Weyers, J., Gastmans, R., Eds.; Springer: Heidelberg, Germany, 1990.
12. Polchinski, J. *String Theory*; Cambridge University Press: Cambridge, UK, 1998.
13. Hong, S.T. Photon intrinsic frequency and size in stringy photon model. *Nucl. Phys. B* **2022**, *976*, 115720. [[CrossRef](#)]
14. Dirac, P.A.M. An extensible model of the electron. *Proc. R. Soc. Lond. A* **1962**, *268*, 57.
15. Cho, Y.S.; Hong, S.T. Singularities in geodesic surface congruence. *Phys. Rev. D* **2008**, *78*, 067301. [[CrossRef](#)]
16. Cho, Y.S.; Hong, S.T. Dynamics of stringy congruence in early universe. *Phys. Rev. D* **2011**, *83*, 104040. [[CrossRef](#)]
17. Nambu, Y. *Lecture at the Copenhagen Symposium*; University of Chicago: Chicago, IL, USA, 1970; unpublished.
18. Goto, T. Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model. *Prog. Theor. Phys.* **1971**, *46*, 1560. [[CrossRef](#)]
19. Scherk, J.; Schwarz, J.H. Dual models and the geometry of space-time. *Phys. Lett. B* **1974**, *52*, 347. [[CrossRef](#)]
20. Manton, N.S. Geometry of Skyrmions. *Commun. Math. Phys.* **1987**, *111*, 469. [[CrossRef](#)]
21. Jackson, A.D.; Wirzba, A.; Castillejo, L. Two Skyrmions on a hypersphere. *Nucl. Phys. A* **1988**, *486*, 634. [[CrossRef](#)]
22. Dirac, P.A.M. *Lectures on Quantum Mechanics*; Yeshiva University Press: New York, NY, USA, 1964.
23. Lee, T.D. *Particle Physics and Introduction to Field Theory*; Harwood Academic Publishers: London, UK, 1981; p. 477.
24. Neto, J.A. Remarks on the collective quantization of the SU(2) Skyrmion model. *J. Phys. G* **1995**, *21*, 695. [[CrossRef](#)]
25. Candel, A.; Conlon, L. *Foliations I*; American Mathematical Society: Providence, RI, USA, 2000.
26. Rotman, J.J. *An Introduction to Algebraic Topology*; Springer: Heidelberg, Germany, 1988.
27. Weibel, C.A. *An Introduction to Homological Algebra*; Cambridge University Press: Cambridge, UK, 1994.
28. Frankel, T. *The Geometry of Physics*; Cambridge University Press: Cambridge, UK, 1997.
29. Kikkawa, K.; Sato, M.; Uehara, K. Non-linear strings. *Prog. Theor. Phys.* **1977**, *57*, 2101. [[CrossRef](#)]
30. Wald, R.M. *General Relativity*; University of Chicago Press: Chicago, IL, USA, 1984.
31. Scherk, J. An introduction to the theory of dual models and strings. *Rev. Mod. Phys.* **1975**, *47*, 123. [[CrossRef](#)]
32. Berndt, B.C. *Ramanujan's Notebooks: Part 1*; Springer: Heidelberg, Germany, 1985; p. 135.

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