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Comment on the Vacuum Energy Density for $\lambda\phi^4$ Theory in d Spacetime Dimensions

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Abstract: In a recent article we showed that the vacuum energy density in two spacetime dimensions for a wide variety of integrable quantum field theories has the form $\rho_{\rm vac} = -m^2/2\mathfrak{g}$ where m is a physical mass and $\mathfrak{g}$ is a generalized coupling, where in the free field limit $\mathfrak{g} \to 0$, $\rho_{\rm vac}$ diverges. This vacuum energy density has the form $\langle T_{\mu\nu}\rangle = -\rho_{\rm vac} g_{\mu\nu}$, and has previously been considered as a contribution to the stress energy tensor in Einstein's gravity as a "cosmological constant". We speculated that in four spacetime dimensions $\rho_{\rm vac}$ takes a similar form $\rho_{\rm vac} = -m^4/2\mathfrak{g}$, but did not support this idea in any specific model. In this article, we study this problem for $\lambda\phi^4$ theory in d spacetime dimensions. We show how to obtain the *exact* $\rho_{\rm vac}$ for the sinh–Gordon theory in the weak coupling limit by using a saddle point approximation. This calculation indicates that the vacuum energy can be well-defined, positive or negative, without spontaneous symmetry breaking. We also show that $\rho_{\rm vac}$ satisfies a Callan–Symanzik type of renormalization group equation. For the most interesting case physically, $\rho_{\rm vac}$ is positive and can arise from a marginally relevant negative coupling $\mathfrak{g}$ and the vacuum energy flows to zero at low energies.

Keywords: vaccum energy; quantum field theory; cosmological constant

1. Introduction

The so-called "cosmological constant problem" continues to provide serious challenges to our understanding of fundamental physics. Einstein's equations of General Relativity involve the classical stress-energy tensor as a source of gravitation and should include all possible sources of stress energy. Experimental cosmology provides evidence for a very small positive cosmological constant, and the origins of it remain unknown. There are many possibilities that remain to be explored, everything from modified classical General Relativity, quantum fluctuations of the vacuum, to quantum gravity effects at the Planck scale. In this article, we study this problem from the point of view in which it was first stated, namely as originating in quantum vacuum fluctuations, which is where the often-quoted discrepancy by 120 orders of magnitude originated. However, we do not claim any kind of resolution of the problem, since the non-zero cosmological constant may have completely different origins. Nevertheless, it is worthwhile to fully explore this option. We need to say, however, that some researchers believe that this problem cannot be resolved without considering quantum fields in curved spacetime, which is far beyond the scope of this article. See, for instance, the recent articles [1,2] and references therein.

In a semi-classical quantum theory, it is reasonable to suppose that the classical $T_{\mu\nu}$ is replaced by its quantum vacuum expectation value $\langle 0|T_{\mu\nu}|0\rangle$, where $|0\rangle$ is the vacuum state. Based on general coordinate invariance, one expects

$$\langle 0|T_{\mu\nu}|0\rangle = -\rho_{\rm vac}\, g_{\mu\nu} \tag{1}$$

where $g_{\mu\nu}$ is the spacetime metric. In the above equation, the convention for the metric is the signature $g_{\mu\nu} = {\rm diag}(-1,1,1,1)$, i.e., $g_{00} = -g_{ii} = -1$ in Minkowski space. Perhaps the first version of the cosmological constant problem was based on viewing a *free* quantum

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field as a collection of harmonic oscillators of frequency $\omega_k = \sqrt{k^2 + m^2}$, and the vacuum energy is naively the sum of the zero point energies [3,4]:

$$\rho_{\text{vac}} = \int_0^\Lambda \frac{dk}{(2\pi)^3} \, 4\pi k^2 \, \tfrac{1}{2} \sqrt{k^2 + m^2} \approx \frac{\Lambda^4}{16\pi^2} \tag{2}$$

where Λ is an ultraviolet cutoff and we have assumed $\Lambda \gg m$. The problem is that for reasonable values of the cut-off Λ, such as the Planck scale, the above ρ_{vac} is off by roughly 120 orders of magnitude compared to astrophysical measurements. The original problem has evolved to consider a series of phase transitions in the thermal development of the dynamical evolution of the Universe where Λ is a scale of spontaneous symmetry breaking (SSB), such as the electro-weak scale, a supersymmetry breaking scale, or even the QCD scale (see, for example, the review [5] and references therein.) In any case, the corresponding Λ leads to much too high a scale to explain the observed astrophysical value of ρ_{vac}.

We emphasize that we will not consider here quantum fields in curved spacetime, nor quantum gravity; in fact, we neglect gravity entirely. This is the main shortcoming of this article since, as previously mentioned, some physicists feel that one cannot avoid dealing with gravity in order to even attempt to solve the cosmological constant problem. Although ignoring gravity may turn out to be an oversimplification, let us mention that one does not need to understand quantum electrodynamics in curved spacetime in order to understand the cosmic microwave background, so as a first step we can suppose this is true for the cosmological constant itself. We feel this is justified since the original version of the cosmological constant problem was based on divergences in the vacuum energy in pure quantum field theory without gravity, and it is worthwhile to make sense of this since it is not well understood; it is worth exploring whether it can be ruled out or not, and we will argue that it is not. Although perhaps an abuse of terminology, henceforth we refer to "ρ_{vac}" and "cosmological constant" interchangeably, although clearly we are not studying cosmology per se but rather the vacuum energy density of an interacting QFT in flat space.

One should strongly question the above naive computation in (2), since we are accustomed to dealing with divergences in quantum field theory (QFT) in a way that leads to finite physical predictions. Moreover, as already mentioned, the way the problem is stated above, it is actually a QFT problem in the absence of gravity. It is only relevant to gravity when one treats $\langle 0|T_{\mu\nu}|0\rangle$ as a source in Einstein's equations of General Relativity. Thus it would appear that a first step in addressing the vacuum energy density should focus on making mathematical and physical sense of $\langle 0|T_{\mu\nu}|0\rangle$ purely in the context of quantum field theory in flat space, i.e., without gravity. This may or may not resolve the cosmological constant problem, but it is worthwhile exploring with the theoretical tools we have available at the present time. In [6], we studied this problem for integrable quantum field theory in $d = 2$ spacetime dimensions. Although $d = 2$ is considerably simpler, conceptually the problem is essentially the same as in $4d$ since in $2d$ the calculation (2) also leads to a divergent $\rho_{\text{vac}} \approx \Lambda^2/4\pi$. We proposed that interactions can actually fix the above simplistic free field calculation. Using integrability, we were able to exactly calculate ρ_{vac} for a wide variety of models, including massive and massless, and some with and without SSB. The main point is that it is physically meaningful and calculable without quantum gravity. It was found that for all these models

$$\rho_{\text{vac}} = -\frac{m^2}{2\,\mathfrak{g}} \tag{3}$$

exactly, where m is a physically measurable mass scale and $\mathfrak{g}$ an interaction coupling. The main tool that led to this result was Zamolodchikov's analysis of the Thermodynamic Bethe Ansatz (TBA) [7–9], which is a relativistic generalization of Yang–Yang thermodynamics [10]. For many additional references which deal with some specific models, we refer to [6]. For the massive case, in formula (3) $m = m_1$ which is the *physical* mass of the *lightest* particle and $\mathfrak{g}$ is a generalized coupling which is a trigonometric sum over certain resonance angles of the exact two-body S-matrix for the scattering of this lightest particle with itself. See, for example, (16) below. This is ultimately a consequence of the S-matrix bootstrap, which in principle applies in all spacetime dimensions. For massless cases, which are renormalization group flows between two conformal field theories, m can be the scale of SSB. We should add that although we have not considered QFT in general curved spacetime, the TBA formalism that led to (3) does involve QFT on a cylinder, not flat space.

The above $2d$ results led us to suggest [6] that in $4d$,

$$\rho_{\rm vac} = -\frac{m^4}{2\,\mathfrak{g}}\,. \tag{4}$$

In [6], we did not attempt to justify the above $4d$ proposal in any particular model. In this paper, we will do so for $\lambda\phi^4$ theory. We were encouraged to undertake this study by some recent results from a very different approach involving charged black holes and the notion of a Swampland [11,12]. There, it was proposed that

$$\rho_{\rm vac} < \frac{m^4}{2e^2} \tag{5}$$

where m is the mass of a charged particle, and $\alpha = e^2/4\pi$ is the electromagnetic fine structure constant. This is weaker than (4) since it is an upper bound rather than an equality. Remarkably, this is consistent with (4) if m in (5) is the lightest mass particle and $<$ is replaced with $\leq$. In other words the novelty of our proposal (4) is that whereas it is consistent with (5) if m is the lightest mass, it proposes that the lightest mass particle saturates the inequality, leading to an equality. One intriguing aspect of (4) is that if m is for the lightest mass particle and $\mathfrak{g} \approx 1$, then the astrophysically measured value of $\rho_{\rm vac} \approx 10^{-9}$ Joule/m^3 implies the lightest particle has a mass on the order of the expected neutrino masses (0.03 eV)[1].

The main goal of this paper is to understand how to obtain (4) *without* relying on integrability, at least in some approximation. We will also demonstrate that a QFT can have a well-defined cosmological constant even in the absence of spontaneous symmetry breaking. First of all there is no integrability in $4d$ and thus no TBA. Secondly, in the TBA the theory lives on an infinite cylinder of circumference β; in thermal field theory $\beta = 1/T$ where T is the temperature. In [6], we proposed that the cosmological constant $\rho_{\rm vac}$ is the β-independent term in the free energy density; however, in the TBA this term is sometimes tricky to extract since it can mix with terms coming from conformal perturbation theory. On the other hand, it should be possible to compute $\rho_{\rm vac}$ directly in the zero temperature quantum field theory, and this paper shows how to do this for a simple model, namely the $\lambda\phi^4$ theory, in a weak coupling approximation. We chose to study the latter theory since this alternative calculation can be compared with exact results for the sinh–Gordon model at small coupling as a check of the method.

In the next section, we review the exact $\rho_{\rm vac}$ for the sinh–Gordon model which was originally obtained with the help of the TBA. We show how this result can be obtained at weak coupling from a relatively simple calculation without introducing β and the TBA[2]. We then apply this approach to $\lambda\,\phi^4$ theory in d spacetime dimensions and show how to obtain both (3) and (4). An interesting feature is that in order to obtain the correct result one must analytically continue in m^2 from a regime where m^2 is negative and has SSB to a physical region with no SSB, since there is no SSB in the sinh–Gordon model. We will derive a Callan–Symanzik for $\rho_{\rm vac}$ based on the renormalization group for the coupling λ, which leads to an RG flow for $\mathfrak{g}$. The two main cases correspond to whether $\mathfrak{g}$ is marginally relevant or irrelevant. For the marginally relevant case, the cosmological constant *decreases* in the flow to low energies.

2. Generalities for a Scalar Field in Any Spacetime Dimension

In this article, we only consider models of a single scalar field in d spacetime dimensions. The classical theory can be defined by the action in euclidean space

$$\mathcal{S} = \int d^d x \Big(\tfrac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \Big). \tag{6}$$

As usual, we consider the partition function $Z = \mathrm{Tr}\, e^{-\beta H}$ where β is the inverse temperature. From Z, we can calculate the free energy density $\mathcal{F}$, energy density $\mathcal{E}$, and pressure p in the usual manner

$$\mathcal{F} = -p = -\frac{1}{\beta \mathcal{V}} \log Z, \qquad \mathcal{E} = -\frac{1}{\mathcal{V}} \frac{\partial \log Z}{\partial \beta} \tag{7}$$

where $\mathcal{V}$ is the $d-1$ dimensional spatial volume. For arbitrary β, the above equations determine an equation of state relating $\mathcal{E}$ and p, which generally does not correspond to a cosmological constant. However, in [6] it was shown that the β-independent term in $\mathcal{F}$ does correspond to a cosmological constant. Let us show this here in a different manner. First of all, consider an arbitrary shift of $V(\phi)$ by a constant v, $V(\phi) \to V(\phi) + v$. Whereas Z depends on v, correlation functions do not, since v cancels in $\langle \mathcal{O} \rangle = (\int D\phi \, e^{-S} \, \mathcal{O} \,)/Z$.

Let us calculate $\rho_{\rm vac}$ in a saddle point approximation. In the vacuum, ϕ has no dependence on spacetime, so we can ignore the $\partial \phi$ terms. The saddle point is then the value of $\phi = \phi_0$, satisfying

$$\frac{dV(\phi)}{d\phi}\Big|_{\phi=\phi_0} = 0. \tag{8}$$

The action is then

$$\mathcal{S}_0 = \int d^d x \, V(\phi_0) = \mathcal{V} \, \beta \, V(\phi_0) \quad \Longrightarrow \quad Z \approx e^{-\mathcal{V}\beta \, V(\phi_0)}, \tag{9}$$

since in thermal field theory, euclidean time is a circle of circumference β. This implies a β-independent free energy density

$$\mathcal{F} = V(\phi_0). \tag{10}$$

The equation of state corresponds to a cosmological constant (1) since it implies the equation of state $\mathcal{E} = -p$:

$$\mathcal{E} = V(\phi_0), \qquad p = -V(\phi_0). \tag{11}$$

We adopt the standard convention that a positive $\mathcal{E}$ corresponds to negative pressure p:

$$\rho_{\rm vac} = V(\phi_0) \tag{12}$$

in this approximation.

3. The 2*d* Sinh–Gordon Model at Weak Coupling

The sinh–Gordon model is perhaps the simplest integrable and relativistic quantum field theory. It can be defined by the action

$$\mathcal{S} = \int d^2 x \left(\frac{1}{8\pi} (\partial_\mu \phi \, \partial^\mu \phi) + 2\mu \, \cosh(\sqrt{2}\, b\phi) \right). \tag{13}$$

The $1/8\pi$ normalization of the kinetic term is such that the two point function has the standard 2d conformal field theory normalization: $\langle \phi(x) \phi(0) \rangle = -\log x^2$ when $\mu = 0$. The operator $\cosh(\sqrt{2} b\phi)$ is then strongly relevant with scaling dimension $-2b^2$. The spectrum consists of a single particle of mass m. Parameterizing the energy and momentum of a particle in terms of a rapidity θ,

$$E = m\cosh\theta, \qquad p = m\sinh\theta, \tag{14}$$

the exact two-body S-matrix is

$$S(\theta) = \frac{\sinh\theta - i\sin\pi\gamma}{\sinh\theta + i\sin\pi\gamma}, \qquad \gamma \equiv \frac{b^2}{1+b^2}. \tag{15}$$

As explained in [6], the strict $2d$ analog of the $4d$ cosmological constant corresponds to the so-called bulk term in the effective central charge $c(\beta m)$. The latter can be extracted from the TBA, but without some level of difficulty [7–9]. In the TBA, one calculates the free energy on a cylinder of circumference $R = \beta = 1/T$, where T is temperature. However, the exact result is quite simple:

$$\rho_{\rm vac} = \frac{m^2}{8\,\sin\pi\gamma}. \tag{16}$$

Since this result depends only on S-matrix parameters, it must be possible to obtain it directly in the zero temperature quantum field theory, and this is the primary goal of this paper, since doing so can provide insights into the $4d$ cosmological constant problem.

Whereas a shift of the potential by a constant v in the last section would appear to shift the saddle point approximation to $\rho_{\rm vac}$, there is clearly no room for such a shift of the above-quoted (16) vacuum energy for the sinh–Gordon model on a finite cylinder. Once given the S-matrix, the TBA equations are determined, and the coefficient of the bulk term in the free energy on a cylinder is completely fixed. It would be nice to understand this better; however, we suspect it is due to the finite circumference R of the cylinder that renders the problem well defined. This leads us to propose the following principle which eliminates the freedom to shift by v: The only contributions to the stress-energy tensor in Einstein's General Relativity are properties that can be measured in a flat space laboratory[3]. This rather conservative principle solves the usual fine-tuning problems. Moreover, it is consistent with the Casimir effect, in that only changes in the vacuum energy density as a result of changing a geometric modulus, for Casimir it is the separation of the plates, is measurable, since it leads to measurable force. Indeed, the sinh–Gordon result (16) is measurable in the finite geometry of a circle of circumference R. In fact it can even be derived on a lattice [15]. On the other hand the shift by v is in fact not measurable in flat space by any means whatsoever without gravity. We also wish to repeat that the TBA calculations that lead to (16) require studying the theory on a cylinder, which is not flat spacetime.

At small coupling b, one has

$$\lim_{b\to 0}\rho_{\rm vac} = \frac{m^2}{8\pi b^2}. \tag{17}$$

Note that as the couplling $b \to 0$, this is a free field limit, and there is indeed a divergence, which is consistent with (1). This can be obtained in a simple way using results of the last section. The saddle point satisfying (8) is simply $\phi_0 = 0$; thus,

$$\rho_{\rm vac} = 2\mu. \tag{18}$$

The above result does not rely on integrability, and is not exact except in the $b \to 0$ limit. If one allows results from integrability, then the relation between μ and the physical mass m and coupling constant b is known exactly [16]. Since the cosh potential has dimension

$-2b^2$, the scaling dimension of μ is $2+2b^2$; thus, $\mu \propto m^{2+2b^2}$ where m is the renormalized physical mass. The exact relation is

$$\mu = \frac{1}{\pi}\frac{\Gamma(1-b^2)}{\Gamma(b^2)}\,[m\mathcal{Z}(\gamma)]^{2+2b^2}, \quad \text{with} \quad \mathcal{Z}(\gamma) = \frac{1}{8\sqrt{\pi}}\,\gamma^{\gamma}\,(1-\gamma)^{1-\gamma}\,\Gamma\left(\tfrac{1-\gamma}{2}\right)\Gamma\left(\tfrac{\gamma}{2}\right). \tag{19}$$

In the limit $b^2 \to 0$, $\mathcal{Z} \approx 1/4b^2$ which implies

$$\mu \approx \frac{m^2}{16\pi\, b^2}, \tag{20}$$

and this combined with (18) gives the correct limit (17).

In the $b \to 0$ limit, the result (20) can be obtained in a much simpler way without using integrability and this will be useful in the sequel. Expanding the cosh and redefining $\phi \to \sqrt{4\pi}\phi$, the lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + O(\phi^6), \quad \text{with} \quad m^2 = 16\pi b^2\,\mu, \quad \lambda = 128\pi^2 b^4\,\mu. \tag{21}$$

This naturally leads us to the next section where we consider the cosmological constant for $\lambda\phi^4$ theory in d spacetime dimensions in light of the above understanding.

4. $\lambda\phi^4$ Theory in d Spacetime Dimensions

The theory is defined by the euclidean action

$$\mathcal{S} = \int d^d x \left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right). \tag{22}$$

Let $[X]$ denote the scaling dimension of X in mass units. The classical, engineering, dimensions are

$$[m] = 1, \quad [\phi] = (d-2)/2, \quad [\lambda] = 4-d, \quad [\rho_{\rm vac}] = d. \tag{23}$$

4.1. *Saddle Point Approximation*

The saddle point equation leads to

$$\phi_0^2 = -6\,\frac{m^2}{\lambda} \quad \Longrightarrow \quad \rho_{\rm vac} = V(\phi_0) = -\frac{3}{2}\frac{m^4}{\lambda}. \tag{24}$$

As is well known, a non-zero real solution ϕ_0 only exists if m^2 is negative, and there is spontaneous symmetry breaking of the $\phi \to -\phi$ symmetry. Although well-known and simple, what will be new is how to explain the exact result of the sinh–Gordon model from it. It is important to note that in the small b approximation to the sinh–Gordon model (21), m^2 is positive and there is no spontaneous symmetry breaking, but nevertheless it has a *positive* cosmological constant. As we will argue below, in order to explain the 2d result (17) we will need to analytically continue m^2 from negative to positive values[4].

Based on the engineering dimensions (23) let us define a dimensionless coupling $\mathfrak{g}$ as follows:

$$\lambda \equiv 3\,m^{4-d}\,\mathfrak{g}, \tag{25}$$

where by definition m is the true physical mass. The above equation is analogous to the exact sinh–Gordon result (19). Then, $\rho_{\rm vac}$ has the desired form stated in the Introduction for any spacetime dimension d:

$$\rho_{\rm vac} = -\frac{m^d}{2\mathfrak{g}}. \tag{26}$$

One sees that for the saddle point approximation to $\rho_{\rm vac}$ in Section 2, the main features of the exact sinh–Gordon result at small b, including overall factors, are obtained if one analytically continues $m^2 \to -m^2$ which makes $\rho_{\rm vac}$ positive, and identifies $\mathfrak{g} = 4\pi b^2$. The need to analytically continue in m^2 in order to obtain a positive cosmological constant is not completely clear; however, what is clear is that this is what one needs to do to obtain the correct sinh–Gordon result from (21) since the m^2 has the wrong sign for there to be a non-trivial ϕ_0.

4.2. Renormalization Group Considerations

The saddle point approximation to $\rho_{\rm vac}$, namely (26), is not a renormalization group (RG) invariant. For the 2d sinh–Gordon model, with a proper RG prescription, b^2 can be viewed as an RG invariant. In other dimensions, $\mathfrak{g}$ has a non-trivial RG flow, and one needs to investigate the implications of this. Renormalization of $\lambda\phi^4$ theory is well understood (see, for instance, [17]); however, its implications for $\rho_{\rm vac}$ have not been considered previously in much detail, at least to our knowledge. Being related to a correlation function (1), $\rho_{\rm vac}$ satisfies an RG differential equation. This involves absorbing divergences into the parameters m, λ and the normalization of the field ϕ, which necessarily introduces an arbitrary mass scale M, and a specific renormalization prescription which defines physical parameters, such as the actual physical mass of particles. Being a one-point correlation function which is independent of spacetime coordinates, these RG equations for $\rho_{\rm vac}$ are simpler than for general correlation functions. For our purposes, we want m in (26) to be the *physical*, measurable mass of a particle. For this reason, the Callan–Symanzik form of the RG equation is most suitable, since there the arbitrary renormalization scale M is the actual physical mass m. In this prescription, m has dimension 1 with no anomalous corrections [5], and the beta function β_λ for the coupling λ only depends on λ and not m. This RG equation is

$$\left(m\frac{\partial}{\partial m} + \beta_\lambda \frac{\partial}{\partial \lambda} \right) \rho_{\rm vac} = \Gamma_\rho \, \rho_{\rm vac} \tag{27}$$

where $\beta_\lambda = m\partial_m \lambda$, Γ_ρ is the scaling dimension of $\rho_{\rm vac}$, and

$$\beta_\lambda(\lambda) = (4-d)\lambda + O(\lambda^2). \tag{28}$$

Indeed, $\rho_{\rm vac} \propto m^4/\lambda$ as in (24) satisfies the above equation to lowest order with $\Gamma_\rho = d + O(\lambda)$. However, the higher-order corrections to β_λ imply that the beta function for the classically dimensionless $\mathfrak{g}$ is non-zero, and the Callan–Symanzik equation now is

$$\left(m\frac{\partial}{\partial m} + \beta(\mathfrak{g}) \frac{\partial}{\partial \mathfrak{g}} \right) \rho_{\rm vac} = \Gamma_\rho \, \rho_{\rm vac}, \qquad \beta(\mathfrak{g}) \equiv m \frac{\partial \mathfrak{g}}{\partial m}. \tag{29}$$

This is consistent with $\rho_{\rm vac} \propto m^d/\mathfrak{g}$ and $\beta(\mathfrak{g}) = 0$ classically.
Quantum corrections to 1-loop are known [17]

$$\beta(\mathfrak{g}) = m\frac{d\mathfrak{g}}{dm} = -\frac{9}{16\pi^2}\,\mathfrak{g}^2 + O(\mathfrak{g}^3). \tag{30}$$

The RG flow toward low energy corresponds to increasing m. Let us fix $\mathfrak{g} = \mathfrak{g}_0$ at some high energy scale m_0 such as the Planck scale. Then, integrating the one-loop β function (30) one has

$$\mathfrak{g}(m) = \frac{\mathfrak{g}_0}{1 + \frac{9}{16\pi^2}\,\mathfrak{g}_0 \log(m/m_0)}. \tag{31}$$

In any spacetime dimension d there are essentially two generic cases to consider:

Marginally irrelevant. Here, $\mathfrak{g}_0 > 0$, and $\rho_{\rm vac}$ is negative. In the flow to low energies (increasing m), $\mathfrak{g} \to 0$ and $\rho_{\rm vac} \to -\infty$.

Marginally relevant. Here, $\mathfrak{g}_0 < 0$, and ρ_{vac} is positive. In the flow to low energies, $|\mathfrak{g}|$ increases and ρ_{vac} slowly flows to $\rho_{\text{vac}} = 0$ and reaches there at

$$m/m_0 = e^{-16\pi^2/9\mathfrak{g}_0} > 1, \tag{32}$$

then it changes sign. The exponential in (32) implies there can be a very large hierarchy of scales relating the cosmological constant in the UV and IR.

There are some features that specifically depend on the spacetime dimension d:

$\boldsymbol{d = 2}$. Here, $\rho_{\text{vac}} = -m^2/2\mathfrak{g}$. Recall that for the sinh–Gordon model, ρ_{vac} is positive and there is no spontaneous symmetry breaking. Thus, in order to reproduce the known exact result in the sinh–Gordon model at weak coupling, one must analytically continue $m^2 \to -m^2$, which makes $\rho_{\text{vac}} > 0$ and is consistent with no spontaneous symmetry breaking, i.e., $\phi_0 = 0$.

$\boldsymbol{d = 4}$. Here, $\rho_{\text{vac}} = -m^4/2\mathfrak{g}$. Thus the analytic continuation $m^2 \to -m^2$ does not change the sign of ρ_{vac}. A positive cosmological constant requires a marginally relevant coupling $\mathfrak{g}$ that is negative. As explained above, this can occur for asymptotical free theories in the UV, where $\mathfrak{g} \to 0$ and $\rho_{\text{vac}} \to \infty$ at high energy.

5. Concluding Remarks

We have argued that the quantum vacuum expectation value of the stress energy tensor can be well-defined in d spacetime dimensions by including interactions. The main support for our analysis is that it can reproduce the exact, small coupling limit for some integrable quantum field theories in $d = 2$, in particular the sinh–Gordon model. This study could provide insight into the cosmological constant problem since the most well-known version of the problem is an issue of QFT in flat space, where the source of gravitation is the vacuum expectation value $\langle T_{\mu\nu} \rangle$. There are other versions of the problem mentioned in the Introduction, and it is not at all clear this is the origin of the observed cosmological constant, since we have not incorporated gravity. However, the problem studied here is well-motivated and -posed, and essentially decouples the problem from classical and quantum gravity.

Based on insights gained in $2d$, we studied the problem for $\lambda\phi^4$ theory in d spacetime dimensions and motivated the result $\rho_{\text{vac}} = -m^d/2\mathfrak{g}$ in a saddle point approximation. This result does not require spontaneous symmetry breaking. This entails a renormalization group equation satisfied by ρ_{vac} which is naturally of Callan–Symanzik type. For a marginally relevant coupling $\mathfrak{g}$, such as for asymptotically free theories, ρ_{vac} can flow from large positive values to zero, and this flow introduces a large hierarchy of energy scales.

If our analysis proves to be correct, then there are many open avenues for exploration. It would be interesting to try and extend our results to theories with both bosons and fermions as in the Standard Model of particle physics. In fact, based on our analysis of simpler models, conceptually the cosmological constant in the Standard Model is *in principle* computable, but difficult; it is non-perturbative, and perhaps can be computed on a lattice from finite size or temperature effects. The computation of vacuum energy density based on the TBA described in [6] is actually a finite size effect since the formalism involves quantum fields on a cylinder. Indeed, it was shown how to obtain exact results for the vacuum energy density for models such as the sinh–Gordon model from the lattice [15]. In fact, it can in principle be measured in a laboratory through finite size effects, as for the usual Casimir effect.

We have not at all explored the consequences of including ρ_{vac} in the temporal and thermal evolution of the Universe; as already stated, we decoupled the cosmological constant problem from gravity itself and thus cosmology. However, we suggested one scenario wherein $\mathfrak{g}$ is a negative marginally relevant coupling, for instance, for an asymptotically free theory, and $\rho_{\rm vac}$ flows to zero at low energies, indicating a kind of "cosmic freedom" in that the cosmological constant does not dominate at very late times.

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Conflicts of Interest: The authors declare no conflict of interest.

Notes

1 Astronomical data is based on WMAP [13]. The subject of neutrino masses is reviewed in [14].
2 This short article may thus be viewed as an addendum to [6].
3 In plain language, If you can't relate, you don't gravitate!
4 Equation (24) together with the $\lambda\phi^4$ approximation to the sinh–Gordon model (21) leads to $\rho_{\rm vac} = -3\mu$ rather than $\rho_{\rm vac} = 2\mu$ in (18), however this is clearly due to the approximation of the cosh potential with a $\lambda\phi^4$ theory.
5 $\gamma_m = 0$ in the notation in [17].

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