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Effects of Quantum Gravity on Thermodynamic Quantities of Gases around a Novel Neutral Four-Dimensional Gauss–Bonnet Black Hole

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Abstract: Taking the generalized uncertainty principle (GUP) into account, we apply the corrected state density to investigate the entropy density, energy density, pressure and equation of state for the perfect relativistic gases of massless particles with an arbitrary spin of s $\leq$ 2 surrounding a new four-dimensional neutral Gauss–Bonnet black hole. The modifications of these thermodynamic quantities by the gravity correction factor and particle spin are shown, and the expressions have completely different forms from those in flat space-times. For example, the energy density is not proportional to the fourth power of the temperature. In other words, the energy density differs from that of blackbody radiation. The quantum gravity effects reduce these quantities and are proportional to the gravity correction factor. The result that the equation of state is not zero is compatible with the non-vanishing trace of the stress tensor.

Keywords: effects of quantum gravity; Gauss–Bonnet black hole; thermodynamic quantity; equation of state; generalized uncertainty principle

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1. Introduction

Black hole thermodynamics have always been one of the hot topics in theoretical physics. Although great progress has been made in many aspects, there are still many unanswered questions such as the statistical origin of black hole entropy. At the same time, some new topics have arisen with the successful detection of gravitational waves. Recently, the thermodynamic properties of black holes in modified gravity such as Hawking radiation and phase transition have extensively been studied. However, there are few studies on the thermodynamic quantities of gases around a black hole, of which the number of studies taking the GUP into account is even lower. Li [1,2] used the quantization procedure referred to as the Boulware vacuum state [3] and killing time, t, to investigate the thermodynamic quantities around the Schwarzschild and Reissner–Nordström black hole, and found that their forms differ from those in flat space-times because of the appearance of additional spin-dependent terms. Since then, this method was extended to other Einstein gravity black holes [4–9]. However, gravitational interactions were not considered.

More than a decade ago, an idea was proposed by Mead [10] which suggested that the uncertainty principle could be affected by gravity. If the gravity is strong enough, the conventional Heisenberg uncertainty relation is no longer suitable and should be replaced by the inequality of the GUP, $\Delta X \Delta P \geq \frac{\hbar}{2}\left[1+\lambda(\Delta P)^2\right]$, given by theories of quantum gravity such as string theory, loop quantum gravity and quantum geometry [11–27], where ΔX and ΔP are the generalized coordinate and momentum, λ is the gravity correction factor and the order of the Planck area. Based on the GUP, the state density, $\frac{d^3Xd^3Y}{(2\pi\hbar)^3}$, supported by Heisenberg uncertainty relation should be modified to $\frac{d^3Xd^3Y}{(2\pi\hbar)^3(1+\lambda P^2)^3}$. In this paper, we shall make use of the corrected state density to study the thermodynamic quantities for the Weyl neutrino (s = 1/2), electromagnetic (s = 1), mass-less Rarita–Schwinger (s = 3/2) and gravitational (s = 2) fields around a novel modified-gravity black hole. The hole in four-dimensional Gauss–Bonnet space-time was initially proposed by Glavan and Lin [28] and was further proven to be a solution to the well-defined theory developed by Aoki, Gorji and Mukohyama [29,30]. The solution has shed new light on the theory of gravity [31,32], Ads/CFT correspondence and astrophysics. For example, new features were found in holographic superconductors [33], black hole shadows [34], and black hole microstructure [35]. Thermodynamic properties such as horizons, phase transition and entropy of the black hole have been studied [35–37]. It was shown that the range of dynamical stability of the background black hole is constrained by small values of Gauss–Bonnet coupling only [38]. As far as we are aware, there is no research on the thermodynamic quantities of gases around this hole.

2. Four-Dimensional Gauss–Bonnet Gravity Space-Time and Field Equations

The metric of the four-dimensional Gauss–Bonnet gravity reads [28]

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

$$f(r) = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M}{r^3}}\right) \tag{2}$$

where M is the black hole mass, and α is the coupling parameter describing the Gauss–Bonnet contribution to the action. There exists no real solution at short radial distances for which $r^3 < |8\alpha M|$ when $\alpha < 0$, while the metric (1) reduces to the Schwarzschild black hole in General Relativity in the limit $\alpha = 0$. Therefore, we shall focus on the case of $\alpha > 0$ in the following. Solving the equation $f(r) = 0$ yields the horizon radii $r_\pm = M \pm \sqrt{M^2 - \alpha}$. Obviously, Gauss–Bonnet coupling decreases the outer horizon radius and increases the inner one, then thins the unidirectional membrane between them. We note that the inner and outer horizons coincide and the black hole becomes an extreme one when $\alpha = M^2$. The Hawking temperature is

$$T_{\rm H} = \frac{1}{4\pi}f'(r_+) = \frac{1}{4\pi}\left(\frac{6M}{r_+^2 + 2\alpha} - \frac{2}{r_+}\right) \tag{3}$$

where the prime denotes the derivation to r.

The following null tetrad vectors can be established as

$$l^\mu = [f^{-1}(r), 1, 0, 0]; n^\mu = \frac{1}{2}[1, f(r), 0, 0]; m^\mu = \frac{1}{\sqrt{2}r}\left[0, 0, 1, \frac{i}{\sin\theta}\right]. \tag{4}$$

Making use of the Newman–Penrose formula [39], we obtain gain the non-vanishing spin coefficients and component of the Weyl tensor.

$$\begin{aligned} &\alpha = -\beta = -\tfrac{ctg\theta}{2\sqrt{2}r}, \qquad \rho = -\tfrac{1}{r}, \\ &\mu = -\tfrac{1}{2r}f(r), \qquad \gamma = \tfrac{1}{4}f'(r), \\ &\Psi_2 = \tfrac{5}{12}f''(r) + \tfrac{1}{6r}f'(r) + \tfrac{1}{6r^2}[1 - f(r)]. \end{aligned} \tag{5}$$

These show that the metric (1) is of Petrov type D [40].

According to the results obtained by Teukolsky [41], the field equations of the Weyl neutrino, electromagnetic, massless Rarita–Schwinger and gravitational fields can be combined into

$$\begin{cases} \{[D-(2s+1)\rho](\Delta-2s\gamma+\mu) \\ \qquad -[\delta+(2s-2)\alpha](\overline{\delta}-2s\alpha)-(2s-1)(s-1)\Psi_2\}\varphi_p=0 \ \ (p=s); \\ \{[\Delta+(2s-2)\gamma+(2s+1)\mu](D-\rho) \\ \qquad -[\overline{\delta}+(2s-2)\alpha](\delta-2s\alpha)-(2s-1)(s-1)\Psi_2]\varphi_p=0 \ \ (p=-s). \end{cases} \tag{6}$$

where $D=l^\mu\partial_\mu, \Delta=n^\mu\partial_\mu, \delta=m^\mu\partial_\mu$ are the directional derivatives.

Adopting the Wentzel–Kramers–Brillouin (WKB) approach and setting the mode functions as $\varphi_p=\exp[-iEt+iS_p(r,\theta,\varphi)]$, we can obtain from Equation (6)

$$\frac{E^2}{f(r)}-f(r)P_r^2-\frac{1}{r^2}P_\theta^2-\frac{1}{r^2\sin^2\theta}(P_\varphi+p\cos\theta)^2+\eta(r,p)=0 \tag{7}$$

with $P_\mu=\partial_\mu S_p$ being the conjugate momenta and

$$\begin{aligned} \eta(r,p)=&\tfrac{s+p}{2}f''(r)+\tfrac{(s+p)s+p+1}{r}f'(r)+\tfrac{2s}{r^2}f(r) \\ &+2(2s-1)(s-1)\Psi_2-(s-p) \end{aligned} \tag{8}$$

It should be noted that the existence of the spin makes Equation (7) differ from the Hamilton–Jacobi equation of a massless scalar particle, $g^{\mu\nu}P_\mu P_\nu=0$. The momentum of a particle is

$$P^2=\frac{E^2}{f(r)}+\eta(r,p) \tag{9}$$

which reaches infinity at the outer event horizon, hence WKB approximation is valid [42].

3. Thermodynamic Quantities

Taking the GUP into account, the number of eigenstates with an energy smaller than E is modified and reads

$$\mathrm{g}(E)=\frac{1}{(2\pi)^3}\sum_p\int\frac{\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}p_r\mathrm{d}p_\theta\mathrm{d}p_\varphi}{(1+\lambda P^2)^3} \tag{10}$$

It is not difficult to perform the integrals in momentum space or θ and ϕ to obtain

$$g(E)=\frac{2}{3\pi}\sum_p\int\frac{r^2\mathrm{d}r}{\sqrt{f(r)}}\left(\frac{E^2}{f(r)}+\eta(r,p)\right)^{3/2}\left[1-\lambda\left(\frac{E^2}{f(r)}+\eta(r,p)\right)\right]^{-3} \tag{11}$$

The quantum state density (11) is convergent at the event horizon as a result of the non-vanishing λ. This is one of the advantages of the GUP. On the other hand, to obtain the integral r, we employ the framework of 't Hooft [42]. That is, the boundary conditions for the wave function are chosen as $\phi_p=0$ at $r=r_++\varepsilon$ and $r=r_++L$, the region of integration of r is a spherical shell from $r_++\varepsilon$ to r_++L where $0<\varepsilon<<L$. Because the GUP determines a minimum space length, $\Delta X_{\min}=\sqrt{\lambda}$, we set the proper distance of the above spherical shell from the horizon to be $\sqrt{\lambda}$, i.e., $\int_{r_+}^{r_++\varepsilon}\sqrt{f^{-1}(r)}dr=\sqrt{\lambda}$. In other words, the UV cutoff does not need to be introduced manually, which is the other advantage of the GUP.

At the inverse temperature, $\beta_{\mathrm{H}}=T_{\mathrm{H}}^{-1}$, the free energy of a system is given by

$$F=(-1)^{2s}\beta^{-1}\textstyle\sum_\alpha\ln\left[1-(-1)^{2s}e^{-\beta E_\alpha}\right] \tag{12}$$

where α expresses a set of quantum numbers. Considering that the energy distribution is continuous, Equation (12) can be rewritten as

$$F = -\int_0^\infty \frac{g(E)dE}{e^{\beta_{\rm H}E} \mp 1} \tag{13}$$

where the plus sign corresponds to the Fermi case and the minus sign to the Bose one. Making use of the corrected state density (11) to accomplish the E integral, we have

$$\begin{aligned} F = &-\int \frac{r^2dr}{\sqrt{f(r)}}\left[\frac{15+(-1)^{2s}}{16}\frac{2\omega\pi^3}{45\beta_{\rm H}^4[f(r)]^{3/2}} + \frac{3+(-1)^{2s}}{4}\frac{\pi\sum_p \eta(r,p)}{6\beta_{\rm H}^2[f(r)]^{1/2}}\right] \\ &+\lambda\int \frac{r^2dr}{\sqrt{f(r)}}\left[\frac{63+(-1)^{2s}}{64}\frac{16\omega\pi^5}{63\beta_{\rm H}^6[f(r)]^{5/2}} + \frac{15+(-1)^{2s}}{16}\frac{\pi^3\sum_p \eta(r,p)}{3\beta_{\rm H}^4[f(r)]^{3/2}}\right] \end{aligned} \tag{14}$$

where $\omega = \sum_p 1 = 2$ is the degeneracy of the particle spin.

The total entropy and energy of the system of the black hole and spin field read

$$\begin{aligned} S &= \beta_{\rm H}^2\frac{\partial F}{\partial \beta_{\rm H}} \\ &= \int \frac{r^2dr}{\sqrt{f(r)}}\left[\frac{15+(-1)^{2s}}{16}\frac{8\omega\pi^3}{45\beta_{\rm H}^3[f(r)]^{3/2}} + \frac{3+(-1)^{2s}}{4}\frac{\pi\sum_p \eta(r,p)}{3\beta_{\rm H}[f(r)]^{1/2}}\right] \\ &\quad -\lambda\int \frac{r^2dr}{\sqrt{f(r)}}\left[\frac{63+(-1)^{2s}}{64}\frac{32\omega\pi^5}{21\beta_{\rm H}^5[f(r)]^{5/2}} + \frac{15+(-1)^{2s}}{16}\frac{4\pi^3\sum_p \eta(r,p)}{3\beta_{\rm H}^3[f(r)]^{3/2}}\right], \end{aligned} \tag{15}$$

$$\begin{aligned} \mathrm{U} &= \frac{\partial(\beta_{\rm H}F)}{\partial \beta_{\rm H}} \\ &= \int \frac{r^2dr}{\sqrt{f(r)}}\left[\frac{15+(-1)^{2s}}{16}\frac{2\omega\pi^3}{15\beta_{\rm H}^4[f(r)]^{3/2}} + \frac{3+(-1)^{2s}}{4}\frac{\pi\sum_p \eta(r,p)}{6\beta_{\rm H}^2[f(r)]^{1/2}}\right] \\ &\quad -\lambda\int \frac{r^2dr}{\sqrt{f(r)}}\left[\frac{63+(-1)^{2s}}{64}\frac{80\omega\pi^5}{63\beta_{\rm H}^6[f(r)]^{5/2}} + \frac{15+(-1)^{2s}}{16}\frac{\pi^3\sum_p \eta(r,p)}{\beta_{\rm H}^4[f(r)]^{3/2}}\right]. \end{aligned} \tag{16}$$

On the other hand, the total entropy and energy of a system can be given as

$$S = \int \sigma(r)\frac{4\pi r^2dr}{\sqrt{f(r)}}; \quad U = \int \rho(r)4\pi r^2dr, \tag{17}$$

where the spherical volume elements chosen for the black hole are spherically symmetric, and σ and ρ are the entropy density and energy density. The factor $1/\sqrt{f(r)}$ does not appear in the integral for the total energy of the thermal excitations [43].

Comparing Equation (16) with Equation (17) yields

$$\begin{aligned} \sigma(r) = &\frac{15+(-1)^{2s}}{16}\frac{2\omega\pi^2}{45}T^3 + \frac{3+(-1)^{2s}}{4}\frac{\sum_p \eta(r,p)}{12}T \\ &-\lambda\left[\frac{63+(-1)^{2s}}{64}\frac{8\omega\pi^4}{21}T^5 + \frac{15+(-1)^{2s}}{16}\frac{\pi^2\sum_p \eta(r,p)}{3}T^3\right], \end{aligned} \tag{18}$$

$$\begin{aligned} \rho(r) = &\frac{15+(-1)^{2s}}{16}\frac{\omega\pi^2}{30}T^4 + \frac{3+(-1)^{2s}}{4}\frac{\sum_p \eta(r,p)}{24}T^2 \\ &-\lambda\left[\frac{63+(-1)^{2s}}{64}\frac{20\omega\pi^4}{63}T^6 + \frac{15+(-1)^{2s}}{16}\frac{\pi^2\sum_p \eta(r,p)}{4}T^4\right], \end{aligned} \tag{19}$$

where $T(r) = \frac{T_{\rm H}}{\sqrt{f(r)}}$ is the local temperature [44]. Obviously, the expression of the energy density is inconsistent with Stefan–Boltzmann law. Only for the scalar field, our result (19) is reduced to the Stefan–Boltzmann law when $\lambda = 0$.

The pressure, entropy density and energy density satisfy the formula [45]

$$P(r) = \sigma(r)T(r) - \rho(r) \tag{20}$$

Substituting (18) and (19) into (20), we have the pressure equation,

$$\begin{aligned} P(r) = &\tfrac{15+(-1)^{2s}}{16}\tfrac{\omega\pi^2}{90}T^4 + \tfrac{3+(-1)^{2s}}{4}\tfrac{\sum_p \eta(r,p)}{24}T^2 \\ &-\lambda\left[\tfrac{63+(-1)^{2s}}{64}\tfrac{4\omega\pi^4}{63}T^6 + \tfrac{15+(-1)^{2s}}{16}\tfrac{\pi^2\sum_p \eta(r,p)}{12}T^4\right] \end{aligned} \tag{21}$$

and the state equation,

$$\rho(r) - 3P(r) = -\frac{3+(-1)^{2s}}{4}\frac{\sum_p \eta(r,p)}{12}T^2 - \lambda\frac{63+(-1)^{2s}}{64}\frac{8\omega\pi^4}{63}T^6. \tag{22}$$

4. Discussion and Conclusions

We take the GUP into account and use the modified density to calculate the entropy density, energy density, pressure and equation of state for the perfect relativistic gases of massless particles with a spin of s = 1/2, 1, 3/2 and 2 around a four-dimensional neutral Gauss–Bonnet black hole. It is shown through Equations (18), (19) and (21) that the entropy density, energy density and pressure all include four terms. The first term is the exact same as that in a flat space-time, and the second term only depends on the spin value of the particles, while the last two ones are proportional to the gravity correction factor and arise from the state density modified by the GUP. To be more specific, the third term only comes from the gravity correction factor and is spin-independent, while the forth one is the joint contribution of the thermodynamic quantities of the gravitation interactions and spin fields. We note that this cross impact does not appear in the equation of state Equation (22).

It should be noted that the Hamilton–Jacobi equation of a massless scalar particle is recovered if we set η = 0 in Equation (9). Therefore, we can gain the entropy density, energy density, pressure and equation of state for the scalar field only if we set η = 0, s = 0, and ω = 1 in Equations (18), (19), (21) and (22):

$$\sigma(r) = \frac{2\pi^2}{45}T^3 - \lambda\frac{8\pi^4}{21}T^5 \tag{23}$$

$$\rho(r) = \frac{\pi^2}{30}T^4 - \lambda\frac{20\pi^4}{63}T^6 \tag{24}$$

$$P(r) = \frac{\pi^2}{90}T^4 - \lambda\frac{4\pi^4}{63}T^6 \tag{25}$$

$$\rho(r) - 3P(r) = -\lambda\frac{8\pi^4}{63}T^6. \tag{26}$$

Obviously, quantum gravity effects decrease the thermodynamic quantities. Meanwhile, that the equation of state Equation (26) is negative means that the decrease in the energy density is greater than that in the pressure.

That the equations of state Equations (22) and (26) are not zero show the non-vanishing trace of the stress tensor, while the trace anomaly is an unescapable consequence as result of all renormalization schemes of curved space-time.

It is interesting to note that, when r is big enough, $\Psi_2 \to 0$ and $\eta(r,p) \to p - s$. Then, the contribution of the spin only comes from the spin state, $p = -s$.

In conclusion, we utilize the corrected state density according to the GUP to investigate the quantum gravity effects on the thermodynamic quantities such as the entropy density, energy density, pressure and equation of state as well as the effects of spin fields. The quantum gravity effect decreases the thermodynamic quantity and is proportional to the gravity correction factor. In addition, that gravitation affects the state equation and thermodynamic quantities, which was argued by Li and Liu, is shown once again [46].

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References

1. Li, Z.H. State equations for massless spin fields near the event horizon in Schwarzschild spacetime. *Class. Quantum Grav.* **2004**, *21*, 1181.
2. Li, Z.H. Effect of spin on thermodynamical quantities around Reissner–Nordstroem black holes. *Chin. Phys. Lett.* **2005**, *22*, 1321.
3. Boulware, D.G. Quantum field theory in Schwarzschild and Rindler spaces. *Phys. Rev. D* **1975**, *11*, 1404. [CrossRef]
4. Mi, L.Q.; Li, Z.H. Thermodynamical quantities around a RNAdS black hole. *Chin. Phys.* **2006**, *15*, 1184.
5. Li, G.Q. Thermodynamic quantities for gases in static spherically symmetric backgrounds possessing a horizon. *JETP Lett.* **2007**, *86*, 153. [CrossRef]
6. Li, G.Q. Effect of spin on the thermodynamical quantities around dilatonic black hole. *Europhys. Lett.* **2007**, *77*, 10001. [CrossRef]
7. Li, G.Q. Spin-Dependence of Thermodynamic Quantities Around A Horowitz–Strominger Black Hole. *Mod. Phys. Lett. A* **2008**, *23*, 437. [CrossRef]
8. Li, G.Q. Subleading terms of thermodynamic quantities around static spherical black holes. *Chin. Phys. B* **2009**, *18*, 66.
9. Li, G.Q.; Xiao, S.F. State equations for massless spin fields in static spherical spacetime filled with quintessence. *Gen. Relativ. Gravit.* **2010**, *42*, 1719. [CrossRef]
10. Mead, C.A. Possible connection between gravitation and fundamental length. *Phys. Rev. D* **1964**, *135*, 849. [CrossRef]
11. Kempf, A.; Mangano, G.; Mann, R.B. Hilbert space representation of the minimal length uncertainty relation. *Phys. Rev. D* **1995**, *52*, 1108. [CrossRef] [PubMed]
12. Chang, L.N.; Minic, D.M.; Okamura, N.; Takeuchi, T. Effect of the minimal length uncertainty relation on the density of states and the cosmological constant problem. *Phys. Rev. D* **2002**, *65*, 125028. [CrossRef]
13. Gross, D.J. High-energy symmetries of string theory. *Phys. Rev. Lett.* **1988**, *60*, 1229. [CrossRef] [PubMed]
14. Maggiore, M. The algebraic structure of the generalized uncertainty principle. *Phys. Lett. B* **1993**, *319*, 83–86. [CrossRef]
15. Padmanabhan, T. Limitations on the operational definition of spacetime events and quantum gravity. *Class. Quant. Grav.* **1987**, *4*, L107. [CrossRef]
16. Amati, D.; Ciafaloni, M.; Veneziano, G. Can spacetime be probed below the string size? *Phys. Lett. B* **1989**, *216*, 41–47. [CrossRef]
17. Maggiore, M. Quantum groups, gravity, and the generalized uncertainty principle. *Phys. Rev. D* **1994**, *49*, 5182. [CrossRef]
18. Scardigli, F. Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment. *Phys. Lett. B* **1999**, *452*, 39. [CrossRef]
19. Kempf, A.; Mangano, G. Minimal length uncertainty relation and ultraviolet regularization. *Phys. Rev. D* **1997**, *55*, 7909. [CrossRef]
20. Gross, D.J.; Mende, P.F. String theory beyond the Planck scale. *Nucl. Phys. B* **1988**, *303*, 407. [CrossRef]
21. Garay, L.J. Quantum gravity and minimum length. *Int. J. Mod. Phys. A* **1995**, *10*, 145. [CrossRef]
22. Hossenfelder, S.; Bleicher, M.; Hofmann, S.; Ruppert, J.; Scherer, S.; Stöcker, H. Signatures in the Planck regime. *Phys. Lett. B* **2003**, *575*, 85. [CrossRef]
23. Bambi, C.; Urban, F.R. Natural extension of the generalized uncertainty principle. *Class. Quant. Grav.* **2008**, *25*, 095006. [CrossRef]
24. Brau, F.J. Minimal length uncertainty relation and the hydrogen atom. *Phys. A* **1999**, *32*, 7691. [CrossRef]
25. Magueijo, J.; Smolin, L. Lorentz invariance with an invariant energy scale. *Phys. Rev. Lett.* **2002**, *88*, 190403. [CrossRef] [PubMed]
26. Maggiore, M. A generalized uncertainty principle in quantum gravity. *Phys. Lett. B* **1993**, *304*, 65. [CrossRef]
27. Cortes, J.L.; Gamboa, J. Quantum uncertainty in doubly special relativity. *Phys. Rev. D* **2005**, *71*, 065015. [CrossRef]
28. Glavan, D.; Lin, C. Einstein-Gauss-Bonnet gravity in four-dimensional spacetime. *Phys. Rev. Lett.* **2020**, *124*, 081301. [CrossRef]
29. Aoki, K.; Gorji, M.A.; Mizuno, S.; Mukohyama, S. A consistent theory of D $\rightarrow$ 4 Einstein-Gauss-Bonnet gravity. *Phys. Lett. B* **2020**, *810*, 135843. [CrossRef]
30. Aoki, K.; Gorji, M.A.; Mukohyama, S. Inflationary gravitational waves in consistent D $\rightarrow$ 4 Einstein-Gauss-Bonnet gravity. *JCAP* **2021**, *2021*, 54. [CrossRef]
31. Lu, H.; Pang, Y. Horndeski gravity as D $\rightarrow$ 4 limit of Gauss-Bonnet. *Phys. Lett. B* **2020**, *809*, 135717. [CrossRef]
32. Hennigar, R.A.; Kubizňák, D.; Mann, R.B.; Pollack, C.J. On taking the D $\rightarrow$ 4 limit of Gauss-Bonnet gravity: Theory and solutions. *High Energy Phys.* **2020**, *7*, 27. [CrossRef]
33. Qiao, X.Y.; Liang, O.Y.; Wang, D.; Pan, Q.Y.; Jing, J.L. Holographic superconductors in 4D Einstein-Gauss-Bonnet gravity. *JHEP* **2020**, 192. [CrossRef]
34. Guo, M.Y.; Li, P.C. Innermost stable circular orbit and shadow of the 4 D Einstein–Gauss–Bonnet black hole. *Eur. Phys. J. C* **2020**, *80*, 588. [CrossRef]
35. Wei, S.W.; Liu, Y.X. Extended thermodynamics and microstructures of four-dimensional charged Gauss-Bonnet black hole in AdS space. *Phys. Rev. D* **2020**, *101*, 104018. [CrossRef]
36. Fernandes, P.G.S. Charged black holes in AdS spaces in 4D Einstein Gauss-Bonnet gravity. *Phys. Lett. B* **2020**, *805*, 135468. [CrossRef]

37. Hegde, K.; Kumara, A.N.; Rizwan, C.L.A.; Ajith, K.M.; Ali, M.S. Thermodynamics, Phase Transition and Joule Thomson Expansion of novel 4-D Gauss Bonnet AdS Black Hole. *arXiv* **2020**, arXiv:2003.08778.
38. Konoplya, R.A.; Zinhailo, A.F. Quasinormal modes, stability and shadows of a black hole in the 4D Einstein–Gauss–Bonnet gravity. *Eur. Phys. J. C* **2020**, *80*, 1049. [CrossRef]
39. Newman, E.; Penrose, R. An approach to gravitational radiation by a method of spin coefficients. *J. Math. Phys.* **1962**, *3*, 566. [CrossRef]
40. García, A.; Macías, A. *Black Holes as Exact Solutions of the Einstein-Maxwell Equations of Petrov Type D*; Springer Press: Berlin/Heidelberg, Germany, 1998.
41. Teukolsky, S.A.; Press, W.H. Perturbations of a rotating black hole. III-Interaction of the hole with gravitational and electromagnetic radiation. *Astrophys. J.* **1974**, *193*, 443–461. [CrossRef]
42. Hooft, G.T. On the quantum structure of a black hole. *Nucl. Phys. B* **1985**, *256*, 727. [CrossRef]
43. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; Freeman: San Francisco, CA, USA, 1973.
44. Tolman, R.C. *Relativity, Thermodynamics and Cosmology*; Oxford University Press: Oxford, UK, 1934.
45. Unruh, W.G.; Wald, R.M. Acceleration radiation and the generalized second law of thermodynamics. *Phys. Rev. D* **1982**, *25*, 942. [CrossRef]
46. Li, L.X.; Liu, L. Properties of radiation near the black-hole horizon and the second law of thermodynamics. *Phys. Rev. D* **1992**, *46*, 3296. [CrossRef] [PubMed]