Review

Torsion at Different Scales: From Materials to the Universe

Nick E. Mavromatos [1,2,*], Pablo Pais [3,4] and Alfredo Iorio [4]

1   Physics Department, School of Applied Mathematical and Physical Sciences, National Technical University of Athens, 157 80 Athens, Greece
2   Theoretical Particle Physics and Cosmology Group, Department of Physics, King's College London, Strand, London WC2R 2LS, UK
3   Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Casilla 567, Valdivia 5090000, Chile; pais@ipnp.troja.mff.cuni.cz
4   Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 18000 Prague, Czech Republic; alfredo.iorio@mff.cuni.cz
*   Correspondence: nikolaos.mavromatos@kcl.ac.uk

Abstract: The concept of torsion in geometry, although known for a long time, has not gained considerable attention from the physics community until relatively recently, due to its diverse and potentially important applications to a plethora of contexts of physical interest. These range from novel materials, such as graphene and graphene-like materials, to advanced theoretical ideas, such as string theory and supersymmetry/supergravity, and applications thereof in terms of understanding the dark sector of our Universe. This work reviews such applications of torsion at different physical scales.

Keywords: quantum gravity; torsion; supersymmetry and supergravity; analogs; Dirac materials

1. Introduction

Torsion is as important a concept of differential geometry as curvature [1–3]. The latter plays a key role in General Relativity (GR), but the former plays no role at all there. Nonetheless, torsion enters into various contexts and formulations, directing to diverse physical predictions and realisations that span a huge range of length scales—from cosmology to condensed matter and particle physics. Therefore, the related literature is huge, and it is not possible to cover it all in the restricted space of this review.

Here, we focus on specific aspects of torsion, either in the emergent geometric description of the physics of various materials of great interest to condensed matter physics— mainly graphene—or in the spacetime geometry itself, in particular in the early Universe. These two situations correspond to scales that are separated by a huge amount, yet the mathematical properties of torsion appear to be universal. Torsion has important physical effects, in principle experimentally testable, in both scenarios.

Graphene and related materials provide a tabletop realisation of some high-energy scenarios where torsion is associated with (the continuum limit of) the appropriate dislocations in the material. A way to represent the effect of dislocations, in the long wave-length regime, through torsion tensor is to consider a continuum field-theoretic fermionic system in a (2 + 1)-dimensional space with a torsion-full spin-connection.

In the case of fundamental physics, torsion is associated with supergravity (SUGRA) theories or with the geometry of the early Universe (cosmology). We discuss physical aspects of torsion that may affect particle physics phenomenology. In such cases, the (totally antisymmetric component of) torsion corresponds to a dynamical pseudoscalar (axion-like) degree of freedom, which is responsible for giving the vacuum a form encountered in the so-called running vacuum model (RVM) cosmology, characterised by a dynamical inflation without external inflaton fields, but rather due to non-linearities of the underlying gravitational dynamics. Moreover, under some circumstances, the torsion-associated

axions can lead to background configurations that spontaneously violate Lorentz (and CPT) symmetry, leading, in some models with right-handed neutrinos, to lepton asymmetry in the early radiation epoch, that succeeds the exit from inflation.

The structure of the review is as follows. First, in Section 2, we extensively discuss the concept of the torsion tensor in general geometric terms. This has the double scope of introducing our notations but also, and more importantly, of elucidating as many details as possible of the geometry and physics of torsion. The following Section 3 is dedicated to an important illustration of how torsion may affect well known theories, such as quantum electrodynamics (QED), while Section 4 focuses on some ambiguities of the Einstein–Cartan gravity theories and on the Barbero–Immirzi (BI) parameter. In Section 5, we discuss how torsion can be practically realised in a tabletop system, that is graphene. Then, after recalling in Section 6 how standard SUGRAs necessarily include torsion, we discuss in Section 7 a novel type of local supersymmetry (SUSY), without superpartners, whose natural realisation is in graphene. The rather extended Section 8 is dedicated to the important and hot topic of torsion in cosmology. Our concluding remarks and some brief description of other applications of torsion, which are not covered in this review, are given in the last Section 9.

2. Properties of Torsion

As already mentioned, torsion is an old subject [1–3] that goes beyond GR, as it constitutes a more general formalism in the sense that, to obtain Einstein's GR, one needs to impose a constraint to guarantee the absence (vanishing) of torsion tensor in a Riemannian spacetime. Specifically, let $\mathcal{M}$ be a (3 + 1)-dimensional Minkowski-signature curved world-manifold[1], parametrised by coordinates x^μ, where Greek indices $\mu, \nu = 0, \dots 3$ are spacetime volume indices, raised and lowered by the curved metric $g_{\mu\nu} = \eta_{ab} \, e^a{}_\mu \, e^b{}_\nu$, with $e^a{}_\mu$ the vielbein (we also define the inverse vielbein as $E^\mu{}_a \, e^a{}_\nu = \delta^\mu_\nu$, and $E^\mu{}_a \, e^b{}_\mu = \delta^b_a$, such that $g^{\mu\nu} = \eta^{ab} \, E^\mu{}_a \, E^\nu{}_b$ gives the inverse metric tensor). In the above formulae, Latin indices $a, b, \dots = 0, \dots 3$ are (Lorentz) indices on the tangent hyperplane of $\mathcal{M}$ at a given point p (cf. Figure 1), and are raised and lowered by the Minkowski metric η_{ab} (and its inverse η^{ab}), which is the metric of the tangent space $T_p\mathcal{M}$.
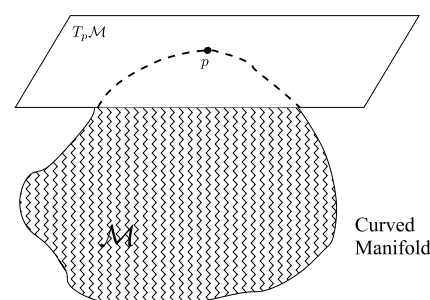

Figure 1. Tangent hyperplane $T_p\mathcal{M}$ at a point p of a curved $(d+1)$-dimensional manifold $\mathcal{M}$, used in the first order formalism of GR to define the vielbein $e^a{}_\mu$ map $\mathcal{M} \rightarrow T_p\mathcal{M}$.

In differential form language [4,5], which we use here often for notational convenience, the torsion two form is defined as [1–3,6]:

$$T^a = \frac{1}{2} \, T^a{}_{\mu\nu} \, dx^\mu \wedge dx^\nu \equiv de^a + \omega^a{}_b \wedge e^b, \tag{1}$$

where in the first equality, we used the definition of a differential two form [4], and the $\wedge$ denotes the exterior product[2], and $\omega^a{}_{b\,\mu}$ is the generalised (contorted) spin connection one form, which can can be split into a part that is torsion-free, $\mathring{\omega}^a{}_{b\,\mu}$, and related to the standard Christoffel symbols of GR, and another part that involves the *contorsion one-form*[3] $\mathcal{K}^a{}_{b\,\mu}$ [2,3]:

$$\omega^a{}_{b\,\mu} = \mathring{\omega}^a{}_{b\,\mu} + \mathcal{K}^a{}_{b\,\mu}. \tag{2}$$

We can use the one-form $\omega^a{}_b$ to define the covariant derivative $D(\omega)$ acting on q-forms $Q^a{}_{\ddot{b}\ldots}$ in this contorted spacetime [6]:

$$D(\omega)\, Q^a{}_{\ddot{b}\ldots} = d\, Q^a{}_{\ddot{b}\ldots} + \omega^a{}_c \wedge Q^c{}_{\ddot{b}\ldots} + \cdots - (-1)^q\, Q^a{}_{\ddot{d}\ldots} \wedge \omega^d{}_b - \ldots \tag{3}$$

It can be readily seen, using the covariant constancy of the Minkowski tangent space metric η^{ab}

$$D(\omega)\, \eta_{ab} = 0, \tag{4}$$

that the spin connection (2) is antisymmetric in its Lorentz indices

$$\omega_{ab} = -\,\omega_{ba}. \tag{5}$$

We also have covariant constancy for the totally antisymmetric Levi–Civita tensor ϵ_{abcd}:

$$D(\omega)\, \epsilon_{abcd} = 0. \tag{6}$$

In this Section, we discuss the generalisation of Einstein–Hilbert action for spacetime geometries with torsion. To this end, we first note that the *generalised* Riemann curvature, or Lorentz curvature, two-form is defined as:

$$R^a{}_b = d\,\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{7}$$

We can write the components of the Lorentz curvature in terms of the Riemann curvature two-form $\mathring{R}^a{}_b$, defined only by the torsion-less spin-connection, i.e., $\mathring{R}^a{}_b = d\,\mathring{\omega}^a{}_b + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b$, and the contorsion $\mathcal{K}^a{}_b$,

$$R^a{}_b = \mathring{R}^a{}_b + D(\mathring{\omega})\,\mathcal{K}^a{}_b + \mathcal{K}^a{}_c\,\mathcal{K}^c{}_b, \tag{8}$$

where the quantity $D(\mathring{\omega})$ denotes the diffeomorphic covariant derivative of GR. From the definition of the covariant derivative (3), we therefore have that the torsion two form is just the covariant derivative of the vielbein,

$$T^a = d\,e^a + \omega^a{}_b \wedge e^b, \tag{9}$$

and [6]

$$D(\omega)\, T^a = R^a{}_b \wedge e^b,$$
$$D(\omega)\, R^a{}_b = 0, \tag{10}$$

where the Equation (10) are the generalisation of the usual Bianchi identity. The two Equations (7) and (9), are known as *Cartan structure equations* [5].

Taking into account that the full diffeomorphic covariant derivative on the vielbein is zero, $D(\omega, \Gamma)\, e^a = 0$, we obtain a relation between the affine connection, $\Gamma^\lambda{}_{\mu\nu}$, and the spin connection (2), $\omega_\mu{}^a{}_b$. In components [6]:

$$D_\mu(\Gamma)\, e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda{}_{\nu\mu}\, e^a_\lambda = -\omega_\mu{}^a{}_b\, e^b_\nu. \tag{11}$$

From (1), (7) and (11), we easily obtain

$$T^a{}_{\mu\nu} = e^a_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \right) \equiv 2\, e^a_\lambda\, \Gamma^\lambda{}_{[\mu\nu]}, \tag{12}$$

where $[\mu\nu]$ indicates antisymmetrisation of the indices.

The relation (12) expresses the essence of torsion, namely that in its presence the affine connection loses its symmetry in its lower indices. The torsion *tensor* is associated with the antisymmetric part (in the lower indices) of the affine connection, which is its only part that transforms as a tensor under general coordinate transformations.

The spin connection then, in general, is torsion-full. If we want a torsion-free connection (that is the case of GR) we need to impose

$$d\, e^a + \omega^a_{\ b} \wedge e^b = 0,$$ (13)

and we have that the antisymmetric (because of (4)) connection is $\omega_{ab} = \mathring{\omega}_{ab}$. In other words, covariant constancy of the metric is a separate request from zero torsion. In fact, in Riemann–Cartan spaces the metric is compatible, hence ω_{ab} is antisymmetric, but torsion is nonzero.

We next remark that the contorsion one-form coefficients $\mathcal{K}^a_{\ bc} = \mathcal{K}^a_{\ b\mu} E^\mu_c$ satisfy $\mathcal{K}^c_{\ ab} = -\mathcal{K}^c_{\ ba}$ and are related to the torsion tensor coefficients $T^a_{\ bc} = T^a_{\ \mu\nu} E^\mu_b E^\nu_c$ via [5]

$$\mathcal{K}_{abc} = -\frac{1}{2}\left(T_{cab} - T_{abc} - T_{bca}\right) \Rightarrow T_{[abc]} = -2\mathcal{K}_{[abc]},$$ (14)

where $[abc]$ denotes total antisymmetrisation. From (14) we can write the torsion tensor in term of contorsion as

$$T^a_{\ bc} = -2\mathcal{K}^a_{[bc]}.$$ (15)

Equations (14) and (15) tell us that both torsion and contorsion tensors carry the same information.

2.1. Geometric Interpretation

Let us now discuss the geometric interpretation of torsion, by parallel transporting the vector v^μ along the direction dx^ν, using the connection $\Gamma^\lambda_{\ \mu\nu}$ that appears in (12)

$$\delta_\parallel v^\mu = v^\mu_\parallel(x + dx) - v^\mu(x) = \Gamma^\mu_{\ \rho\nu}\, v^\rho\, dx^\nu.$$

Then, the covariant derivative $D(\Gamma)\, v$ can be written, in its components, as the difference

$$v^\mu(x + dx) - v^\mu_\parallel(x + dx) = v^\mu(x + dx) - v^\mu(x) - \delta_\parallel v^\mu = \left(\partial_\nu v^\mu - \Gamma^\mu_{\ \rho\nu}v^\rho\right) dx^\nu$$
$$\equiv D_\nu(\Gamma)\, v^\mu\, dx^\nu.$$ (16)

Both curvature and torsion measure the noncommutativity of covariant derivatives of a vector taken along two directions [7],

$$D_\nu(\Gamma)\, D_\rho(\Gamma)\, v^\mu - D_\rho(\Gamma)\, D_\nu(\Gamma)\, v^\mu = R^\mu_{\ \sigma\rho\nu}\, v^\sigma + T^\sigma_{\ \rho\nu}\, D_\sigma(\Gamma)\, v^\mu,$$

where the torsion components have been defined already and the curvature components can be written as

$$R^\mu_{\ \sigma\rho\nu} = \partial_\rho \Gamma^\mu_{\ \sigma\nu} - \partial_\nu \Gamma^\mu_{\ \sigma\rho} + \Gamma^\mu_{\ \lambda\nu}\, \Gamma^\lambda_{\ \sigma\rho} - \Gamma^\mu_{\ \lambda\rho}\, \Gamma^\lambda_{\ \sigma\nu}.$$ (17)

It is remarkable that for a scalar field, φ, such noncommutativity is entirely due to torsion,

$$D_\nu(\Gamma)\, D_\rho(\Gamma)\, \varphi - D_\rho(\Gamma)\, D_\nu(\Gamma)\, \varphi = T^\sigma_{\ \rho\nu}\partial_\sigma\varphi.$$

We introduce the *metric tensor* $g_{\mu\nu}$ when it is necessary to measure angles and distances between events in a spacetime manifold. The line element is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu.$$ (18)

We can define the longitude of any curve on $\mathcal{M}$ by integrating (18).

A very reasonable assumption usually taken is that local distances do not change under parallel transport, i.e.,

$$D_\rho(\Gamma)\, g_{\mu\nu} = 0.$$ (19)

The condition (19) for a linear connection Γ is called *metric compatibility*, which leads to the antisymmetry of the spin-connection (5) [5]. Figure 2 illustrates a geometric interpretation of torsion, with details in the caption.
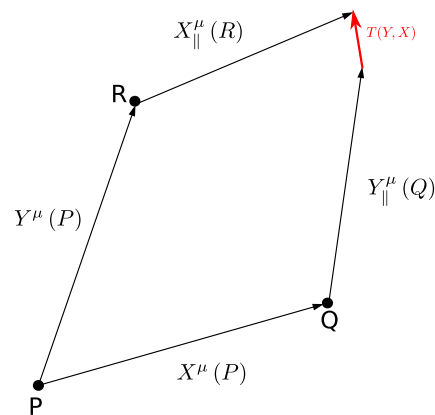

Figure 2. A geometric interpretation of torsion in Riemann–Cartan spaces. Consider two vector fields, X and Y, at a point P. First, parallel-transport X along Y to the infinitesimally close point R. Then, again from P, parallel-transport Y along X to reach a point Q. The failure of the closure of the parallelogram is the geometrical signal of torsion, and its value is the difference $T(X, Y)$ (here in red) between the two resulting vectors. An *n*-dimensional manifold M with a linear connection preserving local distances, i.e., fulfilling condition (19), is called a *Riemann–Cartan space*, denoted by U_n. In Riemannian spaces, V_n, this tensor is assumed to be zero. The picture was inspired by [8] but with the notation of [5], and was taken from [9].

The unique linear metric-compatible torsionless connection, called the *Levi–Civita connection*, can then be obtained from the metric $g_{\mu\nu}$ (see [5] for details)

$$\left\{ \begin{matrix} \mu \\ \nu\,\rho \end{matrix} \right\} = \frac{1}{2}\, g^{\mu\sigma}\, (\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}). \tag{20}$$

The quantities (20) are called *Christoffel symbols*, and the curvature associated with the Levi–Civita connection is the *Riemannian curvature tensor*, denoted by $\mathring{R}^{\mu}{}_{\nu\rho\sigma}$. In this way, the linear connection in a Riemann–Cartan space can be written as

$$\Gamma^{\mu}{}_{\nu\rho} = \left\{ \begin{matrix} \mu \\ \nu\,\rho \end{matrix} \right\} + K^{\mu}{}_{\nu\rho}.$$

2.2. Gravitational Dynamics in Presence of Torsion

The Einstein–Hilbert scalar curvature term corresponding to the generalised contorted Riemann tensor is given by

$$\begin{aligned} \mathcal{S}_{\text{grav}} &= \frac{1}{2\kappa^2} \int d^4x\, \sqrt{-g}\, R = \frac{1}{2\kappa^2} \int R_{ab} \wedge \star(e^a \wedge e^b) \\ &= \frac{1}{2\kappa^2} \int \left(\mathring{R}_{ab} + D(\mathring{\omega})\, \mathcal{K}_{ab} + \mathcal{K}_{ac} \wedge \mathcal{K}^{c}{}_{b} \right) \wedge \star(e^a \wedge e^b) \\ &= \frac{1}{2\kappa^2} \int \left(\mathring{R}_{ab} + \mathcal{K}_{ac} \wedge \mathcal{K}^{c}{}_{b} \right) \wedge \star(e^a \wedge e^b), \end{aligned} \tag{21}$$

where, in the last two equalities, we used form language and took into account the definition of the generalised curvature two form (7) in terms of the contorted spin connection (2). In (21), $\kappa^2 = 8\pi G = M_{\text{Pl}}^{-2}$ is the gravitational constant in four dimensions, which is the inverse of the square of the reduced Planck mass M_{Pl} in units $\hbar = c = 1$ we work throughout. In passing from the second to the third equality we used the fact that the term $D(\mathring{\omega})\, \mathcal{K}_{ab} \wedge \star(e^a \wedge e^b)$ is a total derivative and thus yields, by means of Stoke theorem, a

boundary term that we assume to be zero (we used also the metric compatibility of the spin-connection (5)).

For completeness, we give below the component form of the gravitational action (in the notation of [6]):

$$
S_{\text{grav}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\mathring{R} + \Delta \right),
$$

$$
\Delta \equiv K^{\lambda}{}_{\mu\nu} K^{\nu\mu}{}_{\lambda} - K^{\mu\nu}{}_{\nu} K_{\mu\lambda}{}^{\lambda} = T^{\nu}{}_{\nu\mu} T^{\lambda}{}_{\lambda}{}^{\mu} - \frac{1}{2} T^{\mu}{}_{\nu\lambda} T^{\nu\lambda}{}_{\mu} + \frac{1}{4} T_{\mu\nu\lambda} T^{\mu\nu\lambda}. \tag{22}
$$

Next, we decompose the torsion tensor in its irreducible parts under the Lorentz group [3,6,10],

$$
T_{\mu\nu\rho} = \frac{1}{3} \left(T_\nu g_{\mu\rho} - T_\rho g_{\mu\nu} \right) - \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} S^\sigma + q_{\mu\nu\rho}, \tag{23}
$$

where

$$
T_\mu \equiv T^{\nu}{}_{\mu\nu}, \tag{24}
$$

is the *torsion trace vector*, transforming like a vector,

$$
S_\mu \equiv \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma}, \tag{25}
$$

is the *pseudotrace axial vector* and the antisymmetric tensor $q_{\mu\nu\rho}$ satisfies

$$
q^{\nu}{}_{\rho\nu} = 0 = \epsilon^{\sigma\mu\nu\rho} q_{\mu\nu\rho}. \tag{26}
$$

Thus, we may write the contorsion tensor components as:

$$
\mathcal{K}_{abc} = \frac{1}{2} \epsilon_{abcd} S^d + \widehat{\mathcal{K}}_{abc}, \tag{27}
$$

being $\widehat{\mathcal{K}}_{abc}$, by definition, the difference of $\mathcal{K}_{abc}$ and the first term of (27). This yields for the quantity Δ in (22):

$$
\Delta = \frac{3}{2} S_d S^d + \widehat{\Delta}, \tag{28}
$$

with $\widehat{\Delta}$ being given by the combination appearing in the expression for Δ in (22) in terms of the contorsion tensor, but with K replaced by $\widehat{K}$ [6].

Using the decomposition (23) and the relations (24)–(26) and discarding total derivative terms, the gravitational part of the action can be written as:

$$
S_{\text{grav}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(E_a^\mu E_b^\nu R^{ab}{}_{\mu\nu}(\omega) + \frac{1}{24} S_\mu S^\mu - \frac{2}{3} T_\mu T^\mu + \frac{1}{2} q_{\mu\nu\rho} q^{\mu\nu\rho} \right)
$$

$$
\equiv \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R + \widehat{\Delta} \right) + \frac{3}{4\kappa^2} \int S \wedge \star S, \tag{29}
$$

where in the last line we used mixed notation components/form, following [6], as this will be more convenient for the discussion that follows. For future use, the reader should notice that $\widehat{\Delta}$ is independent of the pseudovector S^d.

An important part of our review will deal with fermionic torsion, that is torsion induced by fermion fields in the theory. Such a feature arises either in certain materials, such as graphene, to be discussed in Section 5, or in fundamental theories, which may play a role in particle physics, such as SUGRA (local SUSY, Section 6), unconventional supersymmetry (USUSY) (Section 7), and string theory (with applications to cosmology, Section 8.1). In the next Section we review such a (quantum) torsion in a fermionic theory corresponding to QED, as an instructive example, which can be generalised to non-Abelian gauge fields as well.

3. (Quantum) Torsion, Axions and Anomalies in Einstein–Cartan Quantum Electrodynamics

Our starting point is a $(3 + 1)$-dimensional QED with torsion (termed, from now on, "contorted QED"), describing the dynamics of a massless Dirac fermion field $\psi(x)$, coupled to a gauged (electromagnetic) U(1) field A_μ, in a curved spacetime with torsion[4]. The action of the model reads [6]:

$$\mathcal{S}_{\text{TorsQED}} = \frac{i}{2} \int d^4x \sqrt{-g} \left[\overline{\psi}(x) \gamma^\mu \mathcal{D}_\mu(\omega, A) \psi(x) - \overline{\mathcal{D}_\mu(\omega, A)\psi(x)} \gamma^\mu \psi(x) \right], \qquad (30)$$

where $\mathcal{D}_\mu(\omega, A) \equiv D_\mu(\omega) - i e A_\mu$ is the diffeomorphic *and* gauge covariant derivative and [2,3]:

$$D_\mu(\omega) = \partial_\mu + i \omega^a{}_{b\mu} \sigma^b{}_a , \quad \sigma^{ab} \equiv \frac{i}{4} [\gamma^a, \gamma^b]. \qquad (31)$$

The quantities γ^a and γ^μ denote the 4×4 Dirac matrices in the tangent space and in the manifold, respectively. On account of (31) and (2) (discussed in Section 2), the action (30) becomes:

$$S_{\text{TorQED}} = S_{\text{QED}}(\mathring{\omega}, A) + \frac{1}{8} \int d^4x \sqrt{-g}\, \overline{\psi}(x) \{\gamma^c, \sigma^{ab}\} \mathcal{K}_{abc}\, \psi(x), \qquad (32)$$

where $S_{\text{QED}}(\mathring{\omega}, A)$ is the standard QED action in a torsion-free curved spacetime and $\{\,,\,\}$ denotes the standard anticommutator. Since the Dirac γ-matrices obey

$$\{\gamma^c, \sigma^{ab}\} = 2\epsilon^{abc}{}_d\, \gamma^d\, \gamma^5,$$

where ϵ^{abcd} is the Levi–Civita tensor in $(3 + 1)$-dimensions, one can prove that it is only the totally antisymmetric part of the torsion that couples to fermionic matter [3]. Indeed, on using (14), we may write (32) in the form

$$S_{\text{TorQED}} = S_{\text{QED}}(\omega, A) - \frac{3}{4} \int d^4x \sqrt{-g}\, S_\mu\, \overline{\psi}\, \gamma^\mu\, \gamma^5\, \psi, \qquad (33)$$

where $S_d = \frac{1}{3}\epsilon^{abc}{}_d\, T_{abc}$ (or in form language $S = \star T$) is the dual pseudovector constructed out of the totally antisymmetric part of the torsion. From (33) we thus observe that only the totally antisymmetric part of the torsion couples to the fermion axial current

$$j^{5\mu} = \overline{\psi}\, \gamma^\mu\, \gamma^5\, \psi. \qquad (34)$$

The $(2 + 1)$-dimensional version of (33) will be our starting point to describe the conductivity electrons in graphene-like materials in a fixed spacetime with torsion (see Section 5). In contorted QED, the Maxwell tensor is defined with respect to the ordinary torsion-free geometry, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = D_\mu(\mathring{\omega})A_\nu - D_\nu(\mathring{\omega})A_\mu$. This way, the Maxwell tensor continues to satisfy the Bianchi identity (in form language $d F = 0$) even in the presence of torsion. Thus the standard Maxwell term, independent from torsion, is added to the action (30) to describe the dynamics of the photon field:

$$S_{\text{Max}} = -\frac{1}{4} \int d^4x \sqrt{-g}\, F_{\mu\nu}\, F^{\mu\nu} = -\frac{1}{2} \int F \wedge \star F, \qquad (35)$$

where $\star$ denotes the Hodge star [4,5].

The dynamics of the gravitational field is described by adding Einstein–Hilbert scalar curvature action (21) (or, equivalently, (22), in component form) of Section 2 to the above actions. By adding (33) to (22), so as to obtain the full gravitational action in a contorted geometry, with QED as its matter content, we obtain from the graviton equations of motion the stress-energy tensor of the theory, which can be decomposed into various components gauge, fermion and torsion-S (the reader should recall that only the totally antisymmetric part of the torsion S couples to matter in the theory):

$$T^A_{\mu\nu} = F_{\mu\lambda}\,F^{\lambda}_{\ \nu} - \frac{1}{2}\,g_{\mu\nu}\,F_{\alpha\beta}\,F^{\alpha\beta},$$

$$T^{\psi}_{\mu\nu} = -\left(\frac{i}{2}\overline{\psi}\gamma_{(\mu}\,\mathcal{D}_{\nu)}\,\psi - (\mathcal{D}_{(\mu}\overline{\psi})\gamma_{\nu)}\,\psi\right) + \frac{3}{4}\,S_{(\mu}\,\overline{\psi}\,\gamma_{\nu)}\,\gamma^5\,\psi,$$

$$T^S_{\mu\nu} = -\frac{3}{2\,\kappa^2}\left(S_{\mu}\,S_{\nu} - \frac{1}{2}\,g_{\mu\nu}\,S_{\alpha}\,S^{\alpha}\right), \tag{36}$$

where $(\dots)$ denotes indices symmetrisation.

Variation of the above gravitational action with respect to the torsion components T^{μ}, $q^{\mu\nu\rho}$ and S^{μ} (cf. (23)), treated as *independent* field variables, leads to the equations of motion (in form language):

$$T^{\mu} = 0, \qquad q_{\mu\nu\rho} = 0, \qquad S = \frac{\kappa^2}{2}\,j^5, \tag{37}$$

respectively, where j^5 is the axial fermion current one form, which in components is given by Equation (34). Thus, classically, only the totally antisymmetric component of the torsion is non vanishing in this Einstein–Cartan theory with fermions. From (2), (14) and (23), we then obtain for the *on-shell* torsion-full spin connection:

$$\omega^{ab}_{\mu} = \mathring{\omega}^{ab}_{\mu} + \frac{\kappa^2}{4}\epsilon^{ab}_{\ \ cd}\,e^c_{\mu}\,j^{5\,d}, \tag{38}$$

thereby associating the torsion part of the connection, induced by the fermions, with the spinor axial current.

We next remark that the equations of motion for the fermion, stemming from (33), imply the gauged Dirac equation with the vector pseudovector S_{μ}, corresponding to the totally antisymmetric torsion component, playing the role of an axial source:

$$i\gamma^{\mu}\,\mathcal{D}_{\mu}(\omega, A)\,\psi = \frac{3}{4}\,S_{\mu}\,\gamma^{\mu}\,\gamma^5\,\psi. \tag{39}$$

Classically, (37) implies a direct substitution of the torsion by the axial fermion current in (36) and (39). Moreover, as a result of the Dirac equation (39), a classical conservation of the axial current follows, $d \star j^5 = 0$. In view of (37), this, in turn, implies a classical conservation of the torsion pseudovector S, that is:

$$d \star S = 0. \tag{40}$$

Because the action is quadratic in S_{μ}, one could integrate it out exactly in a path integral, thus producing repulsive four fermion interactions

$$-\frac{3\,\kappa^2}{16}\int j^5 \wedge \star j^5, \tag{41}$$

which are a characteristic feature of Einstein–Cartan theories.

However, this would *not* be a self consistent procedure in view of the fact that, due to chiral anomalies, the axial fermion current conservation is violated at a quantum level [12–17]. Specifically at one loop one obtains for the divergence of the axial fermion current in a curved spacetime with torsion:

$$d \star j^5 = \frac{e^2}{8\pi^2}\,F \wedge F - \frac{1}{96\pi^2}R^a_{\ b} \wedge R^b_{\ a} \equiv \mathcal{G}(\omega, A). \tag{42}$$

It can be shown [14,18,19] that, by the addition of appropriate counterterms, the torsion contributions to $\mathcal{G}(\omega, A)$ can be removed, and hence one obtains

$$d \star j^5 = \frac{e^2}{8\pi^2}\,F \wedge F - \frac{1}{96\pi^2}\mathring{R}^a_{\ b} \wedge \mathring{R}^b_{\ a} \equiv \mathcal{G}(\mathring{\omega}, A), \tag{43}$$

where only torsion-free quantities appear in the anomaly equation.

Therefore, to consistently integrate over the torsion S_μ in the path integral of the contorted QED, we need to add appropriate counterterms order by order in perturbation theory. This will ensure the conservation law (40) in the quantum theory, despite the presence of the anomaly (43). This can be achieved [6] by implementing (40) as a δ-functional constraint in the path integral, represented by means of a Lagrange multiplier pseudoscalar field Φ:

$$\delta(d \star S) = \int D\Phi \, \exp\left(i \int \Phi \, d \star S\right), \tag{44}$$

thus writing for the S-path integral

$$\begin{aligned}
\mathcal{Z} &\propto \int D S \, \delta(d \star S) \, \exp\left(i \int \left[\frac{3}{4\kappa^2} S \wedge \star S - \frac{3}{4} S \wedge \star j^5\right]\right) \\
&= \int DS \, D\Phi \, \exp\left(i \int \left[\frac{3}{4\kappa^2} S \wedge \star S - \frac{3}{4} S \wedge \star j^5 + \Phi \, d \star S\right]\right).
\end{aligned} \tag{45}$$

The path integral over S can then be performed, making this way the field Φ dynamical. Normalising the kinetic term of Φ, requires the rescaling $\Phi = (3/(2\kappa^2))^{1/2} \, b$. We may write then for the result of the S path-integration [6]:

$$\begin{aligned}
\mathcal{Z} &\propto \int Db \, \exp\left[i \int \left(-\frac{1}{2} db \wedge \star db - \frac{1}{f_b} db \wedge \star j^5 - \frac{1}{2f_b^2} j^5 \wedge \star j^5\right)\right], \\
f_b &\equiv (3\kappa^2/8)^{-1/2},
\end{aligned} \tag{46}$$

which demonstrates the emergence of a massless axion-like degree of freedom $b(x)$ from torsion. The characteristic shift-symmetric coupling of the axion to the axial fermionic current with f_b the corresponding coupling parameter [20]. Using the anomaly Equation (43) we may partially integrate this term to obtain:

$$\mathcal{Z} \propto \int Db \, \exp\left[i \int \left(-\frac{1}{2} db \wedge \star db + \frac{1}{f_b} b \, \mathcal{G}(\omega, A) - \frac{1}{2f_b^2} j^5 \wedge \star j^5\right)\right]. \tag{47}$$

The repulsive four fermion interactions in (46) and (47) are characteristic of Einstein–Cartan theories, as already mentioned, but as we see from (47) this is not the only effect of torsion. One has also the coupling of torsion to anomalies, which induces a coupling of the axion to gauge and gravitational anomaly parts of the theory. The emergence of axionic degrees of freedom from torsion is an important result which will play a crucial role in our cosmological considerations. We have observed that, in the massless chiral QED case, torsion became dynamical, due to anomalies. We stress that the effective field theory (47) guarantees the conservation law (40), and hence the conservation of the axion charge

$$Q_S = \int \star S, \tag{48}$$

order by order in perturbation theory.

Viewed as a gravitational theory, (47) corresponds to a Chern–Simons gravity [21–23], due to the presence of the gravitational anomaly. From a physical point of view, placing the theory on an expanding Universe Friedman–Lemaître–Robertson–Walker (FLRW) background spacetime, we observe that the gravitational anomaly term vanishes [21,23]. However, the gauge chiral anomaly survives. This could have important consequences for the cosmology of the model.

In fact, although above we discussed QED, we could easily consider more general models, with several fermion species, some of which could couple to non-Abelian gauge fields, e.g., the SU(3) colour group of Quantum Chromodynamics (QCD). In such a case, torsion, being gravitational in origin, couples to all fermion species, in a similar way as in the aforementioned QED case, (33), but now the axial current (34) is generalised to include all the fermion species:

$$J^{5\,\mu}_{\text{tot}} = \sum_{i=\text{fermion species}} \overline{\psi}_i\, \gamma^\mu\, \gamma^5\, \psi_i. \tag{49}$$

Chiral anomalies of the axial fermion current as a result of (non-perturbative) instanton effects of the non-Abelian gauge group, e.g., SU(3), during the QCD cosmological era of the Universe, will be responsible for inducing a breaking of the axion shift-symmetry, by generating a potential for the axion *b* of the generic form [20]

$$V(b) = \int d^4x \sqrt{-g}\, \Lambda^4_{\text{QCD}} \left[1 - \cos\left(\frac{b}{f_b}\right)\right], \tag{50}$$

where Λ_{QCD} is the energy scale at which the instantons are a dominant configurations. As we observe from (50) one obtains this way a *mass* for the torsion-induced axion $m_b = \frac{\Lambda^2_{\text{QCD}}}{f_b}$, which can thus play a role of a dark matter component. In this way, we can have a geometric origin of the dark matter component in the Universe [24], which we discuss in Section 8.1, where we describe a more detailed scenario in which such cosmological aspects of torsion are realised in the context of string-inspired cosmologies.

4. Ambiguities in the Einstein–Cartan Theory —The Barbero–Immirzi Parameter

The contorted gravitational actions discussed in the previous Section can be modified by the addition of total derivative topological terms, which do not affect the equations of motion, and hence the associated dynamics. One particular form of such total derivative terms plays an important role in Loop quantum gravity [25,26], a non-perturbative approach to the canonical quantization of gravity. Below, we shall briefly mention such modifications, which, as we shall see, introduce an extra (complex) parameter, β, in the connection, termed "BI parameter", due to its discoverer [27,28]. This is a free parameter of the theory and it may be thought of as the analogue of the instanton angle θ of non-Abelian gauge theories, such as QCD, associated with strong CP violation.

Let us commence our discussion by presenting the case of pure gravity in the first-order formalism. In pure gravity, if torsion is absent, a term in the action linear in the *dual* of the Riemann curvature tensor, $\tilde{\mathring{R}}^{ab}_{\mu\nu} \equiv \epsilon^{ab}_{\ \ cd}\mathring{R}^{cd}_{\mu\nu}$, called the Holst term [29]

$$S_{\text{Holst}} = -\frac{\beta}{4\kappa^2} \int d^4x\, e\, E^\mu_a E^\nu_b\, \tilde{\mathring{R}}^{ab}_{\mu\nu}, \tag{51}$$

where $e = \sqrt{-g}$ is the vielbein determinant, and vanishes identically, as a result of the corresponding Bianchi identity of the Riemann curvature tensor:

$$\mathring{R}_{\alpha\mu\nu\rho} + \mathring{R}_{\alpha\nu\rho\mu} + \mathring{R}_{\alpha\rho\mu\nu} = 0. \tag{52}$$

However, if torsion is present, such a term yields non-trivial contributions, since in that case the Bianchi identity (52) is not valid. In the general case β is a complex parameter, and the reader might worry that in order to guarantee the reality of the effective action one should add the appropriate complex conjugate (i.e., impose reality conditions). As we shall discuss below, however, the effective action contributions in the second-order formalism, obtained from (51) upon decomposing the connection into torsion and torsion-free parts, and using the solutions for the torsion obtained by varying the Holst modification of the general relativity action with respect the independent torsion components, as in the Einstein–Cartan theory discussed previously, are independent of the BI parameter β, which can thus take on any value.

We mention for completeness that the term (51) has been added by Holst [29] to the standard first-order GR Einstein–Hilbert term in the action in order to derive a Hamiltonian formulation of canonical general relativity suggested by Barbero [30,31] from an action. This formulation made use of a real SU(2) connection in general relativity, as opposed to the complex connection introduced by Ashtekar in his canonical formulation of gravity [32]. The link between the two approaches was provided by Immirzi [27,28] who, by means of

a canonical transformation, introduced a finite complex number $\beta \neq 0$ (the *BI parameter*, previously mentioned) in the definition of the connection. When the (otherwise free) parameter takes on the purely imaginary values $\beta = \pm i$, the theory reduces to the self (or anti-self) dual formulation of canonical quantum gravity proposed by Ashtekar [32,33] and Ashtekar–Romano–Tate [34]. The values $\beta = \pm 1$ lead to Barbero's real Hamiltonian formulation of canonical gravity. The Holst modification (51), can then be used to derive these formulations from an effective action, with the coefficient β in (51) playing the role of the complex BI parameter[5].

In the presence of fermions, the Holst modification (51) is *not* a total derivative. Therefore, if added it will lead to the false prediction of "observable effects" of the BI parameter. In particular, following exactly the same procedure as for the Einstein–Cartan theory of the previous Section, and using the decomposition (23) of the torsion in the Holst modification of the Einstein action, obtained by adding (51) to the combined actions (29), (33) and (35), one can derive the following extra contributions in the action (up to total derivatives) [35–38]

$$S_{\text{Holst}} = -\frac{1}{2\kappa^2} \int d^4 x e \left(\frac{\beta}{3} T_\mu S^\mu + \frac{\beta}{2} \epsilon_{\mu\nu\rho\sigma} q_\lambda{}^{\mu\rho} q^{\lambda\nu\sigma} \right). \tag{53}$$

By varying independently the combined actions (29), (33) and (53) with respect to the torsion components, as in the Einstein–Cartan theory, one arrives at the equations:

$$\frac{1}{24\kappa^2} S^\mu + \frac{\beta}{6\kappa^2} T^\mu - \frac{1}{8} j^{5\mu} = 0,$$
$$-4T^\mu + \beta S^\mu = 0,$$
$$q_{\mu\nu\rho} + \beta\epsilon_{\nu\sigma\rho\lambda} q_\mu^{\sigma\lambda} = 0. \tag{54}$$

The solutions of (54) are [35,37]

$$T^\mu = \frac{3\kappa^2}{4} \frac{\beta}{\beta^2 + 1} j^{5\mu}, \quad S^\mu = \frac{3\kappa^2}{\beta^2 + 1} j^{5\mu}, \quad q_{\mu\nu\rho} = 0. \tag{55}$$

Substituting these back into the action, and following the steps carried out for the Einstein–Cartan theory, would lead to a four-fermion induced interaction term of the form [35]

$$S_{j^5 - j^5} = -\frac{3}{16(\beta^2 + 1)} \kappa^2 \int d^4 x e\, j^{5\mu} j_\mu^5. \tag{56}$$

The coupling of this term depends on the BI parameter β, which is in contradiction to its role in the canonical formulation of gravity [27,28], as a free parameter, being implemented by a canonical transformation in the connection field. Moreover, for purely imaginary values of β, such that $|\beta|^2 > 1$, the four fermion interaction is *attractive*. For values of $\beta \to \pm i$ (which corresponds to the well-defined Ashtekar–Romano–Tate theory [34]) the interaction diverges, which presents a puzzle. Furthermore, for values of $|\beta| \to 1^+$ the coupling of the four-fermion interaction is strong. Such strong couplings can lead to the formation of fermion condensates in flat spacetimes, given that the attractive four-fermion effective coupling of (56) in this case is much stronger than the weak gravitational coupling $\kappa^2 \propto G_N$. These features are all in contradiction with the allegedly topological nature of the BI parameter.

The above are indeed pathologies related to the mere addition of a Holst term in a theory with fermions. Such an addition is inconsistent with the first-order formalism, for the simple reason that the Holst term (51) alone is not a total derivative in the presence of fermions, and thus there is no surprise that its addition leads to "observable" effects (56) in the effective action. In addition, as observed in [37], the solution (55) of (54) is mathematically inconsistent, given that the first line of (55) equates a proper vector (T^μ) with an axial one (the axial spinor current $j^{5\mu}$).

The only consistent cases are those for which either $\beta \to 0$ or $\beta \to \infty$. The first is the Einstein–Cartan theory. The second means no torsion, in the sense that in a path integral

formalism, where one integrates over all spin connection configurations, only the zero torsion contributions survive in the partition function, so as to compensate the divergent coefficient. In either case, $T^\mu \to 0$, and the solution (55) reduces to that of the Einstein–Cartan theory (33) and (41). However, this is in sharp contradiction with the arbitrariness of the BI parameter β of the canonical formulation of gravity, which is consistent for every (complex in general) β.

The resolution of the problem was provided by Mercuri [37], who noticed that an appropriate Holst-like modification of a gravity theory in the presence of fermions is possible, if the Holst modification contains additional fermionic-field dependent terms so as to become a total derivative and thus retains its topological nature that characterises such modifications in the torsion-free pure gravity case. The proposed Holst-like term for the torsion-full case of gravity in the presence of fermions contains the Holst term (51) and an *additional* fermion-piece of the form [37] (we ignore the electromagnetic interactions from now on, for brevity, as they do not play an essential role in our arguments):

$$S_{\text{Holst}-\text{fermi}} = \frac{\alpha}{2} \int d^4x\, e \left(\overline{\psi} \gamma^\mu \gamma_5 D_\mu(\omega)\psi + \overline{D_\mu(\omega)\psi} \gamma^\mu \gamma_5 \psi \right), \quad \alpha = \text{const.}, \tag{57}$$

so that the total Holst-like modification is given by the sum $S_{\text{Holst}-\text{total}} \equiv S_{\text{Holst}} + S_{\text{Holst}-\text{fermi}}$.

We next note that the fermionic Holst contributions (57) when combined with the Dirac kinetic terms of the QED action, yield terms of the form (in our relative normalisation with respect to the Einstein terms in the total action):

$$S_{\text{Dirac}-\text{Holst}-\text{fermi}} = \frac{i}{2} \int d^4x\, e \left[\overline{\psi} \gamma^\mu (1 - i\alpha\gamma_5) D_\mu(\omega)\psi + \overline{D_\mu(\omega)\psi} \gamma^\mu (1 - i\alpha\gamma_5)\psi \right]. \tag{58}$$

We thus observe that in the Ashtekar limit [32,33] $\beta = \pm i$, the terms in the parentheses in (58) containing the constant α become the chirality matrices $(1 \pm \gamma_5)/2$ and this is why the specific theory is chiral.

In general, the (complex) parameter α is to be fixed by the requirement that the integrand in $S_{\text{Holst}-\text{total}}$ is a *total derivative*, so that it does not contribute to the equations of motion. It can be readily seen that this is achieved when

$$\alpha = \beta. \tag{59}$$

In that case, one recovers the results of the Einstein–Cartan theory, as far as the torsion decomposition and the second-order final form of the effective action are concerned[6].

4.1. Holst Actions for Fermions and Topological Invariants

A final comment concerns the precise expression of the total derivative term that amounts to the total Holst-like modification $S_{\text{Holst}-\text{total}}$. As discussed in [37], this action can be cast in a form involving (in the integrand) a *topologically invariant density*, the so-called Nieh–Yan topological density [39], which is the only exact form invariant under local Lorentz transformations associated with torsion:

$$S_{\text{Holst}-\text{total}} = -i \frac{\beta}{2} \int d^4x \left[I_{\text{NY}} + \partial_\mu j^{5\mu} \right], \tag{62}$$

with I_{NY} the Nieh–Yan invariant density [39]:

$$I_{\text{NY}} \equiv \epsilon^{\mu\nu\rho\sigma} \left(T_{\mu\nu}{}^a T_{\rho\sigma\,a} - \frac{1}{2} e^a_\mu e^b_\nu R_{\rho\sigma ab}(\omega) \right). \tag{63}$$

Taking into account that, in our case, the torsion-full connection has the form (38), we observe that the first term in I_{NY}, quadratic in the torsion T, vanishes identically, as a result of appropriate Fierz identities. Thus, upon taking into account (38), the Holst-like modification of the gravitational action in this case becomes a total derivative of the form [40]:

$$S_{\text{Holst-total}} = \frac{\mathrm{i}\,\beta}{4} \int d^4x \partial_\mu j^{5\mu} = -\frac{\mathrm{i}\,\beta}{6} \int d^4x \epsilon^{\mu\nu\rho\sigma} \partial_\mu T_{\nu\rho\sigma}(\psi), \tag{64}$$

where the last equality stems from the specific form of torsion in terms of the axial fermion current, implying $2\epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}(\psi) + 3j^{5\mu} = 0$. In general, the Nieh–Yan density is just the divergence of the pseudotrace axial vector associated with torsion, $I_{\text{NY}} = \epsilon^{\mu\nu\rho\sigma} \partial_\mu T_{\nu\rho\sigma}$.

The alert reader can notice that if the axial fermion current is conserved in a theory, then the Holst action (64) vanishes trivially. However, in the case of chiral anomalies, examined above, the axial current is not conserved but its divergence yields the mixed anomaly (42). In that case, by promoting the BI parameter to a canonical pseudoscalar field $\beta \rightarrow \beta(x)$ [38], the Holst term (64) becomes equivalent to the torsion–axion–$j^{5\mu}$ interaction term in (46), upon identifying $\beta(x) = \frac{b(x)}{f_b}$. In this case, the field-prompted BI parameter plays a role analogous to the QCD CP violating parameter [38]. As we have discussed in Section 3. Therefore, this is consistent with the association of torsion with an axion-like dynamical degree of freedom, and thus the works of [38] and [6] lead to equivalent physical results from this point of view [41].

Before closing this subsection, we remark that Holst modifications, along the lines discussed for the spin 1/2 fermions above, are known to exist for higher spin 3/2 fermions, ψ_μ, like gravitinos of SUGRA theories [40,42]. In fact, Holst-like modifications, including fermionic contributions, have been constructed in [40,43] for various supergravities (e.g., $N = 1, 2, 4$), non-trivially extending the spin 1/2 case discussed above. The total derivative nature of these Holst-like actions implies no modifications to the equations of motion. On-shell (local and global) supersymmetries are then preserved. We discuss such issues in Section 6.

4.2. Barbero–Immirzi Parameter as an Axion Field

The classical models described until now in this Section 4 lack the presence of a dynamical pseudoscalar (axion-like) degree of freedom, which, as we have seen in Section 3, is associated with quantum torsion.

Such a pseudoscalar degree of freedom arises in [42,44], which were the first works to promote the BI parameter to a dynamical field, the starting point is the so-called Holst action (51), which by itself is *not* a topological invariant, in contrast to the Nieh–Yan term (63). The work of [42,44] deals with matter free cases. If $\gamma(x)$ represents the BI field, the Holst term now reads (in form language)

$$S_{\text{Holst}} = \frac{1}{2\kappa^2} \int \overline{\gamma}(x)\, e^a \wedge e^b \wedge R_{ab}, \tag{65}$$

where R_{ab} is the curvature two-form, in the presence of torsion, and we used the notation of [44] for the inverse of the BI field $\overline{\gamma}(x) = 1/\gamma(x)$, to distinguish this case from the Kalab–Ramond (KR) axion $b(x)$ in our string-inspired one. The analysis of [42,44] showed that the gravitational sector results in the action

$$\mathcal{S}_{\text{grav+Holst+BI-field}}^{\text{eff}} = \int d^4x \sqrt{-g} \left[-\frac{1}{2\kappa^2} R + \frac{3}{4\kappa^2 \left(\overline{\gamma}^2 + 1\right)} \partial_\mu \overline{\gamma} \partial^\mu \overline{\gamma} \right]. \tag{66}$$

Coupling the theory to fermionic matter [35,36,45] can be achieved by introducing a rather generic non-minimal coupling parameter α, for massless Dirac fermions in the form

$$\mathcal{S}_F = \frac{i}{12} \int \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge \left[(1 - i\alpha)\, \overline{\psi}\gamma^d D(\omega)\, \psi - (1 + i\alpha)\overline{(D(\omega)\,\psi)}\, \gamma^d\, \psi \right], \tag{67}$$

where $\alpha \in \mathbb{R}$ is a constant parameter. The case of constant $\overline{\gamma}$ has been discussed in [35,36] (in fact, Ref. [35] deals with minimally-coupled fermions, i.e., the limit $\alpha = 0$), whilst the work of [37] extended the analysis to coordinate-dependent BI, $\overline{\gamma}(x)$.

The extension of the BI to a coordinate dependent quantity, which is assumed to be a *pseudoscalar field*, implies:

(i) The consistency of (55), given that now the BI parameter being a pseudoscalar field, reinstates the validity of the first of the Equation (55), since the product of its right-hand side is now parity even, and thus transforms as a vector, in agreement with the nature of the left-hand side of the equation.

(ii) Additional terms of interaction of the fermions with the derivative of the BI field $\partial_\mu \overline{\gamma}$:

$$\mathcal{S}_{F\partial\overline{\gamma}} = \frac{1}{2} \int \sqrt{-g} \left(\frac{3}{2(\overline{\gamma}^2 + 1)} \partial^\mu \overline{\gamma} \left[-j_\mu^5 + \alpha\, \overline{\gamma}(x)\, j_\mu \right] \right), \tag{68}$$

with j_μ^5 being the axial current (34) and

$$j_\mu = \overline{\psi}\, \gamma_\mu\, \psi, \tag{69}$$

the vector current.

(iii) Interaction terms of fermions with non-derivative $\overline{\gamma}(x)$ terms:

$$\mathcal{S}_{F-\text{non-deriv}\overline{\gamma}} = \frac{i}{2} \int \sqrt{-g} \left[\left[(1 - i\alpha)\, \overline{\psi}\gamma^d D(\mathring{\Gamma})\, \psi - (1 + i\alpha)\, \overline{(D(\mathring{\Gamma})\psi)}\, \gamma^d\, \psi \right] \right.$$
$$\left. - \int \sqrt{-g}\, \frac{3}{16(\overline{\gamma}^2 + 1)} \left[j_\mu^5 j^{5\mu} - 2\alpha\, \overline{\gamma}\, j_\mu^5 j^\mu - \alpha^2\, j_\mu j^\mu \right], \right. \tag{70}$$

with $D(\mathring{\Gamma})$ being the diffeomorphic covariant derivative, expressed in terms of the torsion-free Christoffel connection, which is the result of [36], as expected, because this term contains non derivative terms of the BI.

The action (70) involves four-fermion interactions with *attractive* channels among the fermions. Such features may play a role in the physics of the early Universe, as we shall discuss in Section 8.2.

We also observe from (70) that the case $\alpha = 0$ (minimal coupling), corresponds to a four-fermion axial-current (56), which however depends on the BI field. Thus, this limiting theory is not equivalent to our string-inspired model, in which the corresponding quantum-torsion-induced four-fermion axial-current-current interaction (56) is independent from the KR axion field $b(x)$, although both cases agree with the sign of that interaction.

A different fermionic action than (70), using non-minimal coupling of fermions with γ^5, has been proposed in [37] as a way to resolve an inconsistency of the Holst action, when coupled to fermions, in the case of constant γ. In that proposal, the $1 + i\alpha$ factor in (70) below, is replaced by the Dirac-self-conjugate quantity $1 - i\,\alpha\,\gamma^5$. The decomposition of the torsion into its irreducible components in the presence of the Holst action with arbitrary (constant) BI parameter, leads to an inconsistency, implying that the vector component of the torsion is proportional to the axial fermion current, and hence this does not transform properly under improper Lorentz transformations. With the aforementioned modification of the fermion action the problem is solved, as demonstrated in [37], upon choosing $\alpha = \overline{\gamma}$, which eliminates the vector component of the torsion. But this inconsistency is valid only if $\overline{\gamma}$ is considered as a constant. Promotion of the BI parameter $\overline{\gamma}$ to a *pseudoscalar* field, $\overline{\gamma}(x)$, resolves this issue, as discussed in [45], given that one obtains in that case consistent results, in the sense that the vector component of the torsion transforms correctly under parity, as a vector, since it contains now, apart from terms proportional to the vector fermionic current (69), also terms proportional to the product of the BI pseudoscalar with the axial fermionic current (34), as well as terms of the form $\overline{\gamma}\partial_\mu\overline{\gamma}$, all transforming properly as vectors under improper Lorentz transformations.

5. Torsion on Graphene

The use of graphene as a tabletop realisation of some high-energy scenarios is now considerably well developed, see, e.g., [46], the review [47] and the contribution [48] to this Special Issue. Let us here recall the main ideas and those features that make graphene a place where torsion is present.

Graphene is an allotrope of carbon and, being a one-atom-thick material, it is the closest to a two-dimensional object in nature. It is fair to say that was theoretically speculated [49,50] and, decades later, it was experimentally found [51]. Its honeycomb lattice is made of two intertwined triangular sub-lattices L_A and L_B, see Figure 3. As is by now well known, this structure is behind a natural description of the electronic properties of graphene in terms of massless, $(2+1)$-dimensional, Dirac quasi-particles.

● = sublattice L_A ○ = sublattice L_B

Figure 3. The honeycomb lattice of graphene, and its two triangular sublattices L_A and L_B. The choice of the basis vectors, $(\vec{a}_1, \vec{a}_2)$ and $(\vec{s}_1, \vec{s}_2, \vec{s}_3)$, is, of course, not unique. Figure taken from [52].

Indeed, starting from the tight-binding Hamiltonian for the conductivity electrons, and considering only near-neighbours contribution[7]

$$H = -t \sum_{\vec{r} \in L_A} \sum_{i=1}^{i=3} \left(a^\dagger(\vec{r}) b(\vec{r} + \vec{s}_i) + b^\dagger(\vec{r} + \vec{s}_i) a(\vec{r}) \right), \tag{71}$$

where t is the nearest-neighbour hopping energy which is approximately 2.8 eV, and $a, a^\dagger (b, b^\dagger)$ are the anticommuting annihilation and creation operators for the planar electrons in the sub-lattice $L_A (L_B)$.

If we Fourier-transform to momentum space, $\vec{k} = (k_x, k_y)$ annihilation and creation operators,

$$a(\vec{r}) = \sum_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}, \ b(\vec{r}) = \sum_{\vec{k}} b_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}, \text{etc}, \tag{72}$$

then

$$H = -t \sum_{\vec{k}} \sum_{i=1}^{i=3} \left(a_{\vec{k}}^\dagger b_{\vec{k}} e^{i\vec{k}\cdot\vec{s}_i} + b_{\vec{k}}^\dagger a_{\vec{k}} e^{-i\vec{k}\cdot\vec{s}_i} \right).$$

Using the conventions for $\vec{s}_i$ of Figure 3, we find that

$$\mathcal{F}(\vec{k}) = -t \sum_{i=1}^{3} e^{i\vec{k}\cdot\vec{s}_i} = -t\, e^{-i\ell k_y} \left[1 + 2 e^{i\frac{3}{2}\ell k_y} \cos\left(\frac{\sqrt{3}}{2} \ell k_x \right) \right], \tag{73}$$

leading to

$$H = \sum_{\vec{k}} \mathcal{F}(\vec{k}) a_{\vec{k}}^\dagger b_{\vec{k}} + \mathcal{F}^*(\vec{k}) b_{\vec{k}}^\dagger a_{\vec{k}}.$$

For graphene, the conduction and valence bands touch at two points[8] $K_{D\pm} = (\pm\frac{4\pi}{3\sqrt{3}\ell}, 0)$, as one can check by finding the zeroes of (73). These points are called *Dirac points*. The dispersion relation $E(\vec{k}) = |f(\vec{k})|$, for $t\ell = 1$, is shown in Figure 4a.
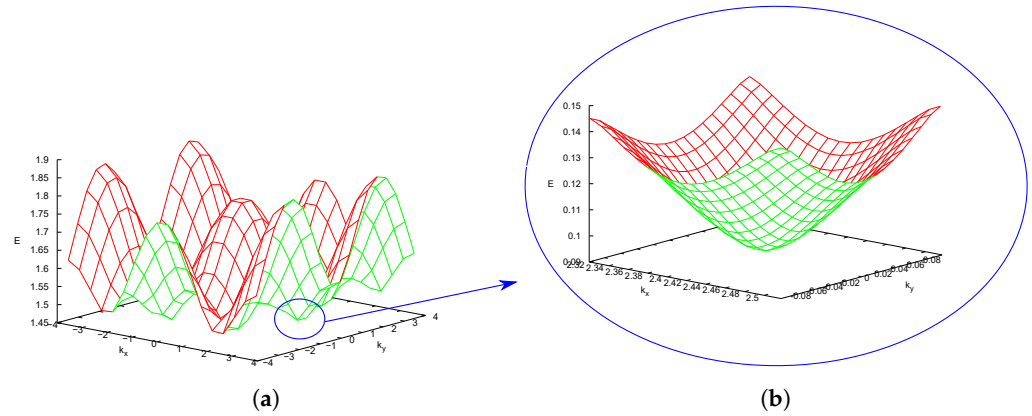
(a) **(b)**

Figure 4. (**a**) The dispersion relation $E(\vec{k})$ for graphene, setting $t\ell = 1$. We only take into account the near neighbours contribution in (71). (**b**) A zoom near the Dirac point K_{D+} showing the linear approximation works well in the low energies regime.

If we expand $\mathcal{F}(\vec{k})$ around the Dirac points, $\vec{k}_\pm = \vec{K}_{D\pm} + \vec{p}$, assuming $|p| \ll |K_D|$, we have

$$
\begin{aligned}
\mathcal{F}_+(\vec{p}) &\equiv f(\vec{k}_+) = v_F(p_x + i\,p_y), \\
\mathcal{F}_-(\vec{p}) &\equiv f(\vec{k}_-) = -v_F(p_x - i\,p_y),
\end{aligned}
$$

where $v_F \equiv \frac{3}{2}t\ell \sim c/300$ is the *Fermi velocity*. We can see from this that the dispersion relations around the Fermi point is

$$
|E_\pm(\vec{p})| = v_F|\vec{p}|, \tag{74}
$$

which is the dispersion relation for a v_F-relativistic massless particle (see Figure 4b).

Defining $a_\pm \equiv a(\vec{k}_\pm)$ and $b_\pm \equiv b(\vec{k}_\pm)$, and arranging the annihilation (creation) operators as a column (row) vector $\psi_\pm = \begin{pmatrix} b_\pm \\ a_\pm \end{pmatrix} ; \psi_\pm^\dagger = \begin{pmatrix} b_\pm^\dagger & a_\pm^\dagger \end{pmatrix}$, then

$$
H = v_F \sum_{\vec{p}} \left[\psi_+^\dagger \vec{\sigma} \cdot \vec{p}\,\psi_+ - \psi_-^\dagger \vec{\sigma}^* \cdot \vec{p}\,\psi_- \right], \tag{75}
$$

where $\vec{\sigma} = (\sigma_1, \sigma_2)$ and $\vec{\sigma}^* = (\sigma_1, -\sigma_2)$, being σ_i the Pauli matrices.

Going back to the configuration space, which is equivalent to make the usual substitution $p^\mu \to -i\partial^\mu$,

$$
H = -i\,v_F \int d^2x \left[\psi_+^\dagger \sigma^\mu \partial_\mu \psi_+ - \psi_-^\dagger \sigma^{*\mu} \partial_\mu \psi_- \right], \tag{76}
$$

where sums turned into integrals because the continuum limit was assumed.

By including time to make the formalism fully relativistic, although with the speed of light c traded for the Fermi velocity v_F, and making the Legendre transform of (76), we obtain the action

$$
S = i\,v_F \int d^3x \bar{\Psi} \gamma^a \partial_a \Psi, \tag{77}
$$

here $x^a = (t, x, y)$, are the flat spacetime coordinates, $\Psi = (\psi_+, \psi_-)$ is a reducible representation for the Fermi field and the γ^a are Dirac matrices in the same reducible representation in three dimensions.

5.1. Torsion as Continuous Limit of Dislocations

Even if we will deal mainly with graphene, the considerations here apply to many other two-dimensional crystals [56]. For the purposes of this work, we can define a *topological defect* as a lattice configuration that cannot be undone by continuous transformations. These are obtained by cutting and sewing the pristine material through what is customarily called a *Volterra process* [57]. Probably, the easiest defects to visualise are the *disclinations*. For this hexagonal lattice, a disclination defect is an *n*-sided polygon with $n \neq 6$, characterised by a *disclination angle s*. When $n = 3, 4, 5$, the defect has a *positive* disclination angle $s = 180°, 120°, 60°$, respectively, whilst for $n = 7, 8, \ldots$, it has a *negative* disclination angle $s = -60°, -120°, \ldots$, respectively. These defects carry intrinsic positive or negative curvature, according to the sign of the corresponding angle *s*, localised at the tip of a conical singularity. In a continuum description, obtained for large samples in the large wavelength regime, one can associate[9] [64,65] to the disclination defect the spin-connection $\omega^{ab}{}_\mu$. Associated to ω is the curvature two-form tensor R^{ab},

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu} \omega^{cb}{}_\nu - \omega^a{}_{c\nu} \omega^{cb}{}_\mu,$$

that we have already met in (7) and in (17).

A *dislocation* can be produced as a dipole of disclinations with zero total curvature. Figure 5 shows a heptagon–pentagon dipole, which in the Volterra process is equivalent to introducing a strip in the lower-half plane, whose width is the *Burgers vector* $\vec{b}$, that characterises this defect.
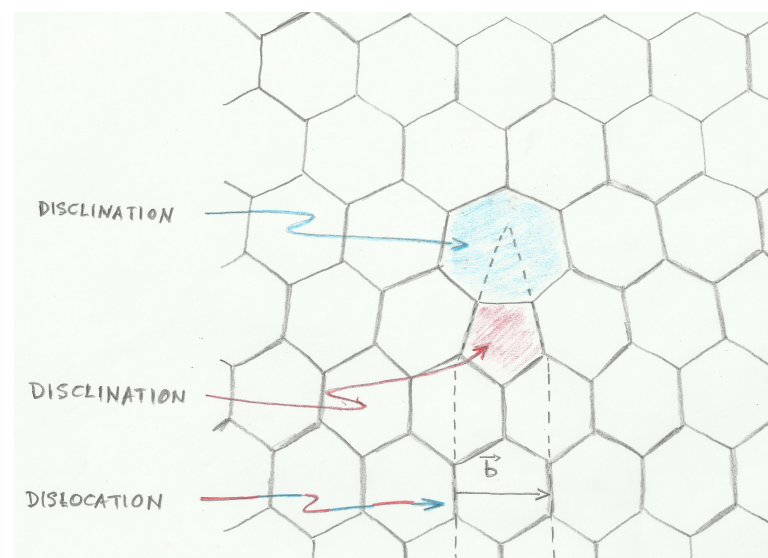

Figure 5. Edge dislocation from two disclinations. Two disclinations, a heptagon, and a pentagon add-up to zero total intrinsic curvature, and make a dislocation with Burgers vector $\vec{b}$, as indicated. In the continuous long wave-length limit, this configuration carries nonzero torsion. Figure taken from [66].

In the continuum limit one can associate the Burgers vector to the torsion tensor [64,65],

$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu, \tag{78}$$

where $T^\rho{}_{\mu\nu} = E_a{}^\rho T^a{}_{\mu\nu}$. On this see our earlier discussion around (1) and (9).

The explicit relation between Burgers vectors and torsion can be written as [67]

$$b^i = \iint_\Sigma T^i_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{79}$$

where the surface Σ has a boundary enclosing the defect. Roughly speaking, the torsion tensor is the surface density of the Burgers vector. Nonetheless, although the relation (79) looks simple, there are subtleties: given a distribution of Burgers vector, there is no simple procedure to assign a torsion tensor to it, even for the simple case of edge dislocations [68].

The smooth way to introduce the effect of dislocations in the long wave-length regime, through torsion tensor, is to consider an action in a $(2 + 1)$-dimensional space with a spin-connection that carries torsion, i.e., a Riemann–Cartan space U_3 [2]. Demanding only Hermiticity and local Lorentz invariance, starting by a simple action

$$S = \frac{\mathrm{i}}{2} v_F \int d^3x \sqrt{|g|} \left(\overline{\Psi}\gamma^\mu \overrightarrow{D}_\mu(\omega)\Psi - \overline{\Psi}\overleftarrow{D}_\mu(\omega)\gamma^\mu\Psi \right), \tag{80}$$

where

$$\overrightarrow{D}_\mu(\omega)\Psi = \partial_\mu\Psi + \frac{1}{8}\omega_\mu^{ab}[\gamma_a, \gamma_b]\Psi, \tag{81}$$

$$\overline{\Psi}\overleftarrow{D}_\mu(\omega) = \partial_\mu\overline{\Psi} - \frac{1}{8}\overline{\Psi}[\gamma_a, \gamma_b]\omega_\mu^{ab}, \tag{82}$$

we obtain, besides possible boundary terms (see details in Appendix A of [69]),

$$S = \mathrm{i}\, v_F \int d^3x \, |e|\, \overline{\Psi}\left(E_a^\mu\, \gamma^a \overrightarrow{D}_\mu(\mathring{\omega}) - \frac{\mathrm{i}}{4}\gamma^5 \frac{\epsilon^{\mu\nu\rho}}{|e|} T_{\mu\nu\rho} \right)\Psi, \tag{83}$$

where $|e| = \sqrt{|g|}$, the covariant derivative is based only on the torsion-free connection, $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2 = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix}$ (we used the conventions for $\gamma^0, \gamma^1, \gamma^2$ that give a γ^5 that *commutes* with the other three gamma matrices[10]), and the contribution due to the torsion is all in the last term through its totally antisymmetric component [69].

We see that the last term couples torsion with the fermionic excitations describing the quasi-particles and is the three-dimensional version of (33), for $A_\mu = 0$. It can be also seen that, to have a nonzero effect, we need $\epsilon^{\mu\nu\rho}T_{\mu\nu\rho} \neq 0$, that requires at least three dimensions. This mathematical fact is behind the obstruction pointed out some time ago leading to the conclusion that, in two-dimensional Dirac materials, torsion can play no physical role [70–72].

To overcome this obstruction, in [69] the time dimension is included in the picture. With this in mind, we have two possibilities that a nonzero Burgers vector gives rise to $\epsilon^{\mu\nu\rho}T_{\mu\nu\rho} \neq 0$:

(i)   a *time-directed* screw dislocation (only possible if the crystal has a time direction)

$$b_t \propto \int \int T_{012} dx \wedge dy, \tag{84}$$

    or

(ii)   an edge dislocation "felt" by an integration along a *spacetime circuit* (only possible if we can actually go around a loop in time), e.g,

$$b_x \propto \int \int T_{102} dt \wedge dy. \tag{85}$$

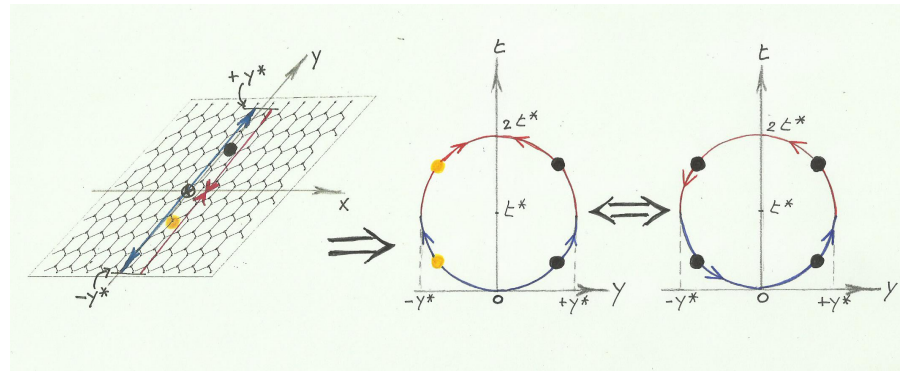This last scenario is depicted in Figure 6.

Figure 6. Idealised *time-loop*. At $t = 0$, the hole (yellow) and the particle (black) start their movements from $y = 0$, in opposite directions. At $t = t^* > 0$, the hole is at position $-y^*$, while the particle is at position $+y^*$, (the blue portion of the circuit). Then they come back to their original position, $y = 0$, at $t = 2t^*$ (red portion of the circuit). On the far right, is depicted an equivalent *time-loop*, where the hole moving forward in time is replaced by a particle moving backward. Figure taken from [69].

5.2. Time-Loops in Graphene

Scenario (i) could be explored in the context of the very intriguing time crystals introduced some time ago [73,74], and nowadays under intense experimental studies [75,76]. Lattices that are discrete in all dimensions, including time, would be an interesting playground to probe quantum gravity ideas [77]. In particular, it would have an impact to explore defect-based models of classical gravity/geometry, see for instance [64,65]. However, here we shall focus only on scenario (ii).

By assuming the Riemann curvature to be zero, $\mathring{R}^\mu{}_{\nu\rho\sigma} = 0$, but nonzero torsion (or contorsion $K^\mu{}_{\nu\rho} \neq 0$), and choosing a frame where $\mathring{\omega}^{ab}_\mu = 0$ (see Appendix B of [69]), the action (83) is

$$S = \mathrm{i}\, v_F \int d^3x |e| \left(\overline{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{\mathrm{i}}{4} \overline{\psi}_+ \phi \psi_+ + \frac{\mathrm{i}}{4} \overline{\psi}_- \phi \psi_- \right), \qquad (86)$$

where $\Psi = (\psi_+, \psi_-)$ and $\phi \equiv \epsilon^{\mu\nu\rho} T_{\mu\nu\rho}/|e|$ is what we call *torsion field*; it is a pseudo-scalar and the three-dimensional version of the S_μ we discussed earlier. Even in the presence of torsion, the two irreducible spinors, ψ_+ and ψ_-, are decoupled (however, with opposite signs).

The pictures in Figure 6 refer to a defect-free honeycomb graphene-like sheet. The presence of a dislocation, with Burgers vector $\vec{b}$ directed along x, would result in a failure to close the loop proportional to $\vec{b}$ [69].

The idea of time-looping is fascinating. The challenge is to bring this idealised picture close to experiments. We present below the first steps in that direction, as taken in [69].

5.3. Towards Spotting Torsion in a Lab

The simplest way to realise the scenario just discussed is to have:

(i) the particle-hole pair required for the time-loop to be excited by an *external electromagnetic field*, and

(ii) that what we shall call *holonomy*—a proper disclination or torsion—provides the non-closure of the loop in the proper direction.

Stated differently, we are searching for *the quantifiable consequences of an holonomy, caused by disclination or torsion in a time-loop*. Only an appropriate combination of (i) and (ii) can yield the desired outcome.

With this in mind, the action governing such microscopic dynamics is

$$S = \mathrm{i} \int d^3x\, |e| \left(\overline{\Psi} \gamma^\mu (\partial_\mu - \mathrm{i} g_{\mathrm{em}} A_\mu) \Psi - \mathrm{i} g_{\mathrm{tor}} \overline{\psi}_+ \phi \psi_+ + \mathrm{i} g_{\mathrm{tor}} \overline{\psi}_- \phi \psi_- \right) \qquad (87)$$

$$\rightarrow \mathrm{i} \int d^3x \left(\overline{\psi} \gamma^\mu \partial_\mu \psi - \mathrm{i} g_{\mathrm{em}}\, \hat{j}^\mu_{\mathrm{em}} A_\mu - \mathrm{i} g_{\mathrm{tor}}\, \hat{j}_{\mathrm{tor}}\, \phi \right) \equiv S_0[\overline{\psi}, \psi] + S_I[A, \phi], \qquad (88)$$

where v_F is taken to be one, while g_{em} and g_{tor} are the electromagnetic and torsion coupling constants, respectively, the latter including the factor $1/4$. In (88) we only have one Dirac point, say $\psi \equiv \psi_+$, as this simplifies calculations, and we focus on flat space, $|e| = 1$. Finally, $\hat{j}^\mu_{em} \equiv \overline{\psi}\gamma^\mu\psi$ and $\hat{j}_{tor} \equiv \overline{\psi}\psi$.

The electromagnetic field is *external*, hence it is a four-vector $A_\mu \equiv (V, A_x, A_y, A_z)$. Nonetheless, the dynamics it induces on the electrons living on the membrane is two-dimensional. Therefore, the effective vector potential may be taken to be $A_\mu \equiv (V, A_x, A_y)$, see, e.g., [78,79]. There are two alternatives to this approach. One is the *reduced QED* of [80,81], where the gauge field propagates on a three-dimensional space and one direction is integrated out to obtain an effective interaction with the electrons, constrained to move on a two-dimensional plane. In this approach a Chern–Simons photon naturally appears (see, e.g., [82,83]). Another approach is to engineer a $(2+1)$-dimensional A_μ by suitably straining the material, see, e.g., [71,72], and [84]. In that case, one usually takes the temporal gauge and $A_x \sim u_{xx} - u_{yy}$, $A_x \sim 2u_{xy}$, where u_{ij} is the strain tensor.

As said, defects here are not dynamical, therefore the torsion field ϕ enters the action as an external field, just like the electromagnetic field. One could, as well, include the effects of the constant ϕ into the unperturbed action, as a mass term $S_0 \to S_m$, see, e.g., [85], where $S_m = i \int d^3x \, \overline{\psi}(\slashed{\partial} - m(\phi))\psi$.

We are in the situation described by the microscopic perturbation,

$$S_I[F_i] = \int d^3x \, \hat{X}_i(\vec{x}, t)F_i(\vec{x}, t), \tag{89}$$

with the system responding through $\hat{X}_i(\vec{x}, t)$ to the external probes $F_i(\vec{x}, t)$. The general goal is then to find

$$\hat{X}_i[F_i], \tag{90}$$

to the extent of predicting a measurable effect of the combined action of the two perturbations $F_i(\vec{x}, t)$: $F_1^{em}(\vec{x}, t) \propto A_\mu(\vec{x}, t)$ that induces the response $\hat{j}^\mu_{em}$, and $F_2^{tor}(\vec{x}, t) \propto \phi(\vec{x}, t)$ inducing the response $\hat{j}_{tor}$:

$$S_I[A, \phi] = \int d^3x \left(\hat{j}^\mu_{em}A_\mu + \hat{j}_{tor}\phi \right), \tag{91}$$

where the couplings, g_{em} and g_{tor}, are absorbed in the respective currents.

With no explicit calculations, simply based on the charge conjugation invariance of the action (88), we can already predict that

$$\chi^{torem}_\mu(x, x') \sim \langle \hat{j}^{em}_\mu(x)\hat{j}^{tor}(x') \rangle \equiv 0, \tag{92}$$

which is just an instance of the Furry's theorem of quantum field theory [86], that in QED reads

$$\chi^{em}_{\mu_1\dots\mu_{2n+1}}(x_1, \dots, x_{2n+1}) \sim \langle \hat{j}^{em}_{\mu_1}(x_1) \cdots \hat{j}^{em}_{\mu_{2n+1}}(x_{2n+1}) \rangle = 0, \tag{93}$$

and for us implies

$$\chi^{torem}_{\mu_1\dots\mu_{2n+1}}(x_1, \dots, x_{2n+1}, y_1, \dots, y_m) \sim \langle \hat{j}^{em}_{\mu_1}(x_1) \cdots \hat{j}^{em}_{\mu_{2n+1}}(x_{2n+1})\hat{j}^{tor}(y_1) \cdots \hat{j}^{tor}(y_m) \rangle = 0. \tag{94}$$

This finding indicates that entering the nonlinear response domain is necessary to observe the desired consequences. High-order harmonic generation (HHG) is a well-established technique that has been used to analyse structural changes in atoms, molecules, and more recently, bulk materials (see, e.g., ref. [87] for a recent overview). Thus, the presence or absence of dislocations will significantly alter the intra-band harmonics in our system, which are controlled by the intra-band (electron-hole) current.

5.4. On the Continuum Description of the Two Inequivalent Dirac Points

We have shown earlier that two Dirac points, associated to the reducible $\Psi = (\psi_+, \psi-)$ [47], are important to treat torsion. Two such points are actually relevant in a broader set of cases. From the material point of view [88], this generally has to do with an extra "valley"

degree of freedom in a pristine material, also called colour index [89]. Things change more drastically when topological defects are present. For instance, to make a fullerene C_{60} form pristine graphene we need twelve pentagons sitting at the vertices of an icosahedron, and this generates colour mismatches, see a discussion of these effects in [90]. There, different magnetic flux are added for each vertex which contain a colour line frustration, pointing out to a "magnetic monopole" at the centre of the molecule structure [90]. Such a "monopole" is associated with the SU(2) symmetry group stemming from the doublet structure of the valley degree of freedom (not to be confused with the doublet structure associated with each valley, which generates the irreducibles $\psi_{\pm}$).

Another instance where both Dirac points are needed for an effective description are *grain boundaries* (GBs). A GB is a line of disclinations of opposite curvature, pentagonal and heptagonal here, arranged in such a way that the two regions (grains) of the membrane match. The two grains have lattice directions that make an angle $\theta/2$ with respect to the direction the lattice would have in the absence of the GB. Different arrangements of the disclinations, always carrying zero total curvature, correspond to different θs, the allowed number of which is of course finite, and related to the discrete symmetries of the lattice (hexagonal here). The most common (stable) being $\theta = 21.8°$, and $\theta = 32.3°$, see, e.g., [91,92]. Other arrangements can be found in [93]. In general, one might expect that the angle of the left grain differs in magnitude from the angle of the right grain, $|\theta_L| \neq |\theta_R|$, nonetheless, high asymmetries are not common, and the symmetric situation depicted in Figure 7 is the one the system tends to on annealing [94].
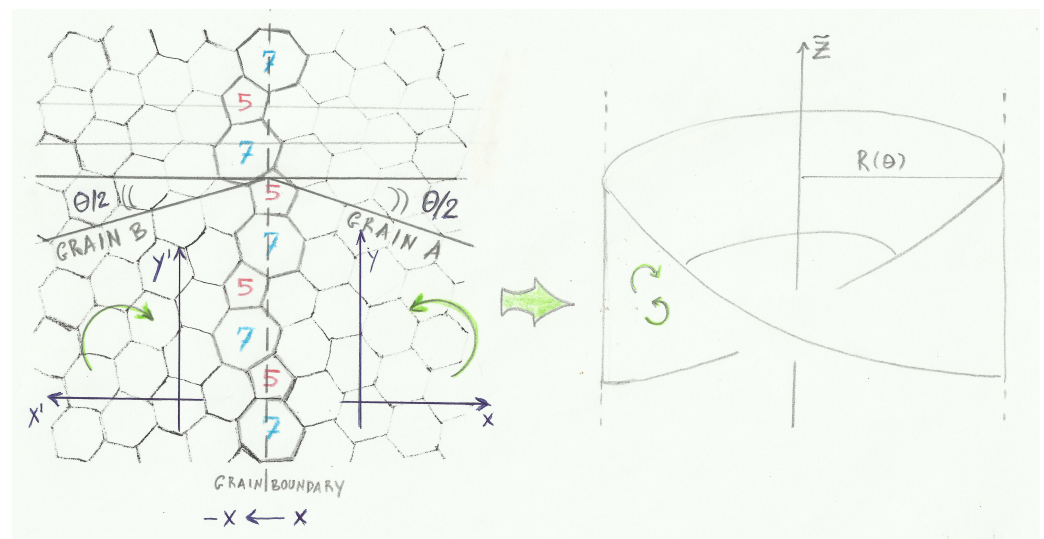

Figure 7. A grain boundary (**left**), and a possible modelling of its effects in a continuum (**right**). This is the prototypical GB, where grain A and grain B are related via a parity ($x \to -x$) transformation. With this, the right-handed frame in grain A is mapped to the left-handed frame in grain B, so that the net effect of a GB is that two orientations coexist on the membrane, and a discontinuous change happens at the boundary. If one wants to trade this discontinuous change for a continuous one, an equivalent coexistence is at work in the non-orientable Möbius strip. One way to quantify the effects of different θs is to relate a varying θ to a varying radius $R(\theta)$ of the Möbius strip. Notice that the third spatial axis is an abstract coordinate, $\tilde{z}$, whose relation with the real z of the embedding space is not specified. Figure taken from [66].

There exists [92,93] a relation (the Frank formula) between θ and the resultant Burgers vector, obtained by adding all Burgers vectors $\vec{b}$s cut by rotating a vector, laying on the GB, of an angle θ with respect to the reference crystal. A possible modelling for this kind of defects was put forward in [66]. That is a four-spinor living on a Möbius strip, see Figure 7, and [66].

6. Torsion in Standard Local Supersymmetry

As a prelude to the Section dedicated to cosmology, we should discuss fermionic (gravitino) torsion in SUGRA models, which can also lead to dynamical breaking of SUGRA. Such models can serve in inducing inflationary scenarios by providing sources for primordial gravitational waves which play a crucial role in inflation, to be discussed in detail in Section 8.

SUGRA theories are Einstein–Cartan theories with fermionic torsion, provided by the gravitino field, $\psi_\mu(x)$, the spin-3/2 (local) supersymmetric fermionic partner of the graviton.

We commence our discussion with the first local SUSY constructed historically, the $(3 + 1)$-dimensional $N = 1$ SUGRA [95–97], which in fact finds a plethora of (conjectural) applications to the phenomenology of particle physics [98]. In the remainder of this Section we shall work in units of the gravitational constant $\kappa = 1$ for brevity.

The spectrum of the unbroken $(3 + 1)$-dimensional $N = 1$ SUGRA is a massless spin 2 graviton field, described by the symmetric tensor field $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, $\mu, \nu = 0, \dots 3$ and a massless gravitino spin 3/2 Rarita–Schwinger Majorana fermion $\psi_\mu(x)$.

The standard action is given by [97]

$$S_{\mathrm{SG1}} = \frac{1}{2} \int d^4x \, \sqrt{-g} \left(\Sigma_{ab}^{\mu\nu} \, R_{\mu\nu}{}^{ab}(\omega) - \epsilon^{\mu\nu\rho\sigma} \, \overline{\psi}_\mu \, \gamma^5 \, \gamma_\nu \, D_\rho(\omega) \, \psi_\sigma \right), \tag{95}$$

where $\Sigma_{ab}^{\mu\nu} = \frac{1}{2} E_{[a}^\mu \, E_{b]}^\nu$ and $D_\mu(\omega) = \partial_\mu + \frac{1}{8} \omega_{ab\,\mu} \, [\gamma^a, \gamma^b]$ is the diffeomorphic covariant derivative, with respect to a spin connection $\omega^a{}_{b\mu}$ which, as we shall discuss below, necessarily contains fermionic (gravitino-induced) torsion.

As shown in [40,43], the action (95) can be augmented by adding to it a total derivative Holst type action, which preserves the on-shell $N = 1$ SUSY for an arbitrary coefficient t:

$$S_{\mathrm{Holst1}} = \mathrm{i} \, \frac{\eta}{2} \int d^4x \, \sqrt{-g} \left(\Sigma_{ab}^{\mu\nu} \, \widetilde{R}_{\mu\nu}{}^{ab}(\omega) - \epsilon^{\mu\nu\rho\sigma} \, \overline{\psi}_\mu \, \gamma_\nu \, D_\rho(\omega) \, \psi_\sigma \right), \tag{96}$$

with $\widetilde{R}_{\mu\nu}{}^{ab}(\omega)$ the dual Lorentz curvature tensor.

Indeed, as demonstrated in [40,43], the combined action

$$S_{\mathrm{total\ SG}} = S_{\mathrm{SG1}} + S_{\mathrm{Holst1}} =$$

$$\frac{1}{2} \int d^4x \left(\sqrt{-g} \left[E_a^\mu \, E_b^\nu \, R^{ab}{}_{\mu\nu} - \frac{t}{2} \, \epsilon^{ab}{}_{cd} \, R^{cd}{}_{\mu\nu} \right] + \epsilon^{\mu\nu\rho\sigma} \, \overline{\psi}_\mu \, \gamma^5 \, \gamma_\rho \, \frac{1 - \mathrm{i}\,\eta\,\gamma^5}{2} \, D_\sigma(\omega) \, \psi_\nu \right), \tag{97}$$

is invariant under the *local* SUSY transformation with infinitesimal Grassmann parameter $\alpha(x)$:

$$\delta\psi_\mu = D_\mu(\omega) \, \alpha, \quad \delta e_\mu^a = \frac{\mathrm{i}}{2} \, \overline{\alpha} \, \gamma^a \, \psi_\mu, \quad \delta B_{ab\mu} = \frac{1}{2} \left(C_{\mu ab} - e_{\mu[a} \, C^c{}_{cb]} \right), \tag{98}$$

where, by definition,

$$C^{\lambda\mu\nu} \equiv \frac{1}{\sqrt{-g}} \, \epsilon^{\mu\nu\rho\sigma} \, \overline{\alpha} \, \gamma^5 \, \gamma^\lambda \, \frac{1 - i\eta\,\gamma^5}{2} \, D_\rho(\omega) \, \psi_\sigma. \tag{99}$$

We remark for completion that in the special case where $\eta = \pm\mathrm{i}$ we obtain Ashtekar's chiral SUGRA extension, while for $\eta = 0$ one recovers the standard $N = 1$ SUGRA transformations.

We next remark that variation of the action (97) with respect to the spin connection, leads to the well-known gravitational equation of motion in first order formalism [95–97], which leads to an expression of the gravitino-induced torsion $T_{\rho\sigma}{}^\mu(\psi)$ in terms of the gravitino fields:

$$D_{[\mu}(\omega) \, e_{\nu]}^a \equiv 2 T_{\mu\nu}{}^a(\psi) = \frac{1}{2} \, \overline{\psi}_\mu \, \gamma^a \, \psi_\nu, \tag{100}$$

with the contorted spin connection being given by:

$$\omega_\mu{}^{ab}(e, \psi) = \mathring{\omega}_\mu{}^{ab}(e) + K_\mu{}^{ab}(\psi), \tag{101}$$

where $\mathring{\omega}_\mu{}^{ab}(e)$ is the torsion-free spin connection (expressible, as in standard GR, in terms of the vielbeins e_μ^a), and $K_\mu{}^{ab}(\psi)$ is the contorsion, given in terms of the gravitino field as:

$$K_{\mu\rho\sigma}(\psi) = \frac{1}{4}\left(\overline{\psi}_\rho\,\gamma_\mu\,\psi_\sigma + \overline{\psi}_\mu\,\gamma_\rho\,\psi_\sigma - \overline{\psi}_\mu\,\gamma_\sigma\,\psi_\rho\right). \tag{102}$$

The parameter η does not enter the expression for the contorsion, which thus assumes the standard form of $N = 1$ SUGRA without the Holst terms.

Substitution of the solution of the torsion equations of motion into the first-order Lagrangian density, corresponding to the action (97), leads to a second-order Lagrangian density that can be written as the sum of the standard $N = 1$ SUGRA Lagrangian density [97] and a total derivative, depending on the gravitino fields only:

$$\mathcal{L}(\text{second order}) = \mathcal{L}_{\text{usual N=1 SUGRA}}(\text{second order}) + \frac{i}{4}\,\eta\,\partial_\mu(\epsilon^{\mu\mu\rho\sigma}\,\overline{\psi}_\nu\,\gamma_\rho\,\psi_\sigma), \tag{103}$$

where the standard *N = 1 SUGRA in the second-order formalism includes four-gravitino terms*,

$$\begin{aligned}
\mathcal{L}_{\text{usual N=1 SUGRA}} &= \sqrt{-g}\,\frac{1}{2}R(e) + \frac{1}{4}\partial_\mu[E_a^\mu\,E_b^\nu\,\sqrt{-g}]\left(\overline{\psi}_\nu\,\gamma^a\,\psi^b - \overline{\psi}_\nu\,\gamma^b\,\psi^a + \overline{\psi}^a\,\gamma_n u\,\psi^b\right) \\
&\quad - \frac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}\overline{\psi}_\mu\,\gamma^5\,\gamma_\nu\left[\partial_\lambda + \frac{1}{2}\omega_\lambda^{ab}(e)\sigma_{ab}\right]\psi_\rho \\
&\quad - \frac{11}{16}\sqrt{-g}\left[(\overline{\psi}_a\,\psi^a)^2 - (\overline{\psi}_b\,\gamma^5\,\psi^b)^2\right] + \frac{33}{64}\,\sqrt{-g}\,(\overline{\psi}_b\,\gamma^5\,\gamma_c\,\gamma^b)^2 \\
&\quad + \text{appropriate auxilliary} - \text{field terms}, \qquad \sigma_{ab} = \frac{i}{4}[\gamma_a,\,\gamma_b], \tag{104}
\end{aligned}$$

and as standard [97] the Lagrangian density is computed by requiring the irreducibility condition:

$$\gamma^\mu\,\psi_\mu = 0, \tag{105}$$

which ensures that the spin is exactly 3/2 and not a mixture of this and lower spins. We note that the four-gravitino terms of (104) have been used in [99,100] in order to discuss, upon appropriate inclusion of Goldstino terms [101][11], the possibility of dynamical breaking of SUGRA, via the formation of condensates of gravitino fields $\sigma_c = \langle\overline{\psi}_\mu\,\psi^\mu\rangle \neq 0$. The gravitino field becomes massive, with mass which can be close to Planck mass, which implies its eventual decoupling from the low-energy (non supersymmetric) theory.

Such scenarios have been used to discuss hill-top inflation, as a consequence of the double-well shape of the effective gravitino potential. Indeed, for small condensates $\kappa^6\,\sigma_c(x) \ll 1$, one may obtain an inflationary epoch, not necessarily slow roll, as the gravitino rolls down towards one of the local minima of its double well potential [103] (cf. Figure 8). Such scenarios will be exploited further in Section 8.1, from the point of view of the generation of gravitational waves in the very early Universe, which can lead to a second inflationary era in such models, that could provide interesting—and compatible with the data—phenomenology/cosmology.

We complete the discussion on $N = 1$ SUGRA as in Einstein–Cartan theory, by noticing that, on using (62), (63) and (103), we may write for the super Holst term in this case [40,43]:

$$S_{\text{Super Holst N=1 SUGRA}}(e, \psi) = -\frac{i\eta}{2}\int d^4x\left[T_{\text{NY}} + \partial_\mu J^\mu(\psi)\right], \tag{108}$$

with

$$J^\mu(\psi) = \epsilon^{\mu\nu\rho\sigma}\,\overline{\psi}_\nu\,\gamma_\rho\,\psi_\sigma, \tag{109}$$

the axial gravitino current, and the Nieh–Yan invariant is given by (63).

Finally, combining the Fierz identity $\epsilon^{\mu\nu\rho\sigma}\,(\overline{\psi}_\mu\,\gamma_a\,\psi_\nu)\,\gamma^a\,\psi_\rho = 0$, with the expression for the N=1 SUGRA torsion $T_{\mu\nu}{}^a(\psi)$ (100), we arrive at $\epsilon^{\mu\nu\rho\sigma}\,T_{\mu\nu a}(\psi)\,T_{\rho\sigma}{}^a(\psi) = 0$, we may write for the on-shell-local-SUSY-preserving Holst term (108):

$$S_{\text{Super Holst N=1 SUGRA}}(e,\psi) = -\frac{i\eta}{4}\int d^4x\,\partial_\mu J^\mu(\psi)] = \frac{i\eta}{2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu T_{\nu\rho\sigma}(\psi). \quad (110)$$

This section is concluded by mentioning that super Holst modifications have been constructed [40] for extended SUGRAs, such as $N=2,4$, following and extending appropriately the $N=1$ case. The spectrum of the $N=2$ SUGRA consists of a massless spin-2 graviton, two massless chiral spin-3/2 gravitinos, $\gamma^5\psi_\mu^I = +\psi_\mu^I$, $\gamma^5\psi_{I\mu} = -\psi_{I\mu}$, $I=1,2$, and an Abelian gauge field A_μ. This is also an Einstein–Cartan theory, with torsion

$$2T_{\mu\nu}{}^a = \frac{1}{2}\left(\overline{\psi}_\mu^I\gamma^a\psi_{I\nu} + \overline{\psi}_{I\mu}\gamma^a\psi_\nu^I\right), \quad (111)$$

and contorsion

$$K_{\mu\rho\sigma} = \frac{1}{4}\left[\overline{\psi}_\rho^I\gamma_\mu\psi_{I\sigma} + \overline{\psi}_\mu^I\gamma_\rho\psi_{I\sigma} - \overline{\psi}_\mu^I\gamma_\sigma\psi_{I\rho} + \text{c.c.}\right], \quad (112)$$

where c.c. denotes complex conjugate, whilst the super Holst term has the form [40]:

$$S_{\text{Super Holst N=2 SUGRA}}(e,\psi) = -\frac{i\eta}{4}\int d^4x\,\partial_\mu J^\mu(\psi)] = \frac{i\eta}{2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu T_{\nu\rho\sigma}(\psi), \quad (113)$$

with $J^\mu(\psi) = \epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\nu^I\gamma_\rho\psi_{I\sigma}$ the axial gravitino current in this case. We observe from (112) that then contorsion is again independent, as in the $N=1$ case, from the super Holst action parameter η.
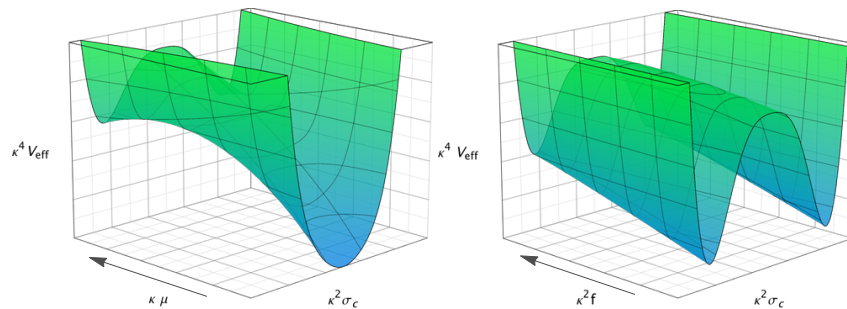

Figure 8. The effective potential of the torsion-induced gravitino condensate $\sigma_c = \langle\overline{\psi}_\mu\psi^\mu\rangle$ in the dynamical breaking of $N=1$ SUGRA scenario of [99], in which, for simplicity, the one-loop-corrected cosmological constant $\Lambda \to 0^+$ (for an analysis with $\Lambda > 0$ see [100] and references therein). The figures show schematically the effect of tuning the inverse-proper-time (renormalisation-group like) scale μ and the scale of SUSY breaking f, whilst holding, respectively, f and μ fixed. The arrows in the respective axes correspond to the direction of increasing μ and f. The reader should note (see left panel) that the double-wall shape of the potential, characteristic of the super-Higgs effect (dynamical SUGRA breaking), appears for values of μ larger than a critical value, in the direction of increasing μ, that is as we flow from Ultraviolet (UV) to infrared (IR) regions. Moreover, as one observes from the right panel of the figure, tuning f allows us to shift the value of the effective potential V_{eff} appropriately so as to attain the correct vacuum structure, that is, non-trivial minima σ_c such that $V_{\text{eff}}(\sigma_c) = \Lambda \to 0^+$. Picture taken from [99].

Finally, we complete the discussion with the $N=2$ gauged SU(4) SUGRA. For our discussion, we restrict our attention only to the relevant part of its spectrum, consisting of massless spin-2 gravitons, four chiral Majorana spin-3/2 gravitinos ψ_μ^I, $I=1,\ldots 4$, in the 4 and $4^\star$ representations of SU(4), and four Majorana chiral gauginos Λ^I, $I=1,\ldots 4$. The torsion of this theory depends on both the gravitino and gaugino fields [40],

$$2T_{\mu\nu}{}^a = 2T_{\mu\nu}{}^a(\psi) + 2T_{\mu\nu}{}^a(\psi) = \frac{1}{2}\overline{\psi}_{[\mu}^I\gamma^a\psi_{\nu]I} + \frac{1}{2\sqrt{-g}}e^{a\rho}\epsilon_{\mu\nu\rho\sigma}\overline{\Lambda}_I\gamma^\sigma\lambda^I, \quad (114)$$

and the contorsion reads

$$K_{\mu\nu\rho} = \frac{1}{4}\left(\overline{\psi}^I_\nu\,\gamma_\mu\,\psi_{\rho\,I} + \overline{\psi}^I_\mu\,\gamma_\nu\,\psi_{\rho\,I} - \overline{\psi}^I_\mu\,\gamma_\rho\,\psi_{\nu\,I} + \text{c.c.}\right) - \frac{1}{4\sqrt{-g}}\,\epsilon_{\mu\nu\rho\sigma}\,\overline{\Lambda}_I\,\gamma^\sigma\,\Lambda^I, \quad (115)$$

which again is independent of the parameter η of the super Holst term, which has the form [40]:

$$S_{\text{Super Holst } N=2 \text{ SUGRA}}(e, \psi) = -\frac{i\eta}{4}\int d^4x\,\partial_\mu[J^\mu(\psi) - J^\mu(\Lambda)]$$

$$= \frac{i\eta}{2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu\left(T_{\nu\rho\sigma}(\psi) - \frac{1}{3}T_{\nu\rho\sigma}(\Lambda)\right), \quad (116)$$

where $J^\mu(\Lambda) = \sqrt{-g}\,\overline{\Lambda}_I\,\gamma^\mu\,\Lambda^I$, and the torsion quantities have been defined in (114).

7. Torsion in Unconventional Supersymmetry

USUSY is an appealing theory where all the fields belong to a one-form connection $\mathbb{A}$, in $(2+1)$ dimensions, and the vielbein is realised in a different way than in standard SUGRA models [104]. It has nontrivial dynamics, and leads to a scenario where local SUSY is absent (although there is still diffeomorphism invariance), but rigid SUSY can survive for certain background geometries. Because there is no local SUSY, there are no SUSY pairings. Likewise, no gauginos are present. The only propagating degrees of freedom are fermionic [105], and the parameters that appear in the model are either dictated by gauge invariance, or arise as integration constants. We take the one-form connection spanned by the Lorentz generators $\mathbb{J}_a$, the SU(2) generators corresponding to the internal gauge symmetry $\mathbb{T}_I$ (or a other internal group generator, including the Abelian U(1)), the supercharges $\overline{\mathbb{Q}}^i$ and $\mathbb{Q}_i$ (note that these last generators contain the index corresponding to the fundamental group of SU(2) as well as the spinors)[12] [106]

$$\mathbb{A} = A^I\mathbb{T}_I + \overline{\psi}^i\!\!\not{e}\,\mathbb{Q}_i + \overline{\mathbb{Q}}^i\!\!\not{e}\,\psi_i + \omega^a\mathbb{J}_a, \quad (117)$$

where $A^I = A^I_\mu dx^\mu$ is the one-form SU(2) connection, $\omega^a = \omega^a{}_\mu dx^\mu$ is the one-form Lorentz connection in $(2+1)$ dimensions, and we defined the one-form $\not{e} \equiv e^a{}_\mu\gamma_a dx^\mu$.

We can construct a three-form Chern–Simons Lagrangian from (117), namely[13]

$$L = \frac{\kappa}{2}\langle\mathbb{A}d\mathbb{A} + \frac{2}{3}\mathbb{A}^3\rangle, \quad (118)$$

where $\langle\dots\rangle$ is the invariant supertrace of $\mathfrak{usp}(2,1|2)$ graded Lie algebra (for the case of internal SU(2) group) and κ is a dimensionless constant. This way, the Lagrangian can be written simply as

$$L = \frac{\kappa}{4}\left(A^I dA_I + \frac{1}{3}\epsilon_{IJK}A^I A^J A^K\right) + \frac{\kappa}{4}\left(\omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c\right) + L_\psi, \quad (119)$$

where the fermionic part is

$$L_\psi = \kappa\overline{\psi}\left(\gamma^\mu\overrightarrow{D}_\mu - \overleftarrow{D}_\mu\gamma^\mu - \frac{i}{2}\epsilon_a{}^{bc}T^a{}_{bc}\right)\psi|e|d^3x.$$

We can see the action (119) possesses also a local scale (Weyl) symmetry. Indeed, by scaling the dreibein and the fermions as

$$e^a{}_\mu \to e^a{}_\mu{}' = \lambda e^a{}_\mu\,, \; \psi \to \psi' = \lambda^{-1}\psi,$$

where $\lambda = \lambda(x)$ is a non-singular function on the spacetime manifold, the action (119) is invariant. This is a consequence of the particular construction of the connection (117), where the fermions always appear along with the dreibein field, forming a composite field.

For the case of the internal group SU(2), the internal index can be interpreted as valley index, making USUSY another good scenario with which to describe the continuous limit of both Dirac points (see details in [66]).

The action of USUSY in $(2 + 1)$ dimensions, for fixed background bosonic fields, apart from possible boundary terms, is obtained from the Chern–Simons three-form for $\mathbb{A}$ with an SU(2) internal gauge group [106]

$$S_{USUSY} = \kappa \int \overline{\psi}^i \left(\gamma^\mu \overset{\circ}{\vec{D}}_\mu - \frac{i}{8} \epsilon_a{}^{bc} T^a{}_{bc} \right) \psi_i |e| d^3 x, \tag{120}$$

where lower case Latin letters, $a, b, \ldots$, represent tangent space Lorentz indices, and $T^a{}_{bc} = T^a{}_{\mu\nu} E^\mu_b E^\nu_c$.

This action immediately points to (83), that is, the action with torsion we have seen emerging in graphene, where we only need to fix the dimensionfull κ to include v_F rather than c. Notice that, as discussed at length in [66] the two Dirac points are both necessary, so that the emergent action is made of two parts, one per Dirac point. This makes possible to have both the internal SU(2) symmetry and torsion, that is necessary for the USUSY description. So far as for similarities between (83) and (120). There are differences, though. The first is the coefficient of the torsion term, which appears in USUSY as an integration constant [104]. The second difference is the index i (here taken as an internal colour index, considering both Dirac points in the model). Both differences are due to the starting point to obtain (83), which is an Hermitian action with local Lorentz invariance in a Riemann–Cartan space. In contrast, the starting point of USUSY is an action with a supergroup $USP(2, 1|2)$ invariance, which is allowed by using another representation for ψ and the Dirac matrices (see details in Appendix B of [66]). In addition, it is also possible to take into account the two Dirac points by using other internal supergroups, such as $OSp(p|2) \times OSp(q|2)$ in this USUSY context [85]. In any case, (83) and (120) and the model proposed in [85] are top-down approaches to describe the ψ electrons in graphene-like systems. Therefore, we should keep in mind these (and others) models to compare them with the results of a real experiment in the lab.

Finally, let us comment that the Bañados–Zanelli–Teitelboim (BTZ) black hole [107], in a pure bosonic vacuum state ($\psi = 0$), is a solution of USUSY [104]. This follows from the fact that the BTZ black hole, whose metric in cylindrical coordinates ($-\infty < t < +\infty$, $0 < r < +\infty$, and $0 \leq \phi \leq 2\pi$) is

$$ds^2 = -N^2(r) \, dt^2 + N^2(r) \, dr^2 + r^2 \left(N^\phi(r) \, dt + d\phi \right)^2, \tag{121}$$

$$N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, N^\phi(r) = \frac{J^2}{2r^2}, \tag{122}$$

can be obtained from a Lorentz-flat connection with torsion [108]. The spectrum of these black holes is given in terms of their mass, M, and angular momentum, J, including the extremal, $M\ell = |J|$ and $M = 0$ cases[14]. We also mention here that the $M = 0$ case could play a very important role in the generalised Uncertainty Principle induced by gravity [110,111], and in Hawking–Unruh phenomenon on graphene and graphene-like materials [112].

8. Torsion in Cosmology

A plethora of precision cosmological data [113], over the past twenty-five years, have indicated that the energy budget of the current cosmological epoch of our (observable) Universe is dominated (by $\sim$95%) by a dark sector of unknown, at present, microscopic origin. If one fits the available data at large scales, corresponding to the modern era of the Universe, within the so-called ΛCDM framework, which consists of a de Sitter Universe (dominated by a positive cosmological constant Λ) and a Cold Dark Matter (CDM) component, then one obtains excellent agreement. On the other hand, there appear to be tensions to such data at smaller scales [114–116], arising either from discrepancies between the value of the Hubble parameter in the modern era obtained from direct observations of nearby galaxies and that inferred by ΛCDM fits ("H_0 tension"), or from discrepancies in the value of the parameter σ_8 characterising galactic growth data between direct observations and ΛCDM fits ("σ_8 tension").

To these tensions, provided of course the latter do not admit more mundane astrophysical explanations or are mere artefacts of relatively low statistics [117], and thus will be absent from future data, one should add theoretical obstacles to the self consistency of the ΛCDM framework, when viewed as a viable gravity model embeddable in microscopic models of quantum gravity, such as string theory [118,119] and its brane extensions [120]. Indeed, the existence of eternal de Sitter horizons, in spacetimes with a constant $\Lambda > 0$, prohibits the definition of asymptotic states, and thus a perturbative scattering S-matrix, which is the cornerstone of perturbative string theories, appears not to be well defined, thus posing problems with the compatibility of a de Sitter spacetime as a consistent background of perturbative strings [121,122]. Such problems extend to fully quantum gravity considerations, when one attempts to embed de Sitter spacetimes in microscopic ultraviolet complete models such as strings or branes, due to the so-called swampland conjectures [123–128], which are violated by the ΛCDM framework.

Barring the (important) possibility of misinterpretation of the Planck data as far as dark energy is concerned, by, e.g., *relaxing* the assumption of homogeneity and isotropy of the Universe at cosmological scales [129,130], one is therefore tempted to seek for theoretical alternatives to ΛCDM, which will not be characterised by a positive constant Λ, but rather having the de Sitter vacuum as a *metastable one*, in such a way that there are no asymptotic in future time de Sitter horizons. The current literature has a plethora of potential theoretical resolutions to the de Sitter Λ problem [131], which simultaneously alleviate the aforementioned tensions in small-scale cosmological data. What we would like to discuss below, in the context of our review, is the potential role of a purely geometric origin of such a metastable dark sector, including both Dark Energy (DE) and Dark Matter (DM), which is associated with the existence of torsion in the geometry of the early Universe [24,41,132].

To this end, we consider as a first example, in the next Subsection, string-inspired cosmologies with chiral anomalies. Our generic discussion in Section 2 on the role of (quantum) torsion in Einstein–Cartan QED [6], will find interesting application in this case. There we argued that, as a generic feature, the torsion degrees of freedom implied the existence of pseudoscalar (axion-like) massless dynamical fields in the spectrum, coupled to chiral anomalies.

8.1. Quantum Torsion in String-Inspired Cosmologies and the Universe Dark Sector

We have seen that in Einstein–Cartan theories, which have been exemplified here by massless contorted QED, torsion conservation (40) introduces an axionic degree of freedom to the system, associated with the totally antisymmetric part of the torsion which is the only part that couples to matter (fermions). The axion-like field becomes a dynamical part of the theory as a result of (chiral) anomalies, otherwise it would decouple from the quantum path integral. A similar situation characterises string-inspired theories in which anomalies are not supposed to be cancelled in the (3 + 1)-dimensional spacetime after string compactification, which, as we shall review below, provide interesting cosmological models [133–136] in which the dark sector of the Universe, including the origin of its inflationary epoch, admits a geometric interpretation.

The starting point of such an approach to cosmology is that the early Universe is described by the (bosonic) gravitational theory of the degrees of freedom that constitute the massless gravitational multiplet of the string (which in the case of superstring is also their ground state). The latter consists of spin-0 dilatons, Φ, spin-2 gravitons $g_{\mu\nu}$, and the spin-1 antisymmetric KR tensor field [118,119] $B_{\mu\nu} = -B_{\nu\mu}$.

Due to an Abelian gauge symmetry that characterises the closed string sector $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_{[\mu} \theta_{\nu]}$, the (3 + 1)-dimensional effective target spacetime action arising in the low-energy limit of strings (compared to the string mass scale M_s) depends only on the totally antisymmetric field strength of the KR field $B_{\mu\nu}$,

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} . \tag{123}$$

As explained in [134], one can assume self consistently a constant dilaton, so that the low-energy particle phenomenology is not affected. In this case, to lowest non-trivial order in a derivative expansion, or equivalent to $\mathcal{O}((\alpha')^0)$, with $\alpha' = M_s^2$ the Regge slope, the effective gravitational action reads [137,138]:

$$S_B = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{6} \mathcal{H}_{\lambda\mu\nu} \mathcal{H}^{\lambda\mu\nu} + \dots \right), \tag{124}$$

where $\mathcal{H}_{\mu\nu\rho} \equiv \kappa^{-1} H_{\mu\nu\rho}$ has dimension [mass]^2, and the ... represent higher derivative terms.

Comparing (124) with (21), one observes that the quadratic in the H-field terms can be viewed as a contorsion, in such a way that the effective action (124) can be expressed in terms of a generalised scalar curvature in a contorted geometry, with a generalised Christoffel symbols:

$$\overline{\Gamma}^\rho_{\mu\nu} = \mathring{\Gamma}^\rho_{\mu\nu} + \frac{\kappa}{\sqrt{3}} \mathcal{H}^\rho_{\mu\nu} \neq \overline{\Gamma}^\rho_{\nu\mu}, \tag{125}$$

where $\mathring{\Gamma}^\rho_{\mu\nu} = \mathring{\Gamma}^\rho_{\nu\mu}$ is the torsion-free Christoffel symbols[15].

The requirement of cancellation of gauge versus gravitational anomalies lead Green and Schwarz [140] to add appropriate counterterms in the effective target space action of strings, expressed by the modification of the field strength of the KR field (123) by the Lorentz (L) and Yang–Mills (Y) gauge Chern–Simons (CS) terms [119]:

$$\mathcal{H} = d\,B + \frac{\alpha'}{8\,\kappa} \left(\Omega_{3\mathrm{L}} - \Omega_{3\mathrm{Y}} \right),$$

$$\Omega_{3\mathrm{L}} = \omega^a{}_c \wedge d\,\omega^c{}_a + \frac{2}{3} \omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_a, \quad \Omega_{3\mathrm{Y}} = A \wedge d\,A + A \wedge A \wedge A, \tag{126}$$

where ω is the standard torsion-free spin connection, and A the non-Abelian gauge fields that characterise strings.

The modification (126) of the KR field strength (123) leads to the following Bianchi identity [119]

$$d\,\mathcal{H} = \frac{\alpha'}{8\,\kappa} \mathrm{Tr} \left(R \wedge R - F \wedge F \right), \tag{127}$$

with $F = d\,A + A \wedge A$ the Yang–Mills field strength two form and $R^a{}_b = d\,\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, the curvature two form and the trace (Tr) is over gauge and Lorentz group indices. The non zero quantity on the right hand side of (127) is the "mixed (gauge and gravitational) quantum anomaly" we have seen previously in the non-conservation of the axial fermion current (43)[16].

In [133], the crucial assumption made was the $(3 + 1)$-dimensional gravitational anomalies are not cancelled in the very early Universe. This was the consequence of the assumption that only fields from the massless gravitational string multiplets characterised the early universe gravitational theory, appearing as external fields. Chiral fermionic matter, radiation and in general gauge fields, which constitute the physical content of the low-energy particle physics models derived from strings, appear as the result of the decay of the false vacuum at the end of inflation in the scenario of [133–136].

In this sense, the gauge fields A in (126) can be set to zero, $A = 0$. In such a case, the Bianchi identity (127) becomes (in component form):

$$\varepsilon_{abc}{}^\mu \mathcal{H}^{abc}{}_{;\mu} = \frac{\alpha'}{32\,\kappa} \sqrt{-g}\, R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma} \equiv -\sqrt{-g}\, \mathcal{G}(\omega), \tag{128}$$

where the semicolon denotes covariant derivative with respect to the standard Christoffel connection, and

$$\varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\, \epsilon_{\mu\nu\rho\sigma}, \quad \varepsilon^{\mu\nu\rho\sigma} = \frac{\mathrm{sgn}(g)}{\sqrt{-g}}\, \epsilon^{\mu\nu\rho\sigma}, \tag{129}$$

with $\epsilon^{0123} = +1$, etc., are the gravitationally covariant Levi–Civita tensor densities, totally antisymmetric in their indices, and the dual is defined as

$$\widetilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\pi}R^{\lambda\pi}{}_{\rho\sigma}. \tag{130}$$

The alert reader should have observed similarities between the contorted QED model, examined in the previous Section 3, and the string inspired gravitational theory, insofar as the constraints imposed by the torsion conservation (40) in the QED case, and the Bianchi constraint (128). They are both exact results that are valid in the quantum theory (the Bianchi (128) is an exact one-loop result due to the nature of the chiral anomalies). In fact the dual of $H^{\mu\nu\rho}$, $\varepsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma}$ plays a role analogous with the pseudovector S_μ of the contorted QED case, associated with the totally antisymmetric component of the torsion. In the string theory example, this is all there is from torsion, as we infer from (125).

Following the contorted QED case, one may implement the Bianchi constraint (128) via a δ-functional in the corresponding path integral, represented by means of an appropriate Lagrange multiplier pseudoscalar field $b(x)$, canonically normalised:

$$\Pi_x \, \delta\Big(\varepsilon^{\mu\nu\rho\sigma} \, \mathcal{H}_{\nu\rho\sigma}(x)_{;\mu} + \mathcal{G}(\omega)\Big) \Rightarrow$$

$$\int \mathcal{D}b \, \exp\Big[i \int d^4x \sqrt{-g} \, \frac{1}{\sqrt{3}} \, b(x)\Big(\varepsilon^{\mu\nu\rho\sigma} \, \mathcal{H}_{\nu\rho\sigma}(x)_{;\mu} - \mathcal{G}(\omega)\Big)\Big]$$

$$= \int \mathcal{D}b \, \exp\Big[-i \int d^4x \sqrt{-g} \Big(\partial^\mu b(x) \frac{1}{\sqrt{3}} \, \epsilon_{\mu\nu\rho\sigma} \, \mathcal{H}^{\nu\rho\sigma} + \frac{b(x)}{\sqrt{3}} \, \mathcal{G}(\omega)\Big)\Big], \tag{131}$$

where to arrive at the second equality we performed partial integration, upon assuming that fields die out properly at spatial infinity, so that no boundary terms arise. We remark at this point that the similarity [41] of the exponent in the right-hand side of the last equality in (131), upon performing a partial integration of the first term, and identifying the anomaly with $\partial_\mu j^{5\mu}$, with the total Holst action (including the Nieh–Yan invariant) (64), in the case where the BI parameter is promoted to a pseudoscalar field [38].

Inserting the identity (131) in the path integral over H of the theory (124), we observe that the equations of motion of the (non-derivative) field H yield $\epsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma} \propto \partial_\mu b$, implying an analogy of the pseudovector field S_μ with $\partial_\mu b$. After path-integrating out the H-torsion, one obtains an effective target space action with a dynamical torsion-induced axion b:

$$S_B^{\text{eff}} = \int d^4x \sqrt{-g}\Big[\frac{1}{2\kappa^2} \, R + \frac{1}{2} \, \partial_\mu b \, \partial^\mu b + \sqrt{\frac{2}{3}} \, \frac{\alpha'}{96 \, \kappa} \, b(x) \, R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma} + \dots\Big], \tag{132}$$

where the dots $\dots$ denote higher derivative terms appearing in the target-space string effective action [6,137,138].

With the exception of the four-fermion interactions, which are absent here, as the theory is bosonic, the action (132) has the same form as the effective action (47), with the pseudoscalar field b having similar origin related to torsion as its contorted QED counterpart. But the action (132) is purely bosonic, and the anomalies here arise from the Green–Schwarz counterterms (126). In the model of [133] these are primordial anomalies, unrelated to chiral matter fermions as in the QED case, but because of the presence of such anomalies, the torsion (through its dual axion field $b(x)$) maintains its non trivial role via its coupling to the gravitational anomaly CS term. The gravitational model (132) is a Chern–Simons modified gravity model [21,23].

The massless axion field $b(x)$ is the so-called string-model independent axion [141], and is one of the many axion fields that string models have. The other axions are due to compactification. The string axions lead to a rich phenomenology and cosmology [142,143].

From our point of view we restrict ourselves to the role of the KR axion in implying a geometric origin of the dark sector of the Universe, including non conventional inflation. Indeed, in [133–136] it was argued that condensation of primordial gravitational waves (GW) leads to a non-vanishing contribution of the gravitational Chern–Simons

term $\langle R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma}\rangle$, where $\langle \dots \rangle$ denote weak graviton condensates associated with primordial chiral GW [144,145]. If one assumes a density of sources for primordial GW, which have been formed in he very early Universe, before the inflationary stage in the model of [135,136], then, the weak quantum graviton calculation of [145], adopted to include densities of GW sources, leads to [146]:

$$\langle R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma}\rangle_{\text{condensate}} \mathcal{N} = \frac{\mathcal{N}(t)}{\sqrt{-g}} \frac{1.1}{\pi^2} \left(\frac{H}{M_{\text{Pl}}}\right)^3 \mu^4 \frac{\dot{b}(t)}{M_s^2} \equiv n_\star \frac{1.1}{\pi^2} \left(\frac{H}{M_{\text{Pl}}}\right)^3 \mu^4 \frac{\dot{b}(t)}{M_s^2}. \quad (133)$$

In the above expression, μ is an ltraviolet (UV) cutoff for the graviton modes entering the chiral GW, and $n_\star \equiv \frac{\mathcal{N}(t)}{\sqrt{-g}}$ denotes the number density (over the proper de Sitter volume) of the sources of GW. Without loss of generality, we may take this density to be (approximately) time independent during the very early universe. The parameter $H(t)$ is the Hubble parameter of a FLRW Universe, which is assumed slowly varying with the cosmic time[17]. The analysis of [133,135] then, shows that there is a metastable de Sitter spacetime emerging, given that the condensate (133) is only mildly depending on cosmic time through $H(t)$ mainly, and thus can be considered approximately constant. It can be shown [133] that, as a consequence of the axion b equations of motion, the existence of a condensate leads to approximately constant $\dot{b}$ during the inflationary period (for which $H \simeq$ constant)

$$\dot{b} \simeq \epsilon H M_{\text{Pl}}, \quad (134)$$

where the overdot denotes a derivative with respect to the cosmic time t. The parameter ϵ is phenomenological and, to satisfy the Planck data [113] on slow-roll inflation, one should set it to $\epsilon = \mathcal{O}(10^{-2})$ [135]. Then, conditions for an approximately constant

$$\langle b(t) R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma}\rangle_{\text{condensate}} \mathcal{N} \simeq \text{constant}, \quad (135)$$

for some period Δt, can be ensured, which then leads to a *metastable* de Sitter spacetime (inflation), with Δt the duration of inflation. Taking into account that the scale of inflation, set by the current Planck data [113] is

$$H_I \lesssim 10^{-5} M_{\text{Pl}}, \quad (136)$$

and that the the number of e-foldings is estimated to be (in single-field models of inflation) $\mathcal{N} = \mathcal{O}(60 - 70)$, these conditions can be stated as:

$$|\bar{b}(t_0)| \gtrsim N_e \sqrt{2\epsilon} M_{\text{Pl}} = \mathcal{O}(10^2) \sqrt{\epsilon} M_{\text{Pl}}, \quad (137)$$

with $b(t_0)$ being the initial value of the axion field at the onset ($t = t_0$) of inflation.

In view of the H-dependence of the condensate the inflation is of the so-called Running-Vacuum-Model (RVM) type [148–153], which involves a time-dependent, rather than a constant de Sitter parameter $\Lambda(t) \propto H^2(t)$, but with a de Sitter equation of state for the vacuum,

$$p_{\text{rvm}} = -\rho_{\text{rvm}}, \quad (138)$$

where p (ρ) denotes pressure (energy) density. In the model of [135], detailed calculations have shown that, in the phase of the GW-induced condensate (133) and (135), the de Sitter–RVM equation of state (138) is satisfied. The corresponding energy density, comprising of contributions from b field (superscript b), the gravitational CS terms (superscript gCS) and the condensate term (superscript Λ), acquires [133,135,136,146] the familiar RVM form [151–153]

$$\rho^{\text{total}} = \rho^b + \rho^{\text{gCS}} + \rho^\Lambda_{\text{condensate}} = -\frac{1}{2} \epsilon M_{\text{Pl}}^2 H^2 + 4.3 \times 10^{10} \sqrt{\epsilon} \frac{|\bar{b}(0)|}{M_{\text{Pl}}} H^4. \quad (139)$$

The important point to notice is that the RVM inflation does not require a fundamental inflaton scalar field, but is due to the non-linear H^4 terms in the respective vacuum energy density (139) [151–153], arising in our case by the form of the condensate (133). Such terms are dominant in the early Universe and drive inflation. During the RVM inflation in our string-inspired CS gravity the H^2 term is negative in contrast to standard RVM formalisms with a smooth evolution from inflation to the current era [151,152]. In our case, it is the CS quadratic curvature corrections to GR that leads to such negative contributions tom the stress-energy tensor, in full analogy to the dilaton–Gauss–Bonnet string-inspired theories [154]. Nevertheless, the dominance of the condensate (i.e., $\mathcal{O}(H^4)$) terms in (139) ensures the positivity of the vacuum energy density during the RVM inflationary era. We stress that the H^4 term in the vacuum energy density (139) arises exclusively from the gravitational anomaly condensate in our string-inspired cosmology. In standard quantum field theories in curved spacetime, RVM energy densities arise after appropriate renormalisation of the quantum matter fields in the FLRW spacetime background but, in such cases, an H^4 term is *not* generated in the vacuum energy density. Instead, one has the generation of order H^6 terms and higher [153,155–158]. Such non linear terms, which will be dominant in the early Universe, can still, of course, drive RVM inflation.

During the final stages of RVM inflation, the decay of the RVM metastable vacuum [151,152] results in the generation of chiral matter fermions in the cosmology model of [133–136] we are analysing here. The chiral fermions would generate their own mixed (gauge and gravitational) chiral anomaly terms through the non conservation of the chiral current (49) over the various chiral fermion species (43). The effective action during such an era will, therefore, contain fermions, which will couple universally to the torsion $H_{\mu\nu\rho}$ via the diffeomorphic covariant derivative. After integrating out the H-field, we arrive at the following effective action including fermions [133]:

$$S^{\text{eff}} = \int d^4x \sqrt{-g} \Big[\frac{1}{2\kappa^2} R + \frac{1}{2} \partial_\mu b \, \partial^\mu b + \sqrt{\frac{2}{3}} \frac{\alpha'}{96\,\kappa} \, b(x) \, R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma} \Big]$$
$$+ S^{\text{Free}}_{\text{Dirac or Majorana}} + \int d^4x \sqrt{-g} \Big[\Big(\mathcal{F}_\mu - \frac{\alpha'}{2\kappa} \sqrt{\frac{3}{2}} \, b \, J^{5\mu}_{\;\;;\mu} \Big) - \frac{3\alpha'^2}{16\,\kappa^2} J^5_\mu J^{5\mu} \Big] + \dots, \quad (140)$$

where the $S^{\text{Free}}_{\text{Dirac or Majorana}}$ fermionic terms denote the standard Dirac or Majorana fermion kinetic terms in a curved spacetime without torsion, and $\mathcal{F}^a = \varepsilon^{abcd} \, e_{b\lambda} \, \partial_d \, e^\lambda_c$.

The gravitational part of the anomaly is assumed in [133] to *cancel* the primordial gravitational anomalies, but the chiral gauge anomalies remain in general. Thus, in [133], we assumed that, at the exit phase from RVM inflation, one has the condition,

$$\partial_\mu \Big[\sqrt{-g} \Big(\sqrt{\frac{3}{8}} \kappa J^{5\mu} - \sqrt{\frac{2}{3}} \frac{\kappa}{96} \mathcal{K}^\mu \Big) \Big] = \sqrt{\frac{3}{8}} \frac{\alpha'}{\kappa} \frac{e^2}{8\pi^2} \sqrt{-g} \, F^{\mu\nu} \widetilde{F}_{\mu\nu}$$
$$+ \sqrt{\frac{3}{8}} \frac{\alpha'}{\kappa} \frac{\alpha_s}{8\pi} \sqrt{-g} \, G^a_{\mu\nu} \widetilde{G}^{a\mu\nu}, \quad (141)$$

where we used the fact that the gravitational CS anomaly is a total derivative of an appropriate topological current $\mathcal{K}^\mu$ [15–17],

$$R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma} = \mathcal{K}^\mu_{\;\;;\mu}, \quad (142)$$

$F_{\mu\nu}$ denotes the electromagnetic U(1) Maxwell tensor, which corresponds to radiation fields in the post inflationary epoch, and $G^a_{\mu\nu}$, $a = 1, \dots, 8$ is the gluon tensor associated with the SU(3) (of colour) strong interactions with (squared) coupling $\alpha_s = g_s^2/(4\pi)$, which dominate the Universe during the QCD epoch, and the $\widetilde{(\dots)}$ denotes the corresponding duals, as usual (cf. (130)), with $\widetilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

At the exit from RVM inflation, it was assumed in [133–136] that no chiral gauge anomalies are dominant. Such dominance comes much later in the post inflationary

Universe evolution. In such a case, it can be shown [133] that the b-field equation of motion implies a scaling of $\dot{b}$ with the temperature as

$$\dot{b} \propto T^3 . \tag{143}$$

In this case, one may obtain an unconventional leptogenesis of the type discussed in [159,160] in theories involving massive sterile right handed neutrinos, as a result of the decay of the latter to standard-model particles in the presence of the Lorentz-violating background (143). Hence, in such scenarios the torsion is also linked to matter-antimatter asymmetry, given that the so-generated lepton asymmetry can be communicated to the baryon sector vial Baryon (B) and Lepton number (L) violating, but B-L conserving sphaleron processes in the standard-model sector [161].

Connection of torsion to DM might be obtained by noting that the QCD dominance era (which in the models of [133,134] comes much after the leptogenesis epoch) might be characterised by SU(3) instanton effects, which in turn break the axionic shift symmetry by inducing appropriate potential, and mass terms, (cf. (50)) for the torsion-induced axion field b, which could play a role as a DM component. The electromagnetic U(1) chiral anomalies may be dominant in the modern eras, and their effects have been discussed in detail in [133].

We also mention for completion that, as a result of the (anomalous) coupling $\dot{b}J^{50}$ (cf. (140)), one obtains a Standard-Model-Extension (SME) situation, with the Lorentz and CPT Violating SME background being provided by $\dot{b}$. It is the latter that is constrained by a plethora of precision experiments, which provide stringent bounds for Lorentz and CPT violation [162]. Using the chiral gauge anomalies at late eras of the Universe, as appearing in (141), the thermal evolution of the Lorentz- and CPT- symmetry-violating torsion-induced background $\dot{b}(T)$ at late eras of the Universe, including the current epoch, has also been estimated in [133], and found to be comfortably consistent with the aforementioned existing bounds of Lorentz and CPT Violation, as well as torsion today [162].

In the above cosmological scenarios, the entire dark sector of the Universe and its cosmological evolution are one way or another linked to some sort of torsion in the geometry. During the very early epochs after the Big bang, it is the gravitino torsion of a SUGRA theory, which the effective string cosmology model of [133,135] is embedded to, that leads to a first inflationary epoch [103], whilst it is the stringy torsion associated with the field strength of the antisymmetric spin-one KR field, which in turn gives rise to the KR axion $b(x)$, that is responsible for the second RVM type inflation, and the eventual cosmological evolution until the present era, during which the field $b(x)$ can also develop a mass, thus becoming a dark-matter candidate. Schematically, such a cosmological evolution is depicted in Figure 9 [136].

Before closing this section, we would like to mention the very recent related work of [163], which explores further the cosmology of Kalb–Ramond-like particles (KRLP), which one encounters in string models, and which contain also massive pseudovector excitations, in addition to the massless pseudoscalar ones, discussed in this review in connection with the totally antisymmetric part of torsion. Although the non-interacting KRLP are related to either pseudoscalar or pseudovector excitations, the interacting massive KRLP can be distinguished from its scalar and vector counterparts, and can have important phenomenological implications for the dark sector of the Universe, which are described in detail in [163].
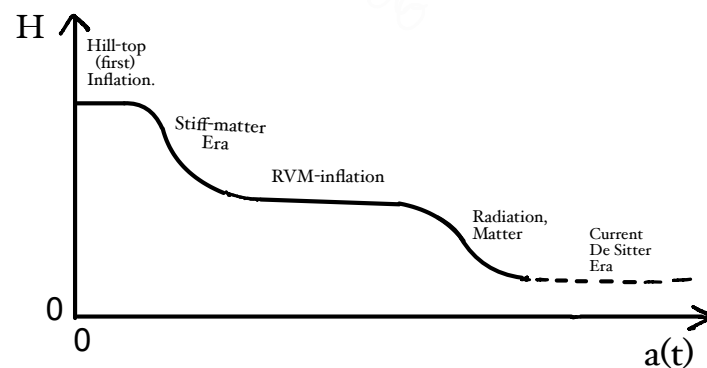
Figure 9. Schematic representation of the RVM cosmological evolution of the contorted cosmological model of [133–136]. The figure depicts the evolution of the Hubble parameter with the scale factor of an expanding stringy-RVM Universe, involving two torsion-induced inflationary eras, interpolated by a stiff KR-axion "matter" epoch: a first hill-top first inflation, which exists immediately after the Big-Bang, and is due to dynamical breaking of SUGRA, as a result of gravitino-torsion-induced condensates of the gravitino field, and second an RVM inflation, due to gravitational anomaly condensates, that are coupled to the torsion-induced KR axion field $b(x)$. The latter can also play the role of a dark matter component during post-RVM inflationary eras. Picture taken from [136].

8.2. Comments on Other Contorted Cosmological Models with a Spin

In the previous Subsection we discussed cosmological models corresponding to the standard generic type of Einstein–Cartan theories with fermionic torsion, involving in their Lagrangian densities repulsive four fermion interactions, of axial-current-current terms $j^5 - j^5$, with fixed coefficient depending on the theory, proportional to the gravitational coupling κ^2. Condensates of such repulsive terms, when formed, have been interpreted as providers of dark energy components in both the early [164] and the late [165] Universe, thus leading to a current-era acceleration of the Universe.

In this subsection, we shall briefly discuss generalisations involving more general four-fermi structures among *chiral* (Weyl) spinors [166], which include vector fermionic currents in addition to the axial ones, in similar spirit to the models (70), but with more general coefficients (on the other, hand, unlike the situation encountered in (70), the BI parameter in [166] is assumed constant, which, as we have discussed in Section 2, and mention below as well, is a problematic feature). Depending on the couplings considered, such fermion self-interactions may conserve or break parity invariance, while they may contribute positively or negatively to the energy density, thus having the feature that they could also be attractive. Thus, such "cosmologies with a spin" [166] exhibit a broad spectrum of possibilities, ranging from cases for which no significant cosmological novelties arise, to cases in which the fermion self-interaction can turn a mass potential into an upside-down Mexican hat potential, leading to cosmologies with a bounce [166,167], without a cosmic singularity.

However, as we shall discuss below, there are some subtleties in the treatment of [166], which, in view of what we discussed in Section 2, require some discussion. Let us first describe the approach of [166]. On defining Dirac spinors $\Psi(x)$ from the chiral ones ξ, χ as

$$\Psi(x) = \begin{pmatrix} \xi(x) \\ \chi(x) \end{pmatrix}, \tag{144}$$

the authors of [166] constructed fermionic field theories in a contorted curved spacetime, with action given by:

$$\mathcal{S}_\Psi[e, \omega^{ab}, \Psi] = \frac{1}{3} \int d^4x \, \epsilon_{abcd} \, e^b \, e^c \left[\frac{1}{2} e^a \, \overline{\Psi} \, \gamma^d D(\omega) \, \Psi - \overline{D(\omega)\Psi} \, \gamma^d \, \Psi \right) + \frac{3}{2} \, T^a (\alpha \, V^d + \beta A^d) \right]$$
$$- \frac{1}{4} \int d^4x \, U \, \epsilon_{abcd} \, e^a \, e^b \, e^c \, e^d + S_{\text{int}}[\xi, \chi, A], \tag{145}$$

where T^a is the torsion two-form, (1), U is a fermion-self-interaction potential which is assumed to be a function of scalars constructed from $\overline{\Psi}\Psi$ and $\overline{\Psi}\gamma^5\Psi$, while S_{int} denotes an interaction term of the chiral spinors ξ, χ with (in general, non-Abelian) gauge fields A. We also defined $V^d = \overline{\Psi}\gamma^d\Psi$ as the vector chiral current, and $A^d = \overline{\Psi}\gamma^5\gamma^d\Psi$ its axial counterpart. Finally, the quantities $\alpha, \beta \in \mathbb{R}$ are real couplings that characterise the model.

The gravitational dynamics, on the other hand, is described by the standard Einstein–Hilbert term plus the Holst action, this is the combination (29) and (51), which in the parametrization and normalisation of [166] is written as:

$$\mathcal{S}_{\text{grav+Holst}} = \frac{1}{2\kappa^2}\int d^4x \left(\epsilon_{abcd} + \frac{1}{\gamma}\eta_{ac}\eta_{bd}\right)e^a\,e^b\,R^{cd}, \tag{146}$$

with R^{ab} being the Riemann curvature two-form, and $\gamma \in \mathbb{R}$ being related to the BI parameter $\beta = -1/\gamma$ (51).

This is not a minimal torsion model, as the generic Einstein–Cartan theories examined before, given that it includes several postulated interaction potentials. Because of this, this model leads to more general four-fermion interactions than the standard Einstein–Cartan theory. The effective four fermion interaction is found by using, as in the standard Einstein–Cartan theories, the Euler–Lagrange equations of motion for the fermions, torsion and gravity fields. By varying the action with respect to the contorted spin connection, we determine the torsion T^a and contorsion K_{abc} for this model [166]:

$$\frac{1}{\kappa^2}\left(\epsilon_{abcd} + \frac{2}{\gamma}\eta_{a[c}\eta_{d]b}\right)T^a\,e^b = \frac{1}{4}\epsilon_{amnp}\,e^a\,e^m\,e^n\,\epsilon^{dp}{}_{cd}\,A_d - \frac{1}{4}\epsilon_{[c|mnq}\,e^m\,e^n e_{|d]}\left(\alpha\,V^q + \beta A^q\right),$$

$$K_{abc} = \kappa^2\frac{\gamma^2}{4\left(\gamma^2+1\right)}\left[\epsilon^d{}_{abc}\frac{1}{2}\left(A_d + \frac{1}{\gamma}(\alpha V_d + \beta A_d)\right) - \frac{1}{\gamma}A_{[b}\eta_{a]c} + \alpha\,V_{[b}\eta_{a]c} + \beta\,A_{[b}\eta_{a]c}\right]. \tag{147}$$

From the graviton (vielbein) and fermion equations of motion, on the other hand, we obtain, respectively:

$$\frac{2}{\kappa^2}\mathring{G}_{\mu\nu} = -\frac{i}{2}e_{d\mu}(\overline{\Psi}\gamma^d\,\mathring{D}_\nu\Psi - \overline{\mathring{D}_\nu\Psi}\,\gamma^d\,\Psi) + \frac{i}{2}e^\sigma_d(\overline{\Psi}\gamma^d\,\mathring{D}_\sigma\Psi - \overline{(\mathring{D}_\sigma\Psi}\,\gamma^d\,\Psi) - g_{\mu\nu}\,W,$$

$$i\gamma^d\,e^\mu_d\,\mathring{D}_\mu\Psi = \frac{\delta W}{\delta\overline{\Psi}}, \tag{148}$$

where $\mathring{G}_{\mu\nu}$ is the standard Einstein tensor of GR; to ease the notation we used $\mathring{D}_\mu \equiv D(\mathring{\omega})_\mu$ and W is the effective four-fermion interaction potential, which depends on the contorsion:

$$W = U + \frac{3\kappa^2}{16}\frac{\gamma^2}{\gamma^2+1}\left[(1-\beta^2+\frac{2}{\gamma}\beta)A_a\,A^a - \alpha^2\,V_a\,V^a - 2\alpha(\beta-\frac{1}{\gamma})\,A_a\,V^a\right]. \tag{149}$$

The mixed axial-vector current term in (149) breaks parity. One should compare these four-fermion interactions with the ones in the models (70), discussed in Section 4.2.

However, the analysis of [166], leading to (149), is not entirely formally correct, as we have explained in Section 2, following the careful analysis of [37]. The presence of the (constant) BI parameter in the effective potential (149) would imply that a parameter that appears in a total derivative term does affect physics at the end. As explained above, this paradox leads also to another inconsistency, that of Equation (55), in which, for non-zero $1/\gamma$, one obtains the inconsistent result that the vector component of torsion is proportional to the pseudovector of the axial current. As we discussed in Section 2, the resolution of this paradox is achieved by considering the addition of the Nieh–Yan topological invariant [39] (62).

We do mention at this stage that, naively, the independence of the potential W on the (constant) BI parameter γ can be achieved in the specific cases

$$\beta = \frac{1}{\gamma} \quad \text{and} \quad \alpha^2 = c_0^2\frac{\gamma^2+1}{\gamma^2}, \tag{150}$$

where $c_0 \in \mathbb{R}$ is an arbitrary real constant. This case preserves parity, since the mixed term $A_a V^a$ in the potential W (149) is absent. In such a case the effective four-fermion interactions become

$$W = U + \frac{3\kappa^2}{16}\left(A_a A^a - c_0^2 V_a V^a\right). \tag{151}$$

This model contains, in addition to the potential term U, the standard repulsive axial-current-current four-fermion interactions of the Einstein–Cartan theory, augmented by vector-current-current four fermion interactions.

Superficially looking at (151), one may think that the contributions to the vacuum energy density due to such interactions could be positive or negative, depending on the relative magnitude of the parameter c_0^2, and in general the terms in (151). However, this is not the case. Indeed, as discussed in [166], for classical spinors, as appropriate for solutions of Euler–Lagrange equations of motion, one may argue that

$$\langle A_a A^a \rangle = -\langle V_a V^a \rangle, \tag{152}$$

given that the axial term is always space-like, while the vector time-like. From (151) and (152) we obtain that in this case $W = U + \frac{\kappa^2}{16}\left(1 + c_0^2\right) A_a A^a$ and, due to the space-like nature of the classical axial-current-current term $\langle A_a A^a \rangle$, the four-fermion interaction is always repulsive, as in the standard Einstein–Cartan theory, but with a coefficient whose magnitude is unconstrained, given the phenomenological nature of the parameter c_0. In that case, one can show that there are no bouncing cosmologies or other effects, such as for instance turning a positive mass potential into a Higgs one, which arose in the treatment of [166]. Nonetheless, doubt is cast on the mathematical consistency of such solutions in view of (55), which is still valid in such special cases, even if the potential (151) is independent from the BI parameter.

The above criticisms, however, may be bypassed in the case one promotes the BI parameter to a pseudoscalar (axion-like) field $1/\gamma \to a(x)$, as discussed previously in Section 4.2. Indeed in such a case, the corresponding effective four-fermion interactions (149) have to be reworked in accordance with the fact that the BI parameter is now a fully fledged pseudoscalar field, as in the case of the action (70). Thus, cosmologies based on such models, with four-fermion interactions that may include *attractive* fermion channels, may justify (some of) the expectations of [168] on the role of torsion-induced *fermion condensates* in the early universe cosmology, which cannot characterise the repulsive terms (56). In this latter respect, in the context of SUGRA theories (cf. Section 6), the torsion-connected four gravitino interactions can also lead, due to the existence of attractive channels, to the formation of appropriate condensates [99,100], which, as we have discussed in Section 8.1, may play an important role in the early eras of string-inspired cosmologies.

9. Concluding Remarks: Other Observational Effects of Torsion

We reviewed various aspects of torsion, both in emergent geometric descriptions of graphene or other Dirac materials, and in fundamental theories of spacetime, especially cosmology. These two scenarios have enormously different scales, yet the physical properties of torsion appear to be universal, and can in principle be appreciated in experiments in both frameworks.

On the cosmological side, we focused on specific string-inspired models in which the totally antisymmetric component of torsion is represented as an axion-like field. Condensates associated with torsion can lead, as we have discussed, to inflationary physics of RVM type, characterised under some conditions, by torsion-induced-axion background that spontaneously violates Lorentz symmetry. Such a situation may leave imprints in the early Universe cosmic microwave background.

In general, however, in generic Einstein–Cartan theories, torsion has more components. In [169], a plethora of tests involving coupling of the various torsion components to fermions in combination with Lorentz violation, in the context of the Standard Model

Extension [162], have been discussed which exhibit sensitivity for some of the pertinent Lorentz-violating parameters down to 10^{-1} GeV.

The presence of torsion may also have important consequences for cosmological observations independent of Lorentz violation. For instance, as discussed in [170], non-zero torsion affects the relation between the angular-diameter (D_A) and luminosity (D_L) distances used in astrophysical/cosmological measurements, such that the quantity $\eta = \frac{D_L}{D_A(1+z)^2} - 1$ is linked to various types of torsion. This may affect low-redshift measurements, and thus contribute to the observed Hubble-parameter (H_0) tensions [171]. Of course, contributions to such tensions, including the growth of structure ones (σ_8) [114–116], can also come, as we discussed in Section 8.1, from the late-Universe RVM cosmology, which the contorted string-inspired models lead to, but the combination of the plethora of late-time cosmological measurements, and details of structure formation [172] can provide information that can distinguish between the quantum string-inspired RVM cosmology and generic torsion models.

Other constraints on late Universe torsion of relevance to our discussion here, namely of associating axions to torsion, come from CP (rather than Lorentz) violation effects in axion-photon cosmic plasma through dynamo primordial-magnetic-field amplification [173] (see also [174] on the role of axion fields), which torsion is a specific species of for cosmic magnetic helicity generation).

An alternative way to probe experimentally the role of torsion is to realise in graphene, or other Dirac materials, the scenarios described in this review. At this time, there is still nothing going on in that direction. There are two steps that will make this enterprise possible. On the theory side, we should identify the best experimental setting to have a precise correspondence between the specific dislocation defects (the nonzero Burgers vectors) and the torsion term in the Dirac action. On the experimental side, we should be able to realise, with the help of suitable external electromagnetic fields, the time-loop that will spot the nonzero torsion in the time direction.

We mention for completeness that we have not covered here certain interesting aspects of torsion, such as those characterising teleparallel theories [175], in which torsion replaces the metric, or the so-called $f(Q)$ gravity theories [176], which involve the non-metricity tensor $Q_{\alpha\mu\nu} = D_\alpha(\omega) g_{\mu\nu} \neq 0$. The interested reader is referred to the rich relevant literature for more details on the formal, phenomenological and cosmological aspects of such models.

We would like also to mention here that, in the current literature, there are several works which deal with topics partially overlapping with those of our review, but from a different perspective than ours.

In a revisited Einstein–Cartan approach to graphene dislocations, in particular wedge disclination in a planar graphene sheet, the authors of [177], studied the properties of its electronic degrees of freedom in a novel approach which relates to elasticity theory, given that the aforementioned disclination is found there. An important novel result, as these authors claim, is the demonstrated explicit dependence of the energy on the elasticity (Poisson's) constant. The works [178,179] examine effects of the thermal Nieh–Yan anomaly terms of the axial fermion current, of the form $\partial_\mu J^{5\mu} \propto T^2 \, T^a \wedge T_a$, where T is the temperature, and T^a the effective/emergent torsion, where the proportionality constant is determined by the geometry and topology of the material, and the number of chiral quantum fields. In the case of Weyl superfluids, the authors show that such anomalous terms characterise the hydrodynamics of a chiral p-wave superfluid, such as ^{3}He-A, or a chiral superconductor.

The role of torsion, when induced by the BI field within Holst and Nieh–Yan formulations, in modified general relativity and bounce cosmology has been studied in a series of works [180–184], which complement our treatment of the H-torsion in this review and related references. In this context, the role of spacetime torsion, sourced from antisymmetric tensor (Kalb–Ramond) fields in various modified gravity theories is discussed in [185–189], including phenomenological aspects, providing potential explanation for the invisibility

of torsion in late eras of the Universe, due to dynamical suppression in its couplings with standard model fields.

Claims on potential connection between Lorentz symmetry breaking and torsion are provided in [190], where the one-loop fermionic effective action in Einstein–Cartan theories, computed by the proper time method, results in a contact interaction term between the two topological terms of the Nieh–Yan topological current (axial vector torsion S_μ in our review) and the Chern–Simons topological current, which is thoroughly determined by the metric. Such terms may lead to spontaneous breaking of Lorentz symmetry, through appropriate vacuum expectation values of S_μ. We note that similar mechanisms for spontaneous breaking of Lorentz symmetry arise in our stringy RVM model [133,135], where the time derivative of the Kalb–Ramond axion acquires a constant vacuum expectation value.

The effects of torsion on gravitational waves in extended theories of gravity, in particular in Einstein–Cartan gravity using the post-Newtonian formalism devised by Blanchet–Damour, that goes beyond the linearised gravitational theory is discussed in the works [191–196].

In [197], the Lorentzian gravitational path integral has been evaluated in the presence of non-vanishing torsion (with the application of the Picard–Lefschetz theory for minisuperspaces corresponding to a number of phenomenological bouncing cosmological models as well as for the inflationary paradigm). In addition, in [198], it was demonstrated that, unlike any other non-trivial modifications of the Einstein gravity, the presence of spacetime torsion does not affect the entropy of a black hole. In [199], a shift-symmetric Galileon model in the presence of spacetime torsion has been constructed, with applications to the study of the evolution of the universe at a cosmological scale. For a wide class of torsional structures, the model leads to late time cosmic acceleration, while the standard results are obtained in the limit of vanishing torsion which is a smooth one.

The role of metric-scalar-torsion couplings and their impact on the growth of matter perturbations in the Universe has been discussed in [200] within the context of an interacting dark-energy scenario in which the matter density of a scalar field that sources a torsion mode ceases to be self-conserved, thereby affecting not only the background cosmological evolution but also the perturbative spectrum of the local inhomogeneities, thus leading to cosmic growth. As argued in [200], the model can become phenomenologically viable.

A rather surprising feature of spacetimes with torsion was pointed out in [201], where the authors, on considering the coupling of fermions in the presence of torsion, have demonstrated the emergence of a possibly new length scale (in analogy to the electroweak theory, as we shall explain below), which turns out to be transplanckian, and actually much larger than the Planck length. The new scale arises as a result of the non-renormalisable, gravitational four-fermion contact interaction, which characterises generic Einstein–Cartan theories, as we discussed repeatedly in this review. The authors of [201], argued that, by augmenting the Einstein–Cartan Lagrangian with suitable kinetic terms quadratic in the torsion and curvature, gives rise to new, massive propagating gravitational degrees of freedom. The whole situation is to be viewed in close analogy to the Fermi's effective four-fermion weal interaction, which is the effective low energy theory of the standard model and arises from virtual exchange of the (emergent) W and Z weak bosons of the electroweak theory.

In an interesting recent work [202], torsion was associated with potentially measurable properties of the electroweak vacuum, in the sense that the latter can be stabilised provided one assumes the metric-affine framework instead of the usual metric formulation of gravity. In this framework the Holst invariant is present since in general the torsion does not vanish and this leads to important physical consequences, according to the claims of the authors of that work. Specifically, by using measured quantities such as the Higgs and top quark masses, the authors claim that, in principle, the Einstein–Cartan theory can be differentiated from the standard General Relativity.

Last but not least, we mention the work of ref. [203], where the authors, with the help of appropriate conformal transformations, explored the use of non-symmetric contorted

connections in "Fisher information geometry". As is well known, the latter corresponds to a probability distribution function ubiquitous in the study of the effective "geometry" entering information theory. They introduced the idea of both metric and torsion playing equal roles in such a context, and studied the corresponding scalar curvature for a few statistical systems, which served as concrete examples for pointing out the relevant properties. As the authors claimed, this study helps to solve some long-standing problems in the field of information geometry, concerning the uniqueness of the Fisher information metric.

Our report would not be complete if we did not mention the role of torsion in the hydrodynamics of a fluid system with spin currents, as discussed in [204]. This could be of interest in the case of, say, heavy-ion collisions in particle physics, where there is experimental evidence for correlations between the spin polarisation of Λ-hyperons and the angular momentum of the quark-gluon plasma in off-centre collisions [205,206] or in the case of liquid metals, where an experimental realisation of spin currents has been demonstrated [207].

A fully consistent theory of spin-current hydrodynamics is currently lacking. In constructing such a theory, the first open issue to be addressed is identifying a canonical spin current. At this stage, we remind the reader that, in a relativistic theory, on a flat background without torsion, Lorentz invariance dictates that energy and momentum are conserved, which, as a result, implies also the conservation of angular momentum. In the absence of torsion, it is always possible to add an improvement term to the energy momentum tensor, $T_{\mu\nu}$ such that the symmetry property $T_{\mu\nu} = T_{\nu\mu}$ follows from an additional equation of motion, implying angular momentum conservation from energy-momentum conservation. Recalling that the angular momentum tensor $\mathcal{J}^{\mu\nu\rho}$ is related to the spin current $S^{\mu\nu\rho}$ via $\mathcal{J}^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - S^{\mu\nu\rho}$, it follows that the spin current suffers from ambiguities due to the possibility of adding improvement terms to the stress tensor. Specifically, by a judicious choice of such terms, it can be set to zero. In the work of [204], it has been argued that one way of dealing with such ambiguity is to couple the theory to an external spin connection with torsion (which is thus independent of the vielbeins). As discussed in that work, the presence of such a background torsion leads to a uniqueness of the spin current by precluding the addition of improvement terms to the stress tensor. After the computation of the spin current, one can set the background torsion to zero, going back to Minkowski spacetime. The presence of torsion plays an important role in ensuring uniqueness, i.e., the absence of ambiguities, in the so-called entropy current that enters the local version of the second law of thermodynamics in the pertinent fluid. The formalism of turning on the background torsion, and eventually turning it off, ensures that the total entropy current is independent of the choice of improvement terms, which in turn resolves some issues regarding the effect of the improvement terms (called *pseudo-gauge transformations*) on the entropy production in the system.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

Notes

[1]   Although the contorted geometry formalism can be generic and valid in $(d+1)$-dimensional spacetime, nonetheless for the sake of concreteness, in this work we shall present the analysis for $d = 3$, and, in the case of graphene, for $d = 2$.

[2]   The action of $\wedge$ on forms is expressed as [4,5]: $f^{(k)} \wedge g^{(\ell)} = (-1)^{k\ell} g^{(\ell)} \wedge f^{(k)}$, where $f^{(k)}$ and $g^{(\ell)}$ are k-forms and ℓ-forms, respectively.

[3]   Some references refer to it as a *contortion tensor* [2]. However, as we are more closely following the terminology of [5], we keep the name *contorsion*. As far as we know, there is no consensus yet about the name of this quantity.

[4]   For a recent study of the massive case, where the focus is on neutrino mixing and oscillations, see [11].

[5]   In the original formulation of Barbero and Immirzi, the BI parameter is $\gamma = 1/\beta$, but this is not important for our purposes.

[6]   Indeed, by applying the decomposition (23) onto (57), prior to imposing (59), we obtain the following extra contribution in the effective action, as compared to the terms discussed previously in the case $\alpha = 0$ [37]:

$$\int d^4 x e \frac{\alpha}{2} T_\mu j^{5\mu}. \tag{60}$$

Including such contributions, and considering the vanishing variations of the total action with respect to the (independent) torsion components, T^μ, S^μ and $q^{\mu\nu\rho}$, we obtain the solution

$$T^\mu = \frac{3\,\kappa^2}{4}\left(\frac{\beta - \alpha}{\beta^2 + 1}\right) j^{5\mu}, \qquad S^\mu = 3\kappa^2 \frac{1 + \alpha\beta}{1 + \beta^2}\, j^{5\mu}, \; q_{\mu\nu\rho} = 0. \tag{61}$$

Clearly, as we discussed above, the first equation is problematic from the point of view of leading to a proportionality relation between a vector and a pseudovector, except in the Einstein–Cartan case $\beta = 0$ and the limit $\alpha = \beta$, where the situation is reduced again to the Einstein–Cartan theory, given that in such a case the Holst-like modification is a total derivative.

[7]   It is possible to include in the description next-to-near neighbour contributions, while keeping a modified Dirac structure [53]. In fact, such modifications allow for the reproduction of scenarios related to generalised uncertainty principles both for commuting coordinates [54] and noncommuting coordinates [55].

[8]   Actually, there are six such points, but the only two shown above are inequivalent under lattice discrete symmetry.

[9]   A deep study of how curvature and torsion emerge in a geometrical approach to quantum gravity, along the lines of how classical elastic-theory emerges from QED, can be found in [58], see also [59]. In those papers, the authors elaborate on a model of quantum gravity inspired by graphene, but independent from it [60,61], see also [62,63]. A review can be found in [48]

[10]   This is due to the reducible, rather than irreducible, representation of the Lorentz group we use

[11]   The Goldstino λ is a Majorana spin $1/2$ fermion which plays the role of the Goldstone-type fermionic mode arising from the spontaneous breaking of global SUSY. To incorporate the relevant dynamics into the dynamically-broken SUGRA scenario, one adds to the SUGRA Lagrangian (104) the terms

$$\mathcal{L}_{\text{golds}} = -f^2 \det\left(\delta^\mu_\nu + i\frac{1}{2\,f^2} \bar{\lambda}\,\gamma^\mu\,\partial_\nu\,\lambda\right) = -f^2 - \frac{1}{2}\,i\,\bar{\lambda}\,\gamma^\mu\,\partial_\mu\lambda + \dots \tag{106}$$

where $f \in \mathbb{R}$ is the energy scale of SUSY breaking, and the $\dots$ denote higher order self-interaction terms of λ. Such a term realises SUSY non linearly in the sense of Volkov and Akulov [102]. After an appropriate gauge fixing (105) the derivative $\partial_\mu\lambda$ can then be absorbed, by a suitable redefinition of the gravitino field ψ_μ in the schematic combination $\psi'_\mu = \psi_\mu + \partial_\mu\lambda$, so that the gravitino field acquires a non zero mass, proportional to the gravitino condensate σ. Then, all that is left from the lagrangian density (106) is a negative cosmological constant term $-f^2 < 0$, and thus the final, gauge fixed, SUGRA lagrangian encoding dynamical breaking of local SUSY, is given by:

$$\mathcal{L}_{\text{total}} = -f^2 + \mathcal{L}_{\text{N=1 SUGRA}}. \tag{107}$$

We shall not give further details here on this dynamical mechanism for SUGRA breaking, referring the interested reader to the literature (see refs. [99,100] and references therein).

[12]   It is possible to add a central extension generator $\mathbb{Z}$ and its corresponding one-form coefficient b [106]. However, we shall not consider this extension in the present work.

[13]   Here, we omitted the wedge notation for the exterior product. For instance, $\mathbb{A}^3$ stands for the three-form $\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}$.

[14]   The case $M = -1$ is the globally anti-De Sitter space, while the other cases are conical singularities [109].

[15]   We note for completeness that, by exploiting local field redefinition ambiguities [6,137–139], which do not affect the perturbative scattering amplitudes, one may extend the above conclusion to the fourth order in derivatives, that is, to the $\mathcal{O}(\alpha'^2)$ effective low-energy action, which includes quadratic curvature terms.

16 We stress once again that the modifications (126) and the right-hand-side of the Bianchi (127) contain the *torsion-free* spin connection, given that, as explained previously, any *H*-torsion contribution can be removed by an appropriate addition of counterterms [18,19].

17 To ensure homogeneity and isotropy conditions, the authors of [135] assumed the existence of a stiff-axion-*b*-dominated era (i.e., with equation of state $w_b = +1$) that succeeds a first hill-top inflation [103] (cf. Figure 9), which is the result of dynamical breaking of local SUSY (SUGRA) right after the Big Bang, that is assumed to characterise the superstring inspired theories. This breaking is achieved by a condensation of the gravitino (supersymmetric partner of gravitons) as a result of the existence of attractive channels in the four-gravitino interactions that characterise the SUGRA Lagrangian due to fermionic torsion [99,100], as discussed in Section 6. As argued in [135,136], unstable domain walls (DW) are formed as a result of the gravitino condensate double well potential (Figure 8), whose degeneracy can be lifted by percolation effects [147]. The non-spherical collapse of such DW leads to primordial GW, which then condense leading to (133).

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