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Modification Study on Quantum Tunneling Radiation of Kinnersley Black Hole

Cong Wang *, Jie Zhang and Yu-Zhen Liu

College of Physics and Electronic Engineering, Qilu Normal University, Jinan 250200, China; zhangjie@qlnu.edu.cn (J.Z.); 20158170@qlnu.edu.cn (Y.-Z.L.)
* Correspondence: wangcong@qlnu.edu.cn

Abstract: In the spacetime of a linearly accelerating Kinnersley black hole, the Lorentz-breaking theory is used to modify the dynamical equations of Dirac particles by selecting gamma matrices and aether-like field vectors in the curved spacetime of this black hole. Using the WKB approximation and black hole quantum tunneling radiation theory, we investigate the characteristics of quantum tunneling radiation in this black hole.By solving the modified spinor field equations, we obtain expressions for the corrected quantum tunneling rate, Hawking temperature, and surface gravitation of the black hole. By studying the particle radial component of the general momentum in this curved spacetime, a new expression for the modified distribution of positive and negative energy levels of Dirac particles, as well as their maximum value of crossing energy level, is obtained. In order to further elucidate the physical significance of the research methodology employed in the article and a series of conclusions obtained, a detailed discussion of the corresponding results is provided in the later sections of this paper.

Keywords: Lorentz-breaking; Kinnersley black hole; Hawking temperature; quantum tunneling rate; Dirac energy levels

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1. Introduction

Currently, Lorentz-breaking modified field theory models are being extensively investigated. A series of meaningful studies have been conducted on the modification effects caused by Lorentz-breaking theories from flat spacetime to curved spacetime [1–16]. In the relevant research papers, modifications are discussed at both the classical and quantum levels [17–25]. Research in quantum gravity theory and quantum field theory suggests that at the microscopic high-energy regime, Lorentz symmetry needs to be appropriately modified. Therefore, Lorentz-breaking theory has attracted widespread attention. A series of studies has shown that investigating the effects of Lorentz-breaking in curved spacetime is a research topic worth attention [26–34]. Studying the effects of Lorentz-breaking theory in curved spacetime requires, first and foremost, modifying the action of scalar fields and spinor fields for different spacetime backgrounds in curved spacetime. This allows for appropriate modifications to the dynamical equations of different particles. Building upon this foundation, research involves studying the quantum tunneling rates of bosons and fermions, as well as the modified Hawking temperature and Bekenstein–Hawking entropy. For bosons, it is necessary to introduce aether-like correction terms. The modified method for the dynamics equation of bosons in the presence of Lorentz-breaking is to first add an aether-like field vector correction term to the scalar field action. Based on this, the variational principle is applied to derive the modified dynamics equation for bosons after Lorentz-breaking. By selecting the appropriate aether-like field vector in curved spacetime, one can study the effects of Lorentz-breaking theory on bosonic tunneling radiation in curved spacetime. Based on this, expressions for various physical quantities such as Hawking temperature and Bekenstein–Hawking entropy of different black holes can be studied. These expressions contain Lorentz-breaking correction terms, making the physical content they represent more enriched. The modification of the action of spinor fields due to Lorentz-breaking is more complex compared to the modification of scalar field action. For fermions, the study of modified dynamical equations becomes more complex. One approach is to directly incorporate correction terms into the fermion dynamical equations based on the Lorentz dispersion relation. Another approach is to utilize the coupling models of spinor fields, introducing correction terms such as the Carroll–Field–Jackiw (CFJ) correction, aether-like field vector correction, and Chiral correction. In this paper, we study the modified forms of physical quantities such as the Hawking temperature and Dirac energy level distribution for general moving black holes using the Kinnersley black hole with arbitrary acceleration, Lorentz-breaking theory, WKB approximation theory, and black hole quantum tunneling radiation theory. Furthermore, we analyze the physical implications of the newly obtained results.

The content of this paper is structured into three sections. The following Section 2 introduces a spinor field coupling model using the gamma matrices γ^μ and γ^5 to incorporate Chiral correction terms, aether-like correction terms, and CFJ correction terms. This model is utilized to modify the dynamical equations for fermions. The following Section 3 focuses on the study of the spacetime metric of the dynamic Kinnersley black hole with arbitrary acceleration. In this section, new expressions for the quantum tunneling rate, black hole temperature, Dirac energy levels, and other physical quantities are derived. The Section 4 of this paper provides a comprehensive discussion on the significance of the research methods employed and the conclusions derived from this study.

2. Spinor Field Coupling Models and Modified Dynamical Equations for Fermions in Curved Spacetime

The application research of Lorentz-breaking theory in curved spacetime is a topic worth attention. Lorentz-breaking theory not only has certain effects on the curved spacetime background, but also influences the behavior of particles moving in curved spacetime. Specifically, Lorentz-breaking theory has an impact on the quantum tunneling radiation near the event horizon of black holes. To study the effects of such influences, it is necessary to investigate the modified form of the particle's dynamics equation in curved spacetime. The expression of the spinor field coupling model, taking into account Lorentz-breaking theory, CFJ correction, aether-like correction, and Chiral correction in four-dimensional curved spacetime is given by [17,35]

$$\mathcal{L} = \int d^4x\sqrt{-g}\bar{\psi}\left[\mathrm{i}\gamma^\mu\partial_{;\mu}\left(1-\frac{a\hbar^2}{m^2}\widetilde{\Box}\right)+\frac{b\hbar}{m}\left(u^\mu\partial_{;\mu}\right)^2+\frac{c}{m\hbar}\gamma^5+\frac{m}{\hbar}\right]\psi. \quad (1)$$

In this modified spinor field action, $\widetilde{\Box}$ represents the d'Alembertian operator in curved spacetime, u^μ is the aether-like field vector, and a, b, c correspond to dimensionless real numbers for the CFJ correction term, the aether-like correction term, and the Chiral correction term, respectively. They satisfy the conditions $\frac{a}{m} \ll 1$, $\frac{b}{m} \ll 1$, $\frac{c}{m} \ll 1$, where m is a characteristic mass scale. In Equation (1), ψ represents the wave function of fermions. Equation (1) describes the modified spinor field equations for spin-1/2 fermions, spin-3/2 fermions, and so on. $\bar{\psi}$ denotes the curved version of ψ. In Equation (1), $d^4x\sqrt{-g}$ represents the volume element in four-dimensional spacetime and is a pseudoscalar. The covariant derivative $\partial_{;\mu}$ and the d'Alembertian operator are respectively expressed as

$$\partial_{;\mu} = \partial_\mu - \frac{\mathrm{i}e}{\hbar}A_\mu + \frac{\mathrm{i}}{2}\Gamma_\mu^{\alpha\beta}\Pi_{\alpha\beta}, \quad (2)$$

$$\Pi_{\alpha\beta} = \frac{\mathrm{i}}{\psi}\left[\gamma^\alpha, \gamma^\beta\right], \quad (3)$$

$$\widetilde{\Box} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^{v}} \right). \tag{4}$$

In Equations (1)–(4), the gamma matrices γ^{μ} and γ^{5} need to be determined based on the specific characteristics of curved spacetime. The conditions that must be satisfied by the determined γ^{μ} and γ^{5} are as follows

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} I, \tag{5}$$

$$\gamma^{\mu}\gamma^{5} + \gamma^{5}\gamma^{\mu} = 0. \tag{6}$$

The γ^{μ} and γ^{5} in Equations (5) and (6) need to be determined based on the specific spacetime metric. Different curved spacetimes have different γ^{μ} and γ^{5} counterparts, which are appropriately chosen once γ^{μ} and γ^{5} have been correctly selected. To obtain the correct results when solving the modified fermions' dynamics equation, it is necessary to include the Lorentz-breaking modification theory. In the equation, I represents the identity matrix. It should be noted that the u^{μ} in Equation (1) is the aether-likefield vector in four-dimensional curved spacetime. The u^{μ} discovered here is different from the aether-like field vector found in flat spacetime. In flat spacetime, when considering Lorentz-breaking modifications, we also encounter the aether-like field vector $\overline{u}^{\mu}$, which is a constant vector. The requirement $\overline{u}^{\mu}\overline{u}_{\mu} =$ const. holds in flat spacetime, but in curved spacetime, u^{μ} is not a constant vector, and the requirements for u^{μ} are as follows

$$u^{\mu} u_{\mu} = \text{ const.} \tag{7}$$

Here, u^{μ} is the aether-like field vector and is not a constant vector. However, it is required that u^{μ} satisfies Equation (7), which necessitates the correct selection of u^{μ} based on the specific characteristics of the curved spacetime. By applying the variational principle and Equation (1), we can obtain the following expression

$$\delta \mathcal{L} = 0. \tag{8}$$

By taking the variation of Equation (1), the variation and integration can be exchanged in order. Considering $\delta\psi = \frac{\partial\psi}{\partial x^{\mu}} \delta x^{\mu}$ and$\delta\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x^{\mu}} \delta x^{\mu}$, we can obtain the following equation

$$\psi i \gamma^{\mu} \partial_{;\mu} \left[\left(1 - \frac{a\hbar^{2}}{m^{2}} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^{v}} \right) \right) + \frac{b\hbar}{m} \left(u^{\mu} \partial_{;\mu} \right)^{2} + \frac{c}{m\hbar} \gamma^{5} + \frac{m}{\hbar} \right] = 0, \tag{9}$$

$$\bar{\psi} i \gamma^{\mu} \partial_{;\mu} \left[\left(1 - \frac{a\hbar^{2}}{m^{2}} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^{v}} \right) \right) + \frac{b\hbar}{m} \left(u^{\mu} \partial_{;\mu} \right)^{2} + \frac{c}{m\hbar} \gamma^{5} + \frac{m}{\hbar} \right] = 0. \tag{10}$$

Equation (10) is the conjugate form of Equation (9). Equation (9) represents the Lorentz-breaking modified fermions dynamics in curved spacetime. For the ψ in Equation (9), the representation of the wave function ψ differs for fermions with different spins. For spin-1/2 fermions, ψ can be represented as follows

$$\psi = \begin{pmatrix} A \\ B \end{pmatrix} e^{\frac{i}{\hbar} S}. \tag{11}$$

The wave function ψ is related to matrices, and S in Equation (11) represents the action for spin-1/2 fermions. For spin-1/2 fermions (also known as Dirac particles), Equation (9) can be referred to as the modified Dirac equation. These particles are fermions associated with gravitons. By substituting Equation (11) into Equation (9), the equation satisfied by spin-1/2 fermions can be simplified to

$$\left[-\gamma^{\mu}\left(\frac{\partial S}{\partial x^{\mu}}\right)\left(1+\frac{a}{m^2}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(g^{\mu\nu}\frac{\partial S}{\partial x^{\nu}}\sqrt{-g}\right)\right)-\frac{b}{m}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}+\frac{c}{m}\gamma^{5}+m+\Gamma(\hbar)\right]\left(\begin{array}{c}A\\B\end{array}\right)=0. \tag{12}$$

$\Gamma(\hbar)$ represents a correction term related to $\hbar$. In semiclassical theory, this term can be neglected to obtain a semiclassical equation. When the index $\mu \to v$ and $v \to \mu$ in Equation (12), the form of the equation remains unchanged. Therefore, multiplying both sides of Equation (12) by$\gamma^{v}\left(\frac{\partial S}{\partial x^{v}}\right)$ and considering that the matrix Equation (12) has a solution if and only if the determinant of its corresponding matrix is zero, we have

$$((1+2a)g^{\mu\nu}-2bu^{\mu}u^{\nu})\left(\frac{\partial S}{\partial x^{\mu}}+eA_{\mu}\right)\left(\frac{\partial S}{\partial x^{\nu}}+eA_{\nu}\right)+2c\gamma_0^5-m^2=0, \tag{13}$$

This is the spinor field equation based on Lorentz-breaking modification. In the equation, $g^{\mu\nu}$ represents the specific contravariant metric tensor in curved spacetime, and u^{μ} and γ^5 are also determined by the specific characteristics of curved spacetime. If a black hole has no electric charge Q and spin-1/2 fermions have mass m, then Equation (2) can be rewritten as

$$\partial_{;\mu}=\partial_{\mu}+\frac{\mathrm{i}}{2}\Gamma_{\mu}^{\alpha\beta}\Pi_{\alpha\beta}. \tag{14}$$

and Equation (13) can be rewritten as

$$(1+2a)g^{\mu\nu}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}-2bu^{\mu}u^{\nu}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}+2c\gamma_0^5+m^2=0, \tag{15}$$

where A_{μ} is the electromagnetic potential generated by the black hole's electric charge. Equations (13) and (15) are both modified field equations that are related to the particle action S. According to the WKB theory and quantum tunneling radiation theory, the tunneling probability of fermions is related to the particle action. Therefore, by solving Equation (13) or (15), we can easily determine the particle action, which facilitates the study of quantum tunneling radiation characteristics of fermions in black hole spacetime. Equations (13) and (15) are the semiclassical field equations for spin-1/2 fermions, modified by Lorentz-breaking, in curved spacetime. Clearly, due to the inclusion of corrections from Lorentz-breaking theory, the semiclassical spinor field equations already contain terms such as aether-like correction, CFJ correction, and Chiral correction. These two equations can be used to study the quantum tunneling radiation characteristics of spin-1/2 fermions in the curved spacetime of black holes. It is important to note that starting from either Equation (18) or Equation (15), the radiation characteristics and the resulting series of results obtained when studying spin-1/2 fermions in black hole spacetime are all relevant results under semiclassical theory. To obtain more precise correction results, it is necessary to consider perturbations of $\hbar$, requiring research with theories beyond semiclassical approaches. The following content pertains to the study of black holes undergoing linear acceleration.

3. The Modification of Lorentz-Breaking Theory on the Quantum Tunneling Radiation of Kinnersley Black Holes under Linear Acceleration Motion

The spacetime line element of Kinnersley black hole is expressed in advanced Eddington–Finkelstein coordinates as [36]

$$ds^2=g_{00}dv^2+2g_{01}dvdr+2g_{02}dvd\theta+2g_{03}dvd\varphi+g_{22}d\theta^2+g_{33}d\varphi^2. \tag{16}$$

where,

$$
\begin{aligned}
g_{00} &= 1 - \frac{2M}{r} - 2\tilde{a}\gamma\cos\theta - (\tilde{b}\sin\varphi + \tilde{c}\cos\varphi - \tilde{a}\sin\theta)^2 r^2 - (\tilde{b}\cos\varphi - \tilde{c}\sin\varphi)^2 r^2\cos^2\theta,\\
g_{01} &= g_{10} = -1,\\
g_{02} &= g_{20} = r^2(\tilde{b}\sin\varphi + \tilde{c}\cos\varphi - \tilde{a}\sin\theta),\\
g_{03} &= \sin\theta\cos\theta(\tilde{b}\cos\varphi - \tilde{c}\sin\varphi)r^2,\\
g_{22} &= -r^2,\\
g_{33} &= -r^2\sin^2\theta.
\end{aligned}
\tag{17}
$$

This black hole has a mass $M = M(v)$, where v represents the advanced Eddington coordinate. The parameters $\tilde{a} = \tilde{a}(v)$, $\tilde{b} = \tilde{b}(v)$, and $\tilde{c} = \tilde{c}(v)$ are all acceleration parameters. $\tilde{a}$ represents the magnitude of linear acceleration, directed towards the north pole. $\tilde{b}$ and $\tilde{c}$ represent the rate of change of acceleration direction.The Equations (16) and (17) describe a Kinnersley black hole undergoing arbitrary accelerated motion. Black holes existing in the universe should be dynamic, and the Kinnersley black hole undergoing arbitrary accelerated motion represents a dynamic black hole. From Equations (16) and (17), the non-zero components of the metric determinant $g = |g_{\mu\nu}|$ and the contravariant metric tensor can be calculated as follows

$$
g = -r^4\sin^2\theta,
\tag{18}
$$

$$
\begin{aligned}
g^{01} &= g^{10} = -1,\\
g^{12} &= g^{21} = -(\tilde{b}\sin\phi + \tilde{c}\cos\phi - \tilde{a}\sin\theta),\\
g^{11} &= \left(1 - \frac{2M}{r}\right) + 2\tilde{a}r\cos\theta,\\
g^{13} &= -\cot\theta(\tilde{b}\cos\phi - \tilde{c}\sin\phi),\\
g^{22} &= -\frac{1}{r^2},\\
g^{33} &= -\frac{1}{r^2\sin^2\theta}.
\end{aligned}
\tag{19}
$$

In Equations (16), (17) and (19), considering $\tilde{b} = 0$ and $\tilde{c} = 0$, $g_{03} = 0$, $g^{13} = 0$, the corresponding expressions correspond to the case of linearly accelerating Kinnersley black holes. For solving Equation (13) in the case of linearly accelerating Kinnersley black holes, we need to find the equation satisfied by the event horizon r_H of the linearly accelerating Kinnersley black hole. In curved four-dimensional spacetime, there exists a special type of hypersurface. This hypersurface corresponds to a three-dimensional surface within the four-dimensional spacetime, and it can be described as $F(x^\mu) = 0$. The normal vector to this hypersurface is defined as $\mathrm{n}_\mu = \frac{\partial F}{\partial x^\mu}$, and the length of this normal vector is defined as $n_\mu n^\mu = g^{\mu\nu}n_\mu n_\nu$. Hypersurfaces that satisfy $n_\mu n^\mu = 0$ are referred to as null hypersurfaces. The event horizon of a black hole belongs to a particular class of null hypersurfaces. The condition for determining the event horizon of the black holes is

$$
g^{\mu\nu}\frac{\partial F}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} = 0.
\tag{20}
$$

Equation (20) is the equation for determining the black hole horizon, which for a general black hole has an inner horizon, an outer horizon, and a cosmological horizon. For the Kinnersley black hole with variable acceleration straight-line motion, there are event horizons and Rindler horizons. The equation for determining the horizon of this black hole can be obtained from Equations (19) and (20) as follows:

$$
\left(1 - \frac{2M}{r}\right) + 2\tilde{a}r\cos\theta - 2\dot{r}_{\mathrm{H}} + 2\tilde{a}\sin r'_{\mathrm{H}} - \left(r'\right)^2 r^{-2} = 0,
\tag{21}
$$

where $\dot{r}_{\rm H} = \frac{\partial r_{\rm H}}{\partial v}$, ${r_{\rm H}}' = \frac{\partial r_{\rm H}}{\partial \theta}$. The event horizon and Rindler horizon of this black hole are determined by Equation (21). To illustrate the uniqueness of this black hole horizon, we can consider the θ when $r_{\rm H}$ reaches peak, resulting in ${r_{\rm H}}' = 0$. The equation then becomes:

$$2\tilde{a}\cos\theta_1 r_{\rm H}^2 - (1 - 2\dot{r}_{\rm H})r_{\rm H} - 2M = 0. \tag{22}$$

From Equation (22), the event horizon $r_{\rm H}$ and Rindler horizon $r_{\rm R}$ of this black hole are obtained as:

$$r_{\rm H} = \frac{(1 - 2\dot{r}_{\rm H}) - \left[(1 - 2\dot{r}_{\rm H})^2 + 16M\tilde{a}\cos\theta_1\right]^{\frac{1}{2}}}{4\tilde{a}\cos\theta_1}, \tag{23}$$

$$r_{\rm R} = \frac{(1 - 2\dot{r}_{\rm H}) + \left[(1 - 2\dot{r}_{\rm H})^2 + 16M\tilde{a}\cos\theta_1\right]^{\frac{1}{2}}}{4\tilde{a}\cos\theta_1}. \tag{24}$$

When considering $\dot{r}_{\rm H} = 0$ and $Ma \ll 1$, we have $r_{\rm H} \sim 2M$, $r_{\rm R} \sim \frac{1}{2a\cos\theta_1}$; therefore, $r_{\rm H}$ is the event horizon of this black hole, and $r_{\rm R}$ is the Rindler horizon of this black hole. In Equation (21), considering $M = 0$, $\tilde{a} \neq 0$, we obtain the Rindler horizon equation for an observer with variable acceleration straight-line motion as:

$$1 + 2\tilde{a}r_{\rm R}\cos\theta - 2\dot{r}_{\rm R} + 2\tilde{a}\sin r'_{\rm R} - \left(r'_{\rm R}\right)^2 r_{\rm R}^{-2} = 0. \tag{25}$$

To illustrate the special significance of the quantum tunneling radiation at the event horizon of this black hole, we will study the characteristics of quantum tunneling radiation at the event horizon below. This research method can also be used to study the characteristics of quantum tunneling radiation at the Rindler horizon. Clearly, the part event horizon $r_{\rm H}$ of the black hole depends on v and θ. In the curved spacetime of the black hole,we take the frame field e_I^μ and choose the gamma matrix γ^μ as follows

$$\gamma^\mu = e_I^\mu \gamma^I. \tag{26}$$

Clearly, γ^μ satisfies the following conditions

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I \tag{27}$$

where, $e_I^\mu e_J^\nu \eta^{IJ} = g^{\mu\nu}$. The choice of γ^5 is

$$\begin{aligned}
\gamma^5 &= \frac{\gamma_0^5 i}{4!}\varepsilon_{\mu\nu\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\
&= \frac{\gamma_0^5 i}{4!}\varepsilon_{IJKL} e_\mu^I e_\nu^J e_\kappa^I e_\lambda^J e_{I'}^\mu \gamma^{I'} e_J^v \gamma^{J'} e_{K'}^\kappa \gamma^{K'} e_{L'}^\lambda \gamma^{L'} \\
&= \gamma_0^5 i\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= \gamma_0^5 \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},
\end{aligned} \tag{28}$$

where γ_0^5 is a real number and γ^5 is a hermitian matrix. The symbol I_2 represents a $2 * 2$ identity matrix, while I represents a $4 * 4$ identity matrix. ε_{IJKL} denotes the Levi-Civita symbol. In the expression for γ^5 , we have $\varepsilon_{0123} = 1$. Therefore, we have the following expressions

$$e_I^\mu\gamma^I\gamma^5 + \gamma^5 e_I^\mu\gamma^I = 0. \tag{29}$$

Therefore, we have the following equation

$$\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = e_I^\mu\gamma^I\gamma^5 + \gamma^5 e_I^\mu\gamma^I = e_I^\mu\left(\gamma^I\gamma^5 + \gamma^5\gamma^I\right) = 0. \tag{30}$$

It can be seen that the chosen γ^{μ} and γ^{5} above are consistent with Equations (5) and (6). Based on the line element and the covariant metric tensor $g_{\mu\nu}$and the contravariant metric tensor $g^{\mu\nu}$of this black hole, the components of the aether-like field vector u^{μ} are accurately chosen as follows

$$\begin{aligned}
u^0 &= \frac{c_0}{\sqrt{g^{11}}g_{01}}, u^0 u_0 = u^0 u^1 g_{01} = \frac{c_0 g^{11}}{g_{01}} \frac{c_1}{g^{11}} g_{01} = c_0 c_1 \text{ (const.)} , \\
u^1 &= c_1 \sqrt{g^{11}}, u^1 u_1 = u^1 u^1 g_{11} = c_1^2 \text{ (const.)} , \\
u^2 &= c_2 \sqrt{g^{22}}, u^2 u_2 = u^2 u^2 g_{22} = c_2^2 \text{ (const.)} , \\
u^3 &= c_3 \sqrt{g^{33}}, u^3 u_3 = u^3 u^3 c_{\varphi}^2 g_{33} = c_3^2 \text{ (const.)} .
\end{aligned} \tag{31}$$

Clearly, u^{ν} fully satisfies Equation (7). Taking into account the axial symmetry of the space-time of a linearly accelerating Kinnersley black hole, we have a Killing vector $(\frac{\partial}{\partial\varphi})^{\alpha}$, which implies $(\frac{\partial S}{\partial\phi}) = n$. With the expressions for γ^{μ}, u^{ν}, and $g^{\mu\nu}$, we substitute Equations (19), (29) and (31) into Equation (13) to obtain

$$\begin{gathered}
(1+2a)\left[2g^{01}\frac{\partial S}{\partial v}\frac{\partial S}{\partial r} + g^{11}\left(\frac{\partial S}{\partial r}\right)^2 + 2g^{12}\left(\frac{\partial S}{\partial r}\right)\left(\frac{\partial S}{\partial \theta}\right) + g^{22}\left(\frac{\partial S}{\partial \theta}\right)^2 + g^{33}n^2\right] - \\
2b\left\{\left(\frac{\partial S}{\partial v}\right)\left[\left(\frac{\partial S}{\partial v}\right)c_0^2\left(g^{11}\right)^{-1} - 2c_0c_1\left(\frac{\partial S}{\partial r}\right) - 2c_0c_2\sqrt{\frac{g^{22}}{g^{11}}}\left(\frac{\partial S}{\partial \theta}\right) - 2c_0c_3 n\sqrt{\frac{g^{33}}{g^{11}}}\right] + \right. \\
\left(\frac{\partial S}{\partial r}\right)\left[c_1 g^{11}\left(\frac{\partial S}{\partial r}\right) + 2c_1c_2\sqrt{g^{11}g^{22}}\left(\frac{\partial S}{\partial \theta}\right) - 2c_1c_3 n\sqrt{g^{11}g^{33}}\right] + \left(\frac{\partial S}{\partial \theta}\right)\left[c_2^2 g^{22}\left(\frac{\partial S}{\partial \theta}\right) + \right. \\
\left.\left. 2c_2c_3 n\sqrt{g^{22}g^{33}}\right] + c_3^2 n^2 g^{33}\right\} + 2c\gamma_{\sigma}^5 + m^2 = 0.
\end{gathered} \tag{32}$$

It is a rather complex process to solve for the particle action S from this equation. Since the metric of this black hole is dynamic, we need to perform a generalized tortoise coordinate transformation on Equation (35) in the following form.

$$\begin{aligned}
r_* &= r + \frac{1}{2\kappa}\ln\left|\frac{r - r_{\rm H}(v,\theta,\varphi)}{r_{\rm H}(v_0,\theta_0,\varphi_0)}\right|, \\
v_* &= v - v_0, \\
\theta_* &= \theta - \theta_0.
\end{aligned} \tag{33}$$

From this transformation, we obtain

$$\begin{aligned}
\frac{\partial}{\partial r} &= \left[1 + \frac{1}{2\kappa(r - r_{\rm H})}\right]\frac{\partial}{\partial r_*}, \\
\frac{\partial}{\partial v} &= \frac{\partial}{\partial v_*} - \frac{\dot{r}_{\rm H}}{2\kappa(r - r_{\rm H})}\frac{\partial}{\partial r_*}, \\
\frac{\partial}{\partial \theta} &= \frac{\partial}{\partial \theta_*} - \frac{r'_H}{2\kappa(r - r_H)}\frac{\partial}{\partial r_*}.
\end{aligned} \tag{34}$$

where $\dot{r}_{\rm H} = \frac{\partial r_{\rm H}}{\partial v}, r'_{\rm H} = \frac{\partial r_H}{\partial \theta}$. Let

$$\begin{aligned}
\frac{\partial S}{\partial v_*} &= -\omega, \\
\frac{\partial S}{\partial \theta_*} &= p_2,
\end{aligned} \tag{35}$$

where ω is the particle energy, p_2 is the generalized momentum in the θ direction, and n is the generalized momentum in the φ direction. Substituting Equations (34) and (35) into Equation (32), we obtain

$$
\begin{aligned}
&(1+2a)\left\{-2\left[\frac{2\kappa(r-r_{\mathrm{H}})+1}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]\left[\frac{\partial S}{\partial v_*}-\frac{\dot{r}_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]+g^{11}\left[\frac{2\kappa(r-r_{\mathrm{H}})+1}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]^2+\right.\\
&\left.2g^{12}\left[\frac{2\kappa(r-r_{\mathrm{H}})+1}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]\left[p_2-\frac{r'_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]-\frac{1}{r^2}\left[p_2-\frac{r'_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]^2-\frac{n^2}{r^2\sin^2\theta}\right\}-\\
&2b\left\{\frac{c_0^2}{g^{11}}\left[\frac{\partial S}{\partial r_*}-\frac{\dot{r}_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]^2-2c_0c_1\left[\frac{2\kappa(r-r_{\mathrm{H}})+1}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]\left[\frac{\partial S}{\partial v_*}-\frac{r_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]-\right.\\
&2c_0c_2\sqrt{\frac{g^{22}}{g^{11}}}\left[\frac{\partial S}{\partial v_*}-\frac{\dot{r}_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]\left[p_2-\frac{r'_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]-2c_0c_3n\sqrt{\frac{g^{33}}{g^{11}}}\left[\frac{\partial S}{\partial v_*}-\frac{\dot{r}_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]+\\
&c_1^2g^{11}\left[\frac{2\kappa(r-r_{\mathrm{H}})+1}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]^2+2c_1c_2\sqrt{g^{11}g^{22}}\left[\frac{2\kappa(r-r_{\mathrm{H}})+1}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]\left[p_2-\frac{r'_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]+\\
&2c_1c_3n\sqrt{g^{11}g^{33}}\left[\frac{2\kappa(r-r_{\mathrm{H}})+1}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]+c_2^2g^{22}\left[p_2-\frac{r'_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]^2+2c_2c_3n\sqrt{g^{22}g^{33}}[p_2-\\
&\left.\frac{r'_{\mathrm{H}}}{2\kappa(r-r_{\mathrm{H}})}\frac{\partial S}{\partial r_*}\right]+c_3^2n^2g^{33}\right\}+2c\gamma_0^5+m^2=0.
\end{aligned}
\tag{36}
$$

After rearranging this equation and neglecting higher-order terms, Equation (34) simplifies to

$$A\left(\frac{\partial S}{\partial r_*}\right)^2-2\left(\omega-\omega'\right)\left(\frac{\partial S}{\partial r_*}\right)+2\kappa(r-r_{\mathrm{H}})B^{-1}c=0, \tag{37}$$

where

$$
\begin{aligned}
A&=\frac{1+2a}{2\kappa(r-r_{\mathrm{H}})B}\left[g^{11}-2\dot{r}_{\mathrm{H}}-2g^{12}r'_{\mathrm{H}}-\left(r'_{\mathrm{H}}\right)^2r^{-2}\right]\\
&=\frac{A'}{2\kappa(r-r_{\mathrm{H}})B},
\end{aligned}
\tag{38}
$$

$$B=1+2a+2b\dot{r}_H-2c_0c_1, \tag{39}$$

$$D=c_2^2g^{22}p_2^2+2c_2c_3n\sqrt{g^{22}g^{33}}p_2+c_3^2g^{33}p_2^2-\frac{p_2}{r^2}-\frac{n}{r^2}+2c\gamma_\sigma^5+m^2, \tag{40}$$

$$\omega'=\frac{1}{B}\left(g^{12}p_2++p_2r'_{\mathrm{H}}r^{-2}\right), \tag{41}$$

$\lim\limits_{r\to r_{\mathrm{H}}}A'=0$, black hole quantum tunneling radiation occurs at the event horizon r_{H}. Therefore, when $r\to r_{\mathrm{H}}$, let

$$\lim_{r\to r_{\mathrm{H}}}A=1. \tag{42}$$

Substituting Equation (38) into Equation (42) and using L'Hospital's rule, we obtain

$$\kappa=\frac{1+2a}{1+2a-2b\dot{r}_{\mathrm{H}}+2c_0c_1}\left[\frac{M}{r_{\mathrm{H}}^2}-\tilde{a}\cos\theta+\frac{\left(r'_{\mathrm{H}}\right)^2}{r_{\mathrm{H}}^3}\right]. \tag{43}$$

From Equation (41), it can be seen that

$$\omega'\big|_{r\to r_{\mathrm{H}}}=\omega_0=\frac{1}{B}\left(g^{12}p_2+p_2r'_{\mathrm{H}}r_{\mathrm{H}}^{-2}\right). \tag{44}$$

Therefore, Equation (37) can be further simplified to

$$\left(\frac{\partial S}{\partial r_*}\right)^2 - 2(\omega - \omega_0)\frac{\partial S}{\partial r_*} = 0. \tag{45}$$

From this equation, we can obtain

$$\frac{\partial S_\pm}{\partial r_*} = (\omega - \omega_0) \pm (\omega - \omega_0). \tag{46}$$

According to Equation (34), we have

$$\begin{aligned}\frac{\partial S_\pm}{\partial r} &= \frac{2\kappa(r - r_{\mathrm{H}}) + 1}{2\kappa(r - r_{\mathrm{H}})}\frac{\partial S_\pm}{\partial r_*} \\ &= \frac{2\kappa(r - r_{\mathrm{H}}) + 1}{2\kappa(r - r_{\mathrm{H}})}[(\omega - \omega_0) \pm (\omega - \omega_0)].\end{aligned} \tag{47}$$

According to the residue theorem, we obtain the value of $S_\pm$ when $r \to r_{\mathrm{H}}$ as

$$S_\pm = \pm\frac{\mathrm{i}\pi}{2\kappa}(\omega - \omega_0), \tag{48}$$

According to the WKB approximation theory and black hole quantum tunneling radiation theory, we can obtain the expression for the quantum tunneling rate of spin-1/2 fermions at the event horizon of this black hole as

$$\Gamma \sim \exp(-2\,\mathrm{ImS}_\pm) = \exp\left[-\frac{2\pi}{\kappa}(\omega - \omega_0)\right] = \exp\left(-\frac{\omega - \omega_0}{T_{\mathrm{H}}}\right). \tag{49}$$

where κ is given by Equation (43), thus

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi} = \frac{1 + 2a}{2\pi(1 + 2a - 2b\dot{r}_{\mathrm{H}} + 2c_0c_1)}\left[\frac{M}{r_{\mathrm{H}}^2} - \tilde{a}\cos\theta - \left(r'_{\mathrm{H}}\right)^2 r_{\mathrm{H}}^{-3}\right]. \tag{50}$$

Clearly, T_{H} is the Hawking temperature at the horizon r_{H} of this black hole and is a new expression based on the Lorentz-breaking modification. T_{H} is related to the modified CFJ term coefficient a, as well as the Chiral term coefficients b, c_0, and c_1 of the aether-like correction terms. The parameter κ in Equation (49) is given by Equation (40) and represents the surface gravity of the black hole's event horizon, which affects the quantum tunneling rate of the black hole. The coefficients a, b, c_0, and c_1 have certain effects on the quantum tunneling rate. The parameter ω_0 in Equation (45) is given by Equation (38) and represents a chemical potential-like characteristic that is related to $\tilde{a}$. The coefficients a, b, c_0, and c_1 have an impact on ω_0, while $\dot{r}_{\mathrm{H}}, r'_{\mathrm{H}}$ also exert influence on ω_0. However, the coefficient c of the Chiral correction term does not affect the Hawking temperature. In order to investigate the influence of the Chiral correction term on the Dirac energy levels, we study the quantum nonthermal effects of this black hole.

Within the interval $r_{\mathrm{H}} < r < \infty$, according to Equation (37), we can obtain the expression for the r-component of the generalized momentum $p_r \sim \frac{\partial S}{\partial r_*}$ of a Dirac particle in this black hole curved spacetime as

$$p_r = \frac{\partial S}{\partial r_*} \sim \frac{4\kappa(r - r_{\mathrm{H}})B}{2A'}\left\{(\omega - \omega') \pm \left[\left((\omega - \omega')^2\right) - \frac{A'D}{B^2}\right]^{\frac{1}{2}}\right\}, \tag{51}$$

When $r_{\mathrm{H}} < r < \infty$, p_r should be a real number. Therefore, it is required that the expression inside the square root in Equation (50) satisfies the following conditions

$$(\omega - \omega')^2 - \frac{A'D}{B^2} \geq 0. \tag{52}$$

From this, we can obtain

$$\omega_{\pm} = \omega' \pm \frac{1}{B}\sqrt{A'D}, \tag{53}$$

$$\lim_{r \to r_{\mathrm{H}}} \omega_{\pm} = \lim_{r \to r_{\mathrm{H}}} \left[\omega' \pm \frac{1}{B}\sqrt{A'D} \right] = \omega_0. \tag{54}$$

Neglecting the small quantity c/m^2, we can deduce from Equations (52) and (53), as well as Equations (38)–(41), that

$$\omega_{\pm}|_{r \to \infty} \sim \pm\sqrt{2c\gamma_{\sigma}^5 + m^2} = \pm m\sqrt{1 + \frac{2c\gamma_{\sigma}^5}{m^2}} \approx \pm m. \tag{55}$$

From this, we can obtain the distribution of positive and negative energy levels for the particles as

$$\begin{aligned} \omega &> \omega_+ \\ \omega &< \omega_- \end{aligned} \tag{56}$$

The energy range of the radiated particles is

$$m < \omega \leq \omega_0. \tag{57}$$

Equation (54) indicates the crossing appearance of energy levels ω_+ and ω_- with a maximum value of ω_0. Due to the crossing of energy levels of Dirac particles, quantum nonthermal radiation will be generated. From Equation (53) and the expressions for B, D, and A', it can be observed that the distribution of Dirac energy levels $\omega_{\pm}$ in a linearly accelerating black hole spacetime is dependent on $\dot{r}_{\mathrm{H}}$, r'_{H} , a ,b , c , γ_{σ}^5 , c_0 , c_1 , c_2, and c_3. The expression for $\omega_{\pm}$ indicates that the coefficients c and γ_{σ}^5 associated with the Chiral correction term have an impact on the Dirac energy levels $\omega_{\pm}$, albeit a small one. Let d represent the corresponding forbidden zone width of the Dirac energy level. In the region far from the event horizon,

$$d|_{r \to \infty} = (\omega_+ - \omega_-)|_{r \to \infty} \approx 2m. \tag{58}$$

Neglecting higher-order terms, this width $d|_{r \to \infty}$ is consistent with the width in flat spacetime. The conclusions drawn above pertain to spin-1/2 fermions and Dirac particles. When studying the quantum tunneling radiation of spin-1/2 fermions in a black hole curved spacetime considering the Lorentz-breaking theory, the Chiral correction term coefficient must inevitably appear. From Equations (54) and (57), it can be seen that the maximum energy of spin-1/2 fermions radiated by a linearly accelerating Kinnersley black hole is ω_0 , and the energy range of radiated spin-1/2 fermions is given by Equation (57). This quantum radiation characteristic of the black hole is independent of the Hawking temperature, and this radiation that is independent of the black hole temperature is referred to as nonthermal radiation. The physical interpretation of Equations (54) and (57) is that there are Dirac particle energy levels crossing near the event horizon of this linearly accelerating Kinnersley black hole, which leads to the phenomenon of vacuum negative energy state particle tunneling. Spin-1/2 fermions (emitted particles) with energies satisfying Equation (57) will be radiated away to distant regions. These radiated particles come from the alternating region of positive and negative energy levels. The energy of these radiated particles comes from the kinetic energy of the linearly accelerating Kinnersley black hole. This nonthermal radiation effect of the black hole is known as the Starobinsky–Unruh effect, which is actually the spontaneous radiation (Misner super-radiance) of the black hole. It should be noted that the above description is specific to the linearly accelerating Kinnersley black hole. For different types of black holes, the expression for the Misner super-radiance condition $m < \omega \leq \omega_0$ varies, mainly due to the different expressions for ω_0. If the black hole has rotational and charged characteristics, the energy of the particles radiated by the black hole's nonthermal quantum effect also comes from the rotational and electromagnetic field energy.

4. Discussion

Taking into account the modified form of the semiclassical dynamical equation for Lorentz-breaking spin-1/2 fermions as shown in Equation (13), we apply this modification to the spacetime of a linearly accelerating Kinnersley black hole and investigate the quantum tunneling radiation characteristics of this black hole. We obtain the modified expressions for the quantum tunneling probability, the black hole's Hawking temperature, and the gravitational force on the event horizon surface. We also obtain the particle energy level distribution characteristics associated with quantum nonthermal radiation and the energy range of the particles emitted by quantum nonthermal radiation. In order to demonstrate the correctness of the research methods used and the series of conclusions obtained, it is necessary to provide necessary explanations and discussions. When $\tilde{a} = 0$, Equation (50) implies that the Hawking temperature at the event horizon of the black hole is

$$\begin{aligned} T_{\mathrm{H}} &= \frac{1+2a}{2\pi(1+2a-2b\dot{r}_{\mathrm{H}}+2c_0c_1)}\left[\frac{M}{r_{\mathrm{H}}^2}+(r'_{\mathrm{H}})^2 r_{\mathrm{H}}^{-3}\right] \\ &= \frac{1+2a}{2\pi(1+2a-2b\dot{r}_{\mathrm{H}}+2c_0c_1)}\left[\frac{M}{r_{\mathrm{H}}^2}-\frac{1}{r_{\mathrm{H}}}\left(1-\frac{2M}{r_{\mathrm{H}}}-2\dot{r}_{\mathrm{H}}\right)\right]. \end{aligned} \tag{59}$$

The value of r_{H} in Equation (59) is no longer determined by Equation (26), but rather by the following equation after substituting $g^{11} = -\left(1-\frac{2M}{r}\right)$,

$$\left(1-\frac{2\mathrm{M}}{r_{\mathrm{H}}}\right)-2\dot{r}_{\mathrm{H}} = 0. \tag{60}$$

In this particular case, ω_0 in Equations (47) and (57) can be expressed as

$$\omega_0 = \frac{p_2 r_{\mathrm{H}}{}' r_{\mathrm{H}}^{-2}}{1+2a+2b\dot{r}_{\mathrm{H}}-2c_0c_1}. \tag{61}$$

When $\tilde{a} = 0$,

$$ds^2 = \left(1-\frac{2M}{r}\right)dv^2 - 2dvdr - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2. \tag{62}$$

This is the line element of the Vaidya black hole spacetime [36], which possesses spherical symmetry. In this particular case, the quantum tunneling rate of spin-1/2 fermions after Lorentz-breaking modification, the surface gravity of the black hole event horizon, and the Hawking temperature at the event horizon all simplify to their corresponding simplest forms for the Vaidya black hole. Consequently, κ in Equation (42) and T_{H} in Equation (49) are, respectively, given by

$$\kappa_{\mathrm{v}} = \frac{1+2a}{1+2a-2b\dot{r}_{\mathrm{H}}+2c_0c_1}\frac{1-2\dot{r}_{\mathrm{H}}}{2r_{\mathrm{H}}}, \tag{63}$$

$$T_{\mathrm{v}} = \frac{\kappa}{2\pi} = \frac{1+2a}{2\pi(1+2a-2b\dot{r}_{\mathrm{H}}+2c_0c_1)}\frac{1-2\dot{r}_{\mathrm{H}}}{2r_{\mathrm{H}}}. \tag{64}$$

The quantum tunneling rate corresponding to this temperature is

$$\Gamma_{\mathrm{v}} \sim \exp\left(-\frac{\omega}{T_{\mathrm{v}}}\right). \tag{65}$$

Equation (65) represents the quantum tunneling probability of a Lorentz-breaking modified Vaidya black hole. It should be noted that in the case where in the (54) and (57) equations, $\omega_0 = 0$, this indicates that the black hole has degenerated into a Schwarzschild-like black hole. In this scenario, there is no crossing phenomenon of the Dirac energy levels $\omega_{\pm}$, hence no nonthermal radiation occurs. Equation (57) disappears automatically, and Equation (53) degenerates to

$$\omega_{\pm} = \pm \frac{1}{1 + 2a + 2b\dot{r}_{\mathrm{H}} + 2c_0 c_1} \left[\left(\frac{2M}{r} - 1 - 2\dot{r}_{\mathrm{H}} \right) D \right]^{\frac{1}{2}}. \tag{66}$$

When r is very large, $\omega_{\pm}$ remains consistent with Equation (58). However, when $\omega_0 = 0$, Equation (66) indicates that $T_{\mathrm{v}} \neq 0$. For this temperature, it is the Hawking temperature at the event horizon of Schwarzschild-like black hole. The quantum tunneling rate corresponding to this temperature, which accounts for the thermal effects, is given by Equation (67). In other words, Γ_S represents the expression for the modified Vaidya black hole's quantum tunneling radiation rate corresponding to T_{v}.

In black hole physics, the black hole entropy is closely related to the black hole's Hawking temperature. If we denote the Bekenstein–Hawking entropy of the black hole as S_{BH} and the Bekenstein–Hawking entropy change as ΔS_{BH}, then Equation (52) can be expressed as

$$\Gamma \sim \exp(\Delta \mathrm{S}_{\mathrm{BH}}), \tag{67}$$

If we use $\mathrm{S}'_{\mathrm{BH}}$ to represent the Bekenstein–Hawking entropy of a Vaidya black hole, and use $\Delta\mathrm{S}'_{\mathrm{BH}}$ to represent the Bekenstein–Hawking entropy change, then Equation (65) can be interpreted as follows:

$$\Gamma_{\mathrm{v}} \sim \exp\left(\Delta \mathrm{S}'_{\mathrm{BH}}\right). \tag{68}$$

The above discussion pertains to fermions and explores the quantum tunneling radiation and related issues in linearly accelerating black holes considering the Lorentz-breaking theory. However, the same modification method cannot be applied to study the relevant topics concerning bosons. The study of the modified characteristics of quantum tunneling radiation for bosons requires starting from the modified scalar field action. Furthermore, it is important to note that the selection of γ^{μ} and γ^5 and proving their correctness is a crucial process. The specific characteristics of different curved spacetimes need to be studied in detail. Only in this way can we obtain accurate and valuable conclusions.

It should be further noted that in the aforementioned study, the focus is on the dynamically curved spacetime of the Kinnersley black hole, specifically the curved spacetime background of a Kinnersley black hole undergoing linear acceleration. The same methods can be applied to obtain results for any accelerated black hole. The series of results obtained above are fundamental, reliable, and valuable as references. Considering the impact of Lorentz-breaking theories on the spacetime background is also a topic worth investigating. Ref. [37] considered modifications to the spacetime background caused by Lorentz-breaking theories in Einstein–Bumblebee gravity, and explored the Kerr–Sen-like black hole in Bumblebee gravity. Such investigations into modified spacetime backgrounds are worthy of our attention. Furthermore, this paper examines the quantum tunneling radiation characteristics of Dirac particles influenced by Lorentz-breaking theories at the event horizon of the accelerating linear motion of the Kinnersley black hole. Similar methods can be applied to investigate the related phenomena for spin-3/2,... fermions. However, for bosons, since there will be no occurrence of Chiral correction term and bosons wave function $\varphi = \varphi_0 e^{\frac{i}{\hbar}S}$, the study of modified quantum tunneling radiation characteristics for bosons at the event horizon of black holes would be simpler.

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