

Review

Noncompactified Kaluza–Klein Gravity

Seyed Meraj Mousavi Rasouli ^{1,2,*} , Shahram Jalalzadeh ³  and Paulo Moniz ¹ 

¹ Departamento de Física, Centro de Matemática e Aplicações (CMA-UBI), Universidade da Beira Interior, Rua Marquês d'Avila e Bolama, 6200-001 Covilhã, Portugal

² Department of Physics, Qazvin Branch, Islamic Azad University, Qazvin 341851416, Iran

³ Departamento de Física, Universidade Federal de Pernambuco, Recife 52171-900, Brazil

* Correspondence: mrasouli@ubi.pt

Abstract: We present a brief description of noncompactified higher-dimensional theories from the perspective of general relativity. More concretely, the Space–Time–Matter theory, or Induced Matter theory, and the reduction procedure used to construct the modified Brans–Dicke theory and the modified Sáez–Ballester theory are briefly explained. Finally, we apply the latter to the Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological models in arbitrary dimensions and analyze the corresponding solutions.

Keywords: higher dimensions; noncompactified Kaluza–Klein theories; induced–matter theory; space–time–matter theory; modified Brans–Dicke theory; modified Sáez–Ballester theory; noncompactified cosmology

1. Introduction

Since matter or the source of spacetime and fields are fundamental concepts in classical field theories, the Einstein tensor is expressed in terms of spacetime geometry and matter by the corresponding energy–momentum density tensor. The Einstein field equations connect these two fundamentals. As a result, in general relativity (GR), the distribution of matter determines how spacetime is shaped. On the other hand, one could interpret Einstein's field equations in a different way and assert that geometry creates matter. One of Einstein's objectives was to develop a gravitational theory in which the idea of matter is abandoned in favor of pure fields [1]. According to Einstein, unified field theory is a gravitational theory in which matter is absorbed into the field itself, leading to a set of homogeneous partial differential equations. Many extensions of Einstein's framework have been made to extract matter from pure geometry. One of Einstein's intriguing extensions is the suggestion that our four-dimensional spacetime (called a membrane or brane) is a submanifold embedded in a higher dimensional ambient space (bulk). This idea first appeared in papers by Kaluza and Klein, who proposed uniting gravity and electromagnetism. In Kaluza–Klein's (KK) theory, the extra dimension serves only a formal purpose, and the components of the ambient space metric tensor are independent of the coordinate associated with the extra dimension.

In the last two decades, there has been much appeal in the concept of extra dimensions where ordinary matter is constrained to a brane. Early examples of this methodology can be found in the works of Maia [2], Joseph [3], Akama [4], Rubakov–Shaposhnikov [5], and Visser [6]. Wesson's theory [7], which states that the geometry of the bulk space generates matter on the brane, is the basis for a revised KK approach to unified field theory. This theory differs from the traditional KK scenario due to the noncompact extra dimension and the absence of matter in the five-dimensional bulk space. This theory is called the Induced Matter Theory (IMT) because the effective four-dimensional matter results from the bulk's geometry. In other words, in IMT, the four-dimensional induced matter curves the four-dimensional hypersurface, while the five-dimensional bulk space is Ricci-flat.



Citation: Rasouli, S.M.M.; Jalalzadeh, S.; Moniz, P. Noncompactified Kaluza–Klein Gravity. *Universe* **2022**, *8*, 431. <https://doi.org/10.3390/universe8080431>

Academic Editors: Emmanuel N. Saridakis and Panayiotis Stavrinos

Received: 22 July 2022

Accepted: 18 August 2022

Published: 21 August 2022

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Recently, instead of GR, by applying the scalar–tensor theories as underlying frameworks, another extended version of the noncompactified KK gravity has also been established [8,9], which will be the main content of the this review paper.

For other extensions of Wesson’s theory that have assumed an arbitrary number of noncompact extra dimensions, see [10–12].

Furthermore, the authors of Ref. [13] have obtained interesting IMT cosmological solutions by assuming a conformally flat bulk space. The Weyl tensor of the bulk space vanishes in this case. As noted above, this restriction is in the spirit of IMT. The energy conditions associated with the model of Ref. [13] have been also investigated in detail in [14]. All matter fields in 4-dimensional spacetime are induced from the bulk, according to IMT, and the path of particles in the bulk space is null. Therefore, the bulk can be assumed to be empty. We can have some black holes in the bulk space if we only assume that the bulk is Ricci-flat. Consequently, since black holes have mass, we must have some kind of mass in the bulk, which contradicts the IMT axioms.

One of the consequences of IMT is that the mass of the particles varies from point-to-point in spacetime. Indeed, it is well known that in unified field theories, Mach’s principle is satisfied [15]. Thus, the mass of particles may be affected by the distribution of matter fields in the Universe or the curvature of spacetime. As a result, it is not surprising that Wesson’s IMT is a Machian theory [16] and the particle’s mass is not constant. Wesson used dimensional analysis to introduce the relation $l = Gm/c^2$ (where G and c are the Newton gravitational constant and the speed of light, respectively) between the fifth coordinate, l , and the mass of the test particles, m , to demonstrate mass variation [17,18], and for this reason, he initially called his theory Space–Time–Matter (STM) theory [19]. The authors of Ref. [20] have shown that if we use the above equation to calculate the mass variation of the primordial nucleosynthetic particles and our time and compare it to the variation of mass obtained from nucleosynthesis bounds, the results do not agree. However, in Ref. [21], it was shown that if the induced mass is defined correctly, the variation of mass obtained from IMT agrees with the mass variation bound obtained from the Hot Big Bang.

Let us also mention one of the more intriguing futures of IMT based on quantum mechanics. Particles and waves, for example, have been shown to be merely different representations of the same underlying geometry and may be the same thing viewed in different ways [22]. Wesson [23] has also derived a form of the Heisenberg relation that applies to real and virtual particles using five-dimensional IMT. (For an extension of this idea to brane gravity, please see [24].) In this regard, the authors of [25] derived the induced Einstein equation on the perturbed brane by contracting the Gauss–Codazzi equations. They studied the Einstein equation on the perturbed brane of an FRW universe as a non-perturbed brane embedded in a five-dimensional flat bulk space. They demonstrated that the induced field equations correspond to the semiclassical Einstein equation. This means that the classical fluctuations of the perturbed brane can be interpreted as matter field quantum fluctuations.

Inspired by the IMT, the authors of [8,9] have shown that the field equations associated with two different types of scalar–tensor theories, namely the Sáez–Ballester (SB) and Brans–Dicke (BD) theories, in $(D + 1)$ -dimensions are equivalent to their D -dimensional counterparts with an effective matter field and a potential. These reduced gravitational models have been nominated for the modified Sáez–Ballester theory (MSBT) [9,26] and the modified Brans–Dicke theory (MBDT) [8], respectively. It is important to mention that in both frameworks, introducing a non-zero induced scalar potential whose shape is provided by the reduction process up to a constant of integration, is crucial to reconstruct the corresponding theory on the hypersurface.

Now let us focus on the BD theory and the MBDT. One of the attractive aspects of the BD theory is that the scalar field is not an ad hoc assumption but rather a fundamental component. In cosmological models that use BD theory as a background theory, the scalar field can act as quintessence or K-essence, resulting in an accelerated scale factor [27]. Such cosmological models, however, have flaws that contradict the principles of the original ver-

sion of the BD theory, experimental data, and energy conditions. More concretely, in some of these models, the BD coupling parameter is assumed to be a variable constant [28]; in others, an ad hoc scalar potential is added [29]. Concerning related investigations within the standard BD theory as well as its modified version, see also [30–40]. Nevertheless, it should be noted that the MBDT is free of the shortcomings mentioned above. In particular, using the MBDT framework in cosmology does not necessitate using an ad hoc scalar potential or a variable BD coupling parameter to obtain an accelerated scale factor. Furthermore, unlike other modified BD cosmological models, the FLRW-MBDT cosmologies eliminate the inconsistency in the value of the BD coupling parameter associated with an accelerated expansion of a matter-dominated universe and a decelerated radiation-dominated epoch [8]. In this review paper, we will not present any cosmological application of the MBDT. We recommend readers look at the detailed study of such models in [41–46].

The main goal in constructing the Sáez–Ballester (SB) scalar–tensor theory [47] was to solve the problem of missing matter of the universe. The Einstein–Hilbert action was supplemented in this framework by a non-canonical kinetic term containing a coupling parameter. Both a scalar potential and a cosmological term are absent from the original SB theory. In one particular instance, it is possible to find appropriate transformations that allow the SB action to be reduced to the corresponding one with a canonical kinetic term [48]. In contrast to the latter, the SB theory includes the action of ordinary matter that is not coupled to the SB scalar field. Various cosmological models have been established in the context of the SB theory to study open problems in both the classical and quantum regimes [49–54]. Applying the similar reduction procedure used to construct the MBDT, without assuming the presence of the higher dimensional matter fields and imposing the cylindricity condition on the extra coordinate, it has been shown that the SB field equations associated to the bulk split into four sets of effective field equations on any hypersurface orthogonal to the extra dimension [9].

In the next section, we will review the IMT and present short discussion about this framework. In Section 3, we present a brief review of the MBDT and MSBT. In Section 4, as an application of the reduced SB theory, we will review the solutions of the FLRW-MSBT cosmology. Finally, in Section 5, in addition to a brief summary, we will include other useful discussions.

2. Five-Dimensional Ricci–Flat Space and the Effective Field Equations in Four Dimensions

In this section, we present an overview and important notes on the framework established in [55].

Assuming some postulates and applying an appropriate reduction procedure, the field equation of general relativity (GR) can be set up on a four-dimensional hypersurface. The five-dimensional manifold M_5

$$dS^2 = \mathcal{G}_{ab}(x^c)dx^a dx^b, \tag{1}$$

in which our universe is locally and isometrically embedded can be, at least locally, taken as

$$dS^2 = g_{\mu\nu}(x^\alpha, l)dx^\mu dx^\nu + \epsilon\psi^2(x^\alpha, l)dl^2. \tag{2}$$

In this section, the Latin and Greek indices run from zero to four, and to three, respectively; l is an extra noncompact coordinate, $\psi = \psi(x^\alpha, l)$ is a scalar field and $\epsilon = \pm 1$ (where $\epsilon^2 = 1$) is introduced such that we can take the extra dimension as either time-like or space-like.¹ Up to now, we have indeed expressed the ‘Postulate I’ of the Wesson’s scheme [56].

Remark. Before reviewing the effective field equations, let us mention a few important points. In the IMT, similar to the Kaluza’s approach, the same definitions for the five-dimensional Christoffel symbols and Ricci tensor have been used [7]. However, unlike the compactified KK theory, the authors

of IMT did not want to geometrize the electromagnetic field. Moreover, the derivatives with respect to the extra coordinate were not assumed to be zero.² We must emphasize that the frameworks to be presented in this paper recover from a fully noncompactified approach without making any a priori assumptions about the nature of the extra-dimensional manifold.

It is straightforward to show that the expressions for the $\alpha\beta$ -, $\alpha 4$ - and 44 -parts of the Ricci tensor $R_{ab}^{(5)}$ are:

$$R_{\alpha\beta}^{(5)} = R_{\alpha\beta}^{(4)} - \frac{\mathcal{D}_\alpha \mathcal{D}_\beta \psi}{\psi} + \frac{\epsilon}{2\psi^2} \left(\frac{\psi^* g_{\alpha\beta}^*}{\psi} - g_{\alpha\beta}^{**} + g^{\lambda\mu} g_{\alpha\lambda}^* g_{\beta\mu}^* - \frac{1}{2} g^{\mu\nu} g_{\mu\nu}^* g_{\alpha\beta}^* \right), \tag{3}$$

$$R_{4\alpha}^{(5)} = \psi \mathcal{D}_\beta P^\beta_{\alpha}, \tag{4}$$

$$R_{44}^{(5)} = -\epsilon \psi \mathcal{D}^2 \psi - \frac{1}{4} g^{*\lambda\beta} g_{\lambda\beta} - \frac{1}{2} g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{\psi^*}{2\psi} g^{\lambda\beta} g_{\lambda\beta}^*, \tag{5}$$

where

$$P_{\alpha\beta} \equiv \frac{1}{2\psi} \left(g_{\alpha\beta}^* - g_{\alpha\beta} g^{\mu\nu} g_{\mu\nu}^* \right), \tag{6}$$

$A \equiv \frac{\partial A}{\partial t}$, \mathcal{D}_α is the covariant derivative on the hypersurface and $\mathcal{D}^2 \equiv \mathcal{D}^\alpha \mathcal{D}_\alpha$.

In the IMT [7,19,55], it was presumed that there is no higher-dimensional ordinary matter; therefore $G_{ab}^{(5)} = 0$ (where $G_{ab}^{(5)}$ are the components of the Einstein tensor in five dimensions), or equivalently, $R_{ab}^{(5)} = 0$, which is known as the ‘Postulate II’ of the Wesson’s scheme [56]. Therefore, by defining a hypersurface Σ_4 , for which $l = l_0 = \text{constant}$, and

$$g_{\mu\nu}(x^\alpha) = \mathcal{G}^{(4)}(x^\alpha, l_0), \tag{7}$$

Equations (3)–(5) reduce to

$$R_{\alpha\beta}^{(4)} = \frac{\mathcal{D}_\alpha \mathcal{D}_\beta \psi}{\psi} - \frac{\epsilon}{2\psi^2} \left[\frac{\psi^* g_{\alpha\beta}^*}{\psi} - g_{\alpha\beta}^{**} + g^{\lambda\mu} g_{\alpha\lambda}^* g_{\beta\mu}^* - \frac{1}{2} g^{\mu\nu} g_{\mu\nu}^* g_{\alpha\beta}^* \right], \tag{8}$$

$$\mathcal{D}_\beta P^\beta_{\alpha} = 0, \tag{9}$$

$$\epsilon \psi \mathcal{D}^2 \psi = -\frac{1}{4} g^{*\lambda\beta} g_{\lambda\beta} - \frac{1}{2} g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{\psi^*}{2\psi} g^{\lambda\beta} g_{\lambda\beta}^*. \tag{10}$$

Equations (8)–(10) “form the basis of five-dimensional noncompactified KK theory” [7]. The interpretation of these equations in four dimensions and their applications in cosmology and astrophysics have been extensively presented in the literature, see for instance, [7,19]. In what follows, let us briefly analyze one of the effective field equations.

Using Equations (8) and (10), and $(\delta^\mu_\nu)_{,4} = 0 = g^{\mu\beta} g^{\sigma\lambda} g_{\lambda\beta}^* g_{\sigma\lambda}^{*\mu\sigma} + g_{\mu\sigma}^* g_{\mu\sigma}^*$, we obtain an expression for the Ricci scalar $R^{(4)} = g^{\alpha\beta} R_{\alpha\beta}^{(4)}$:

$$R^{(4)} = \frac{\epsilon}{4\psi^2} \left[g^{*\mu\nu} g_{\mu\nu} + \left(g^{\mu\nu} g_{\mu\nu}^* \right)^2 \right]. \tag{11}$$

Defining an induced energy-momentum tensor in four dimensions as $T_{\alpha\beta}^{[IMT]} \equiv R_{\alpha\beta}^{(4)} - 1/2 R^{(4)} g_{\alpha\beta}$, from using Equations (8) and (11), one can easily obtain

$$T_{\alpha\beta}^{[IMT]} \equiv \frac{\mathcal{D}_\alpha \mathcal{D}_\beta \psi}{\psi} - \frac{\epsilon}{2\psi^2} \left(\frac{\psi g_{\alpha\beta}^{**}}{\psi} - g_{\alpha\beta}^{**} + g^{\lambda\mu} g_{\alpha\lambda}^* g_{\beta\mu}^* - \frac{1}{2} g^{\mu\nu} g_{\mu\nu}^* g_{\alpha\beta}^* \right) - \frac{\epsilon g_{\alpha\beta}}{8\psi^2} \left[g^{*\mu\nu} g_{\mu\nu} + (g^{\mu\nu} g_{\mu\nu}^*)^2 \right]. \tag{12}$$

In summary, according to the ‘Postulate III’ [56], we use the above expression as a four-dimensional energy momentum tensor associated with our universe; hence the Einstein field equations on a four-dimensional hypersurface, $G_{\alpha\beta}^{(4)} = T_{\alpha\beta}^{[IMT]}$, are automatically contained in the corresponding five-dimensional vacuum equations $G_{ab}^{(5)} = 0$. In this regard, $T_{\alpha\beta}^{[IMT]}$, which describes a matter as a manifestation of pure geometry in higher-dimensional spacetime, has been interpreted as the energy momentum tensor of an *induced-matter* in the KK theory.

3. Modified Scalar–Tensor Theories

In this section, we again take the line element (2) and the noncompact extra dimensions. Moreover, we will consider the Remark mentioned in the previous section as well as the following one

Remark. Equations (3)–(6) are again valid for this framework as well as that which will be introduced in the next subsection. However, since in this and the subsequent subsection we relate the equations associated with $(D + 1)$ -dimensional spacetime to their corresponding D -dimensional counterparts, the indices 4α and 4λ should be replaced by $D\alpha$ and DD , respectively. Moreover, the superscripts (5) should be replaced with $(D + 1)$. Furthermore, due to the presence of the scalar field as well as the higher-dimensional matter in the bulk, equations $G_{ab}^{(5)} = 0$ or equivalently, $R_{ab}^{(5)} = 0$ are generally no longer valid. More precisely, as we will show below, not only are Equations (8)–(12) generalized, but we will also show that when the wave equation of each framework is reduced on the hypersurface, it involves induced scalar potential.

In addition, we will consider the following generalizations. (i) Instead of deriving the modified field equations in four dimensions, we want to obtain them in arbitrary dimensions. Therefore, we assume that the Latin and Greek indices run from zero to D , and to $D - 1$, respectively. Moreover, \mathcal{G} and $R^{(D+1)}$, respectively, are the determinant and the Ricci scalar of the $(D + 1)$ -dimensional metric \mathcal{G}_{ab} ; ∇_a stands for the covariant derivative in $(D + 1)$ -dimensional spacetime, and $\nabla^2 \equiv \nabla_a \nabla^a$. The Lagrangian $L_{\text{matt}}^{(D+1)}$ describes ordinary matter in the $(D + 1)$ -dimensional spacetime. (ii) Rather than considering the GR as a background theory, let us consider two different types of scalar–tensor theories: in the first, the scalar field is minimally coupled to gravity, while in the second, the scalar field is non-minimally coupled. (iii) In order to apply a generalized reduction method, we consider, in addition to the presence of the scalar field in the action, a higher-dimensional ordinary matter³ $L_{\text{matt}}^{(D+1)}$.

3.1. Modified Sáez–Ballester Theory in Arbitrary Dimensions

Let us give a brief overview of the framework established in [9], see also [26,48]. We consider a generalized version⁴ of the SB action proposed in [47] in $(D + 1)$ -dimensional spacetime as

$$\mathcal{S}_{\text{SB}}^{(D+1)} = \int d^{D+1}x \sqrt{|\mathcal{G}|} \left[R^{(D+1)} - \mathcal{W} \phi^n \mathcal{G}^{ab} (\nabla_a \phi)(\nabla_b \phi) + \chi L_{\text{matt}}^{(D+1)} \right], \tag{13}$$

where ϕ is the SB scalar field; \mathcal{W} , n are dimensionless parameters of the model; $\chi = 8\pi$, and we used the same units taken in [9]. We should emphasize that there is no scalar potential in action (13).

One can easily show that the equations of motion corresponding to (13) are:

$$G_{ab}^{(D+1)} = \mathcal{W}\phi^n \left[(\nabla_a\phi)(\nabla_b\phi) - \frac{1}{2}\mathcal{G}_{ab}(\nabla^c\phi)(\nabla_c\phi) \right] + \chi T_{ab}^{(D+1)} \tag{14}$$

and

$$2\phi^n \nabla^2\phi + n\phi^{n-1}(\nabla_a\phi)(\nabla^a\phi) = 0, \tag{15}$$

where $G_{ab}^{(D+1)}$ and $T_{ab}^{(D+1)}$, respectively, denote the Einstein tensor and the energy momentum tensor (of any ordinary matter field) in $(D + 1)$ -dimensions. We should emphasize that $T_{ab}^{(D+1)}$ does not depend on ϕ , and it is therefore identically conserved. Equation (14) leads to

$$R^{(D+1)} = \mathcal{W}\phi^n(\nabla_a\phi)(\nabla^a\phi) - \frac{2\chi}{D-1}T^{(D+1)}, \tag{16}$$

where $T^{(D+1)} = \mathcal{G}^{ab}T_{ab}^{(D+1)}$.

Before deriving the effective field equations, let us write some useful equations that will be used later:

$$\nabla_\mu \nabla_\nu \phi = \mathcal{D}_\mu \mathcal{D}_\nu \phi + \frac{\epsilon \phi^{**} g_{\mu\nu}^*}{2\psi^2}, \tag{17}$$

$$\nabla^2\phi = \mathcal{D}^2\phi + \frac{(\mathcal{D}_\alpha\psi)(\mathcal{D}^\alpha\phi)}{\psi} + \frac{\epsilon}{\psi^2} \left[\phi^{**} + \phi^* \left(\frac{g^{\mu\nu} g_{\mu\nu}^*}{2} - \frac{\psi^*}{\psi} \right) \right], \tag{18}$$

$$(\nabla^a\phi)(\nabla_a\phi) = (\mathcal{D}^\alpha\phi)(\mathcal{D}_\alpha\phi) + \epsilon \left(\frac{\phi^*}{\psi} \right)^2, \tag{19}$$

$$\nabla_D \nabla_D \phi = \epsilon\psi(\mathcal{D}_\alpha\psi)(\mathcal{D}^\alpha\phi) + \phi^{**} - \left(\frac{\psi^*}{\psi} \right)^* \phi. \tag{20}$$

Letting $a \rightarrow \mu$ and $b \rightarrow \nu$ in Equation (14) yields the D -dimensional counterpart of the corresponding $(D + 1)$ -dimensional quantity:

$$G_{\mu\nu}^{(D+1)} = \mathcal{W}\phi^n \left[(\mathcal{D}_\mu\phi)(\mathcal{D}_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\mathcal{D}_\alpha\phi)(\mathcal{D}^\alpha\phi) \right] - \frac{\epsilon\mathcal{W}\phi^n}{2} \left(\frac{\phi^*}{\psi} \right)^2 g_{\mu\nu} + \chi T_{\mu\nu}^{(D+1)}, \tag{21}$$

where we have used (19).

Now we are going to derive the equations associated with the MSBT⁵

1. Let us first obtain a dynamical equation for the scalar field ψ , i.e., an extended version of (10), which, in turn, will be applied to retrieve other modified equations. Letting $a \rightarrow D$ and $b \rightarrow D$ in Equation (14), we obtain

$$R_{DD}^{(D+1)} = \left(\frac{\epsilon\chi\psi^2}{1-D} \right) T^{(D+1)} + \chi T_{DD}^{(D+1)} + \mathcal{W}\phi^n \left(\frac{\phi^*}{\psi} \right)^2, \tag{22}$$

where we used Equation (16). Equating relations (5) (please see the Remarks) and (22), we retrieve

$$\frac{\mathcal{D}^2\psi}{\psi} = -\frac{\epsilon}{2\psi^2} \left[g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{1}{2} g^{*\lambda\beta} g_{\lambda\beta} - \frac{g^{\lambda\beta} g_{\lambda\beta}^* \psi}{\psi} \right] - \epsilon \mathcal{W} \phi^n \left(\frac{\phi}{\psi} \right)^2 + \chi \left[\frac{T^{(D+1)}}{D-1} - \frac{\epsilon T_{DD}^{(D+1)}}{\psi^2} \right], \tag{23}$$

which is one of our effective equations. It can be seen that in a special case where $\phi = \text{constant}$ and the higher-dimensional ordinary matter is absent, Equation (23) reduces to its IMT counterpart.

- Let us now construct the Einstein tensor on the hypersurface. Concretely, we want to retrieve the counterpart of (14) on a D -dimensional hypersurface. In this regard, by taking the SB theory into account, we first relate the Ricci scalars $R^{(D+1)}$ and $R^{(D)}$:

$$R^{(D+1)} = \mathcal{G}^{ab} R_{ab}^{(D+1)} = \mathcal{G}^{\alpha\beta} R_{\alpha\beta}^{(D+1)} + \mathcal{G}^{DD} R_{DD}^{(D+1)}. \tag{24}$$

Substituting $R_{\alpha\beta}^{(D+1)}$ and $R_{DD}^{(D+1)}$ from relations (3) and (5), respectively (where we respect the expressions presented in the Remarks) into Equation (24), and then using Equation (23), after some manipulations, we obtain

$$R^{(D+1)} = R^{(D)} - \frac{\epsilon}{4\psi^2} \left[\left(g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 + g^{*\alpha\beta} g_{\alpha\beta} \right] + \frac{2\epsilon \mathcal{W} \phi^n \phi^2}{\psi^2} + 2\chi \left(\frac{\epsilon T_{DD}^{(D+1)}}{\psi^2} - \frac{T^{(D+1)}}{D-1} \right). \tag{25}$$

Now, we proceed as follows. By substituting $R_{\mu\nu}^{(D+1)}$ and $R^{(D+1)}$ from relations (3) and (25) into

$$G_{\mu\nu}^{(D+1)} = R_{\mu\nu}^{(D+1)} - \frac{1}{2} G^{\mu\nu} R^{(D+1)}, \tag{26}$$

and equating the result with (21), we can easily construct the Einstein tensor on a D -dimensional hypersurface:

$$\begin{aligned} G_{\mu\nu}^{(D)} &= \mathcal{W} \phi^n \left[(\mathcal{D}_\mu \phi)(\mathcal{D}_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\mathcal{D}_\alpha \phi)(\mathcal{D}^\alpha \phi) \right] + \chi \left(F_{\mu\nu} + T_{\mu\nu}^{[\text{MSBT}]} \right) - \frac{1}{2} g_{\mu\nu} V(\phi) \\ &\equiv \mathcal{W} \phi^n \left[(\mathcal{D}_\mu \phi)(\mathcal{D}_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\mathcal{D}_\alpha \phi)(\mathcal{D}^\alpha \phi) \right] + \chi T_{\mu\nu}^{(D)[\text{eff}]} - \frac{1}{2} g_{\mu\nu} V(\phi), \end{aligned} \tag{27}$$

where the induced scalar potential $V(\phi)$ should be obtained from the differential Equation (32).

Let us also introduce the effective matter in Equation (27):

- $F_{\mu\nu}$ is the effective matter induced from the $(D + 1)$ -dimensional ordinary energy momentum tensor:

$$F_{\mu\nu} \equiv T_{\mu\nu}^{(D+1)} + g_{\mu\nu} \left[\frac{\epsilon T_{DD}^{(D+1)}}{\psi^2} - \frac{T^{(D+1)}}{D-1} \right]. \tag{28}$$

Obviously, assuming a bulk without a higher-dimensional ordinary matter, i.e., $L_{\text{matt}}^{(D+1)} = 0$, then $F_{\mu\nu}$ vanishes.

- $T_{\mu\nu}^{[\text{MSBT}]}$ is an induced energy momentum tensor associated with our herein MSBT framework, which, in turn, has three components:

$$\chi T_{\mu\nu}^{[\text{MSBT}]} = T_{\mu\nu}^{[\text{IMT}]} + T_{\mu\nu}^{[\phi]} + \frac{1}{2}g_{\mu\nu}V(\phi), \tag{29}$$

where

$$T_{\mu\nu}^{[\phi]} \equiv \left[\frac{1}{2}\epsilon\mathcal{W}\phi^n \left(\frac{\phi}{\psi} \right)^2 \right] g_{\mu\nu}. \tag{30}$$

Here $T_{\mu\nu}^{[\text{IMT}]}$ is exactly the same quantity introduced in the previous section, see Equation (12).

3. We now obtain the reduced wave equation on a D -dimensional hypersurface, i.e., the counterpart to Equation (15). Substituting $\nabla^2\phi$ and $(\nabla^c\phi)(\nabla_c\phi)$ from relations (18) and (19) into (15), we obtain

$$2\phi^n\mathcal{D}^2\phi + n\phi^{n-1}(\mathcal{D}_\alpha\phi)(\mathcal{D}^\alpha\phi) - \frac{1}{\mathcal{W}}\frac{dV(\phi)}{d\phi} = 0, \tag{31}$$

where

$$\frac{dV(\phi)}{d\phi} \equiv -\frac{2\mathcal{W}\phi^n}{\psi^2} \left\{ \psi(\mathcal{D}_\alpha\psi)(\mathcal{D}^\alpha\phi) + \frac{n\epsilon}{2}\left(\frac{\phi}{\psi}\right)^2 + \epsilon \left[\phi^{**} + \phi^* \left(\frac{1}{2}g^{\mu\nu*}g_{\mu\nu}^* - \frac{\psi}{\psi} \right) \right] \right\}. \tag{32}$$

4. Finally, let us obtain the last equation, which is an extended version of (9). In this sense, substituting $a = D$ and $b = \mu$ into Equation (14), we obtain

$$G_{D\mu}^{(D+1)} = R_{D\mu}^{(D+1)} = \chi T_{D\mu}^{(D+1)} + \mathcal{W}\phi^n\phi^*(\mathcal{D}_\alpha\phi). \tag{33}$$

By equating (33) and (4) (which is obtained directly from metric (2); see also the second Remark), we obtain an extended dynamical equation for $P_{\alpha\beta}$ in MSBT:

$$\psi P^\beta_{\mu;\beta} = \chi T_{D\mu}^{(D+1)} + \mathcal{W}\phi^n\phi^*(\mathcal{D}_\mu\phi), \tag{34}$$

where $P_{\alpha\beta}$ is given by (6).

It is worth mentioning that Equations (27) and (31) are obtained from the action

$$S_{\text{SB}}^{(D)} = \int d^Dx \sqrt{-g} \left[R^{(D)} - \mathcal{W}\phi^n g^{\alpha\beta} (\mathcal{D}_\alpha\phi)(\mathcal{D}_\beta\phi) - V(\phi) + \chi L_{\text{matt}}^{(D)} \right], \tag{35}$$

which is a generalized action of the SB theory with a scalar potential and

$$\sqrt{-g} T_{\mu\nu}^{(D)|\text{eff}} \equiv 2 \frac{\delta S_{\text{matt}}}{\delta g^{\mu\nu}}, \tag{36}$$

where $S_{\text{matt}} = \int d^Dx \sqrt{-g} L_{\text{matt}}^{(D)}$ is the action corresponding to the matter fields in D dimensions. We should emphasize that the induced energy momentum tensor is covariantly conserved, namely, $\mathcal{D}_\beta T_\alpha^{(D)|\text{eff}\beta} = 0$.

3.2. Modified Brans–Dicke Theory in Arbitrary Dimensions

In this subsection, we will present a brief overview of the modified Brans–Dicke theory (MBDT). The action in $(D + 1)$ dimensions in the Jordan frame associated with the Brans–Dicke (BD) theory can be written as

$$S_{\text{BD}}^{(D+1)} = \int d^{D+1}x \sqrt{|\mathcal{G}|} \left[\varphi R^{(D+1)} - \frac{\omega}{\varphi} \mathcal{G}^{ab} (\nabla_a \varphi)(\nabla_b \varphi) + 16\pi L_{\text{matt}}^{(D+1)} \right], \tag{37}$$

where φ and ω are the BD scalar field and an adjustable dimensionless parameter (called the BD coupling parameter), respectively⁶

The equations of motion corresponding to the action (37) can be written as

$$G_{ab}^{(D+1)} = \frac{8\pi}{\varphi} T_{ab}^{(D+1)} + \frac{\omega}{\varphi^2} \left[(\nabla_a \varphi)(\nabla_b \varphi) - \frac{1}{2} \mathcal{G}_{ab} (\nabla^c \varphi)(\nabla_c \varphi) \right] + \frac{1}{\varphi} (\nabla_a \nabla_b \varphi - \mathcal{G}_{ab} \nabla^2 \varphi), \tag{38}$$

and

$$\frac{2\omega}{\varphi} \nabla^2 \varphi - \frac{\omega}{\varphi^2} \mathcal{G}^{ab} (\nabla_a \varphi)(\nabla_b \varphi) + R^{(D+1)} = 0. \tag{39}$$

From Equation (38), we easily obtain the Ricci scalar:

$$R^{(D+1)} = -\frac{16\pi T^{(D+1)}}{(D-1)\varphi} + \frac{\omega}{\varphi^2} (\nabla^c \varphi)(\nabla_c \varphi) + \frac{2D}{D-1} \frac{\nabla^2 \varphi}{\varphi}, \tag{40}$$

where $T^{(D+1)} = \mathcal{G}^{ab} T_{ab}^{(D+1)}$.

Substituting (40) into (39) yields⁷

$$\nabla^2 \varphi = \frac{8\pi T^{(D+1)}}{(D-1)\omega + D}. \tag{41}$$

By applying a reduction procedure similar to that in the previous subsection, we can set up the effective equations associated with the D -dimensional hypersurface. For a detailed investigation of such a method in the context of the BD theory, see [8]. We therefore refrain from presenting the details of this approach and confine ourselves to a brief summary of the results. More precisely, considering the metric (2) and using an appropriate reduction procedure, it has been show that Equations (38) and (41) generate four sets of modified field equations on a D -dimensional hypersurface [8]. These field equations are:

1. An equation for the scalar field ψ is:

$$\begin{aligned} \frac{\mathcal{D}^2 \psi}{\psi} = & -\frac{(\mathcal{D}_\alpha \psi)(\mathcal{D}^\alpha \varphi)}{\psi \varphi} - \frac{\epsilon}{2\psi^2} \left(g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{1}{2} g^{*\lambda\beta} g_{\lambda\beta} - \frac{g^{\lambda\beta} g_{\lambda\beta}^* \psi}{\psi} \right) \\ & - \frac{\epsilon}{\psi^2 \varphi} \left[\overset{**}{\varphi} + \overset{*}{\varphi} \left(\frac{\omega \overset{*}{\varphi}}{\varphi} - \frac{\overset{*}{\psi}}{\psi} \right) \right] + \frac{8\pi}{\varphi} \left[\frac{(\omega + 1) T^{(D+1)}}{(D-1)\omega + D} - \frac{\epsilon T_{DD}^{(D+1)}}{\psi^2} \right]. \end{aligned} \tag{42}$$

2. The other effective field equations are the counterpart Equations of (38) and (41):

$$\begin{aligned} G_{\mu\nu}^{(D)} = & \frac{8\pi T_{\mu\nu}^{(D)|\text{eff}}}{\varphi} + \frac{\omega}{\varphi^2} \left[(\mathcal{D}_\mu \varphi)(\mathcal{D}_\nu \varphi) - \frac{1}{2} g_{\mu\nu} (\mathcal{D}_\alpha \varphi)(\mathcal{D}^\alpha \varphi) \right] \\ & + \frac{1}{\varphi} \left(\mathcal{D}_\mu \mathcal{D}_\nu \varphi - g_{\mu\nu} \mathcal{D}^2 \varphi \right) - g_{\mu\nu} \frac{V(\varphi)}{2\varphi}. \end{aligned} \tag{43}$$

In Equation (43), the induced scalar potential $V(\varphi)$ is obtained from a differential equation, see Equation (48); the effective energy-momentum tensor $T_{\mu\nu}^{(D)[\text{eff}]}$ consists of two parts: $T_{\mu\nu}^{(D)[\text{eff}]} \equiv E_{\mu\nu} + T_{\mu\nu}^{[\text{MBDT}]}$ with

$$E_{\mu\nu} \equiv T_{\mu\nu}^{(D+1)} - g_{\mu\nu} \left[\frac{(\omega + 1)T^{(D+1)}}{(D - 1)\omega + D} - \frac{\epsilon T_{DD}^{(D+1)}}{\psi^2} \right], \tag{44}$$

$$\frac{8\pi}{\varphi} T_{\mu\nu}^{[\text{MBDT}]} \equiv T_{\mu\nu}^{[\text{IMT}]} + \frac{1}{\varphi} T_{\mu\nu}^{[\varphi]} + \frac{V(\varphi)}{2\varphi} g_{\mu\nu}. \tag{45}$$

We should note that in Equation (45) $T_{\mu\nu}^{[\text{IMT}]}$ is exactly the same induced matter introduced in the IMT, but $T_{\mu\nu}^{[\varphi]}$ is given by

$$T_{\mu\nu}^{[\varphi]} \equiv \frac{\epsilon \varphi^*}{2\psi^2} \left[g_{\mu\nu}^* + g_{\mu\nu} \left(\frac{\omega \varphi^*}{\varphi} - g^{\alpha\beta} g_{\alpha\beta}^* \right) \right], \tag{46}$$

which depends on the first derivatives of the BD scalar field with respect to l . Concretely, it is another induced energy momentum tensor on the hypersurface arising due to the presence of φ .

3. The wave equation of the MBDT is:

$$\mathcal{D}^2 \varphi = \frac{1}{(D - 2)\omega + (D - 1)} \left[8\pi T^{(D)[\text{eff}]} + \left(\frac{D - 2}{2} \right) \varphi \frac{dV(\varphi)}{d\varphi} - \frac{D}{2} V(\varphi) \right], \tag{47}$$

where $V(\varphi)$ is obtained from⁸

$$\begin{aligned} \varphi \frac{dV(\varphi)}{d\varphi} \equiv & -2(\omega + 1) \left[\frac{(\mathcal{D}_\alpha \psi)(\mathcal{D}^\alpha \varphi)}{\psi} + \frac{\epsilon}{\psi^2} \left(\frac{\varphi^* \psi^*}{\varphi} - \frac{\psi \varphi^*}{\psi} \right) \right] \\ & - \frac{\epsilon \omega \varphi^*}{\psi^2} \left[\frac{\varphi^*}{\varphi} + g^{\mu\nu} g_{\mu\nu}^* \right] + \frac{\epsilon \varphi}{4\psi^2} \left[g^{*\alpha\beta} g_{\alpha\beta} + (g^{\alpha\beta} g_{\alpha\beta}^*)^2 \right] \\ & + 16\pi \left[\frac{(\omega + 1)T^{(D+1)}}{(D - 1)\omega + D} - \frac{\epsilon T_{DD}^{(D+1)}}{\psi^2} \right]. \end{aligned} \tag{48}$$

4. A counterpart to the conservation Equation (9) introduced in the IMT is:

$$G_{\alpha D}^{(D+1)} = \psi \mathcal{D}_\beta P^\beta_\alpha = \frac{8\pi T_{\alpha D}^{(D+1)}}{\varphi} + \frac{\omega \varphi^* (\mathcal{D}_\alpha \varphi)}{\varphi^2} + \frac{\mathcal{D}_\alpha \varphi^*}{\varphi} - \frac{g_{\alpha\lambda}^* (\mathcal{D}^\lambda \varphi)}{2\varphi} - \frac{\varphi^* (\mathcal{D}_\alpha \psi)}{\varphi \psi}. \tag{49}$$

In summary, considering the metric (2) as the background geometry, by applying the reduction procedure introduced in [8], Equations (38) and (41) split into four sets of effective Equations (42), (43), (47) and (49) on a D -dimensional hypersurface. It should be noted that both the induced energy-momentum tensor and the induced scalar potential in the context of the MBDT are obtained from specific equations and therefore have specific types with respect to the phenomenological models. However, we should emphasize that these quantities can be considered as fundamental rather than some ad hoc phenomenological assumptions.

It is seen that Equations (43) and (47) are identical to those of the BD theory obtained from the action

$$\mathcal{S}_{\text{BD}}^{(D)} = \int d^D x \sqrt{-g} \left[\varphi R^{(D)} - \frac{\omega}{\varphi} g^{\alpha\beta} (\mathcal{D}_\alpha \varphi)(\mathcal{D}_\beta \varphi) - V(\varphi) + 16\pi L_{\text{matt}}^{(D)} \right], \tag{50}$$

where specifically $\sqrt{-g} (E_{\mu\nu} + T_{\mu\nu}^{\text{[MBDT]}}) \equiv 2\delta \left(\int d^D x \sqrt{-g} L_{\text{matt}}^{(D)} \right) / \delta g^{\alpha\beta}$ such that $T_{\mu\nu}^{(D)[\text{eff}]} = E_{\mu\nu} + T_{\mu\nu}^{\text{[MBDT]}}$ stands for the Lagrangian associated with the matter in D dimensions.

4. FLRW-MSBT Cosmology

Our main goal in this review paper has been to provide only a very brief overview of the noncompactified KK gravity frameworks and their applications in cosmology. Therefore, in this section, we confine ourselves to a single important cosmological application, namely the FLRW cosmology in one of our modified models. To study other applications, the reader is referred to the related work referenced in this paper, see, e.g., [13,41,43–45] and references therein.

As a cosmological application of the MSBT framework constructed in the previous section, let us present a review of the model investigated in [9]. The spatially flat FLRW universe in a $(D + 1)$ -dimensions was considered in Ref. [9]:

$$dS^2 = -dt^2 + a^2(t) \left[\sum_{i=1}^{D-1} (dx^i)^2 \right] + \epsilon \psi^2(t) dl^2. \tag{51}$$

In Equation (51), t, x^i (where $i = 1, 2, \dots, D - 1$), and $a(t)$ stand for the cosmic time, the Cartesian coordinates, and the scale factor, respectively. Moreover, let us assume there is no higher-dimensional matter, and a, ϕ , and ψ to be depending on the cosmic time only.

Therefore, Equations (14) and (15) for the metric (51) yield

$$\frac{D-2}{2} \left(\frac{\dot{a}}{a} \right)^2 + \left(\frac{\dot{a}}{a} \right) \left(\frac{\dot{\psi}}{\psi} \right) = \frac{\mathcal{W}}{2(D-1)} \left(\phi^{\frac{n}{2}} \dot{\phi} \right)^2, \tag{52}$$

$$\frac{\ddot{a}}{a} + \frac{D-3}{2} \left(\frac{\dot{a}}{a} \right)^2 + \left(\frac{\dot{a}}{a} \right) \left(\frac{\dot{\psi}}{\psi} \right) + \frac{\ddot{\psi}}{\psi} = -\frac{\mathcal{W}}{2(D-2)} \left(\phi^{\frac{n}{2}} \dot{\phi} \right)^2, \tag{53}$$

$$\frac{\ddot{a}}{a} + \frac{D-2}{2} \left(\frac{\dot{a}}{a} \right)^2 = -\frac{\mathcal{W}}{2(D-1)} \left(\phi^{\frac{n}{2}} \dot{\phi} \right)^2, \tag{54}$$

$$\ddot{\phi} + \left[(D-1) \frac{\dot{a}}{a} + \frac{n}{2} \left(\frac{\dot{\phi}}{\phi} \right) + \frac{\dot{\psi}}{\psi} \right] \dot{\phi} = 0, \tag{55}$$

where $\dot{A} \equiv dA/dt$ for any arbitrary variable $A = A(t)$.

Only three of the Equations (52)–(55) are independent, from which we have to determine the three unknowns $a(t), \phi(t)$, and $\psi(t)$.

Let us first focus on a simple case: if the SB scalar field ϕ takes only constant values, we can easily obtain a unique solution as

$$dS^2 = -dt^2 + \left(C_1 t^{\frac{2}{D}} \right)^2 \left[\sum_{i=1}^{D-1} (dx^i)^2 \right] + \epsilon \left(C_2 t^{\frac{2}{D}-1} \right)^2 dl^2, \tag{56}$$

where C_1 and C_2 are constants of integration. It is worth noting that (56) is the only solution in the context of general relativity when an empty universe is described by a $(D + 1)$ -dimensional spatially flat FLRW metric.

Whereas for the general case, employing Equations (52), (53) and (55), two constants of motion are retrieved:

$$a^{D-1} \phi^{\frac{n}{2}} \dot{\phi} \psi = c_1, \tag{57}$$

$$a^{D-1} \dot{\psi} = c_2, \tag{58}$$

where c_1 and c_2 are constants of integration. Then, from using Equations (57) and (58), we can determine ψ as a function of ϕ :

$$\psi(\phi) = \begin{cases} \psi_i \exp\left(\frac{2\beta}{n+2} \phi^{\frac{n+2}{2}}\right), & \text{for } n \neq -2, \\ \psi_i \phi^\beta, & \text{for } n = -2, \end{cases} \tag{59}$$

where $\beta \equiv \frac{c_2}{c_1}$, and ψ_i is an integration constant. We will assume $c_1 \neq 0$ and $a^{D-1} \psi \neq 0$. Moreover, to obtain a as a function of ϕ , we substitute ψ from Equation (59) into Equation (52):

$$a(\phi) = \begin{cases} a_i \exp\left[\frac{2\gamma}{n+2} \phi^{\frac{n+2}{2}}\right] & \text{for } n \neq -2, \\ a_i \phi^\gamma & \text{for } n = -2. \end{cases} \tag{60}$$

In Equation (60), a_i is a constant of integration and

$$\gamma \equiv \frac{1}{D-2} \left[-\beta \pm \sqrt{\beta^2 + \left(\frac{D-2}{D-1}\right) \mathcal{W}} \right], \tag{61}$$

which yields real values provided that

$$\mathcal{W} \geq -\left(\frac{D-1}{D-2}\right) \beta^2. \tag{62}$$

In order to obtain the unknowns in terms of the cosmic time, we substitute ψ and a , respectively, from Equations (59) and (60) into Equation (57). Such a procedure yields

$$\begin{cases} \dot{\phi} \phi^{\frac{n}{2}} \exp\left[\frac{2f}{n+2} \phi^{\frac{n+2}{2}}\right] = \frac{c_1 a_i^{1-D}}{\psi_i}, & \text{for } n \neq -2, \\ \dot{\phi} \phi^f = \frac{c_1 a_i^{1-D}}{\psi_i}, & \text{for } n = -2, \end{cases} \tag{63}$$

where

$$f \equiv (D-1)\gamma + \beta. \tag{64}$$

We should note that both γ and f depend on D . Depending on whether f is zero or not, we will have two different solutions for each of the above differential equations. In what follows, we will analyze them separately.

Case I: $f \equiv (D-1)\gamma + \beta = 0$

For this case, relation (64) yields

$$\gamma = -\frac{\beta}{D-1}, \quad \mathcal{W} = -\left(\frac{D}{D-1}\right) \beta^2. \tag{65}$$

Moreover, from using Equation (63), we obtain

$$\phi(t) = \begin{cases} \left[\frac{(n+2)(1-D)h(t-t_i)}{2\beta} \right]^{\frac{2}{n+2}}, & \text{for } n \neq -2, \\ \exp\left[\frac{(1-D)h(t-t_i)}{\beta} \right], & \text{for } n = -2, \end{cases} \tag{66}$$

where t_i is an integration constant and

$$h \equiv \frac{c_1 \beta a_i^{1-D}}{(1-D)\psi_i}. \tag{67}$$

We should note that h does not depend on the cosmic time, but we emphasize that it is a function of D , i.e., $h = h(D)$. Moreover, substituting the scalar field from (66) into (59) and (60), we obtain the other unknowns in terms of the cosmic time:

$$a(t) = a_i \exp[h(t - t_i)], \quad \forall n \tag{68}$$

$$\psi(t) = \psi_i \exp[(1 - D)h(t - t_i)], \quad \forall n. \tag{69}$$

For Case I, it is seen that among the unknowns, only $\phi(t)$ explicitly depends on n . Substituting $a(t)$ and $\psi(t)$ from Equations (68) and (69) into (51), we eventually obtain

$$dS^2 = -dt^2 + a_i^2 \exp[2(t - t_i)] \sum_{i=1}^{D-1} (dx^i)^2 + \epsilon \psi_i^2 \exp[2(1 - D)h(t - t_i)] dl^2, \quad \forall n. \tag{70}$$

Case II: $f \equiv (D - 1)\gamma + \beta \neq 0$

For this case, from using Equation (61), \mathcal{W} can be expressed as

$$\mathcal{W} = (D - 1)\gamma[2\beta + (D - 2)\gamma]. \tag{71}$$

In order to obtain the SB scalar field in terms of the cosmic time, we integrate both sides of Equation (63) over dt , which yields

$$\phi(t) = \begin{cases} \left\{ \frac{n+2}{2f} \ln [\tilde{h}(t - t_i)] \right\}^{\frac{2}{n+2}}, & \text{for } n \neq -2, \\ [\tilde{h}(t - t_i)]^{\frac{1}{f}}, & \text{for } n = -2, \end{cases} \tag{72}$$

where

$$\tilde{h} \equiv \frac{c_1 f}{a_i^{D-1} \psi_i}. \tag{73}$$

Moreover, by substituting $\phi(t)$ from Equation (72) into Equations (59) and (60), we obtain

$$a(t) = a_i [\tilde{h}(t - t_i)]^r, \quad \forall n \tag{74}$$

$$\psi(t) = \psi_i [\tilde{h}(t - t_i)]^m, \quad \forall n, \tag{75}$$

where

$$r \equiv \frac{\gamma}{f}, \quad m \equiv \frac{\beta}{f}, \quad m + (D - 1)r = 1. \tag{76}$$

Then, we can rewrite relation (71) as

$$\mathcal{W} = (D - 1)f^2r[2m + (D - 2)r]. \tag{77}$$

In what follows, to construct the cosmological solutions on the D -dimensional hypersurface and to analyze the reduced cosmological dynamics, we will use the MSBT framework presented in the previous section.

Employing Equation (29) for metric (51), we can easily obtain the components of the induced energy momentum tensor:

$$\rho_{\text{SB}} \equiv -T_0^{0[\text{MSBT}]} = \frac{1}{\chi} \left[\frac{\ddot{\psi}}{\psi} - \frac{V(\phi)}{2} \right], \tag{78}$$

$$p_{\text{SB}} \equiv T_i^{i[\text{MSBT}]} = -\frac{1}{\chi} \left[\frac{\dot{a}\dot{\psi}}{a\psi} - \frac{V(\phi)}{2} \right], \tag{79}$$

where $i = 1, 2, 3, \dots, D - 1$ (with no sum), and ρ_{SB} and p_{SB} are the induced energy density and pressure, respectively. Moreover, we use Equation (32) to obtain the induced scalar potential:

$$\left. \frac{dV}{d\phi} \right|_{\Sigma_0} = 2\mathcal{W}\phi^n \dot{\phi} \left(\frac{\dot{\psi}}{\psi} \right). \tag{80}$$

In order to eliminate $\dot{\phi}$ and $\dot{\psi}$ in favor of the other variables of the model, we use Equations (57) and (58):

$$\left. \frac{dV}{d\phi} \right|_{\Sigma_0} = 2c_1^2\beta\mathcal{W}a^{2(1-D)}\phi^{\frac{n}{2}}\psi^{-2}. \tag{81}$$

Finally, by substituting a and ψ from Equations (59) and (60) into Equation (81), we obtain

$$\left. \frac{dV}{d\phi} \right|_{\Sigma_0} = \begin{cases} V_0\phi^{-[1+2f]}, & \text{for } n = -2, \\ V_0\phi^{\frac{n}{2}} \exp \left[-\frac{4f}{n+2}\phi^{\frac{n+2}{2}} \right], & \text{for } n \neq -2, \end{cases} \tag{82}$$

where

$$V_0 \equiv 2c_1^2\beta\mathcal{W}a_i^{2(1-D)}\psi_i^{-2}. \tag{83}$$

One sees that the differential Equation (82) depends on the values that f takes on. More precisely, we must continue our discussions for two different cases, I and II, separately. Such a procedure will be carried out later. Here are some important comments.

Equations (27) and (31) corresponding to the D -dimensional spatially flat FLRW metric read

$$\frac{(D - 1)(D - 2)}{2}H^2 = \chi\rho_{\text{SB}} + \rho_\phi \equiv \rho_{\text{tot}}, \tag{84}$$

$$(D - 2)\frac{\ddot{a}}{a} + \frac{(D - 2)(D - 3)}{2}H^2 = -(\chi p_{\text{SB}} + p_\phi) \equiv -p_{\text{tot}}, \tag{85}$$

$$2\phi^n \ddot{\phi} + 2(D - 1)H\phi^n \dot{\phi} + n\phi^{n-1}\dot{\phi}^2 + \frac{1}{\mathcal{W}} \left. \frac{dV}{d\phi} \right|_{\Sigma_0} = 0, \tag{86}$$

where $H \equiv \dot{a}/a$ denotes the Hubble parameter, ρ_{SB} and p_{SB} and $\left. \frac{dV}{d\phi} \right|_{\Sigma_0}$ are given by Equations (78), (79) and (80), respectively. Moreover, ρ_ϕ and p_ϕ stand for the energy density and pressure associated with the SB scalar field ϕ :

$$\rho_\phi \equiv \frac{1}{2} [\mathcal{W}\phi^n \dot{\phi}^2 + V(\phi)], \tag{87}$$

$$p_\phi \equiv \frac{1}{2} [\mathcal{W}\phi^n \dot{\phi}^2 - V(\phi)]. \tag{88}$$

Employing Equations (84) and (85), one can easily show

$$\frac{\ddot{a}}{a} = -\frac{1}{(D-1)(D-2)} [(D-3)\rho_{tot} + (D-1)p_{tot}]. \tag{89}$$

For later usage, let us also introduce the following equation of state (EoS) and density parameters:

$$W_{SB} \equiv \frac{p_{SB}}{\rho_{SB}}, \quad W_\phi \equiv \frac{p_\phi}{\rho_\phi}, \quad W_{tot} \equiv \frac{p_{tot}}{\rho_{tot}} = \frac{\chi p_{SB} + p_\phi}{\chi \rho_{SB} + \rho_\phi}, \tag{90}$$

$$\Omega_{SB} \equiv \frac{2\chi}{(D-1)(D-2)} \frac{\rho_{SB}}{H^2}, \tag{91}$$

$$\Omega_\phi \equiv \frac{2}{(D-1)(D-2)} \frac{\rho_\phi}{H^2}. \tag{92}$$

Employing these parameters, it is seen that Equation (84) can be written as $\Omega_{SB} + \Omega_\phi = 1$.

In the following, we will observe that each equation of (82), due to whether the quantity f vanishes or not, in turn generates two different functions of the scalar field, which are analyzed separately.

4.1. Case (I) D-Dimensional Solutions with $f = (D-1)\gamma + \beta = 0$

For this case, by solving Equation (82), we obtain

$$V(\phi) = \begin{cases} \frac{2V_0}{n+2} \phi^{\frac{n+2}{2}}, & \text{for } n \neq -2, \\ V_0 \ln\left(\frac{\phi}{\phi_i}\right), & \text{for } n = -2. \end{cases} \tag{93}$$

In Equation (93), ϕ_i is an integration constant and

$$V_0 = 2\beta D(1-D)h^2, \tag{94}$$

is obtained from using Equations (65), (67) and (83). Substituting $\phi(t)$ from Equation (66) into (93), we eventually obtain the corresponding induced scalar potential in terms of the cosmic time:

$$V(t) = V_0(1-D)h\beta^{-1}(t-t_i) = 2D(1-D)^2h^3(t-t_i), \quad \forall n, \tag{95}$$

where h is given by Equation (67).

Now, having the scale factors $a(t)$ and $\psi(t)$ from (68) and (69) as well as the scalar potential in terms of the cosmic time, we substitute them into (78) and (79), which yields

$$\chi\rho_{\text{SB}} \equiv -\chi T_0^{0[\text{MSBT}]} = (1 - D)^2 h^2 [-Dh(t - t_i) + 1], \tag{96}$$

$$\chi p_{\text{SB}} \equiv \chi T_i^{i[\text{MSBT}]} = (1 - D)h^2 [D(1 - D)h(t - t_i) - 1]. \tag{97}$$

Employing Equations (68), (96) and (97), it is easy to show that the quantity $\dot{\rho}_{\text{SB}} + (D - 1)H(\rho_{\text{SB}} + p_{\text{SB}})$ vanishes identically. More concretely, the conservation of the induced energy momentum tensor holds in MSBT, which is one of the distinctive features of the framework.

Moreover, substituting $\phi(t)$ and $V(t)$ from relations (66) and (95) into (87) and (88), we obtain

$$\rho_\phi = \frac{1}{2}D(1 - D)h^2 [1 + 2(1 - D)h(t - t_i)], \quad \forall n, \tag{98}$$

$$p_\phi = \frac{1}{2}D(1 - D)h^2 [1 - 2(1 - D)h(t - t_i)], \quad \forall n, \tag{99}$$

from which we obtain

$$W_\phi = \frac{1 - 2(D - 1)h(t - t_i)}{1 + 2(D - 1)h(t - t_i)}, \quad \forall n. \tag{100}$$

In summary, the SB cosmological model in D -dimensional hypersurface derived from the $(D + 1)$ -dimensional solutions associated with Case I are:

$$ds^2 = -dt^2 + a_i^2 \exp \left[2\sqrt{\frac{\Lambda}{D - 1}}(t - t_i) \right] \left[\sum_{i=1}^{D-1} (dx^i)^2 \right], \tag{101}$$

$$\rho_{\text{tot}} = \frac{1}{2}(D - 2)\Lambda, \quad p_{\text{tot}} = -\frac{1}{2}(D - 2)\Lambda, \quad W_{\text{tot}} = -1, \quad \forall n, \tag{102}$$

where

$$\Lambda = \Lambda(D) \equiv \frac{1}{D - 1} \left(\frac{c_1 \beta}{a_i^{D-1} \psi_i} \right)^2 = \text{constant} > 0, \quad \text{for } D > 1. \tag{103}$$

We note that in deriving the total energy density, we have used Equations (84), (85) and (96)–(99).

Relation (103) implies that the quantity $\Lambda(D) > 0$ can be interpreted as a cosmological constant, which, in turn, emerges from combining varying induced matter fields, see relations (96)–(99). Let us look at $\Lambda(D)$ from a different perspective. With Equation (69), the expression in parenthesis of Equation (103) can be replaced by $\dot{\psi}/\psi$, i.e., we obtain

$$\Lambda(D) = \frac{1}{D - 1} \left(\frac{\dot{\psi}}{\psi} \right)^2 = \text{constant}, \quad \forall t, \tag{104}$$

which holds forever. Equation (104) implies that the value of the so-called cosmological constant depends not only on the number of spatial dimensions, but also on the squared expansion rate associated with the extra dimension. Finally, it should be mentioned that the solution belonging to Case I describes an exponentially expanding universe, which is analogous to the usual de Sitter solution.

It is worth making a few more comments about our solution here. Let us consider the canonical metric

$$dS^2 = \frac{l^2}{L^2} \tilde{g}_{\mu\nu}(x^\alpha, l) dx^\mu dx^\nu - dl^2, \tag{105}$$

where $L > 0$ is a constant, and $\tilde{g}_{\mu\nu} = \text{diag}(1, -a^2(t), -a^2(t), \dots, -a^2(t))$. We also assume a special case where $T_{\mu\nu}^{(D+1)} = 0$, and the SB scalar field depends only on t . Therefore, we obtain

$$\chi T_{\mu\nu}^{(D)\text{[eff]}} = \frac{1}{2L^2} (D^2 - 3D + 2 + l_0^2 V_0) \tilde{g}_{\mu\nu}, \tag{106}$$

where $V_0 = \text{constant}$ is obtained from (32), and $l_0 = \text{constant}$ is the value of l on the hypersurface. For this case, from Equation (27) we obtain

$$G_{\mu\nu}^{(D)} = \left(\frac{D^2 - 3D + 2}{2L^2} \right) \tilde{g}_{\mu\nu} + \mathcal{W}\phi^n \left[(\mathcal{D}_\mu\phi)(\mathcal{D}_\nu\phi) - \frac{l_0^2}{2L^2} (\mathcal{D}_\alpha\phi)(\mathcal{D}^\alpha\phi)\tilde{g}_{\mu\nu} \right]. \tag{107}$$

Applying the framework presented in the preceding section, the cosmological solutions corresponding to the metric (105) are

$$a(t) = a_i \exp \left[\frac{(D-1)(t-t_i)}{L} \right], \tag{108}$$

$$\phi(t) = \begin{cases} \left\{ \frac{CL(n+2)}{2(1-D)} \exp \left[\frac{(1-D)(t-t_i)}{L} \right] \right\}^{\frac{2}{n+2}}, & \text{for } n \neq -2, \\ \left\{ \exp \left[\exp \frac{(1-D)(t-t_i)}{L} \right] \right\}^{\frac{CL}{1-b}}, & \text{for } n = -2, \end{cases} \tag{109}$$

where t_i and a_i are integration constants, and $C \equiv a^{D-1} \phi^{\frac{n}{2}} \dot{\phi}$ is a constant of motion. In a special case where $\phi = \text{constant}$, from using Equation (107), we obtain $G_{\mu\nu} = \Lambda(D)\tilde{g}_{\mu\nu}$, where $\Lambda = \Lambda(D) \equiv (D^2 - 3D + 2)/(2L^2)$. To study the latter in four dimensions in more detail, see [59,60] and the references therein.

4.2. Case (II) D-Dimensional Solutions with $f \equiv \beta + (D-1)\gamma \neq 0$

The induced scalar potential assigned to this case is obtained from solving the differential Equation (82):

$$V(\phi) = \begin{cases} -\frac{V_0}{2f} \exp \left[\frac{-4f}{n+2} \phi^{\frac{n+2}{2}} \right], & \text{for } n \neq -2, \\ -\frac{V_0}{2f} \phi^{-2f}, & \text{for } n = -2. \end{cases} \tag{110}$$

Equation (110) can be expressed in terms of t with a single relation for all values of n :

$$V(t) = -\frac{(D-1)mr[1+m-r]}{(t-t_i)^2}, \quad \forall n, \tag{111}$$

where we have used (72).

Substituting $a(t)$, $\psi(t)$ and $V(t)$ from relations (74), (75), and (111) into (78) and (79), we obtain

$$\rho_{\text{SB}} = -\frac{D(D-1)mr^2}{2\chi(t-t_i)^2}, \quad p_{\text{SB}} = -\frac{Dmr(1+m)}{2\chi(t-t_i)^2}, \quad \forall n, \tag{112}$$

which yields a barotropic fluid:

$$p_{\text{SB}} = W_{\text{SB}}\rho_{\text{SB}}, \quad W_{\text{SB}} = \frac{1+m}{(D-1)r}, \quad \forall n. \tag{113}$$

Moreover, ρ_ϕ and p_ϕ are obtained from substituting $\phi(t)$ and $V(t)$ from (72) and (111) into (87) and (88):

$$\mathcal{W}\phi^n \dot{\phi}^2 = \frac{(D-1)r(1+m-r)}{2(t-t_i)^2}, \tag{114}$$

$$\rho_\phi = \frac{\mathcal{W}(1-m)}{2f^2(t-t_i)^2} = \frac{[(D-1)r]^2[2m+(D-2)r]}{2(t-t_i)^2}, \tag{115}$$

$$p_\phi = \frac{\mathcal{W}(1+m)}{2f^2(t-t_i)^2} = \frac{[(D-1)r][2m+(D-1)r][2m+(D-2)r]}{2(t-t_i)^2}, \tag{116}$$

where the first relation was written for later use. Therefore, we obtain

$$W_\phi = \frac{1+m}{1-m} = \frac{(D-1)r+2m}{(D-1)r}, \tag{117}$$

$$\Omega_\phi = \frac{(1-m)(1+m-r)}{(D-2)r} = \frac{(D-1)[(D-2)r+2m]}{(D-2)[(D-1)r+m]}. \tag{118}$$

Now, the total energy density and pressure are obtained easily by substituting ρ_{SB} , p_{SB} , ρ_ϕ , and p_ϕ from relations (112), (115), and (116) into corresponding definitions:

$$\rho_{tot} = \frac{(D-1)(D-2)r^2}{2(t-t_i)^2}, \quad p_{tot} = \frac{(D-2)r(1+m)}{2(t-t_i)^2}, \quad W_{tot} = \frac{1+m}{(D-1)r} \quad \forall n, \tag{119}$$

which implies that the total induced matter is also a barotropic obtained from adding two other barotropic matter fluids. Let us also write another useful equation:

$$\frac{\ddot{a}}{a} = -\frac{r[m+(D-2)r]}{(t-t_i)^2}, \quad \forall n, \tag{120}$$

which has been obtained from substituting ρ_{tot} and p_{tot} from relations (119) into Equation (89).

In what follows, let us analyze the solutions of Case II.

- It is straightforward to show that the corresponding conservation equation is satisfied for our herein three matter fields. Namely,

$$\dot{\rho}_{SB} + 3H(\rho_{SB} + p_{SB}) = 0, \quad \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \quad \dot{\rho}_{tot} + 3H(\rho_{tot} + p_{tot}) = 0. \tag{121}$$

- If we assume that the induced matter plays the role of an ordinary matter in the universe, it is better to check at least the satisfaction of the weak energy condition for it. Let us be more specific. Relations (112) imply that for satisfying $\rho_{SB} \geq 0$, m must take negative values. Moreover, in order to satisfy $\rho_{SB} + p_{SB} = -\frac{Dmr}{\chi}(t-t_i)^2 \geq 0$, r and m must take positive and negative values, respectively. Therefore, if $a(t)$ increases and $\psi(t)$ decreases with cosmic time, then the weak energy condition will be satisfied.
- It is seen that $W_{tot} = W_{SB} = W_\phi$, which can also be fulfilled under approximation conditions with $|p_\phi| \ll \chi|\rho_{SB}|$ and/or $\chi|p_{SB}| \ll |\rho_\phi|$.
- Let us now focus on Equation (120). Before continuing our discussions, we want to mention that we will focus on the solutions with $D > 3$. Moreover, we assume that the constant coefficients $a_i \tilde{h}^r$ and $\psi_i \tilde{h}^m$ that appeared in relations (74) and (75) always take positive values. Furthermore, we do not need to write $\forall n$ in front of the equations since they apply to all n .

In what follows, we present two different cases.

Case IIa: $\gamma < 0, \beta + (D-2)\gamma > 0$

For this case, we obtain

$$-\frac{\beta}{D-2} < \gamma < 0, \quad \beta > 0, \quad \gamma < f, \tag{122}$$

$$2(D-1)\beta\gamma < \mathcal{W} < (D-1)\beta\gamma < 0.$$

Moreover, for this case, we find $-\beta/(D-2) < f < \beta$, namely, f takes positive as well as negative values. However, in order to have an expanding universe, relation (74) yields $r = \gamma/f > 0$. Therefore, from using (74) and (122), we find $f(D) < 0$. We eventually conclude that $f(D)$ must be constrained as

$$-\frac{\beta}{D-2} < f(D) < 0. \tag{123}$$

The above conditions lead to a cosmological model with $a > 0$ and $\ddot{a} > 0$. However, the extra dimension in such a model shrinks with cosmic time, which is favorable within the Kaluza–Klein frameworks [7].

Let us also determine the allowed ranges of the energy density, pressure, and density parameters. According to conditions (122) and (123) together with $f + \beta > 0$ (that is satisfied for $D > 3$), we find

$$\begin{aligned} \rho_{\text{SB}} > 0, \quad p_{\text{SB}} < 0, \quad W_{\text{SB}} < 0, \quad \Omega_{\text{SB}} > 0, \\ \rho_{\phi} < 0, \quad p_{\phi} > 0, \quad W_{\phi} < 0, \quad \Omega_{\phi} < 0, \\ \rho_{\text{tot}} > 0, \quad p_{\text{tot}} < 0, \quad W_{\text{tot}} < 0. \end{aligned} \tag{124}$$

Moreover, in this case, both the potential $V(t)$ and the kinetic term $\mathcal{W}\phi^n\dot{\phi}^2$ take negative values forever.

Case IIb: $\gamma > 0, \beta + (D-2)\gamma < 0$

In this case, we obtain

$$0 < \gamma < -\frac{\beta}{D-2}, \quad \beta < 0, \quad \gamma > f, \tag{125}$$

$$2(D-1)\beta\gamma < \mathcal{W} < (D-1)\beta\gamma < 0.$$

Moreover, we find $\beta < f(D) < -\beta/(D-2)$, which implies that f takes both positive and negative values. Using the similar procedure mentioned above, we eventually obtain:

$$0 < f(D) < -\frac{\beta}{(D-2)}. \tag{126}$$

Admitting the above constraints together with assumption $D > 3$, we can easily show that the inequalities (124) hold among the corresponding physical quantities in this case.

In summary, FLRW-MSBT cosmological solutions associated with this case (Case II where $f \neq 0$), by removing the parameter m , can be written as

$$ds^2 = dS^2|_{\Sigma_y} = -dt^2 + a_i^2[\tilde{h}(t-t_i)]^{2r} \left[\sum_{i=1}^{D-1} (dx^i)^2 \right], \tag{127}$$

$$\rho_{\text{SB}} = -\frac{D(D-1)[1-(D-1)r]r^2}{2\chi(t-t_i)^2}, \quad p_{\text{SB}} = W_{\text{SB}}\rho_{\text{SB}}, \tag{128}$$

$$V(t) = -\frac{(D-1)r(2-Dr)[1-(D-1)r]}{(t-t_i)^2}, \tag{129}$$

where scalar field $\phi(t)$ is given by (72). Moreover, employing relations (76), (77), and (113), W_{SB} and \mathcal{W} are rewritten as

$$W_{SB} = \frac{2}{(D-1)r} - 1, \tag{130}$$

$$\mathcal{W} = \frac{(D-1)r\beta^2(2-Dr)}{[1-(D-1)r]^2}. \tag{131}$$

Concerning the power-law solution, we see that the scale factor accelerates whenever the deceleration parameter $q = -a\ddot{a}/(\dot{a})^2$ takes negative values, which for our model, we need to have $r > 1$. More concretely, all the three EoS parameters should be less than $(3-D)/(D-1)$. (For instance, for $D = 4$, we obtain $W_{SB} (= W_{tot} = W_\phi) < -1/3$.) Let us assume $D > 3$. We therefore find that both ρ_{SB} and ρ_{tot} take positive values, whilst the corresponding pressures take negative values. Consequently, both play the role of dark energy. (For investigations on dark energy models in the context of scalar-tensor theories, see, for instance, [61,62] and related papers.) On the other hand, we can easily show $\rho_{SB} + p_{SB} \geq 0$ and $\rho_{tot} + p_{tot} \geq 0$. The above expressions indicate that the weak energy condition is satisfied for both matters. However, regarding the matter associated with the scalar field, we obtain other properties. Let us be more precise. Assuming $r > 1$, from relation (131), we obtain $\mathcal{W} < 0$. Moreover, Equations (114)–(118) indicate $\mathcal{W}\phi^n\dot{\phi}^2 < 0$, $\rho_\phi < 0$, $p_\phi > 0$, and $\Omega_\phi < 0$, which can be considered as a dark energy [63]. It is worth noting that for this case (i.e., Case II where $f \neq 0$), m takes negative values, which implies that the extra dimension decreases with cosmic time.

Let us see under what conditions we obtain a decelerating scale factor. Assuming that the induced matter is analogous to an ordinary matter with an EoS parameter W constrained as $0 \leq W \leq 1$ (which in particular can be a matter-dominated, radiation-dominated, or stiff fluid with $W = 0, 1/(D-1), 1$ in a D -dimensional hypersurface). From relation (130), we therefore obtain $1/(D-1) \leq r \leq 2/(D-1)$, which is, for $D > 3$, associated with a decelerating universe. Moreover, we obtain $-1 \leq m \leq 0$, i.e., the extra dimension shrinks with cosmic time. It is easy to show that the weak energy condition is satisfied for both the induced and total matters. Whilst, according to (114)–(116), we find that the \mathcal{W} , ρ_ϕ and p_ϕ (the quantities associated with the SB scalar field) in the ranges $2/D < r \leq 2/(D-1)$ and $1/(D-1) \leq r < 2/D$ take negative and positive values, respectively, and they vanish when $r = 2/D$. In addition, note that inequality (62) is satisfied for the allowed ranges that yield the decelerating as well as accelerating scale factor, see [9].

5. Conclusions and Discussion

Since the theories discussed in this paper can be considered as alternative theories to GR, let us take a look at some other theories in this category. Concretely, there are the following methods to alter GR: by changing the matter source, the underlying geometry, the gravitational action functional, or all three, for example, by taking into account scalar field contributions or adding an exotic energy source component to the standard field equations. The latter approach recently attracted much interest and substituted any arbitrary function for the standard Einstein–Hilbert action. This function could be a Ricci scalar R ($f(R)$ gravity), a scalar torsion T ($f(T)$ gravity), a G ($f(G)$ gravity), and $f(R, G)$ gravity), or it could be the inclusion of another matter field Lagrangian along with some geometrical features. Because of the importance of $f(R)$ models in cosmology, the $f(R)$ theory of gravity is regarded as the best modification among the others. It has been hypothesized that cosmic acceleration can be achieved by replacing the Einstein–Hilbert action with a generic Ricci scalar function, $f(R)$. As an extension of $f(R)$ modified theories of gravity, the explicit coupling of any arbitrary function of the Ricci scalar with the matter Lagrangian density has been proposed [64]. Because of the interaction, the massive particles’ motion is nongeodesic, and an extra force orthogonal to the four velocities results. Nojiri and Odintsov [65] analyzed many modified gravity theories that are viewed as gravitational alternatives to dark energy. Several authors [66–77] have examined $f(R)$, $f(G)$, $f(R, T)$, and $f(R, G)$ gravity in various contexts. Shamir [78] provided a theoretically viable $f(R)$ gravity model that illustrates the unification of early-time inflation and late-time acceleration. In addi-

tion, the extension of $f(R, T)$ gravity to KK spacetime has been investigated by several authors [79–83]. One may want to add Einstein–Dilaton–Gauss–Bonnet gravity [84–86].

Induced–Metter Theory is one of several ideas (for example, Weylian geometry, Finsler geometry, and braneworld gravity) for modifying the underlying geometry.

In 1921, Kaluza proposed a unification of electromagnetism and gravity in the context of a four-dimensional hypersurface wrapped in a five-dimensional bulk space. Klein imposed the cylindricity condition and completed the Kaluza theory by proposing a circular topology for the fifth dimension. The extra coordinate has no bearing on the elements of the five-dimensional metric tensor and serves only a formal purpose in the KK theory. To obtain much better performance of the unified theory of gravitation and electromagnetism, Einstein and Mayer thought of Kaluza’s idea from the posture that the spacetime continuum could be a four-dimensional one but possessing vectors (and tensors) with a fifth component. In 1938, Einstein and Bergmann generalized the KK theory. During this work, the condition of cylindricity (resembling the existence of a five-dimensional Killing vector) is replaced by the idea that the space is periodically closed concerning the fifth coordinate.

In recent decades, the KK theory of extra dimensions, in which matter is confined within a lower-dimensional hypersurface, has attracted much attention. Wesson’s induced matter theory, also known as the spacetime matter theory, is based on a revised KK approach. It is worthwhile to highlight that P. S. Wesson and J. Ponce de Leon’s papers are among the original works [55,87,88]. Overduin and Wesson’s work [7] and Wesson’s monograph [19] are particularly significant. These studies demonstrated the possibility of dropping the cylindrical assumption, the revised KK theory’s suitability for addressing cosmological issues, and the presence of an induced electromagnetic field on the brane. One of the most significant achievements of the induced matter theory is the elegant demonstration of the geometric origin of matter. More precisely, Paul Wesson and his colleagues considered our universe as a brane embedded in a five-dimensional bulk space and showed that the latter’s geometry, which is warped but devoid of matter, induces the matter on the brane. The IMT includes papers on dark matter, dark energy [89,90], the induced unified theory of gravity and electromagnetism, successful cosmological models [91], cosmological models with a variable cosmological constant [92,93], and test particles in higher-dimensional models [94–98].

It is worth reviewing the significant results demonstrated in [99] regarding the equivalence between STM and Z_2 -symmetric brane-world theories [100–102]. (i) In both frameworks, matter fields emerge only by assuming the metric as a function of the extra variables, so no matter arises for metrics that do not depend on l . Therefore, brane models incorporating the concept that matter can be viewed as an effect of geometry of bulk are the ultimate goal of STM theory. (ii) It was pointed out that the motion of the test particles has similar properties in both theories. (iii) From a theoretical perspective, both theories employ two opposing approaches to explain the same problem. More concretely, the goal of brane theory is to employ physical information from the brane to reconstruct the generating bulk, whereas in the STM theory, the physics on the hypersurface is constructed from bulk. (iv) From a practical point of view, employing a concrete example, it was demonstrated that solely the interpretation of the effective matter quantities is different in these frameworks [99]. It should also be noted that the main fundamental difference between these frameworks lies in their motivation to introduce an extra dimension.

If, instead of GR, the BD (SB) theory is considered as an underlying framework, then MBDT (MSBT) is established by the same reduction procedure used to construct the STM theory; for a detailed study see [8,9]. The two sets of the field equations associated with the BD (SB) theory in $(D + 1)$ -dimensional spacetime, by employing the reduction procedure, split into four sets on the any D -dimensional hypersurface orthogonal to the extra dimension. One pair of these sets, whose scalar potential and matter fields emerge from the geometry, are the modified version of their corresponding counterparts of the standard BD (SB) framework consisting a scalar potential. Another pair, which has no

equivalent in the mentioned standard scalar–tensor theories, is associated with the scalar field of the extra dimension, ψ and the modified version of the conservation Equation introduced in the STM theory. Inspired by [99], the equivalence between the MBDT (MSBT) and the BD (SB) brane models can also be investigated.

Let us focus on some special cases. (i) As mentioned, the induced scalar potential is one of the fundamental and inseparable sector of the MBDT and MSBT, which vanishes only for a few special cases. For instance, in the MBDT, without loss of generality, it vanishes when $\omega = -1$, $T_{DD}^{(D+1)} = 0$ and l is a cyclic coordinate. Assuming the latter condition and $\psi = \text{constant}$, we obtain a zero scalar potential in the MSBT. (ii) When $L_{\text{matt}}^{(D+1)} = 0$ and the BD and SB scalar fields take constant values, then both the MSBT and MBDT reduce to the corresponding framework constructed in [103]. It is worth noting that the authors of [103] generalized the IMT to arbitrary dimensions. More concretely, they applied the same procedure of IMT to relate a $(D + 1)$ -dimensional vacuum space with a D -dimensional hypersurface with sources and eventually obtained the same effective field equations of IMT. Moreover, they employed this formalism to investigate the relation between lower-dimensional gravity frameworks and the four-dimensional vacuum GR. It has been claimed that the former might be more easily quantized than the usual GR [7,103]. (iii) In both the MBDT and MSBT, when the corresponding scalar fields take constant values and $T_{D\mu}^{(D+1)} = 0$, we obtain $P^{\beta}_{\mu;\beta} = 0$.

In this review paper, we detail only the FLRW-MSBT model among various cosmological models studied in the context of MBDT and MSBT. (For a very brief review of modified gravity and cosmology, see [104].) Regarding the FLRW-MBDT cosmology, let us mention that the accelerated epoch of the matter-dominated universe is not only consistent with the decelerated radiation-dominated epoch, but also with more recent cosmological data [8].

Finally, we should note that the MBDT and MSBT have been applied less as background framework in research. We believe that they have great potential to probe open problems, and we therefore will include them in our future research.

Author Contributions: Conceptualization, S.M.M.R., S.J. and P.M.; Formal analysis, S.M.M.R., S.J. and P.M.; Methodology, S.M.M.R., S.J. and P.M.; Writing—original draft, S.M.M.R., S.J. and P.M.; Writing—review and editing, S.M.M.R., S.J., and P.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We appreciate the anonymous reviewers for their valuable comments. SMMR and PVM acknowledge the FCT grants UID-B-MAT/00212/2020 and UID-P-MAT/00212/2020 at CMA-UBI plus the COST Action CA18108 (Quantum gravity phenomenology in the multi-messenger approach).

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations have been used in this manuscript:

BD	Brans–Dicke
FLRW	Friedmann–Lemaître–Robertson–Walker
GR	General Relativity
IMT	Induced–Matter Theory
KK	Kaluza–Klein
MBDT	Modified Brans–Dicke Theory
MSBT	Modified Sáez–Ballester Theory
SB	Sáez–Ballester
STM	Space–Time–Matter

Notes

- 1 To keep in touch with the original works discussed in each (sub)section, let us apply the same units contained within. For example, in this section we use the same units as in [55].
- 2 It is worth noting that noncompact extra dimensions have also been adopted within compactified KK theory as an approach to incorporate chiral fermions into the theory and to organize a vanishing four-dimensional cosmological constant, see e.g., [57,58]. However, these frameworks have adapted the Klein's mechanism of harmonic expansion, i.e., a finite volume has been assumed for the compact manifold [7].
- 3 It is worth mentioning that in the IMT [55], an apparent vacuum bulk was taken into account. (In this paper, a 'vacuum' spacetime means that 'ordinary matter' does not exist.) As mentioned earlier, however, we will consider a non-vanishing energy momentum tensor, $L_{\text{matt}}^{(D+1)} \neq 0$, to construct an extended version of the corresponding reduced framework. One is free to impose higher-dimensional ordinary matter fields that, in turn, invoke an intricate Kaluza–Klein framework. It should be emphasized that with such a procedure there is a risk that introduces numerous degrees of freedom, so that the corresponding model can no longer be tested. In this respect, we usually assume a vacuum bulk in cosmological applications.
- 4 In the original SB theory, scalar potential was not added to the action [47].
- 5 In [9], the general coordinate free framework has been used to construct the MSBT. However, in this paper, we employ a different particular approach.
- 6 In $(D + 1)$ -dimensions, assuming $\omega > -D/(D - 1)$, it will be possible to go from the Jordan frame to the Einstein frame by conformal transformations.
- 7 We should note that Equations (38)–(40) and (38)–(41) are valid not only for the metric (2) but also for any other more generalized metrics.
- 8 We must emphasize that Equations (45), (47) and (48) (for $D \neq 4$) are the modified version of those introduced in [8].

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