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Solving Linear Tensor Equations II: Including Parity Odd Terms in Four Dimensions

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Abstract: In this paper, focusing on 4-dimensional space, we extend our previous results of solving linear tensor equations. In particular, we consider a 30-parameter linear tensor equation for the unknown tensor component $N_{\alpha\mu\nu}$ in terms of the known component (source) $B_{\alpha\mu\nu}$. The extension also included the parity even linear terms in $N_{\alpha\mu\nu}$ (and the associated traces), which are formed by contracting the latter with the 4-dimensional Levi-Civita pseudotensor. Assuming generic non-degeneracy conditions and following a step-by-step procedure, we show how to explicitly solve for the unknown tensor field component $N_{\alpha\mu\nu}$ and, consequently, derive its unique and exact solution in terms of the component $B_{\alpha\mu\nu}$.

Keywords: tensor equations, parity odd, metric-affine

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1. Introduction

Extending the results we obtained in [1], we consider the most general linear tensor equation in four dimensions, in which all of the possible parity odd (i.e., contractions with the Levi-Civita pseudotensor) combinations of a third-rank tensor[1] are included on top of the even combinations. As we show in the proceeding discussion, this general linear tensor equation involved 30 parameters and consisted of 15 distinct combinations of the unknown third-rank tensor field $N_{\alpha\mu\nu}$. The task is to prescribe a method for solving the aforementioned tensor field in terms of a given (known) component[2]. These linear tensor equations appear in quadratic metric-affine gravity theories [6,7] and their solutions produce a distortion tensor[3] in terms of a hypermomentum tensor. Having solved the distortion, spacetime torsion and non-metricity can then be readily computed.

By solving the distortion in terms of the connection, we can then see exactly how the microscopic properties of matter (encoded in the hypermomentum tensor) produce the non-Riemannian degrees of freedom of torsion and non-metricity. Therefore, it is of great importance to have a prescribed method for solving equations of this kind, since direct physical implications can be drawn from the solutions[4].

Here, we only present a simple application of the results, which could possibly be applied to other branches of physics as well. We start with some basic definitions and, subsequently, state and prove our theorem.

2. Definitions

We now present some basic definitions that are used throughout this paper. We consider a 4-dimensional differentiable manifold endowed with a metric g and an affine connection ∇, namely $(g, \nabla, \mathcal{M})$. The signature of the metric is denoted by s. In addition, $N_{\alpha\mu\nu}$ denotes the component of an arbitrary third-rank tensor field and $\varepsilon^{\alpha\beta\kappa\lambda}$ is the component of the 4-dimensional Levi-Civita pseudotensor.

Definition 1. *We define the first, second and third contractions of* $N_{\alpha\mu\nu}$*, respectively, according to:*

$$N^{(1)}_{\mu} := N_{\alpha\beta\mu}g^{\alpha\beta} \ \ , \ \ N^{(2)}_{\mu} := N_{\alpha\mu\beta}g^{\alpha\beta} \ \ , \ \ N^{(3)}_{\mu} := N_{\mu\alpha\beta}g^{\alpha\beta} \tag{1}$$

Definition 2. *By contracting* $N_{\alpha\mu\nu}$ *with the Levi-Civita pseudotensor, three parity odd combinations are formed, as follows:*

$$M^{(1)}_{\lambda\alpha\beta} := N_{\mu\nu\lambda}\varepsilon^{\mu\nu}{}_{\alpha\beta} \ \ , \ \ M^{(2)}_{\nu\alpha\beta} := N_{\mu\nu\lambda}\varepsilon^{\mu\lambda}{}_{\alpha\beta} \ \ , \ \ M^{(3)}_{\mu\alpha\beta} := N_{\mu\nu\lambda}\varepsilon^{\nu\lambda}{}_{\alpha\beta} \tag{2}$$

Note that by this construction, last pair of indices of each $M^{(i)}$ *value was antisymmetric. In addition, a fourth pseudotrace could be constructed as:*

$$M^{\alpha} = N^{(4)\alpha} := \varepsilon^{\alpha\mu\nu\lambda}N_{\mu\nu\lambda} \tag{3}$$

Note also that further contractions of the $M^{(i)}$ *values did not produce any new traces since* $g^{\mu\nu}M^{(1)}_{\mu\nu\alpha} = -g^{\mu\nu}M^{(2)}_{\mu\nu\alpha} = g^{\mu\nu}M^{(3)}_{\mu\nu\alpha} = -M_{\alpha}$ *and the rest are either identically vanishing or proportional to the latter. Using the above definitions, we now state and prove the main results of this paper.*

3. The Theorem

Theorem 1. *In a 4-dimensional space of signature s, we consider the 30-parameter tensor equation:*

$$\begin{aligned} a_1N_{\alpha\mu\nu} + a_2N_{\nu\alpha\mu} + a_3N_{\mu\nu\alpha} + a_4N_{\alpha\nu\mu} + a_5N_{\nu\mu\alpha} + a_6N_{\mu\alpha\nu} + \sum_{i=1}^{3}\left(a_{7i}N^{(i)}_{\mu}g_{\alpha\nu} + a_{8i}N^{(i)}_{\nu}g_{\alpha\mu} + a_{9i}N^{(i)}_{\alpha}g_{\mu\nu}\right) \\ +b_{11}M^{(1)}_{\alpha\mu\nu} + b_{12}M^{(1)}_{\nu\alpha\mu} + b_{13}M^{(1)}_{\mu\nu\alpha} + b_{21}M^{(2)}_{\alpha\mu\nu} + b_{22}M^{(2)}_{\nu\alpha\mu} + b_{23}M^{(2)}_{\mu\nu\alpha} + b_{31}M^{(3)}_{\alpha\mu\nu} + b_{32}M^{(3)}_{\nu\alpha\mu} + b_{33}M^{(3)}_{\mu\nu\alpha} \\ +\varepsilon_{\rho\alpha\mu\nu}\left(b_1N^{(1)\rho} + b_2N^{(2)\rho} + b_3N^{(3)\rho}\right) + c_1M_{\mu}g_{\alpha\nu} + c_2M_{\nu}g_{\alpha\mu} + c_3M_{\alpha}g_{\mu\nu} = B_{\alpha\mu\nu} \end{aligned} \tag{4}$$

where a_i*,* a_{ji} $i = 1, 2, ..., 6$*,* $j = 7, 8, 9$ *are scalars,* b_{kl}*,* c_m *are pseudoscalars,* $B_{\alpha\mu\nu}$ *is a given (known) tensor and* $N_{\alpha\mu\nu}$ *is the component of the unknown tensor*[5] *N. We define the matrices:*

$$A := \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} & \alpha_{18} & \alpha_{19} & \alpha_{1,10} & \alpha_{1,11} & \alpha_{1,12} & \alpha_{1,13} & \alpha_{1,14} & \alpha_{1,15} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} & \alpha_{27} & \alpha_{28} & \alpha_{29} & \alpha_{2,10} & \alpha_{2,11} & \alpha_{1,12} & \alpha_{2,13} & \alpha_{2,14} & \alpha_{2,15} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} & \alpha_{37} & \alpha_{38} & \alpha_{39} & \alpha_{3,10} & \alpha_{3,11} & \alpha_{1,12} & \alpha_{1,13} & \alpha_{1,14} & \alpha_{1,15} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & \alpha_{47} & \alpha_{48} & \alpha_{49} & \alpha_{4,10} & \alpha_{4,11} & \alpha_{4,12} & \alpha_{4,13} & \alpha_{4,14} & \alpha_{4,15} \\
\alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} & \alpha_{57} & \alpha_{58} & \alpha_{59} & \alpha_{5,10} & \alpha_{5,11} & \alpha_{5,12} & \alpha_{5,13} & \alpha_{5,14} & \alpha_{5,15} \\
\alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} & \alpha_{65} & \alpha_{66} & \alpha_{67} & \alpha_{68} & \alpha_{69} & \alpha_{6,10} & \alpha_{6,11} & \alpha_{6,12} & \alpha_{6,13} & \alpha_{6,14} & \alpha_{6,15} \\
\alpha_{71} & \alpha_{72} & \alpha_{73} & \alpha_{74} & \alpha_{75} & \alpha_{76} & \alpha_{77} & \alpha_{78} & \alpha_{79} & \alpha_{7,10} & \alpha_{7,11} & \alpha_{7,12} & \alpha_{7,13} & \alpha_{7,14} & \alpha_{7,15} \\
\alpha_{81} & \alpha_{82} & \alpha_{83} & \alpha_{84} & \alpha_{85} & \alpha_{86} & \alpha_{87} & \alpha_{88} & \alpha_{89} & \alpha_{8,10} & \alpha_{8,11} & \alpha_{8,12} & \alpha_{8,13} & \alpha_{8,14} & \alpha_{8,15} \\
\alpha_{91} & \alpha_{92} & \alpha_{93} & \alpha_{94} & \alpha_{95} & \alpha_{96} & \alpha_{97} & \alpha_{98} & \alpha_{99} & \alpha_{9,10} & \alpha_{9,11} & \alpha_{9,12} & \alpha_{9,13} & \alpha_{9,14} & \alpha_{9,15} \\
\alpha_{10,1} & \alpha_{10,2} & \alpha_{10,3} & \alpha_{10,4} & \alpha_{10,5} & \alpha_{10,6} & \alpha_{10,7} & \alpha_{10,8} & \alpha_{10,9} & \alpha_{10,10} & \alpha_{10,11} & \alpha_{10,12} & \alpha_{10,13} & \alpha_{10,14} & \alpha_{10,15} \\
\alpha_{11,1} & \alpha_{11,2} & \alpha_{11,3} & \alpha_{11,4} & \alpha_{11,5} & \alpha_{11,6} & \alpha_{11,7} & \alpha_{11,8} & \alpha_{11,9} & \alpha_{11,10} & \alpha_{11,11} & \alpha_{11,12} & \alpha_{11,13} & \alpha_{11,14} & \alpha_{11,15} \\
\alpha_{12,1} & \alpha_{12,2} & \alpha_{12,3} & \alpha_{12,4} & \alpha_{12,5} & \alpha_{12,6} & \alpha_{12,7} & \alpha_{12,8} & \alpha_{12,9} & \alpha_{12,10} & \alpha_{12,11} & \alpha_{12,12} & \alpha_{12,13} & \alpha_{12,14} & \alpha_{12,15} \\
\alpha_{13,1} & \alpha_{13,2} & \alpha_{13,3} & \alpha_{14,4} & \alpha_{14,5} & \alpha_{14,6} & \alpha_{14,7} & \alpha_{14,8} & \alpha_{14,9} & \alpha_{1,10} & \alpha_{1,11} & \alpha_{1,12} & \alpha_{1,13} & \alpha_{1,14} & \alpha_{1,15} \\
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} & \alpha_{18} & \alpha_{19} & \alpha_{1,10} & \alpha_{14,11} & \alpha_{14,12} & \alpha_{14,13} & \alpha_{14,14} & \alpha_{14,15} \\
\alpha_{15,1} & \alpha_{15,2} & \alpha_{15,3} & \alpha_{15,4} & \alpha_{15,5} & \alpha_{15,6} & \alpha_{15,7} & \alpha_{15,8} & \alpha_{15,9} & \alpha_{15,10} & \alpha_{15,11} & \alpha_{15,12} & \alpha_{15,13} & \alpha_{15,14} & \alpha_{15,15}
\end{pmatrix} \tag{5}$$

and

$$\Gamma := \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} \end{pmatrix} \tag{6}$$

where the α_{ij} and γ_{ij} values are linear combinations of the original a_i, a_{ji}, b_j and c_k values (see Appendix B). Then, given that both of these matrices are non-singular, i.e., when

$$det(A) \neq 0 \;\; and \;\; det(\Gamma) \neq 0 \tag{7}$$

holds true, then the general and unique solution of (4) reads:

$$\begin{aligned} N_{\alpha\mu\nu} = {} & \tilde{\alpha}_{11}\hat{B}_{\alpha\mu\nu} + \tilde{\alpha}_{12}\hat{B}_{\nu\alpha\mu} + \tilde{\alpha}_{13}\hat{B}_{\mu\nu\alpha} + \tilde{\alpha}_{14}\hat{B}_{\alpha\nu\mu} + \tilde{\alpha}_{15}\hat{B}_{\nu\mu\alpha} + \tilde{\alpha}_{16}\hat{B}_{\mu\alpha\nu} + \tilde{\alpha}_{17}\check{B}_{\alpha\mu\nu} + \tilde{\alpha}_{18}\check{B}_{\mu\nu\alpha} + \tilde{\alpha}_{19}\check{B}_{\nu\alpha\mu} \\ & + \tilde{\alpha}_{1,10}\bar{B}_{\alpha\mu\nu} + \tilde{\alpha}_{1,11}\bar{B}_{\mu\nu\alpha} + \tilde{\alpha}_{1,12}\bar{B}_{\nu\alpha\mu} + \tilde{\alpha}_{1,13}\mathring{B}_{\alpha\mu\nu} + \tilde{\alpha}_{1,14}\mathring{B}_{\mu\nu\alpha} + \tilde{\alpha}_{1,15}\mathring{B}_{\nu\alpha\mu} \end{aligned} \tag{8}$$

where the $\tilde{\alpha}_{1i}$ values are the first-row elements of the inverse matrix A^{-1} and

$$\hat{B}_{\alpha\mu\nu} = B_{\alpha\mu\nu} - \sum_{i=1}^{4}\sum_{j=1}^{4}\Big(a_{7i}\tilde{\gamma}_{ij}B^{(j)}_{\mu}g_{\alpha\nu} + a_{8i}\tilde{\gamma}_{ij}B^{(j)}_{\nu}g_{\alpha\mu} + a_{9i}\tilde{\gamma}_{ij}B^{(j)}_{\alpha}g_{\mu\nu}\Big) - \varepsilon^{\rho}{}_{\alpha\mu\nu}\sum_{i=1}^{3}\sum_{j=1}^{4}b_i\tilde{\gamma}_{ij}B^{(j)}_{\rho} \tag{9}$$

$$\check{B}_{\alpha\mu\nu} = \varepsilon^{\beta\gamma}{}_{\alpha\mu}\hat{B}_{\beta\gamma\nu} - 2(-1)^s\sum_{j=1}^{4}\Big[(b_{21}+b_{23}+b_{31}+b_{33})\tilde{\gamma}_{1j} + (b_{11}+b_{13}-b_{31}-b_{33})\tilde{\gamma}_{2j} - (b_{11}+b_{13}+b_{21}+b_{23})\tilde{\gamma}_{3j}\Big]B^{(j)}_{[\alpha}g_{\mu]\nu} \tag{10}$$

$$\bar{B}_{\alpha\mu\nu} := \varepsilon^{\beta\gamma}{}_{\alpha\nu}\hat{B}_{\beta\mu\gamma} - 2(-1)^s\sum_{j=1}^{4}\Big[(b_{21}+b_{22}+b_{31}+b_{32})\tilde{\gamma}_{1j} + (b_{11}+b_{12}-b_{31}-b_{32})\tilde{\gamma}_{2j} - (b_{11}+b_{12}+b_{21}+b_{22})\tilde{\gamma}_{3j}\Big]B^{(j)}_{[\alpha}g_{\mu]\nu} \tag{11}$$

$$\mathring{B}_{\alpha\mu\nu} := \varepsilon^{\beta\gamma}{}_{\mu\nu}\mathring{B}_{\alpha\beta\gamma} - 2(-1)^{s+1}\sum_{j=1}^{4}\Big[(b_{22}+b_{23}+b_{32}+b_{33})\tilde{\gamma}_{1j} + (b_{12}+b_{13}-b_{32}-b_{33})\tilde{\gamma}_{2j} - (b_{22}+b_{23}+b_{12}+b_{13})\tilde{\gamma}_{3j}\Big]B^{(j)}_{[\mu}g_{\nu]\alpha} \tag{12}$$

where $\tilde{\gamma}_{ij}$ values are the elements of the inverse matrix Γ^{-1}.

Proof. The first step consists of removing the traces (and pseudotraces) of N and expressing them in terms of the corresponding traces of the known tensor B. To this end, we perform four distinct operations on (4), i.e., we contract the latter with $g^{\alpha\mu}$, $g^{\alpha\nu}$, $g^{\mu\nu}$ and $\varepsilon^{\lambda\alpha\mu\nu}$, which (after some renaming of the indices) produces the system:

$$\gamma_{11}N^{(1)}_{\mu} + \gamma_{12}N^{(2)}_{\mu} + \gamma_{13}N^{(3)}_{\mu} + \gamma_{14}N^{(4)}_{\mu} = B^{(1)}_{\mu} \tag{13}$$

$$\gamma_{21}N^{(1)}_{\mu} + \gamma_{22}N^{(2)}_{\mu} + \gamma_{23}N^{(3)}_{\mu} + \gamma_{24}N^{(4)}_{\mu} = B^{(2)}_{\mu} \tag{14}$$

$$\gamma_{31}N^{(1)}_{\mu} + \gamma_{32}N^{(2)}_{\mu} + \gamma_{33}N^{(3)}_{\mu} + \gamma_{34}N^{(4)}_{\mu} = B^{(3)}_{\mu} \tag{15}$$

$$\gamma_{41}N^{(1)}_{\mu} + \gamma_{42}N^{(2)}_{\mu} + \gamma_{43}N^{(3)}_{\mu} + \gamma_{44}N^{(4)}_{\mu} = B^{(4)}_{\mu} \tag{16}$$

where γ_{ij} values are linear combinations of the initial 27 parameters, the exact form of which we present in the Appendix A. We also used the notation $M_\mu = N^{(4)}_{\mu}$. The above is a system of four equations with four unknowns, which we could express in matrix form as:

$$\Gamma X = Y \tag{17}$$

where we define the columns $X := (N^{(1)}_{\mu}, N^{(2)}_{\mu}, N^{(3)}_{\mu}, N^{(4)}_{\mu})^T$ and $Y := (B^{(1)}_{\mu}, B^{(2)}_{\mu}, B^{(3)}_{\mu}, B^{(4)}_{\mu})^T$ along with the matrix:

$$\Gamma := \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} \end{pmatrix} \tag{18}$$

Now, given that the matrix Γ is not degenerate, i.e., when

$$det(\Gamma) \neq 0 \tag{19}$$

then its inverse Γ^{-1} exists. Then, multiplying (17) with the latter, we obtained:

$$X = \Gamma^{-1} Y \tag{20}$$

which relate the unknown traces $N_\mu^{(i)}$ to the known traces $(B_\mu^{(i)})$. More specifically, the component form of the above matrix equation produces the relationship:

$$N_\mu^{(i)} = \sum_{j=1}^{4} \tilde{\gamma}_{ij} B_\mu^{(j)} \tag{21}$$

where $\tilde{\gamma}_{ij}$ values are the elements of the inverse matrix Γ^{-1}. As a result, we have fully eliminated the traces of N in terms of the traces of B. We could then substitute the last equation back into (4) to obtain:

$$\begin{aligned} &a_1 N_{\alpha\mu\nu} + a_2 N_{\nu\alpha\mu} + a_3 N_{\mu\nu\alpha} + a_4 N_{\alpha\nu\mu} + a_5 N_{\nu\mu\alpha} + a_6 N_{\mu\alpha\nu} \\ &+ b_{11} M^{(1)}_{\alpha\mu\nu} + b_{12} M^{(1)}_{\nu\alpha\mu} + b_{13} M^{(1)}_{\mu\nu\alpha} + b_{21} M^{(2)}_{\alpha\mu\nu} + b_{22} M^{(2)}_{\nu\alpha\mu} + b_{23} M^{(2)}_{\mu\nu\alpha} + b_{31} M^{(3)}_{\alpha\mu\nu} + b_{32} M^{(3)}_{\nu\alpha\mu} + b_{33} M^{(3)}_{\mu\nu\alpha} = \hat{B}_{\alpha\mu\nu} \end{aligned} \tag{22}$$

where

$$\hat{B}_{\alpha\mu\nu} = B_{\alpha\mu\nu} - \sum_{i=1}^{4}\sum_{j=1}^{4} \Big(a_{7i}\tilde{\gamma}_{ij} B_\mu^{(j)} g_{\alpha\nu} + a_{8i}\tilde{\gamma}_{ij} B_\nu^{(j)} g_{\alpha\mu} + a_{9i}\tilde{\gamma}_{ij} B_\alpha^{(j)} g_{\mu\nu} \Big) - \varepsilon^{\rho}{}_{\alpha\mu\nu} \sum_{i=1}^{3}\sum_{j=1}^{4} b_i \tilde{\gamma}_{ij} B_\rho^{(j)} \tag{23}$$

and we renamed $c_1 = a_{74}$, $c_2 = a_{84}$ and $c_3 = a_{94}$ in order to obtain a more compact form. However, this only solved 4 of the 19 unknown combinations as they appear in (4), with 15 more remaining; thus, we need 15 more equations in order to fully solve for N. To this end, we then consider the five possible independent permutations of (22), which (including (22) itself and using shorthand sum notation)[6] read:

$$a_1 N_{\alpha\mu\nu} + a_2 N_{\nu\alpha\mu} + a_3 N_{\mu\nu\alpha} + a_4 N_{\alpha\nu\mu} + a_5 N_{\nu\mu\alpha} + a_6 N_{\mu\alpha\nu} + \sum_{i=1}^{3} \Big(b_{i1} M^{(i)}_{\alpha\mu\nu} + b_{i2} M^{(i)}_{\nu\alpha\mu} + b_{i3} M^{(i)}_{\mu\nu\alpha} \Big) = \hat{B}_{\alpha\mu\nu} \tag{24}$$

$$a_1 N_{\nu\alpha\mu} + a_2 N_{\mu\nu\alpha} + a_3 N_{\alpha\mu\nu} + a_4 N_{\mu\alpha\nu} + a_5 N_{\alpha\nu\mu} + a_6 N_{\nu\mu\alpha} + \sum_{i=1}^{3} \Big(b_{i1} M^{(i)}_{\nu\alpha\mu} + b_{i2} M^{(i)}_{\mu\nu\alpha} + b_{i3} M^{(i)}_{\alpha\mu\nu} \Big) = \hat{B}_{\nu\alpha\mu} \tag{25}$$

$$a_1 N_{\mu\nu\alpha} + a_2 N_{\alpha\mu\nu} + a_3 N_{\nu\alpha\mu} + a_4 N_{\nu\mu\alpha} + a_5 N_{\mu\alpha\nu} + a_6 N_{\alpha\nu\mu} + \sum_{i=1}^{3} \Big(b_{i1} M^{(i)}_{\mu\nu\alpha} + b_{i2} M^{(i)}_{\alpha\mu\nu} + b_{i3} M^{(i)}_{\nu\alpha\mu} \Big) = \hat{B}_{\mu\nu\alpha} \tag{26}$$

$$a_1 N_{\alpha\nu\mu} + a_2 N_{\mu\alpha\nu} + a_3 N_{\nu\mu\alpha} + a_4 N_{\alpha\mu\nu} + a_5 N_{\mu\nu\alpha} + a_6 N_{\nu\alpha\mu} + \sum_{i=1}^{3} \Big(b_{i1} M^{(i)}_{\alpha\nu\mu} + b_{i2} M^{(i)}_{\mu\alpha\nu} + b_{i3} M^{(i)}_{\nu\mu\alpha} \Big) = \hat{B}_{\alpha\nu\mu} \tag{27}$$

$$a_1 N_{\nu\mu\alpha} + a_2 N_{\alpha\nu\mu} + a_3 N_{\mu\alpha\nu} + a_4 N_{\nu\alpha\mu} + a_5 N_{\alpha\mu\nu} + a_6 N_{\mu\nu\alpha} + \sum_{i=1}^{3} \Big(b_{i1} M^{(i)}_{\nu\mu\alpha} + b_{i2} M^{(i)}_{\alpha\nu\mu} + b_{i3} M^{(i)}_{\mu\alpha\nu} \Big) = \hat{B}_{\nu\mu\alpha} \tag{28}$$

$$a_1 N_{\mu\alpha\nu} + a_2 N_{\nu\mu\alpha} + a_3 N_{\alpha\nu\mu} + a_4 N_{\mu\nu\alpha} + a_5 N_{\nu\alpha\mu} + a_6 N_{\alpha\mu\nu} + \sum_{i=1}^{3} \Big(b_{i1} M^{(i)}_{\mu\alpha\nu} + b_{i2} M^{(i)}_{\nu\mu\alpha} + b_{i3} M^{(i)}_{\alpha\nu\mu} \Big) = \hat{B}_{\mu\alpha\nu} \tag{29}$$

In this way, we gather six equations but still have nine more to go. Continuing, we then contract (22) with $\varepsilon^{\alpha\mu}{}_{\beta\gamma}$ and by using the identity $\varepsilon_{\alpha\mu\beta\gamma}\varepsilon^{\kappa\lambda\rho\sigma} = (-1)^s 4! \delta^{\kappa}_{[\alpha}\delta^{\lambda}_{\mu}\delta^{\rho}_{\beta}\delta^{\sigma}_{\gamma]}$, its contractions and some long calculations, we finally arrive at (some useful identities are given in Appendix C):

$$\begin{aligned} &(a_1 - a_6) M^{(1)}_{\nu\beta\gamma} + (a_4 - a_3) M^{(2)}_{\nu\beta\gamma} + (a_2 - a_5) M^{(3)}_{\nu\beta\gamma} + 2(-1)^{s+1}(b_{21} + b_{23} + b_{31} + b_{33}) N^{(1)}_{[\beta} g_{\gamma]\nu} \\ &+ 2(-1)^{s+1}(b_{11} + b_{13} - b_{31} - b_{33}) N^{(2)}_{[\beta} g_{\gamma]\nu} - 2(-1)^{s+1}(b_{11} + b_{13} + b_{21} + b_{23}) N^{(3)}_{[\beta} g_{\gamma]\nu} \\ &+ 2(-1)^s \Big[-(b_{11} + b_{13}) + 2b_{12} \Big] N_{[\beta\gamma]\nu} + 2(-1)^s \Big[-(b_{21} + b_{23}) + 2b_{22} \Big] N_{[\beta|\nu|\gamma]} + 2(-1)^s \Big[-(b_{31} + b_{33}) + 2b_{32} \Big] N_{\nu[\beta\gamma]} \\ &= \varepsilon^{\alpha\mu}{}_{\beta\gamma} \hat{B}_{\alpha\mu\nu} \end{aligned} \tag{30}$$

We then rename the indices $\beta \to \alpha$, $\gamma \to \mu$ and use (21) to remove the traces of N. After some rearranging, we end up with:

$$
\begin{aligned}
&(-1)^s(2b_{12} - b_{11} - b_{13})N_{\alpha\mu\nu} + (-1)^s(2b_{32} - b_{31} - b_{33})N_{\nu\alpha\mu} - (-1)^s(2b_{22} - b_{21} - b_{23})N_{\mu\nu\alpha} \\
&+(-1)^s(2b_{22} - b_{21} - b_{23})N_{\alpha\nu\mu} - (-1)^s(2b_{32} - b_{31} - b_{33})N_{\nu\mu\alpha} - (-1)^s(2b_{12} - b_{11} - b_{13})N_{\mu\alpha\nu} \\
&+(a_1 - a_6)M^{(1)}_{\nu\alpha\mu} + (a_4 - a_3)M^{(2)}_{\nu\alpha\mu} + (a_2 - a_5)M^{(3)}_{\nu\alpha\mu} = \breve{B}_{\alpha\mu\nu}
\end{aligned} \tag{31}
$$

where we set

$$
\breve{B}_{\alpha\mu\nu} = \varepsilon^{\beta\gamma}{}_{\alpha\mu}\hat{B}_{\beta\gamma\nu} - 2(-1)^s\sum_{j=1}^{4}\Big[(b_{21} + b_{23} + b_{31} + b_{33})\tilde{\gamma}_{1j} + (b_{11} + b_{13} - b_{31} - b_{33})\tilde{\gamma}_{2j} - (b_{11} + b_{13} + b_{21} + b_{23})\tilde{\gamma}_{3j}\Big]B^{(j)}_{[\alpha}g_{\mu]\nu} \tag{32}
$$

Next, we consider the index permutation $\alpha \to \mu \to \nu \to \alpha$ in (31), firstly once and then two successive times to obtain two more equations:

$$
\begin{aligned}
&(-1)^s(2b_{12} - b_{11} - b_{13})N_{\mu\nu\alpha} + (-1)^s(2b_{32} - b_{31} - b_{33})N_{\alpha\mu\nu} - (-1)^s(2b_{22} - b_{21} - b_{23})N_{\nu\alpha\mu} \\
&+(-1)^s(2b_{22} - b_{21} - b_{23})N_{\mu\alpha\nu} - (-1)^s(2b_{32} - b_{31} - b_{33})N_{\alpha\nu\mu} - (-1)^s(2b_{12} - b_{11} - b_{13})N_{\nu\mu\alpha} \\
&+(a_1 - a_6)M^{(1)}_{\alpha\mu\nu} + (a_4 - a_3)M^{(2)}_{\alpha\mu\nu} + (a_2 - a_5)M^{(3)}_{\alpha\mu\nu} = \breve{B}_{\mu\nu\alpha}
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
&(-1)^s(2b_{12} - b_{11} - b_{13})N_{\nu\alpha\mu} + (-1)^s(2b_{32} - b_{31} - b_{33})N_{\mu\nu\alpha} - (-1)^s(2b_{22} - b_{21} - b_{23})N_{\alpha\mu\nu} \\
&+(-1)^s(2b_{22} - b_{21} - b_{23})N_{\nu\mu\alpha} - (-1)^s(2b_{32} - b_{31} - b_{33})N_{\mu\alpha\nu} - (-1)^s(2b_{12} - b_{11} - b_{13})N_{\alpha\nu\mu} \\
&+(a_1 - a_6)M^{(1)}_{\mu\nu\alpha} + (a_4 - a_3)M^{(2)}_{\mu\nu\alpha} + (a_2 - a_5)M^{(3)}_{\mu\nu\alpha} = \breve{B}_{\nu\alpha\mu}
\end{aligned} \tag{34}
$$

In the same manner, we contract (22) once with $\varepsilon^{\alpha\nu}{}_{\beta\gamma}$ and another time with $\varepsilon^{\mu\nu}{}_{\beta\gamma}$ and again, after performing one and two successive permutations of the indices for each case, we gather six more equations:

$$
\begin{aligned}
&(-1)^s(b_{21} + b_{22} - 2b_{23})N_{\alpha\mu\nu} - (-1)^s(b_{11} + b_{12} - 2b_{13})N_{\nu\alpha\mu} - (-1)^s(b_{31} + b_{32} - 2b_{33})N_{\mu\nu\alpha} \\
&+(-1)^s(b_{11} + b_{12} - 2b_{13})N_{\alpha\nu\mu} - (-1)^s(b_{21} + b_{22} - 2b_{23})N_{\nu\mu\alpha} + (-1)^s(b_{31} + b_{32} - 2b_{33})N_{\mu\alpha\nu} \\
&+(a_4 - a_2)M^{(1)}_{\mu\alpha\nu} + (a_1 - a_5)M^{(2)}_{\mu\alpha\nu} + (a_6 - a_3)M^{(3)}_{\mu\alpha\nu} = \bar{B}_{\alpha\mu\nu}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&(-1)^s(b_{21} + b_{22} - 2b_{23})N_{\mu\nu\alpha} - (-1)^s(b_{11} + b_{12} - 2b_{13})N_{\alpha\mu\nu} - (-1)^s(b_{31} + b_{32} - 2b_{33})N_{\nu\alpha\mu} \\
&+(-1)^s(b_{11} + b_{12} - 2b_{13})N_{\mu\alpha\nu} - (-1)^s(b_{21} + b_{22} - 2b_{23})N_{\alpha\nu\mu} + (-1)^s(b_{31} + b_{32} - 2b_{33})N_{\nu\mu\alpha} \\
&+(a_4 - a_2)M^{(1)}_{\nu\mu\alpha} + (a_1 - a_5)M^{(2)}_{\nu\mu\alpha} + (a_6 - a_3)M^{(3)}_{\nu\mu\alpha} = \bar{B}_{\mu\nu\alpha}
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&(-1)^s(b_{21} + b_{22} - 2b_{23})N_{\nu\alpha\mu} - (-1)^s(b_{11} + b_{12} - 2b_{13})N_{\mu\nu\alpha} - (-1)^s(b_{31} + b_{32} - 2b_{33})N_{\alpha\mu\nu} \\
&+(-1)^s(b_{11} + b_{12} - 2b_{13})N_{\nu\mu\alpha} - (-1)^s(b_{21} + b_{22} - 2b_{23})N_{\mu\alpha\nu} + (-1)^s(b_{31} + b_{32} - 2b_{33})N_{\alpha\nu\mu} \\
&+(a_4 - a_2)M^{(1)}_{\alpha\nu\mu} + (a_1 - a_5)M^{(2)}_{\alpha\nu\mu} + (a_6 - a_3)M^{(3)}_{\alpha\nu\mu} = \bar{B}_{\nu\alpha\mu}
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&(-1)^s(2b_{31} - b_{32} - b_{33})N_{\alpha\mu\nu} - (-1)^s(2b_{21} - b_{22} - b_{23})N_{\nu\alpha\mu} + (-1)^s(2b_{11} - b_{12} - b_{13})N_{\mu\nu\alpha} \\
&-(-1)^s(2b_{31} - b_{32} - b_{33})N_{\alpha\nu\mu} - (-1)^s(2b_{11} - b_{12} - b_{13})N_{\nu\mu\alpha} + (-1)^s(2b_{21} - b_{22} - b_{23})N_{\mu\alpha\nu} \\
&+(a_3 - a_5)M^{(1)}_{\alpha\mu\nu} + (a_6 - a_2)M^{(2)}_{\alpha\mu\nu} + (a_1 - a_4)M^{(3)}_{\alpha\mu\nu} = \mathring{B}_{\alpha\mu\nu}
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&(-1)^s(2b_{31} - b_{32} - b_{33})N_{\mu\nu\alpha} - (-1)^s(2b_{21} - b_{22} - b_{23})N_{\alpha\mu\nu} + (-1)^s(2b_{11} - b_{12} - b_{13})N_{\nu\alpha\mu} \\
&-(-1)^s(2b_{31} - b_{32} - b_{33})N_{\mu\alpha\nu} - (-1)^s(2b_{11} - b_{12} - b_{13})N_{\alpha\nu\mu} + (-1)^s(2b_{21} - b_{22} - b_{23})N_{\nu\mu\alpha} \\
&+(a_3 - a_5)M^{(1)}_{\mu\nu\alpha} + (a_6 - a_2)M^{(2)}_{\mu\nu\alpha} + (a_1 - a_4)M^{(3)}_{\mu\nu\alpha} = \mathring{B}_{\mu\nu\alpha}
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&(-1)^s(2b_{31}-b_{32}-b_{33})N_{\nu\alpha\mu}-(-1)^s(2b_{21}-b_{22}-b_{23})N_{\mu\nu\alpha}+(-1)^s(2b_{11}-b_{12}-b_{13})N_{\alpha\mu\nu}\\
&-(-1)^s(2b_{31}-b_{32}-b_{33})N_{\nu\mu\alpha}-(-1)^s(2b_{11}-b_{12}-b_{13})N_{\mu\alpha\nu}+(-1)^s(2b_{21}-b_{22}-b_{23})N_{\alpha\nu\mu}\\
&+(a_3-a_5)M^{(1)}_{\nu\alpha\mu}+(a_6-a_2)M^{(2)}_{\nu\alpha\mu}+(a_1-a_4)M^{(3)}_{\nu\alpha\mu}=\mathring{B}_{\nu\alpha\mu}
\end{aligned}
\tag{40}
$$

where we set

$$
\bar{B}_{\alpha\mu\nu}:=\varepsilon^{\beta\gamma}{}_{\alpha\nu}\hat{B}_{\beta\mu\gamma}-2(-1)^s\sum_{j=1}^{4}\Big[(b_{21}+b_{22}+b_{31}+b_{32})\tilde{\gamma}_{1j}+(b_{11}+b_{12}-b_{31}-b_{32})\tilde{\gamma}_{2j}-(b_{11}+b_{12}+b_{21}+b_{22})\tilde{\gamma}_{3j}\Big]B^{(j)}_{[\alpha}g_{\mu]\nu}
\tag{41}
$$

and

$$
\mathring{B}_{\alpha\mu\nu}:=\varepsilon^{\beta\gamma}{}_{\mu\nu}\mathring{B}_{\alpha\beta\gamma}-2(-1)^{s+1}\sum_{j=1}^{4}\Big[(b_{22}+b_{23}+b_{32}+b_{33})\tilde{\gamma}_{1j}+(b_{12}+b_{13}-b_{32}-b_{33})\tilde{\gamma}_{2j}-(b_{22}+b_{23}+b_{12}+b_{13})\tilde{\gamma}_{3j}\Big]B^{(j)}_{[\mu}g_{\nu]\alpha}
\tag{42}
$$

We then place Equations (24)–(29) and (31)–(40) into that exact order and express the system of the above 15 equations in matrix form as:

$$
A\mathcal{N}=\mathcal{B}
\tag{43}
$$

where A is the 15×15 matrix of coefficients (see Appendix B). We also define the columns:

$$
\mathcal{N}=\Big(N_{\alpha\mu\nu},N_{\nu\alpha\mu},N_{\mu\nu\alpha},N_{\alpha\nu\mu},N_{\nu\mu\alpha},N_{\mu\alpha\nu},M^{(1)}_{\alpha\mu\nu},M^{(1)}_{\nu\alpha\mu},M^{(1)}_{\mu\nu\alpha},M^{(2)}_{\alpha\mu\nu},M^{(2)}_{\nu\alpha\mu},M^{(2)}_{\mu\nu\alpha},M^{(3)}_{\alpha\mu\nu},M^{(3)}_{\nu\alpha\mu},M^{(3)}_{\mu\nu\alpha}\Big)^T
\tag{44}
$$

as well as

$$
\mathcal{B}=\Big(\hat{B}_{\alpha\mu\nu},\hat{B}_{\nu\alpha\mu},\hat{B}_{\mu\nu\alpha},\hat{B}_{\alpha\nu\mu},\hat{B}_{\nu\mu\alpha},\hat{B}_{\mu\alpha\nu},\check{B}_{\alpha\mu\nu},\check{B}_{\mu\nu\alpha},\check{B}_{\nu\alpha\mu},\bar{B}_{\alpha\mu\nu},\bar{B}_{\mu\nu\alpha},\bar{B}_{\nu\alpha\mu},\mathring{B}_{\alpha\mu\nu},\mathring{B}_{\mu\nu\alpha},\mathring{B}_{\nu\alpha\mu}\Big)^T
\tag{45}
$$

Then, given that the matrix A is non-singular, namely

$$
det(A)\neq 0
\tag{46}
$$

we can formally multiply the above matrix equation with A^{-1} to obtain:

$$
\mathcal{N}=A^{-1}\mathcal{B}
\tag{47}
$$

Finally, by equating the first elements of the latter column equation, we arrive at the stated result:

$$
\begin{aligned}
N_{\alpha\mu\nu}=\;&\tilde{\alpha}_{11}\hat{B}_{\alpha\mu\nu}+\tilde{\alpha}_{12}\hat{B}_{\nu\alpha\mu}+\tilde{\alpha}_{13}\hat{B}_{\mu\nu\alpha}+\tilde{\alpha}_{14}\hat{B}_{\alpha\nu\mu}+\tilde{\alpha}_{15}\hat{B}_{\nu\mu\alpha}+\tilde{\alpha}_{16}\hat{B}_{\mu\alpha\nu}+\tilde{\alpha}_{17}\check{B}_{\alpha\mu\nu}+\tilde{\alpha}_{18}\check{B}_{\mu\nu\alpha}+\tilde{\alpha}_{19}\check{B}_{\nu\alpha\mu}\\
&+\tilde{\alpha}_{1,10}\bar{B}_{\alpha\mu\nu}+\tilde{\alpha}_{1,11}\bar{B}_{\mu\nu\alpha}+\tilde{\alpha}_{1,12}\bar{B}_{\nu\alpha\mu}+\tilde{\alpha}_{1,13}\mathring{B}_{\alpha\mu\nu}+\tilde{\alpha}_{1,14}\mathring{B}_{\mu\nu\alpha}+\tilde{\alpha}_{1,15}\mathring{B}_{\nu\alpha\mu}
\end{aligned}
\tag{48}
$$

where $\tilde{\alpha}_{ij}$ values are the elements of the inverse matrix A^{-1}. The matrix is the exact and unique solution of (4), provided that the non-degeneracy conditions (19) and (46) were satisfied. □

Evidently, when $B_\alpha=0$ and $det(A)\neq 0$, along with $det(\Gamma)\neq 0$, the unique solution of (4) is always $N_{\alpha\mu\nu}=0$. More precisely, we showed the following.

Corollary 1. *When $B_{\alpha\mu\nu}=0$ and both matrices A and Γ are non-singular, then the unique solution of (4) is $N_{\alpha\mu\nu}=0$.*

4. Conclusions

We considered the most general linear tensor equation of a third-rank tensor N in terms of a given source B in four dimensions. The latter was a 30-parameter tensor equation, as given by (4). By following a step-by-step procedure and given two rather general non-degeneracy conditions among the parameters, we provided the unique and exact solution of the component $N_{\alpha\mu\nu}$ in terms of the known component $B_{\alpha\mu\nu}$, its dualizations and its contractions. The solution is given by the expression (48). An immediate conclusion is that, provided the non-degeneracy conditions hold and in the absence of sources (i.e., when $B_{\alpha\mu\nu} = 0$), $N_{\alpha\mu\nu} = 0$ is the only (unique) solution of the 30-parameter tensor equation. As a final remark, let us note that our results have a natural application in metric-affine theories of gravity (see Appendix D) but could also be applied to other physical situations just as well.

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Appendix A. The γ_{ij} Values

The relationships between the elements of Γ and the 30 initial parameters read:

$$
\begin{gathered}
\gamma_{11} = a_1 + a_3 + a_{71} + 4a_{81} + a_{91} \ , \ \gamma_{12} = a_2 + a_4 + a_{72} + 4a_{82} + a_{92} \ , \ \gamma_{13} = a_5 + a_6 + a_{73} + 4a_{83} + a_{93} \\
\gamma_{21} = a_2 + a_5 + 4a_{71} + a_{81} + a_{91} \ , \ \gamma_{22} = a_1 + a_6 + 4a_{72} + a_{82} + a_{92} \ , \ \gamma_{23} = a_3 + a_4 + 4a_{73} + a_{83} + a_{93} \\
\gamma_{31} = a_5 + a_6 + a_{71} + a_{81} + 4a_{91} \ , \ \gamma_{32} = a_3 + a_4 + a_{72} + a_{82} + 4a_{92} \ , \ \gamma_{31} = a_1 + a_2 + a_{73} + a_{83} + 4a_{93} \\
\gamma_{41} = -2(-1)^s \Big(b_{21} + b_{22} + b_{23} + b_{31} + b_{32} + b_{33} - 3b_1 \Big) \ , \ \gamma_{42} = -2(-1)^s \Big(b_{11} + b_{12} + b_{13} - b_{31} - b_{32} - b_{33} - 3b_2 \Big) \\
\gamma_{43} = 2(-1)^s \Big(b_{11} + b_{12} + b_{13} + b_{21} + b_{22} + b_{23} + 3b_3 \Big) \ , \ \gamma_{44} = a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \\
\gamma_{14} = c_1 + 4c_2 + c_3 - b_{12} - b_{13} - b_{21} - b_{23} - b_{31} - b_{33} \ , \ \gamma_{24} = 4c_1 + c_2 + c_3 + b_{11} + b_{22} - b_{12} - b_{21} - b_{31} - b_{32} \\
\gamma_{34} = c_1 + c_2 + 4c_3 + b_{12} - b_{13} - b_{22} + b_{23} + b_{32} - b_{33}
\end{gathered} \tag{A1}
$$

These were the elements of the 4×4 matrix Γ.

Appendix B. The α_{ij} Values

By placing Equations (24)–(29) and (31)–(40) one after another in that exact order, the coefficients of the combinations of the tensor N in each equation represent the rows of matrix A (with the system of equations written in matrix form $A\mathcal{N} = \mathcal{B}$), i.e., the coefficients in (24) were the first-row elements of A, namely:

$$
\begin{gathered}
\alpha_{11} = a_1 \ , \ \alpha_{12} = a_2 \ , \ \alpha_{13} = a_3 \ , \ \alpha_{14} = a_4 \ , \ \alpha_{15} = a_5 \ , \ \alpha_{16} = a_6 \ , \ \alpha_{17} = b_{11} \ , \ \alpha_{18} = b_{12}, \ \alpha_{19} = b_{13} \\
\alpha_{1,10} = b_{21} \ , \ \alpha_{1,11} = b_{22}, \ \alpha_{1,12} = b_{23} \ , \ \alpha_{1,13} = b_{31} \ , \ \alpha_{1,14} = b_{32} \ , \ \alpha_{1,15} = b_{33}
\end{gathered} \tag{A2}
$$

Of course, the same goes for every other row, with the last one being (the coefficients of the row corresponding to (40)):

$$
\begin{gathered}
\alpha_{15,1} = (-1)^s(2b_{11} - b_{12} - b_{13}) \ , \ \alpha_{15,2} = (-1)^s(2b_{31} - b_{32} - b_{33}) \ , \ \alpha_{15,3} = -(-1)^s(2b_{21} - b_{22} - b_{23}) \\
\alpha_{15,4} = (-1)^s(2b_{21} - b_{22} - b_{23}) \ , \ \alpha_{15,5} = -(-1)^s(2b_{31} - b_{32} - b_{33}) \ , \ \alpha_{15,6} = -(-1)^s(2b_{11} - b_{12} - b_{13}) \\
\alpha_{15,7} = 0 \ , \ \alpha_{15,8} = a_3 - a_5 \ , \ \alpha_{15,9} = 0 \ , \ \alpha_{15,10} = 0 \ , \ \alpha_{15,11} = a_6 - a_2 \ , \ \alpha_{15,12} = 0 \\
\alpha_{15,3} = 0 \ , \ \alpha_{15,14} = a_1 - a_4 \ , \ \alpha_{15,15} = 0
\end{gathered} \tag{A3}
$$

Appendix C. Details on the Derivations

We now provide some additional information regarding the calculations that we used for the proof. Firstly, using the asymmetry of ε and the last pair of indices of the $M^{(i)}_{\alpha\mu\nu}$ values, we trivially find:

$$\varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(i)}_{\alpha\mu\nu} = \varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(i)}_{\mu\nu\alpha}\,, \quad \forall\, i = 1,2,3 \tag{A4}$$

Continuing, we computed:

$$\varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(1)}_{\alpha\mu\nu} = 2(-1)^{s+1}\Big[\Big(N^{(2)}_{[\beta} - N^{(3)}_{[\beta}\Big)g_{\gamma]\nu} + N_{[\beta\gamma]\nu}\Big] \tag{A5}$$

$$\varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(2)}_{\alpha\mu\nu} = 2(-1)^{s+1}\Big[\Big(N^{(1)}_{[\beta} - N^{(3)}_{[\beta}\Big)g_{\gamma]\nu} + N_{[\beta|\nu|\gamma]}\Big] \tag{A6}$$

$$\varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(3)}_{\alpha\mu\nu} = 2(-1)^{s+1}\Big[\Big(N^{(1)}_{[\beta} - N^{(2)}_{[\beta}\Big)g_{\gamma]\nu} + N_{\nu[\beta\gamma]}\Big] \tag{A7}$$

The identity $\varepsilon_{\mu\alpha\beta\gamma}\varepsilon^{\mu\kappa\lambda\rho} = (-1)^s 3!\delta^{\kappa}_{[\alpha}\delta^{\lambda}_{\beta}\delta^{\rho}_{\gamma]}$ was of great use here. In the same manner, we find:

$$\varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(1)}_{\nu\alpha\mu} = 4(-1)^s N_{[\beta\gamma]\nu} \tag{A8}$$

$$\varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(2)}_{\nu\alpha\mu} = 4(-1)^s N_{[\beta|\nu|\gamma]} \tag{A9}$$

$$\varepsilon^{\alpha\mu}{}_{\beta\gamma}M^{(3)}_{\nu\alpha\mu} = 4(-1)^s N_{\nu[\beta\gamma]} \tag{A10}$$

which were useful for proving our theorem.

Appendix D. Application of the Theorem

Here, we illustrate how our theorem can be applied to physics. Probably the most natural place to apply the theorem is in metric-affine gravity theories. In particular, in any given metric-affine theory, the affine-connection plays a fundamental role, along with the metric, and its exact form has to be found in order to study the dynamics of the theory. To appreciate the usefulness of our results, we considered the full quadratic (parity even + parity odd) metric-affine gravity theory, as given by the 17-parameter action:

$$\begin{aligned} S[g,\Gamma,\Phi] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[&R + b_1 S_{\alpha\mu\nu}S^{\alpha\mu\nu} + b_2 S_{\alpha\mu\nu}S^{\mu\nu\alpha} + b_3 S_\mu S^\mu \\ & a_1 Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} + a_2 Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} + a_3 Q_\mu Q^\mu + a_4 q_\mu q^\mu + a_5 Q_\mu q^\mu \\ & + c_1 Q_{\alpha\mu\nu}S^{\alpha\mu\nu} + c_2 Q_\mu S^\mu + c_3 q_\mu S^\mu \\ & + a_6 \varepsilon^{\alpha\beta\gamma\delta}Q_{\alpha\beta\mu}Q_{\gamma\delta}{}^{\mu} + b_5 S_\mu t^\mu + b_6 \varepsilon^{\alpha\beta\gamma\delta}S_{\alpha\beta\mu}S_{\gamma\delta}{}^{\mu} \\ & c_4 Q_\mu t^\mu + c_5 q^\mu t_\mu + c_6 \varepsilon^{\alpha\beta\gamma\delta}Q_{\alpha\beta\mu}S_{\gamma\delta}{}^{\mu}\Big] + S_M[g,\Gamma,\Phi] \end{aligned} \tag{A11}$$

The connection field equations for the above theory read (see [9] for details):

$$\begin{aligned} &\Big(\frac{Q_\lambda}{2} + 2S_\lambda\Big)g^{\mu\nu} - Q_\lambda{}^{\mu\nu} - 2S_\lambda{}^{\mu\nu} + \Big(q^\mu - \frac{Q^\mu}{2} - 2S^\mu\Big)\delta^\nu_\lambda + 4a_1 Q^{\nu\mu}{}_\lambda + 2a_2(Q^{\mu\nu}{}_\lambda + Q_\lambda{}^{\mu\nu}) + 2b_1 S^{\mu\nu}{}_\lambda \\ &+ 2b_2 S_\lambda{}^{[\mu\nu]} + c_1\Big(S^{\nu\mu}{}_\lambda - S_\lambda{}^{\nu\mu} + Q^{[\mu\nu]}{}_\lambda\Big) + \delta^\mu_\lambda\Big(4a_3 Q^\nu + 2a_5 q^\nu + 2c_2 S^\nu\Big) + \delta^\nu_\lambda\Big(a_5 Q^\mu + 2a_4 q^\mu + c_3 S^\mu\Big) \\ &+ g^{\mu\nu}\Big(a_5 Q_\lambda + 2a_4 q_\lambda + c_3 S_\lambda\Big) + \Big(c_2 Q^{[\mu} + c_3 q^{[\mu} + 2b_3 S^{[\mu}\Big)\delta^{\nu]}_\lambda \\ &+ (-2a_6 + c_6)\varepsilon^{\mu\nu\alpha\beta}Q_{\alpha\beta\lambda} + (2b_6 - c_6)\varepsilon^{\mu\nu\alpha\beta}S_{\alpha\beta\lambda} - 2a_6\varepsilon_\lambda{}^{\nu\alpha\beta}Q_{\alpha\beta}{}^{\mu} - c_6\varepsilon_\lambda{}^{\nu\alpha\beta}S_{\alpha\beta}{}^{\mu} \\ &+ \varepsilon^{\alpha\mu\nu}{}_\lambda(b_5 S_\alpha + c_4 Q_\alpha + c_5 q_\alpha) + \Big(\frac{b_5}{2} + c_5\Big)t^\mu\delta^\nu_\lambda + \Big(-\frac{b_5}{2} + 2c_4\Big)t^\nu\delta^\mu_\lambda + c_5 g^{\mu\nu}t_\lambda = \kappa\Delta_\lambda{}^{\mu\nu} \end{aligned} \tag{A12}$$

where $\Delta_{\lambda}{}^{\mu\nu} := \frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}}$ is the hypermomentum source and $Q_{\alpha\mu\nu} = -\nabla_\alpha g_{\mu\nu}$ and $S_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}{}_{[\mu\nu]}$ are the spacetime non-metricity and torsion, respectively. The latter could be expressed in terms of the distortion tensor $N_{\alpha\mu\nu}$[7] through the relationships:

$$Q_{\alpha\mu\nu} := -\nabla_\alpha g_{\mu\nu} \tag{A13}$$

$$S_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}{}_{[\mu\nu]} \tag{A14}$$

Using both of these, we could then express everything on the left-hand side of (A12) in terms of the distortion and its contractions, thereby ending up with:

$$\begin{aligned}
&(4a_1 + b_1 - c_1)N_{\alpha\mu\nu} + \Big(-1 + 2a_2 + \frac{c_1 + b_2}{2}\Big)N_{\nu\alpha\mu} + \Big(-1 + 2a_2 + \frac{c_1 + b_2}{2}\Big)N_{\mu\nu\alpha} \\
&+(2a_2 - b_1 + c_1)N_{\alpha\nu\mu} + \Big(2a_2 - \frac{b_2}{2}\Big)N_{\nu\mu\alpha} + \Big(4a_1 - \frac{b_2}{2} - c_1\Big)N_{\mu\alpha\nu} \\
&+\Big(2a_5 + c_2 - \frac{b_3 + c_3}{2}\Big)g_{\nu\alpha}N^{(1)}_{\mu} + \Big(2a_4 + \frac{b_3}{2} + c_3\Big)g_{\nu\alpha}N^{(2)}_{\mu} + \Big(1 + 2a_4 + \frac{c_3}{2}\Big)g_{\nu\alpha}N^{(3)}_{\mu} \\
&+\Big(8a_3 - 2c_2 + \frac{b_3}{2}\Big)g_{\mu\alpha}N^{(1)}_{\nu} + \Big(2a_5 + c_2 - \frac{c_3 + b_3}{2}\Big)g_{\mu\alpha}N^{(2)}_{\mu} + \Big(2a_5 - \frac{c_3}{2}\Big)g_{\mu\alpha}N^{(3)}_{\nu} \\
&+\Big(2a_5 - \frac{c_3}{2}\Big)g_{\mu\nu}N^{(1)}_{\alpha} + \Big(2a_4 + \frac{c_3}{2} + 1\Big)g_{\mu\nu}N^{(2)}_{\alpha} + 2a_4 g_{\mu\nu}N^{(3)}_{\alpha} \\
&-(-2a_6 + c_6)\varepsilon^{\mu\nu\alpha\beta}N_{\alpha\lambda\beta} - (-2a_6 + c_6)\varepsilon^{\mu\nu\alpha\beta}N_{\lambda\alpha\beta} + (2b_6 - c_6)\varepsilon^{\mu\nu\alpha\beta}N_{\alpha\beta\lambda} \\
&+2a_6\varepsilon_{\lambda}{}^{\nu\alpha\beta}\Big(N_{\alpha}{}^{\mu}{}_{\beta} + N^{\mu}{}_{\alpha\beta}\Big) - c_6\varepsilon_{\lambda}{}^{\nu\alpha\beta}N^{\mu}{}_{\alpha\beta} + \varepsilon^{\alpha\mu\nu}{}_{\lambda}\Big(2c_4 - \frac{b_5}{2}\Big)N^{(1)}_{\alpha} \\
&+\Big(c_5 + \frac{b_5}{2}\Big)\varepsilon^{\alpha\mu\nu}{}_{\lambda}N^{(2)}_{\alpha} + c_5\varepsilon^{\alpha\mu\nu}{}_{\lambda}N^{(3)}_{\alpha} + \Big(\frac{b_5}{2} + c_5\Big)M^{\mu}\delta^{\nu}_{\lambda} + \Big(-\frac{b_5}{2} + 2c_4\Big)M^{\nu}\delta^{\mu}_{\lambda} + c_5 g^{\mu\nu}M_{\lambda} = \kappa\Delta_{\alpha\mu\nu}
\end{aligned} \tag{A15}$$

It could then be seen that the above equation represents a special case of our master equation (4) and, therefore, our general results here could prove to be quite useful for the study of such theories. Using our theorem, we could then go on and write the exact relationship of $N_{\alpha\mu\nu}$ in terms of the hypermomentum source $\Delta_{\alpha\mu\nu}$. However, this goes beyond the scope of this study and interested readers are referred to ([9]) for a detailed analysis of the above quadratic 17-parameter theory.

Notes

1. For applications of third-rank tensors in physics, see [2] and [3]. In addition, a good review of tensor calculus is found in [4].
2. Resorting to a decomposition scheme would not work given the complexity of the 30-parameter tensor equation. In addition to tensors of a rank greater than two, there is no unique irreducible decomposition [5].
3. A distortion tensor is a deviation of the affine connection from the usual Levi-Civita pseudotensor. See, for instance, [7,8].
4. For instance, in the Einstein–Cartan gravity theory, it can be seen that the spin of matter produces spacetime torsion. In this study, our intention was to generalize these results.
5. Of course, the results hold true even when $N_{\alpha\mu\nu}$ is the component of a tensor density instead a connection because $B_{\alpha\mu\nu}$ is also of the same kind.
6. See also [1].
7. A distortion tensor measures the difference between the affine connection and the usual Levi-Civita pseudotensor.

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