

## Article

# Are Quantum-Classical Hybrids Compatible with Ontological Cellular Automata?

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**Abstract:** Based on the concept of ontological states and their dynamical evolution by permutations, as assumed in the Cellular Automaton Interpretation (CAI) of quantum mechanics, we address the issue of whether quantum-classical hybrids can be described consistently in this framework. We consider chains of ‘classical’ two-state Ising spins and their discrete deterministic dynamics as an ontological model with an unitary evolution operator generated by pair-exchange interactions. A simple error mechanism is identified, which turns them into quantum mechanical objects: chains of qubits. Consequently, an interaction between a quantum mechanical and a ‘classical’ chain can be introduced and its consequences for this quantum-classical hybrid can be studied. We found that such hybrid character of composites, generally, does not persist under interactions and, therefore, cannot be upheld consistently, or even as a fundamental notion *à la* Copenhagen interpretation, within CAI.

**Keywords:** quantum-classical hybrid system; cellular automaton; ising spin; qubit; ontological state; Baker–Campbell–Hausdorff formula; quantum mechanics



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## 1. Introduction

Our present aim is to apply recent results illustrating the Cellular Automaton Interpretation of quantum mechanics [1–6] and reconsider the status of Quantum-Classical Hybrids [7], in particular. Can such hybrids exist and consistently be described in a theoretical framework according to this interpretation? We summarise necessary ingredients here, as well as in the following sections, which will ultimately lead to a negative answer to this question.

### 1.1. Quantum-Classical Hybrids—A Reminder

Quantum-Classical Hybrids (QCH) have been of interest since the early days when quantum theory obtained its canonical formulation, as represented in well-known textbooks [8,9].

In the beginning, to separate a composite system into a *classical* part and a *quantum-mechanical* part, as far as their respective states and dynamical behaviour are concerned, may have been mostly of practical interest as an approximation method for otherwise intractable situations [10]—which, to this day, continues to be essential in quantum chemical studies of complex molecules, even biomolecules, and their reactions; similarly, for example, in nuclear reactions, etc.

However, eventually this topic also gained attention from the perspective of foundational issues in quantum theory [11,12]. In particular, numerous and varied attempts to understand and resolve the infamous *measurement problem* of quantum mechanics (QM) have continued to flourish, which we shall not review here.<sup>1</sup>—they all, in one way or another, try to deal with the separation of measurement readings of a classical apparatus, i.e., corresponding to non-superposed pointer states, from the quantum evolution of the object under study. No consensus has been reached as to how to resolve this conundrum *within* quantum theory—yet the Cellular Automaton Interpretation of QM does offer an elegant resolution, to be mentioned in Section 1.2.

Lately, questions of the existence (or not) of QCH have newly moved into focus, since the fundamentally important debate about the either classical or quantum mechanical nature of gravitation is soon to be accompanied by (possibly table-top) high-precision experiments, e.g., [15–19].

Presently, we are interested in considering the consistency or inconsistency of QCH on the basis of the Cellular Automaton Interpretation of QM. In the past, there have been numerous attempts to provide a theoretically satisfactory description of QCH, trying to embed the formalisms of QM and classical mechanics into a common framework, especially along the lines of ref. [20] (with references to earlier work there). However, *interacting* QCH present numerous features where consistency can fail. An almost complete list of essential consistency checks has been presented and passed in ref. [7]. Additional issues, however, have been discussed in refs. [21–24], for example.

### 1.2. Cellular Automaton Interpretation of QM in a Nutshell

According to the Cellular Automaton Interpretation (CAI) of QM, the evolution of the Universe is deterministic and happens in discrete steps [1].

*Ontological States* ( $\mathcal{OS}$ ) are the discrete physical states the Universe can be in. At each step of its evolution, the Universe is in a definite state, which we may denote, for example, by

$$|A\rangle, |B\rangle, |C\rangle, \dots;$$

for simplicity, we assume a finite (or countably infinite) number  $N$  of states.

No *superpositions* of the  $\mathcal{OS}$  exist physically “out there”. It is only by *permutations* that  $\mathcal{OS}$  evolve into  $\mathcal{OS}$ , e.g.,

$$|A\rangle \rightarrow |B\rangle \rightarrow |C\rangle \rightarrow \dots$$

Thus, the set of  $\mathcal{OS}$  forms a *preferred basis*. We may declare this basis to be orthonormal and define an associated *Hilbert space*.

Diagonal operators on the  $\mathcal{OS}$  are *beables* and their eigenvalues correspond to the labels  $A, B, C, \dots$  by which we labelled the ontological states.

Mathematically, by unitary transformations of the preferred basis, we arrive at *Quantum States*  $\mathcal{QS}$  as superpositions of  $\mathcal{OS}$ . These constructs serve to perform physics with the help of mathematical language, according to CAI.

An important property of  $\mathcal{QS}$  follows immediately from the evolution of  $\mathcal{OS}$  by permutations. Referring to the above example, a generic quantum state,

$$|Q\rangle := \alpha|A\rangle + \beta|B\rangle + \dots, \quad |\alpha|^2 + |\beta|^2 + \dots = 1, \quad (1)$$

evolves accordingly:

$$|Q\rangle \rightarrow \alpha|B\rangle + \beta|C\rangle + \dots, \quad (2)$$

where the amplitudes  $\alpha, \beta$ , etc., are conserved. This conservation law has been termed the *conservation of ontology*.

If we choose the amplitudes introduced in the  $\mathcal{QS}$  of Equation (1) to encode probabilities  $|\alpha|^2, |\beta|^2$ , etc. of the  $\mathcal{OS}$   $|A\rangle, |B\rangle$ , etc., respectively, to present the initial state of the evolution step (2), then we have the *Born rule* of QM built right into the formalism. Notably, the *conservation of ontology* under evolution is an essential ingredient.

Completing this sketch of CAI, *Classical States* ( $\mathcal{CS}$ ) are considered as probabilistic distributions of  $\mathcal{OS}$ . Typically, they describe physical macrosystems, which cannot be represented in terms of individual  $\mathcal{OS}$ .

Experiments that “happen” repeatedly—such as apparently repeating evolutions of a sufficiently but always incompletely isolated part of the Universe—pick up different initial conditions regarding  $\mathcal{OS}$ . Therefore, the *classical apparatus* in an experiment must generally be expected to yield different pointer positions as outcomes. Due to the *conservation of ontology*, the probability of a particular outcome directly reflects the probability of having

a particular  $\mathcal{OS}$  as initial condition, since it evolves by permutations of elements of this preferred basis only, cf. (2).

Thus, according to CAI, the apparent *reduction or collapse* to a single pointer position as a measurement result arises due to the intermediary use of *quantum mechanical* (superposition) *states*, when describing what in reality are evolving  $\mathcal{OS}$  that differ in different runs of an experiment.

Summarising, the Cellular Automaton Interpretation suggests building *discrete deterministic dynamical models* with evolution generated by *permutations of ontological states*. The periodicity of trajectories in the space of states, i.e., without fusion or fission, turns them into QM models, as has been demonstrated in several studies before, see, e.g., refs. [1–3,25–28].<sup>2</sup>

We have seen how from rather parsimonious and simple assumptions about the underlying ontology one arrives at essential features of QM. While other aspects will be discussed in the following, we conclude here with a few remarks concerning open problems of CAI that need to be addressed in the future.

First of all, it turns out to be difficult to incorporate *interactions* into previously non-interacting models. This problem, however, cannot be sidestepped if anything like an interacting quantum field theory, especially the paradigmatic Standard Model, is to be explained by an ontological underpinning along the lines of CAI. Furthermore, the relation of the dynamics generated by permutations of ontological states to a *Hamiltonian* that determines the unitary evolution operator as in QM is generally not straightforward to work out. Some glimpses of this will be encountered in what follows.

In Sections 2 and 3, we mainly reformulate results obtained in refs. [4–6] in such a way that our argument against the consistency of quantum-classical hybrids, according to CAI, follows easily, as given in Section 4. Conclusions are presented in Section 5.

## 2. Permutations of Ontological States

Let  $N$  objects,  $A_1, A_2, \dots, A_N$  (“states”), be mapped in  $N$  steps onto one another, involving *all* states. Suitably arranging the sequence of states, this may always be represented by an *unitary*  $N \times N$  matrix with *one* off-diagonal phase per column and row and zero elsewhere:

$$\hat{U}_N := \begin{pmatrix} 0 & . & . & 0 & e^{i\phi_N} \\ e^{i\phi_1} & 0 & . & . & 0 \\ 0 & e^{i\phi_2} & . & . & . \\ . & . & . & . & . \\ 0 & . & 0 & e^{i\phi_{N-1}} & 0 \end{pmatrix}. \quad (3)$$

We associate with this particular representation a basis of vectors called the *standard basis*. It is easy to see that the matrix  $\hat{U}_N$  has the significant property:

$$(\hat{U}_N)^N = e^{i\sum_{k=1}^N \phi_k} \mathbf{1}. \quad (4)$$

This implies that the *Hamiltonian*, defined by the relation  $\hat{U}_N = e^{-i\hat{H}_N T}$ , can be immediately diagonalised to give the diagonal matrix elements ( $n = 1, \dots, N$ ):

$$(\hat{H}_N)_{nn} = \text{diag} \left( \frac{1}{NT} (2\pi(n-1) - \sum_{k=1}^N \phi_k) \right), \quad (5)$$

which refer to the *diagonal basis*. In the following, the arbitrary phase angles  $\phi_k$  play no role, so we set them to zero.

Naturally, the diagonal and standard bases are unitarily related. The transformation between the bases can be obtained explicitly by a discrete Fourier transformation [6]. Using this result, in turn, one evaluates the Hamiltonian for the standard basis, to find the diagonal and off-diagonal matrix elements, respectively:

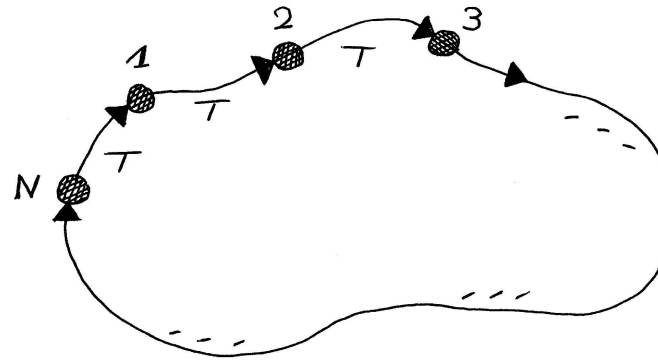
$$(\hat{H}_N)_{nn} = \frac{\pi}{NT} (N-1), \quad n = 1, \dots, N, \quad (6)$$

and,

$$(\hat{H}_N)_{n \neq m} = \frac{\pi}{NT} \left( -1 + i \cot \left( \frac{\pi}{N} (n - m) \right) \right), \quad n, m = 1, \dots, N, \quad (7)$$

and  $\hat{H}_N = \hat{H}_N^\dagger$ .

A *cogwheel model* describes the discrete deterministic dynamics by a unitary permutation matrix  $\hat{U}_N$  [1]. A sketch is shown in Figure 1.



**Figure 1.** The cogwheel model of a system jumping periodically through  $N$  states, with  $T$  denoting the time it takes per jump.

It may come as a surprise, however, in the limits  $N \rightarrow \infty$ ,  $T \rightarrow 0$ , with  $\omega := 1/NT$  fixed, the simple cogwheel model describes the *quantum harmonic oscillator* or systems that can be related to it [1,25,26]. Cellular automata which lead to more general quantum models in the continuum limit are studied in ref. [27].

In the next sections we will apply the dynamics of cogwheel models to study the behaviour of particular Ising spin systems. We choose them as classical discrete deterministic many-body systems evolving by permutations of their states, in line with the picture of ontological cellular automata.

### 3. Ising Spin Chains

Let a one-dimensional spin chain with a periodic boundary condition be composed of  $2S + 1$  classical two-state spins, which represent  $2^{2S}$  OS. We denote the spins and their values by, respectively:

$$s_k = \pm 1 = \uparrow, \downarrow, \quad k = 1, \dots, 2S + 1, \quad s_{2S+1} \equiv s_1, \quad (8)$$

where the first and last spin are identified.

Permutations of the OS of the chain will be generated by *spin exchange* transpositions acting on pairs of spins:

$$\hat{P}_{ij} |s_i, s_j\rangle := |s_j, s_i\rangle, \quad \hat{P}_{2S \ 2S+1} \equiv \hat{P}_{2S \ 1}, \quad (9)$$

with  $\hat{P}_{ij} \equiv \hat{P}_{ji}$ ; from here on we shall often use the *ket* notation familiar from QM to write down states of two or more Ising spins. These transpositions have the properties  $\hat{P}^2 = \mathbf{1}$  and  $[\hat{P}_{ij}, \hat{P}_{jk}] \neq 0$ , for  $i, j, k$  different from each other.

Furthermore, it is a well-known but crucial fact that spin-exchange operators or transpositions can be expressed in terms of a vector  $\hat{\sigma}$  formed by the three Pauli matrices  $\hat{\sigma}^{x,y,z}$ :

$$\hat{P}_{ij} = (\hat{\sigma}_i \cdot \hat{\sigma}_j + \mathbf{1})/2. \quad (10)$$

Finally, we define the *dynamics* of the spin chain by a particular choice of the unitary matrix  $\hat{U}$  incorporating the transpositions that produce permutations of the  $\mathcal{OS}$ :

$$\hat{U} := \prod_{k=1}^S \hat{P}_{2k-1, 2k} \prod_{l=1}^S \hat{P}_{2l, 2l+1} =: \exp(-i\hat{H}T), \quad (11)$$

with the associated Hamiltonian  $\hat{H}$  to be extracted next. This evolution operator respects several conservation laws, among them translation invariance, as discussed in ref. [6].

We note that  $\hat{U}$  distinguishes *even/odd pairs*  $ij$ , according to whether  $i$  of  $\hat{P}_{i<j}$  is even/odd. Following Equation (11), all even pairs are updated first, which is followed by the update of all odd pairs, when  $\hat{U}$  is applied to a given state once. All even pair transpositions  $\hat{P}_{2l, 2l+1}$  commute with each other but not with the odd ones and vice versa. Therefore, the effect of certain transposition is felt by neighbouring pairs only when  $\hat{U}$  is applied a second time. This implements a *final signal velocity* for the propagation of perturbations along the chain.

### 3.1. Extracting the Spin-Exchange Hamiltonian

Consider a generic initial  $\mathcal{OS}$  of a chain of  $2S$  Ising spins,

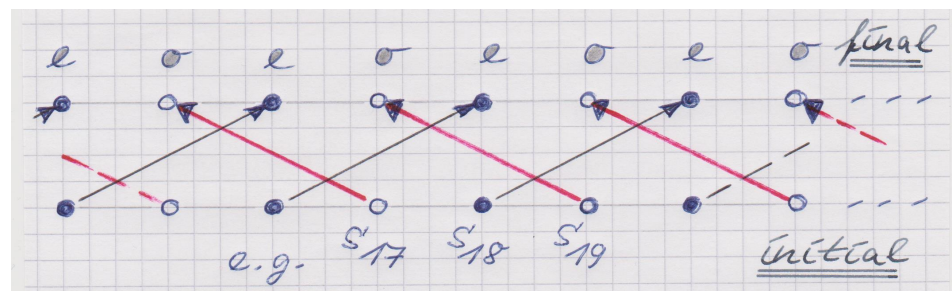
$$|\psi\rangle := |s_1, s_2, s_3, s_4, s_5, \dots, s_{2S-1}, s_{2S}\rangle, \quad (12)$$

where the periodic boundary condition means that  $s_1 \equiv s_{2S+1}$ ; such that the rightmost transposition in  $\hat{U}$  of Equation (11),  $P_{2s, 2s+1}$  can meaningfully be applied to the state.

Applying  $\hat{U}$  once, we observe that the resulting evolution of the state is very simple, which reflects the conservation laws present. This is illustrated in Figure 2. Consecutive updates lead to repetitions of this pattern of *leftmovers* and *rightmovers*.

If we follow the motion of a particular left- or rightmover, it will consecutively pass all odd or even sites, respectively, and continue along the chain periodically. Thus, we find in this model a composition of cogwheels, cf. Section 2, which is produced by nearest-neighbour transpositions in the classical spin chain. Since after  $S$  updates the initial  $\mathcal{OS}$  of the chain is always recovered, the latter itself behaves like an  $S$ -state cogwheel.

In ref. [6], we indicated how this motion can be encoded in two discrete *two-component field equations*. These, in turn, can be mapped one-to-one on  $(1+1)$ -dimensional partial differential equations for bandwidth-limited classical fields with the help of *Sampling Theory*, e.g., similarly to in Ref. [27].



**Figure 2.** One-step update by  $\hat{U}$ , Equation (11), of a piece of an initial spin chain state (**bottom**), cf. Equation (12). The state of a spin on site  $k$ ,  $s_k$ , jumps two sites to the left (**right**) if  $k$  is odd (even), which produces the final state (**top**).

We now briefly outline how the Hamiltonian  $\hat{H}$  is extracted, once we have recognised that any  $\mathcal{OS}$  of the chain evolves like a state in a  $S$ -state cogwheel, however, with an internal structure of  $2S$  Ising spins and nearest-neighbour exchange interactions.

Since the overall behaviour is that of a *S-state cogwheel*, the Hamiltonian must be related to  $\hat{H}_{N=S}$ , cf. Equations (6) and (7). Following simple examples treated explicitly in refs. [4,5], it was shown in ref. [6] how to ‘dress’  $\hat{H}_{N=S}$ , in order to obtain  $\hat{H}$ . Namely,

the latter has to act on the spin variables of a generic state  $|\psi\rangle$ , Equation (12), in one-to-one correspondence to the former. This results in:

$$\hat{H} = \sum_{n=1}^S (\hat{H}_S)_{n1} \hat{U}^{n-1} \quad (13)$$

$$= \frac{\pi}{T} \left( \mathbf{1} + \frac{i}{S} \sum_{n=1}^{S-1} \cot\left(\frac{\pi}{S}n\right) \hat{U}^n \right), \text{ for } \hat{U}|\psi\rangle \neq |\psi\rangle \quad (14)$$

$$= \frac{\pi}{T} \left( \mathbf{1} + \frac{i}{2S} \sum_{n=1}^{S-1} \cot\left(\frac{\pi}{S}n\right) (\hat{U}^n - (\hat{U}^\dagger)^n) \right). \quad (15)$$

Instead, for states with  $\hat{U}|\psi_0\rangle = |\psi_0\rangle$ , one consistently finds that  $\hat{H}|\psi_0\rangle = 0$  for these static *zero modes*— examples of these have all spins either up or down or all leftmovers up (down) and all rightmovers down (up). Taking into account that  $\hat{U}^{S-k} = (\hat{U}^\dagger)^k$ , which follows from  $\hat{U}^k \hat{U}^{S-k} = \mathbf{1} = \hat{U}^k (\hat{U}^\dagger)^k$ , for  $1 \leq k \leq S$ , and the symmetry of the cot-function, the final result is given in a manifestly self-adjoint form.

### 3.2. Remarks

In ref. [6] several comments were made about the interpretation of the Hamiltonian  $\hat{H}$  that acts on  $\mathcal{OS}$  of the Ising spin chain—this concerns physical properties (degeneracy and magnetisation of states, translation invariance), relation to a discrete field theory, and that formally the results summarised in this section yield an interesting, especially terminating *Baker–Campbell–Hausdorff formula*:

$$\hat{U} = \exp \left( -i\pi \left( \mathbf{1} + \frac{i}{S} \sum_{n=1}^{S-1} \cot\left(\frac{\pi}{S}n\right) \hat{U}^n \right) \right), \text{ for } \hat{U}|\psi\rangle \neq |\psi\rangle, \quad (16)$$

with  $\hat{U}$  of Equation (11).

We emphasise that the unitary operator  $\hat{U}$ , Equation (11), with  $\hat{H}$  of Equations (13)–(15), has been constructed to evolve in a discrete, deterministic way stated by the *classical Ising spin model with exchange interactions*. Most importantly, these  $\mathcal{OS}$  evolve by permutations (transpositions) of the spin variables *without ever forming superposition states*, in accordance with CAI (cf. Section 1.2).

However, if numerical constants of the Hamiltonian, especially in Equation (15), are only slightly perturbed, this becomes a genuine *quantum mechanical operator*. Generally, this will produce superposition states of  $\mathcal{OS}$ , which would necessitate a corresponding Hilbert space of  $2S$  2-state quantum spins for its description, i.e., QM of *qubits “by mistake”* [6].— Such perturbations could, for example, be caused by neglected or integrated out *fast high-energy degrees of freedom*, as suggested on other grounds recently [3].

In any case, small errors in  $\hat{H}$ , or  $\hat{U}$ , tend to have a QM effect. For illustration, consider a long chain, with  $S \gg 1$ , and approximate  $\hat{H}$ , Equation (15), roughly by the leading terms:

$$\hat{H} \approx \frac{\pi}{T} \left( \mathbf{1} + \frac{i}{\pi} (\hat{U} - \hat{U}^\dagger) \right). \quad (17)$$

Then, choosing a state,  $|\psi_{\downarrow\downarrow}\rangle := |\dots, \uparrow, \uparrow, \downarrow, \downarrow^e, \downarrow^o, \uparrow, \uparrow, \dots\rangle$ , where the two indicated down spins are located on neighbouring even and odd sites, we obtain by definition of  $\hat{U}$ , Equation (11):

$$\begin{aligned} (\hat{U} - \hat{U}^\dagger) |\psi_{\downarrow\downarrow}\rangle &= |\dots, \uparrow, \uparrow, \downarrow, \uparrow^e, \uparrow^o, \downarrow, \uparrow, \uparrow, \dots\rangle \\ &\quad - |\dots, \uparrow, \downarrow, \uparrow, \uparrow^e, \uparrow^o, \downarrow, \downarrow, \uparrow, \dots\rangle, \end{aligned} \quad (18)$$

and correspondingly  $\hat{H}|\psi_{\downarrow\downarrow}\rangle$ , by Equation (17). This reminds of an *entangled Bell state*; it is not an  $\mathcal{OS}$  but a quantum mechanical superposition state instead.



#### 4. Inconsistency of Quantum-Classical Hybrids in CAI

Next, we employ our observations in order to study the (in)consistency of Quantum-Classical Hybrids (QCH), cf. Section 1.1, when confronted with CAI by way of a simple but representative example.

We consider *two* Ising spin chains for this purpose, one which has become *quantum mechanical* by introducing superpositions of its  $\mathcal{OS}$ , e.g., by a mechanism similar to the one discussed above, and one which is *classical* in the form of a probability distribution of  $\mathcal{OS}$ , cf. Section 1.2. In the simplest case, the latter may be sharply peaked on a single  $\mathcal{OS}$ .

Assuming a temporary interaction, which acts in between two particular updates of both chains, it should be of the ontological kind, i.e., involve only permutations (transpositions) of the Ising spins of the two chains. To be definite, we choose the exchange interaction illustrated in Figure 3.

Let the initial *quantum state* of one of the two chains be given by the superposition:

$$|Q\rangle = \alpha |\dots, a_3, a_4, a_5, a_6, \dots\rangle + \beta |\dots, b_3, b_4, b_5, b_6, \dots\rangle, \quad (19)$$

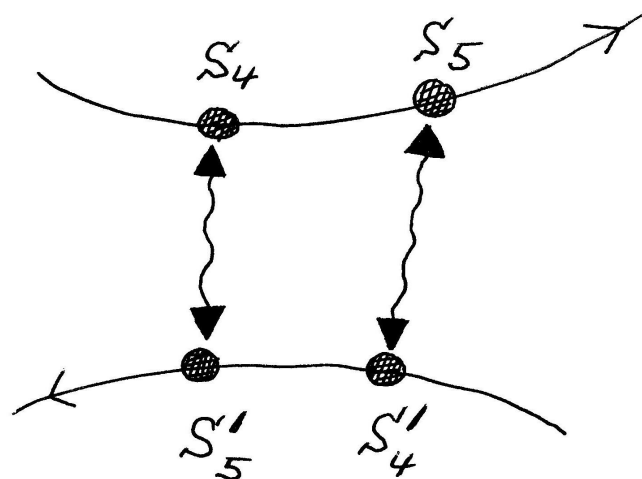
with  $|\alpha|^2 + |\beta|^2 = 1$  and  $a_k, b_k$  denoting specific values of the respective Ising spins. Furthermore, let the initial *classical state* of the second chain be given sharply by an  $\mathcal{OS}$ :

$$|C\rangle = |\dots, s'_3, s'_4, s'_5, s'_6, \dots\rangle. \quad (20)$$

Applying the interaction  $\hat{\mathcal{I}}$  depicted in Figure 3, the QCH consisting of both chains is transformed into:

$$\begin{aligned} \hat{\mathcal{I}}|Q\rangle|C\rangle = & \alpha |\dots, a_3, s'_5, s'_4, a_6, \dots\rangle |\dots, s'_3, a_5, a_4, s'_6, \dots\rangle \\ & + \beta |\dots, b_3, s'_5, s'_4, b_6, \dots\rangle |\dots, s'_3, b_5, b_4, s'_6, \dots\rangle. \end{aligned} \quad (21)$$

Thus, such an interaction is a bilinear map  $\hat{\mathcal{I}} : \{\mathcal{OS}\} \times \{\mathcal{OS}\}' \rightarrow \mathcal{T}$  on the direct product of spaces spanned by the preferred bases of ontological states of both chains. However, due to the linearity, linear superpositions of  $\mathcal{OS}$  from the quantum state of one of the chains, e.g., Equation (19), are transferred by the interaction into the target space  $\mathcal{T}$ , which becomes the *Hilbert space* generated by the tensor product of those bases,  $\mathcal{T} = \{\mathcal{OS}\} \otimes \{\mathcal{OS}\}'$ .



**Figure 3.** The Ising spins at sites 4 and 5 of two chains, with spins labelled by  $s_k$  and  $s'_k$ , respectively, interact by exchanging spins  $s_4$  and  $s'_5$  and spins  $s'_4$  and  $s_5$ . Thus, the states of a pair of *rightmovers* are exchanged with those of a pair of *leftmovers* by this momentum-conserving interaction, cf. Section 3.1.

A curious case arises if both states in the superposition (19) differ only in one (or two)  $a, b$  pair(s) out of  $a_4, a_5, b_4, b_5$ , but are identical otherwise: then, the feature of being in a

quantum superposition state and in a classical state ( $\mathcal{OS}$ ), respectively, is swapped between both chains.

However, generally, the result of Equation (21) presents an *entangled quantum state* involving both chains. In particular, the classical state  $|C\rangle$  of the second chain is changed in such a way that it cannot be factored out any more.

Thus, the interaction changes the character of the composite of two Ising chains from a *quantum-classical hybrid* to an overall *quantum state*. This result follows similarly, if the quantum state in Equation (19) is replaced by a more complicated superposition or the sharp classical state of the second chain is more generally replaced by a probabilistic mixture of  $\mathcal{OS}$  initially, cf. Section 1.2, which can be incorporated in a density matrix.

We conclude that in the framework of CAI quantum-classical hybrids are *not* a consistent construct. In hindsight, this was to be expected, since classical states are of an ontological kind, while quantum states are mathematical constructs of epistemological character, according to the Cellular Automaton Interpretation of QM [1], see also ref. [37].

Perhaps, the long-standing difficulties faced by all attempts to find a satisfactory description of quantum-classical composites in quantum theory, cf. Section 1.1 and see, e.g., refs. [7,21–24] with further references, have indicated already that such attempts are fundamentally flawed. It is interesting that CAI reveals this in a very straightforward way, as we have seen.

## 5. Conclusions

We have reviewed the concept of and interest in Quantum-Classical Hybrids (QCH), i.e., composites of at least one classical and one quantum mechanical part, followed by a sketch of the Cellular Automaton Interpretation (CAI) of quantum mechanics.

We elaborated some aspects of the latter in models of chains consisting of two-state Ising spins, which interact by transpositions of spins; that is, by permutations of the states of a chain. The latter are considered as *ontological states* ( $\mathcal{OS}$ ), existing “out there” in this model Universe. The resulting deterministic dynamics are generated by fine-tuned self-adjoint Hamiltonian operators, much like in QM.

Perturbations of such Hamiltonians generally necessitate an enlarged state space for the ensuing approximate description of the model. It is the Hilbert space that admits superposition states of the  $\mathcal{OS}$  as an *epistemic* mathematical construct. This is where *quantum states* enter.

Based on these observations, the description of QCH can be embedded in CAI, as shown in Section 4. Then, the essential interaction between classical and quantum mechanical components can be introduced, as we have illustrated by a generic example, and its effect on the quantum-classical composite structure be analysed.

This reveals that QCH of interacting quantum mechanical and ontological degrees of freedom are *inconsistent*: Unavoidably, interactions,  $\hat{T} : \{\mathcal{OS}\} \times \{\mathcal{OS}\}' \rightarrow \{\mathcal{OS}\} \otimes \{\mathcal{OS}\}'$ , which bilinearly map  $\mathcal{OS}$  of both chains to tensor product states in a resulting Hilbert space, transfer quantumness in the form of (entangled) superpositions to the assumed classical sector of a QCH. Thus, the hybrid feature is lost through interactions and the concept of QCH loses its meaning within CAI, which is intended to address the hypothesis that the Universe is fundamentally of discrete deterministic character.

Several concluding remarks are in order here:

Since we have demonstrated *inconsistency of quantum-classical hybrids* from the perspective of the Cellular Automaton Interpretation of quantum mechanics, this should, of course, not detract from their practical importance in approximation schemes for otherwise still impossible studies of complex objects, cf. Section 1.1.

It may be obvious that cellular automaton models related to CAI fall into the class of *superdeterministic* dynamical models, which recently have found renewed interest—especially, in order to show that the verdict of Bell’s theorem against realistic local hidden variable theories can be circumvented [1,38–40].



The interaction between two chains that was instrumental for our present argument, of course, has been chosen ad hoc as an example. It is left for future work to construct models along these lines that may approximate physically relevant continuum field theories. It would be very interesting to understand the relation between such *interactions* and *computational gates*.

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**Conflicts of Interest:** The author declare that he has no conflict of interest.

## Notes

- <sup>1</sup> However, for a review and a recent study, see the refs. [13,14].
- <sup>2</sup> We mention that various demonstrations exist of how to arrive at QM with statistical arguments or with stochastic or dissipative modifications of *classical deterministic* dynamics, i.e., without postulating an underlying ontology, such as [29–36].

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