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Weak Gravitation in the 4+1 Formalism

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Abstract: The 4+1 formalism in general relativity (GR) prescribes field equations for the spacetime metric $\gamma_{\mu\nu}(x,\tau)$ which is local in the spacetime coordinates x and evolves according to an external "worldtime" τ. This formalism extends to GR the Stueckelberg Horwitz Piron (SHP) framework, developed to address the various issues known as the problem of time as they appear in electrodynamics. SHP field theories exhibit a formal 5D symmetry on (x,τ) that is strategically broken to 4+1 representations of the Lorentz group, resulting in a manifestly covariant canonical formalism describing the τ-evolution of spacetime structures as an initial value problem. Einstein equations for $\gamma_{\mu\nu}(x,\tau)$ are found by constructing a 5D pseudo-manifold (combining 4D geometry and τ-dynamics) and performing the natural foliation under broken 5D symmetry. This paper discusses weak gravitation in the 4+1 formalism, demonstrating the natural decomposition of the field equations into first-order evolution equations for the unconstrained 4D metric, and the propagation of constraints associated with the Bianchi identity.

Keywords: general relativity; numerical relativity; Stueckelberg Horwitz Piron (SHP) framework

1. Introduction

The "problem of time" most commonly refers to the difficulty of quantizing canonical formulations of general relativity (GR) in which time x^0 is both the parameter of system evolution for initial value problems and a dynamical coordinate of the spacetime to be found as a solution of such problems. The Stueckelberg Horwitz Piron (SHP) framework [1–8] in relativity seeks to overcome these difficulties, as they find expression in classical electrodynamics, introducing a chronological time τ as a physical quantity distinct [9] from coordinate time x^0. Particles and fields, defined locally with respect to some coordinate description of spacetime, evolve under the monotonic advance of τ, a quantity external to the spacetime manifold. A classical event $x^\mu(\tau)$ or a quantum state $\psi(x,\tau)$ is observed at spacetime location x^μ but occurs at chronological time τ, characterizing the temporal ordering of events. Fields and potentials, including the electromagnetic field $f_{\alpha\beta}(x,\tau)$ and metric $g_{\alpha\beta}(x,\tau)$, are made invariant under gauge transformations depending on both x and τ, indices in the first half of the Greek alphabet take values $\alpha,\beta,\gamma,\delta = 0,1,2,3,5$ and the remaining Greek letters take values $\lambda,\mu,\nu = 0,1,2,3$. These field theories will exhibit a formal 5D symmetry containing O(3,1)—possibly O(4,1) or O(3,2)—but matter terms break any higher symmetry to 4+1 representations of Lorentz symmetry.

Building on the insights of SHP electrodynamics, the 4+1 formalism for gravitation [10] was constructed by conflating the 4D geometry of spacetime $\mathcal{M}$ with its evolution under τ to form a 5D pseudo-spacetime [11], writing 5D Einstein equations for the resulting manifold, and strategically breaking the 5D symmetry when setting the Einstein tensor (geometry) equal to the O(3,1) covariant energy–momentum tensor (matter). The decomposition of the symmetry-broken Einstein equations to 4+1 results in first-order differential equations for the metric $\gamma_{\mu\nu}(x,\tau)$ and the extrinsic curvature $K_{\mu\nu}(x,\tau)$ of $\mathcal{M}$. In this paper, we discuss the linearized equations for weak gravitation in the 4+1 formalism, leading to a straightforward derivation of the 4+1 differential equations and offering a directly intuitive interpretation of their meaning.

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Section 2 provides a brief overview of the SHP framework for electrodynamics and gravitation. In section 3 we obtain wave equations for the weak field approximation in SHP and modify the field equations to break the 5D symmetry. Section 4 summarizes the 4+1 formalism for GR obtained by projection onto the spacetime as a 4D hypersurface of a 5D pseudo-spacetime. In section 5 we specify the 4+1 formalism for weak fields and discuss the relationship between the wave equations and the first order evolution equations and their constraints.

2. Overview of the Stueckelberg Horwitz Piron Framework

The standard Feynman-Stueckelberg interpretation of antiparticles as particles traveling backward in time required Stueckelberg to introduce [1,2] two closely related innovations. Clearly the evolution of such particles cannot be parameterized by the time coordinate which is now allowed to reverse direction, and so Stueckelberg introduced an evolution parameter we designate τ. Writing $\dot{x}^\mu = dx^\mu/d\tau$ for indices $\mu, \nu, \lambda = 0,1,2,3$, an event trajectory is observed as a particle when $E = Mc\dot{x}^0 > 0$, or as an antiparticle when $E = Mc\dot{x}^0 < 0$. Thus, a continuous pair creation/annihilation process entails two sign changes of the squared interval

$$c^2\, ds^2(\tau) = -\eta_{\mu\nu} dx^\mu dx^\nu = -\dot{x}^2(\tau)\, d\tau^2 \qquad\qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,. \tag{1}$$

Stueckelberg's interpretation of antiparticles requires that ds^2 be unconstrained, thus demoting the notion of fixed mass shells $mc^2 ds^2(\tau)$ from a priori constraint to a posteriori conserved quantity relevant to τ-independent interactions. Since ds^2 is not positive definite, Stueckelberg concluded that τ cannot be identified with the proper time of the motion.

Horwitz and Piron [3] took a similar approach in their canonical relativistic mechanics for the many-body problem, leading to solutions for relativistic generalizations of the classical central force problems, quantum mechanical potential scattering and bound states [12–17]. Stueckelberg found that classical particle/antiparticle processes require that electromagnetic field $F^{\mu\nu}$ must be supplemented by a vector field strength G^μ. Such a field is also required in accounting for known phenomenology in the radiative transitions of the bound states found by Horwitz and Arshansky [18–20]. Sa'ad, Horwitz, and Arshansky [4] derived the vector interaction from the gauge theory associated with the canonical system. Beginning with the action for a free particle

$$S = \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu \tag{2}$$

which leads to

$$M\ddot{x}^\mu = 0 \qquad\qquad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M\dot{x}_\mu \qquad\qquad -\frac{p^2}{2M} = M\dot{x}^2 = \text{constant} \tag{3}$$

the action is made maximally U(1) gauge invariant (see also [21]) by writing

$$S_{\mathrm{SHP}} = \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c}\dot{x}^\mu a_\mu(x,\tau) + \frac{e}{c} c_5 a_5(x,\tau) \tag{4}$$

$$= \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c}\dot{x}^\beta a_\beta(x,\tau) \tag{5}$$

where $\alpha, \beta, \gamma = 0,1,2,3,5$, and $x^5 = c_5 \tau$ in analogy to $x^0 = ct$. This theory recovers Maxwell electrodynamics [22] when $c_5 \ll c$ and we will generally neglect $(c_5/c)^2$. Notice that for a pure gauge potential $a_\alpha = \partial_\alpha \Lambda(x,\tau)$, the interaction is just a total τ-derivative. The Lorentz force [23] found from this action is

$$M\ddot{x}_\mu = \frac{e}{c}\left(\dot{x}^\nu f_{\mu\nu} + c_5 f_{\mu 5}\right) = \frac{e}{c}\dot{x}^\beta f_{\mu\beta} \tag{6}$$

$$\frac{d}{d\tau}\left(-\frac{1}{2}M\dot{x}^\mu \dot{x}_\mu\right) = c_5\frac{e}{c}\dot{x}^\beta f_{5\beta} \tag{7}$$

with field strength

$$f_{\alpha\beta}(x,\tau) = \partial_\alpha a_\beta - \partial_\beta a_\alpha \ . \tag{8}$$

The mass term in (5) breaks the apparent 5D symmetry of the interaction term $\dot{x}^\beta a_\beta(x,\tau)$ and the SHP action leads to an electrodynamics that differs from a 5D Maxwell theory. We notice that (7) describes mass exchange between particles and fields, determining the condition for non-conservation of proper time. Nevertheless, the total mass, energy, and momentum of particles and fields are conserved [23]. The kinetic term for the field is of the general form

$$S_{\text{field}} = \int d\tau\, d^4x\, f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) \tag{9}$$

in which we raise the five-index of $f_{\alpha\beta}$ suggesting a metric element η^{55}. However, if we view the Lagrangian density as

$$f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) = f^{\mu\nu}(x,\tau) f_{\mu\nu}(x,\tau) + 2\sigma f^\mu_{\ 5}(x,\tau) f_{\mu 5}(x,\tau) \tag{10}$$

then $\eta^{55} = \sigma = \pm 1$ is merely the choice of sign for the vector-vector term. The notation η^{55} is a purely formal convenience, with 5-components denoting O(3,1) scalars, not to be treated as elements of a 5D tensor. Similarly, x^5 is an external parameter and not a timelike coordinate or a dynamical variable; $\dot{x}^5 = c_5$ is a constant scalar. Nevertheless, it is convenient to write

$$\eta_{\alpha\beta} = \text{diag}(-1,1,1,1,\sigma) \tag{11}$$

in the form of a 5D flat space metric.

In this framework $x^\mu(\tau)$ is an irreversible event, occurring at time τ with spacetime coordinates x^μ, and we denote by $\mathcal{M}(\tau)$ the 4D block universe consisting of all spacetime events occurring at τ. The evolution of these events is generated by a scalar Hamiltonian K, so that $\mathcal{M}(\tau)$ occurring at τ evolves to an infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$ occurring at $\tau + d\tau$. This permits the configuration of spacetime, including the past and future of $x^0 = ct$, to change infinitesimally from moment to moment in τ, and so the metric $\gamma_{\mu\nu}(x,\tau)$ of $\mathcal{M}(\tau)$ must evolve with τ. A τ-independent metric would play the role of an absolute background field in this framework, inconsistent with the goals of general relativity.

To find field equations for $\gamma_{\mu\nu}(x,\tau)$ we extend the methods of SHP electrodynamics and generalize the 3+1 formalism as applied in Arnowitt Deser Misner (ADM) geometrodynamics [24]. That is, we interpret the electrodynamic action (5) as exhibiting a symmetry breaking in the matter term

$$S_{\text{5D}} = \int d\tau\, \frac{1}{2}M\dot{x}^\beta \dot{x}_\beta + \frac{e}{c}\dot{x}^\beta a_\beta \quad \longrightarrow \quad S_{\text{SHP}} = \int d\tau\, \frac{1}{2}M\dot{x}^\mu \dot{x}_\mu + \frac{e}{c}\dot{x}^\beta a_\beta(x,\tau) \tag{12}$$

and approach the metric in a similar way, by posing 5D Einstein equations whose energy/matter terms, when joined to the geometrical field terms, break the higher symmetry to 4+1 representations of the Lorentz group. The metric $g_{\alpha\beta}(x,\tau)$ determines the squared interval

$$dX^2 = g_{\alpha\beta}(x,\tau) dX^\alpha dX^\beta \qquad\qquad dX = (x_2,\tau_2) - (x_1,\tau_1) \tag{13}$$

in a pseudo-spacetime $\mathcal{M}_5 = \mathcal{M} \times R$ formed by embedding the 4D hypersurfaces $\mathcal{M}(\tau)$. Unlike a 5D spacetime with symmetry O(4,1) or O(3,2), the symmetries of $\mathcal{M}_5$ can be seen by taking

$$\delta X = (x_2(\tau_1 + \delta\tau) - x_1(\tau_1), \delta\tau) \approx (\delta x(\tau_1) + \dot{x}_1 \delta\tau, \delta\tau) \tag{14}$$

representing the 4D spacetime geometry of $\delta x^\mu(\tau) \in \mathcal{M}(\tau)$ and the canonical evolution between the points $X_1 = (x_1, \tau_1) \in \mathcal{M}(\tau_1)$ and $X_2 = (x_2, \tau_2) \in \mathcal{M}(\tau_2)$. After posing 5D Einstein equations on $\mathcal{M}_5$ and breaking the 5D symmetry to O(3,1), the natural foliation (see also [25,26]) recovers the embedded spacetime hypersurfaces $\mathcal{M}(\tau)$, decomposing the field equations into a τ-evolution problem for the spacetime metric $\gamma_{\mu\nu}(x,\tau)$ and intrinsic curvature $K_{\mu\nu}(x,\tau)$.

Direct application of the Euler–Lagrange equations to the free particle Lagrangian

$$L = \frac{1}{2} M g_{\alpha\beta}(x,\tau) \dot{x}^\alpha \dot{x}^\beta \tag{15}$$

leads to equations of motion

$$0 = \frac{D\dot{x}^\mu}{D\tau} = \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \qquad\qquad 0 = \frac{D\dot{x}^5}{D\tau} = \ddot{x}^5 + \Gamma^5_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{16}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the 5D Christoffel symbol found from $g_{\alpha\beta}(x,\tau)$. However, because x^5 is not a dynamical variable in the SHP framework, the equation on the right must be replaced by $x^5(\tau) \equiv c_5\tau \longrightarrow \ddot{x}^5 \equiv 0$, a first example of breaking 5D symmetry to 4+1. In τ-equilibrium, where $\gamma_{\mu\nu}(x)$ becomes τ-independent and $g_{\alpha 5} = 0$, (16) reduces to

$$0 = \frac{D\dot{x}^\mu}{D\tau} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\rho} \dot{x}^\lambda \dot{x}^\rho \tag{17}$$

which has been studied extensively by Horwitz [7,8] and will not be discussed here.

For simplicity, we treat matter as a non-thermodynamic (zero-pressure) dust of events evolving geodesically under (16). Denoting by $n(x,\tau)$ the number of events per spacetime volume, the five-component event current is

$$j^\alpha(x,\tau) = \rho(x,\tau)\dot{x}^\alpha(\tau) = Mn(x,\tau)\dot{x}^\alpha(\tau) \tag{18}$$

the mass–energy–momentum tensor [4,27] is

$$T^{\alpha\beta} = Mn\dot{x}^\alpha \dot{x}^\beta = \rho \dot{x}^\alpha \dot{x}^\beta \longrightarrow \begin{cases} T^{\mu\nu} = Mn\dot{x}^\mu \dot{x}^\nu = \rho \dot{x}^\mu \dot{x}^\nu \\ T^{5\beta} = \dot{x}^5 \dot{x}^\beta \rho = c_5 j^\beta \,. \end{cases} \tag{19}$$

The current satisfies the continuity equation

$$\nabla_\alpha j^\alpha = \frac{\partial j^\alpha}{\partial x^\alpha} + j^\gamma \Gamma^\alpha_{\gamma\alpha} = \frac{\partial \rho}{\partial \tau} + \nabla_\mu j^\mu = 0 \tag{20}$$

which relates the event density as a function of τ to the flow of the event 4-current into spacetime. Similarly, $T^{\alpha\beta}$ is conserved by virtue of (16) and (20), suggesting that the 4D Einstein equations be extended to

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \tag{21}$$

with 5D Ricci tensor $R_{\alpha\beta}$ and Ricci scalar R obtained from $g_{\alpha\beta}$. To approach the breaking of the 5D symmetry exhibited in (21), we first consider insights from the linearized weak field theory.

3. Linearized Field Equations for Weak Fields

Up to a certain stage, linearization of the Einstein equations for SHP requires no more than replacement of 4D indices λ, μ, ν with 5D indices α, β, γ in the standard derivation. Posing the local metric as a small perturbation of the flat metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad \longrightarrow \qquad \partial_\gamma g_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} \qquad \left(h_{\alpha\beta}\right)^2 \approx 0 \tag{22}$$

$$\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1, \sigma) \tag{23}$$

the Ricci tensor reduces to linear terms in $h_{\alpha\beta}$

$$R_{\alpha\beta} \approx \frac{1}{2}\left(\partial_\beta \partial_\gamma h^\gamma_\alpha + \partial_\alpha \partial_\gamma h^\gamma_\beta - \partial^\gamma \partial_\gamma h_{\alpha\beta} - \partial_\alpha \partial_\beta h\right) . \tag{24}$$

Invariance of the Ricci tensor under a translation $x'^\alpha = x^\alpha + \Lambda^\alpha(x, \tau)$ permits us to apply the 5D Lorenz gauge condition

$$\partial^\beta \bar{h}_{\alpha\beta} = \partial^\beta \left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h\right) = 0 \qquad \longrightarrow \qquad \partial^\beta h_{\alpha\beta} = \frac{1}{2}\partial_\alpha h \tag{25}$$

leading to the Einstein tensor in the form

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} R = -\frac{1}{2}\partial^\gamma \partial_\gamma \left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h\right) = -\frac{1}{2}\partial^\gamma \partial_\gamma \bar{h}_{\alpha\beta} \tag{26}$$

and providing the 5D wave equation

$$-\partial^\gamma \partial_\gamma \bar{h}_{\alpha\beta} = -\left(\partial^\mu \partial_\mu + \partial^5 \partial_5\right) \bar{h}_{\alpha\beta} = \frac{16\pi G}{c^4} T_{\alpha\beta} . \tag{27}$$

This equation has the Green's function [28]

$$G(x, \tau) = \frac{1}{2\pi}\delta(x^2)\delta(\tau) + \frac{c_5}{2\pi^2}\frac{\partial}{\partial x^2}\theta(-\eta_{55} g_{\alpha\beta} x^\alpha x^\beta)\frac{1}{\sqrt{-\eta_{55} g_{\alpha\beta} x^\alpha x^\beta}} \tag{28}$$

in which the first term is instantaneous in τ and dominates at long distance for many problems, leading to the generic approximate solution

$$\bar{h}_{\alpha\beta}(x, \tau) \approx \frac{4G}{c^4}\int d^3x' \frac{T_{\alpha\beta}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}', \tau\right)}{|\mathbf{x}-\mathbf{x}'|} = \frac{4G}{c^4}\dot{x}_\alpha \dot{x}_\beta \int d^3x' \frac{\rho\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}', \tau\right)}{|\mathbf{x}-\mathbf{x}'|} \tag{29}$$

for known mass density $\rho(x, \tau)$ and event velocity $\dot{x}_\alpha$ freely falling under (16). Choosing a spacetime event density $\rho(x - q(\tau))$ centered on a trajectory $q^\mu(\tau)$, and writing $\xi^\alpha = \dot{x}^\alpha / c$ the mass–energy–momentum tensor is

$$T^{\alpha\beta} = m\rho(x, \tau)\dot{q}^\alpha \dot{q}^\beta = mc^2 \xi^\alpha \xi^\beta \rho(x - q(\tau)) \tag{30}$$

producing the metric perturbation

$$\bar{h}_{\alpha\beta}(x, \tau) = \frac{4Gm}{c^2 r(\tau)}\xi_\alpha \xi_\beta \tag{31}$$

where

$$\frac{1}{r(\tau)} = \int d^3x' \frac{\rho\left(t - \frac{q^0(\tau)}{c} - \frac{|\mathbf{x}-\mathbf{x}'+\mathbf{q}(\tau)|}{c}, \mathbf{x}' - \mathbf{q}(\tau)\right)}{|\mathbf{x}-\mathbf{x}'+\mathbf{q}(\tau)|} \tag{32}$$

has units of inverse distance. In particular, taking $q(\tau) = (c\tau, \mathbf{0}, c_5\tau)$ and $\rho(x) = \delta^3(\mathbf{x})$, describing an event distributed around the t-axis in its rest frame, we have $\xi = (1, \mathbf{0}, c_5/c)$ and

$$\tilde{h}_{\alpha\beta}(x,\tau) = \frac{4Gm}{c^2|\mathbf{x}|}\xi_\alpha\xi_\beta = \frac{4Gm}{c^2 r}\left(\delta^0_\alpha + \xi^5\delta^5_\alpha\right)\left(\delta^0_\beta + \xi^5\delta^5_\beta\right) . \tag{33}$$

To obtain the perturbed metric $h_{\alpha\beta}$ from $\tilde{h}_{\alpha\beta}$ we rearrange

$$h_{\alpha\beta} = \tilde{h}_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}h \tag{34}$$

we find the trace h from

$$\tilde{h} = \eta^{\alpha\beta}\tilde{h}_{\alpha\beta} = \eta^{\alpha\beta}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right) = \left(1 - \frac{1}{2}\eta^{\alpha\beta}\eta_{\alpha\beta}\right)h . \tag{35}$$

However, since $\eta^{\alpha\beta}\eta_{\alpha\beta} = 5 \longrightarrow h = -(2/3)\,\tilde{h}$, we will be led to the solution

$$\begin{aligned} h_{00} &= \tilde{h}_{00} - \frac{1}{3}\eta_{00}\tilde{h} = \frac{2}{3}\frac{4Gm}{c^2 r} \qquad & h_{05} &= \tilde{h}_{05} - \frac{1}{3}\eta_{05}\tilde{h} = \frac{2}{3}\sigma\xi_5\frac{4Gm}{c^2 r} \\ h_{ij} &= \tilde{h}_{ij} - \frac{1}{3}\eta_{ij}\tilde{h} = \frac{1}{3}\delta_{ij}\frac{4Gm}{c^2 r} \qquad & h_{55} &= \tilde{h}_{55} - \frac{1}{3}\eta_{55}\tilde{h} = \frac{1}{3}\sigma\frac{4Gm}{c^2 r} \end{aligned} \tag{36}$$

where $i, j = 1, 2, 3$ and we have neglected terms in $\xi_5^2 \approx 0$. This metric structure, where $h_{00} = 2h_{ii}$ and $h_{00} = \pm 2h_{55}$, is not consistent with gravitational phenomenology. In particular we expect $|h_{55}| \ll |h_{00}|$.

To obtain a reasonable solution we must break the 5D symmetry in the relationship between the 5D Einstein tensor and the source term. Writing the linearized Einstein equations as

$$R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta} \tag{37}$$

we take the trace

$$R\left(1 - \frac{1}{2}\eta^{\alpha\beta}\eta_{\alpha\beta}\right) = \frac{8\pi G}{c^4}\left(\eta^{\mu\nu}T_{\mu\nu} + \eta^{55}T_{55}\right) \tag{38}$$

leading to the trace-reversed form

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left[T_{\alpha\beta} + \frac{\frac{1}{2}\eta_{\alpha\beta}}{1 - \frac{1}{2}\eta^{\alpha\beta}\eta_{\alpha\beta}}\left(\eta^{\mu\nu}T_{\mu\nu} + \eta^{55}T_{55}\right)\right] . \tag{39}$$

As in SHP electrodynamics, we treat $\eta_{55} = \sigma$ as a notational device rather than a feature of physical matter, and so we replace

$$\eta^{\alpha\beta} \to \bar{\eta}^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1, 0) \tag{40}$$

in the source terms on the RHS leading to the O(3,1)-covariant field equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}\right) \qquad\qquad R_{5\alpha} = \frac{8\pi G}{c^4}T_{5\alpha} \tag{41}$$

where $T = \eta^{\mu\nu}T_{\mu\nu}$. These modified field equations lead to the wave equations

$$-\,\partial^\gamma\partial_\gamma h_{\mu\nu} = \frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}\right) \qquad\qquad -\,\partial^\gamma\partial_\gamma h_{5\alpha} = \frac{16\pi G}{c^4}T_{5\alpha} \tag{42}$$

which for the perturbation (33) have the solution

$$\begin{aligned} h_{00} &= \tilde{h}_{00} - \frac{1}{2}\eta_{00}\tilde{h} = \frac{2Gm}{c^2 r} & \qquad h_{05} &= \tilde{h}_{05} = \sigma\mathcal{G}[T_{50}] = \sigma\xi_5 \frac{4Gm}{c^2 r} \\ h_{ij} &= \tilde{h}_{ij} - \frac{1}{2}\eta_{ij}\tilde{h} = \delta_{ij}\frac{2Gm}{c^2 r} & \qquad h_{55} &= \tilde{h}_{55} = \sigma\mathcal{G}[T_{55}] = \sigma\xi_5^2 \frac{4Gm}{c^2 r} \ . \end{aligned} \tag{43}$$

Writing

$$1 + \frac{2Gm}{c^2 r} \approx \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} \tag{44}$$

the spacetime part of the metric becomes

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2Gm}{c^2 r}\right), \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}\delta_{ij}\right) \tag{45}$$

while

$$g_{55} = \sigma\left(1 + \sigma\xi_5^2 \frac{2Gm}{c^2 r}\right) \approx \sigma \ . \tag{46}$$

As we saw in the equations of motion (16) for an event, this approach respects the 5D geometry of the fields, as expressed through the Ricci tensor, but breaks the 5D symmetry of the physics to 4+1 in setting the equality between $R_{\alpha\beta}$ and the source $T_{\alpha\beta}$.

4. Overview of 4+1 Formalism

The 3+1 formalism, including approaches such as ADM [24], decomposes the Einstein equations into an initial value problem for the metric and extrinsic curvature of 3-space, parameterized by the time coordinate t. This decomposition begins by choosing a time direction in the 4D spacetime $\mathcal{M}$, defining a foliation onto a collection of spacelike 3D hypersurfaces. By projecting spacetime structures onto this foliation, one finds a pair of first order differential equations for the t-evolution of space along with a pair of constraints that must be met by the initial conditions.

For the SHP approach to GR, the 3+1 formalism has been extended to 4+1 by choice of τ as the time direction, foliation of the pseudo-spacetime $\mathcal{M}_5$, and decomposition of the symmetry-broken Einstein equations (41) into an initial value problem. In this section we summarize the 4+1 formalism. A detailed presentation can be found in [10].

The pseudo-spacetime $\mathcal{M}_5$ introduced in Section 2 is defined by the injective mapping $\Phi : \mathcal{M} \longrightarrow \mathcal{M}_5 = \mathcal{M} \times R$ with the natural foliation to level surfaces of the scalar field $S(X) = X^5/c_5 = \tau$

$$\Sigma_\tau = \left\{ X \in \mathcal{M}_5 \,\middle|\, S(X) = \tau \right\} \ . \tag{47}$$

The normalized gradient of $S(X)$

$$n_\alpha = \sigma \frac{1}{\sqrt{|g^{55}|}}\,\partial_\alpha S(X) = \sigma \frac{1}{\sqrt{|g^{55}|}}\,\delta^5_\alpha \qquad\qquad g^{\alpha\beta} n_\alpha n_\beta = \sigma \tag{48}$$

is normal to Σ_τ because $S(X) =$ constant for $X \in \Sigma_\tau$. The vectors $\mathbf{g}_\mu$ with components

$$(\mathbf{g}_\mu)^\alpha = \partial_\mu \Phi^\alpha = \left(\frac{\partial X^\alpha}{\partial x^\mu}\right)_\tau = \delta^\alpha_\mu \tag{49}$$

form a coordinate frame for the tangent space $\mathcal{T}(\Sigma_\tau) \subset \mathcal{T}(\mathcal{M}_5)$ and a fifth basis vector for $\mathcal{T}(\mathcal{M}_5)$ may be chosen as the linear combination of n and $\{\mathbf{g}_\mu\}$ prescribed by

$$\mathbf{g}_5 = N^\mu \mathbf{g}_\mu + N n \tag{50}$$

often called the ADM parameterization. The 4-vector N^μ generalizes the shift 3-vector in 3+1 formalisms and N is the lapse function. Designating $\gamma_{\mu\nu} = g_{\mu\nu} = \mathbf{g}_\mu \cdot \mathbf{g}_\nu$ we find a generalization of the ADM metric decomposition through $g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta$

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} \qquad g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix} \tag{51}$$

which puts the unit normal into the form

$$\begin{aligned} n^\alpha &= \frac{1}{N}\left(-N^\mu \mathbf{g}_\mu + \mathbf{g}_5\right)^\alpha = \frac{1}{N}\left(-N^\mu \delta^\alpha_\mu + \delta^\alpha_5\right) \\ n_\alpha &= \sigma \frac{1}{\sqrt{|g^{55}|}} \delta^5_\alpha = \sigma N \delta^5_\alpha = \sigma N \left(\mathbf{g}^5\right)_\alpha \end{aligned} \tag{52}$$

where the second expression is implicit in parameterization (50) through $n_\alpha = g_{\alpha\beta} n^\beta$. The projection operator onto $\mathcal{T}(\Sigma_\tau)$ is

$$P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta \qquad P^{\alpha\beta} = g^{\alpha\beta} - \sigma n^\alpha n^\beta \qquad P_{\alpha\gamma} P^{\gamma\beta} = P^\beta_\alpha = \delta^\beta_\alpha - \sigma n_\alpha n^\beta \tag{53}$$

with completeness relations

$$g_{\alpha\beta} = P_{\alpha\beta} + \sigma n_\alpha n_\beta \qquad \delta^\alpha_\beta = P^\alpha_\beta + \sigma n^\alpha n_\beta \,. \tag{54}$$

In particular, the spacetime components are

$$\gamma_{\mu\nu} = g_{\alpha\beta} E^\alpha_\mu E^\beta_\nu = \left(P_{\alpha\beta} + \sigma n_\alpha n_\beta\right) E^\alpha_\mu E^\beta_\nu = P_{\alpha\beta} E^\alpha_\mu E^\beta_\nu = P_{\mu\nu} \tag{55}$$

showing that the projector $P_{\alpha\beta}$ when restricted to Σ_τ acts as the 4D metric $\gamma_{\mu\nu}$.

The 5D covariant derivative ∇_γ compatible with $g_{\alpha\beta}$ is associated with the Christoffel connection $\Gamma^\gamma_{\delta\beta}$ which appeared in the geodesic equations (16). The projected covariant derivative on $\mathcal{T}(\Sigma_\tau)$ compatible with $P_{\alpha\beta}$ (and hence $\gamma_{\mu\nu}$) is denoted D_α

$$(DX)_{\alpha\beta_1\cdots\beta_n} = P^{\alpha'}_\alpha P^{\beta'_1}_{\beta_1} \cdots P^{\beta'_n}_{\beta_n} \left(\nabla_{\alpha'} X_{\beta'_1\cdots\beta'_n}\right) . \tag{56}$$

These covariant derivatives lead to the curvature and projected curvature tensors

$$[\nabla_\beta, \nabla_\alpha] X_\gamma = X_\delta R^\delta_{\gamma\alpha\beta} \qquad [D_\beta, D_\alpha] X_a = X_\delta \bar{R}^\delta_{\gamma\alpha\beta} \tag{57}$$

along with the extrinsic curvature defined by

$$K_{\alpha\beta} = -P^\gamma_\alpha P^\delta_\beta \nabla_\delta n_\gamma = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N \,. \tag{58}$$

The spacetime part of the projected curvature $\bar{R}^\rho_{\lambda\mu\nu}$ is the 4D intrinsic curvature for $\mathcal{M}$ and $K_{\mu\nu}$ characterizes the evolution of the unit normal to $\mathcal{T}(\Sigma_\tau)$.

We may decompose the Riemann tensor into a sum of projections on $\mathcal{T}(\Sigma_\tau)$ and n_α, by using the completeness relation (54) to write

$$R^\gamma{}_{\delta\alpha\beta} = \left(P^{\alpha'}_\alpha + \sigma n_\alpha n^{\alpha'}\right)\left(P^{\beta'}_\beta + \sigma n_\beta n^{\beta'}\right)\left(P^\gamma_{\gamma'} + \sigma n^\gamma n_{\gamma'}\right)\left(P^{\delta'}_\delta + \sigma n_\delta n^{\delta'}\right) R^{\gamma'}{}_{\delta'\alpha'\beta'} \tag{59}$$

leading to terms of the type

$$\left(P^{\alpha'}{}_\alpha P^{\beta'}{}_\beta P^\gamma{}_{\gamma'} P^{\delta'}{}_\delta\right) R^{\gamma'}{}_{\delta'\alpha'\beta'} \qquad \left(P^\gamma_{\gamma'} P^{\alpha'}{}_\alpha P^{\beta'}{}_\beta\right) n^\delta \, R^{\gamma'}_{\delta\alpha'\beta'} \qquad \left(P_{\alpha\alpha'} P^{\beta'}_\beta\right) n^{\gamma'} n^\delta \, R^{\alpha'}_{\delta\beta'\gamma'} \tag{60}$$

because the antisymmetry of the Riemann tensor leads to $n^{\beta'} n^{\gamma'} n^{\delta}\, R^{\alpha'}_{\delta\beta'\gamma'} = 0$. Using (53) in the second of (57) provides the Gauss relation

$$R^{\mu}{}_{\nu\lambda\rho} = \bar{R}^{\mu}{}_{\nu\lambda\rho} - \sigma\left(K^{\mu}_{\lambda} K_{\rho\nu} - K^{\mu}_{\rho} K_{\lambda\nu}\right) . \tag{61}$$

This provides $R^{\mu}{}_{\nu\lambda\rho}$ (the spacetime components of the 5D intrinsic curvature) in terms of the 4D intrinsic curvature $\bar{R}^{\mu}{}_{\nu\lambda\rho}$ and the extrinsic curvature $K_{\rho\nu}$ (which collects the 5-components of $\Gamma^{\gamma}_{\delta\beta}$ not present in $\bar{R}^{\mu}{}_{\nu\lambda\rho}$). Replacing $X_\delta = n_\delta$ in the first of (57) and projecting the three remaining indices onto $\mathcal{T}(\Sigma_\tau)$ leads to the Codazzi relation

$$\left(P^{\alpha'}_{\alpha} P^{\beta'}_{\beta} P^{\gamma'}_{\gamma}\right) n_\delta\, R^{\delta}_{\gamma'\alpha'\beta'} = D_\beta K_{\alpha\gamma} - D_\alpha K_{\beta\gamma} \tag{62}$$

and similarly projecting onto $n^{\gamma'}$ and n^{δ} leads to

$$\left(P_{\alpha\alpha'} P^{\beta'}_{\beta}\right) n^{\gamma'} n^{\delta}\, R^{\alpha'}_{\delta\beta'\gamma'} = -K^{\gamma}_{\alpha} K_{\gamma\beta} - \sigma \frac{1}{N} D_\beta D_\alpha N + P^{\alpha'}_{\alpha} P^{\beta'}_{\beta} n^{\gamma'} \nabla_{\gamma'} K_{\alpha'\beta'} . \tag{63}$$

Equations (61)–(63) generalize the corresponding relations in the 3+1 formalism and play a central role in decomposing the Einstein equations into evolution equations.

To formulate an initial value problem we seek the τ-derivatives of the metric $\gamma_{\alpha\beta} = P_{\alpha\beta}$ and extrinsic curvature $K_{\alpha\beta}$. Introducing the normal evolution vector $m = Nn$ and writing (50) as

$$\mathbf{g}_5 = N^{\mu} \mathbf{g}_{\mu} + Nn = \mathbf{N} + m \tag{64}$$

we find the Lie derivative along m as

$$\mathcal{L}_m = \mathcal{L}_{\mathbf{g}_5} - \mathcal{L}_{\mathbf{N}} \tag{65}$$

and since $(\mathbf{g}_5)^{\gamma} = \delta^{\gamma}_5$, we obtain

$$\mathcal{L}_{\mathbf{g}_5} A_{\alpha\beta} = \delta^{\gamma}_5 \partial_\gamma A_{\alpha\beta} + A_{\gamma\beta} \partial_\alpha \delta^{\gamma}_5 + A_{\alpha\gamma} \partial_\beta \delta^{\gamma}_5 = \partial_5 A_{\alpha\beta} = \frac{1}{c_5} \partial_\tau A_{\alpha\beta} . \tag{66}$$

The Lie derivative of the metric $\gamma_{\alpha\beta}$ along m is

$$\mathcal{L}_m\, \gamma_{\alpha\beta} = m^{\gamma} \nabla_\gamma \gamma_{\alpha\beta} + \gamma_{\gamma\beta} \nabla_\alpha m^{\gamma} + \gamma_{\alpha\gamma} \nabla_\beta m^{\gamma} \tag{67}$$

which may be evaluated using (53) for $P_{\alpha\beta} = \gamma_{\alpha\beta}$ in the first term and using (58) to obtain

$$\mathcal{L}_m\, \gamma_{\alpha\beta} = -2NK_{\alpha\beta} \quad \longrightarrow \quad \frac{1}{c_5} \partial_\tau \gamma_{\mu\nu} = \mathcal{L}_{\mathbf{N}}\, \gamma_{\mu\nu} - 2NK_{\mu\nu} \tag{68}$$

as the evolution equation for the metric. The Lie derivative of $K_{\alpha\beta}$ is

$$\mathcal{L}_m\, K_{\alpha\beta} = m^{\gamma} \nabla_\gamma K_{\alpha\beta} + K_{\gamma\beta} \nabla_\alpha m^{\gamma} + K_{\alpha\gamma} \nabla_\beta m^{\gamma} . \tag{69}$$

Again using (58) to evaluate $\nabla_\alpha m^{\gamma}$ and recalling (63) results in

$$\frac{1}{N} \mathcal{L}_m\, K_{\alpha\beta} + \frac{1}{N} D_\alpha D_\beta N + K_{\alpha\gamma} K^{\gamma}{}_{\beta} = \left(P_{\alpha\alpha'}\, P^{\beta'}{}_{\beta}\right) n^{\gamma} n^{\delta}\, R^{\alpha'}{}_{\delta\beta'\gamma} \tag{70}$$

so that using the Gauss relation (61) we can put (70) into the form

$$P^{\alpha'}{}_{\alpha} P^{\beta'}{}_{\beta} R_{\ \alpha'\beta'} = \sigma \frac{1}{N} \mathcal{L}_m\, K_{\alpha\beta} + \sigma \frac{1}{N} D_\alpha D_\beta N + \bar{R}_{\alpha\beta} - \sigma K K_{\alpha\beta} + \sigma 2 K^{\delta}_{\alpha} K_{\beta\delta} . \tag{71}$$

In this expression only the Ricci tensor on the LHS refers to the 5D geometry of $\mathcal{M}_5$ and may be eliminated using the Einstein field equations. Recalling the trace-reversed form (39) we have in curved space

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left[T_{\alpha\beta} + \frac{\frac{1}{2}g_{\alpha\beta}}{1-\frac{1}{2}g^{\alpha\beta}g_{\alpha\beta}}\left(g^{\mu\nu}T_{\mu\nu} + g^{55}T_{55}\right)\right] \tag{72}$$

in which $g_{\alpha\beta}$ on the RHS must be replaced with a symmetry-broken form, just as we saw in the linearized theory. Breaking the symmetry for the local metric is best achieved in a vielbein formulation of GR, as discussed in a forthcoming paper. In the linearized theory, we replace $g_{\alpha\beta} \to \eta_{\alpha\beta} \to \bar{\eta}_{\alpha\beta}$ and will continue here by writing

$$R_{\alpha\beta} \approx \frac{8\pi G}{c^4}\left[T_{\alpha\beta} - \frac{1}{2}\bar{\eta}_{\alpha\beta}\bar{T}\right] \tag{73}$$

as an approximation, where again $\bar{T} = \eta^{\mu\nu}T_{\mu\nu}$. We decompose the source term by projecting

$$T_{\alpha\beta} = T_{\alpha'\beta'}\left(P_\alpha^{\alpha'} + \sigma n^{\alpha'}n_\alpha\right)\left(P_\beta^{\beta'} + \sigma n^{\beta'}n_\beta\right) = S_{\alpha\beta} + 2\sigma n_\alpha p_\beta + n_\alpha n_\beta \kappa \tag{74}$$

where

$$S_{\alpha\beta} = P_\alpha^{\alpha'}P_\beta^{\beta'}T_{\alpha'\beta'} \qquad p_\beta = -n^{\alpha'}P_\beta^{\beta'}T_{\alpha'\beta'} \qquad \kappa = n^\alpha n^\beta T_{\alpha\beta} \tag{75}$$

so that $S_{\mu\nu}$ corresponds to the 4D energy–momentum tensor $T_{\mu\nu}$, p_μ corresponds to the mass current into the μ direction $T_{5\mu}$, and κ corresponds to the scalar mass density T_{55}. It is useful in this context to regard mass as a quantity expressing the dynamical independence of energy and momentum, providing a variable relation between them. In this notation, $\bar{T} = \eta^{\mu\nu}T_{\mu\nu} = \eta^{\mu\nu}S_{\mu\nu} = S$. Finally, the projected Ricci tensor becomes

$$P^{\alpha'}_{\ \alpha}P^{\beta'}_{\ \beta}R_{\ \alpha'\beta'} \approx P^{\alpha'}_{\ \alpha}P^{\beta'}_{\ \beta}\frac{8\pi G}{c^4}\left[T_{\alpha'\beta'} - \frac{1}{2}\bar{\eta}_{\alpha'\beta'}\bar{T}\right] = \frac{8\pi G}{c^4}\left[S_{\alpha\beta} - \frac{1}{2}\bar{\eta}_{\alpha\beta}S\right] \tag{76}$$

providing an evolution equation for the extrinsic curvature

$$\begin{aligned}\left(\frac{1}{c_5}\mathcal{L}_\tau - \mathcal{L}_{\mathbf{N}}\right)K_{\mu\nu} &= -D_\mu D_\nu N \\ &+N\left[-\sigma\bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_\mu^\lambda K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right)\right].\end{aligned} \tag{77}$$

The double projection of the unbroken 5D field equation onto the time direction n is

$$\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)n^\alpha n^\beta = \frac{8\pi G}{c^4}T_{\alpha\beta}n^\alpha n^\beta \qquad \longrightarrow \qquad R_{\alpha\beta}n^\alpha n^\beta - \frac{1}{2}\sigma R = \frac{8\pi G}{c^4}\kappa \tag{78}$$

which using the Gauss relation (61) becomes

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma\frac{8\pi G}{c^4}\kappa\,. \tag{79}$$

This expression, called the Hamiltonian constraint, applies to the mass density of the gravitational field, not the energy density as in 4D GR. The mixed projection with $P^{\beta'}_{\ \beta}n^\alpha$

$$n^\alpha P^{\beta'}_{\ \beta}\left(R_{\alpha\beta'} - \frac{1}{2}g_{\alpha\beta'}R\right) = n^\alpha P^{\beta'}_{\ \beta}\frac{8\pi G}{c^4}T_{\alpha\beta'} \longrightarrow P^{\beta'}_{\ \beta}n^\alpha R_{\alpha\beta'} - \frac{1}{2}g_{\alpha\beta'}n^\alpha P^{\beta'}_{\ \beta}R = -\frac{8\pi G}{c^4}p_\beta \tag{80}$$

is combined with the Codazzi relation (62) and $g_{\alpha\beta'} n^{\alpha} P^{\beta'}_{\ \beta} = n^{\alpha} P_{\alpha\beta} = 0$ to obtain

$$D_{\mu} K^{\mu}_{\nu} - D_{\nu} K = \frac{8\pi G}{c^4} p_{\nu} \tag{81}$$

which is called the momentum constraint, referring to the flow of mass into the field. Together, the evolution Equations (68) and (77) and constraints (79) and (81) are the 4+1 decomposition of the SHP Einstein equations. Notice however that the Hamiltonian constraint does not reflect the breaking of 5D symmetry to O(3,1) and will be corrected in Section 5.

The evolution equations and constraints contain only objects defined on Σ_{τ}. Unlike the evolution equations, the constraints contain no τ-derivatives. If they are satisfied by the initial conditions, they will be satisfied for all τ. The constraining relationship is said to propagate, rather than evolving under second order differential equations.

5. The 4+1 Decomposition for the Linearized Theory

Under 4+1 decomposition, the metric is written

$$\|g_{\alpha\beta}\| = \begin{bmatrix} g_{\mu\nu} & g_{\mu 5} \\ g_{\mu 5} & g_{55} \end{bmatrix} = \begin{bmatrix} \eta_{\mu\nu} + h_{\mu\nu} & h_{\mu 5} \\ h_{\mu 5} & \eta_{55} + h_{55} \end{bmatrix} \tag{82}$$

allowing us to identify

$$\|g_{\alpha\beta}\| = \begin{bmatrix} \gamma_{\mu\nu} & N_{\mu} \\ N_{\mu} & \sigma N^2 + \gamma_{\mu\nu} N^{\mu} N^{\nu} \end{bmatrix} = \begin{bmatrix} \eta_{\mu\nu} + h_{\mu\nu} & h_{\mu 5} \\ h_{\mu 5} & \eta_{55} + h_{55} \end{bmatrix} \tag{83}$$

from which

$$\begin{gathered} \sigma N^2 + \gamma_{\mu\nu} N^{\mu} N^{\nu} = \sigma N^2 + \left(\eta_{\mu\nu} + h_{\mu\nu}\right) h_5^{\mu} h_5^{\nu} \approx \sigma N^2 \longrightarrow \sigma N^2 = \sigma + h_{55} \\ N = \sqrt{1 + \sigma h_{55}} \approx 1 + \frac{1}{2} \sigma h_{55} \, . \end{gathered} \tag{84}$$

Then

$$\left\|g^{\alpha\beta}\right\| = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \dfrac{1}{N^2} N^{\mu} N^{\nu} & -\sigma \dfrac{1}{N^2} N^{\mu} \\ -\sigma \dfrac{1}{N^2} N^{\mu} & \sigma \dfrac{1}{N^2} \end{bmatrix} \approx \begin{bmatrix} \eta^{\lambda\nu} - h^{\mu\nu} & -\sigma h_5^{\mu} \\ -\sigma h_5^{\mu} & \sigma(1 - \sigma h_{55}) \end{bmatrix} \tag{85}$$

and the unit normal is

$$n_{\alpha} = \sigma N \delta^5_{\alpha} = \sigma \sqrt{1 + \sigma h_{55}} \delta^5_{\alpha} = \sigma \left(1 + \frac{1}{2} \sigma h_{55}\right) \delta^5_{\alpha} \tag{86}$$

$$n^{\alpha} = -h_5^{\mu} \delta^{\alpha}_{\mu} + \left(1 - \frac{1}{2} \sigma h_{55}\right) \delta^{\alpha}_5 \, . \tag{87}$$

Discarding terms of the order $\left(h_{\alpha\beta}\right)^2 \approx 0$, the Lie derivative of the metric reduces to

$$\mathcal{L}_{\mathbf{N}} \gamma_{\mu\nu} = D_{\mu} N_{\nu} + D_{\nu} N_{\mu} \approx \partial_{\mu} N_{\nu} + \partial_{\nu} N_{\mu} = \partial_{\mu} h_{5\nu} + \partial_{\nu} h_{5\mu} \tag{88}$$

and we may neglect the Lie derivative $\mathcal{L}_{\mathbf{N}} K_{\mu\nu}$

$$\mathcal{L}_{\mathbf{N}} K_{\mu\nu} = N^{\lambda} \partial_{\lambda} K_{\mu\nu} + K_{\lambda\nu} \partial_{\mu} N^{\lambda} + K_{\mu\lambda} \partial_{\nu} N^{\lambda} \propto \left(h_{\alpha\beta}\right)^2 \approx 0 \tag{89}$$

along with terms quadratic in $K_{\mu\nu}$. Writing

$$-2NK_{\mu\nu} \approx -2\left(1+\frac{1}{2}\sigma h_{55}\right)K_{\mu\nu} = -2K_{\mu\nu} \tag{90}$$

the evolution Equation (68) for $\gamma_{\mu\nu}$ becomes

$$\frac{1}{c_5}\partial_\tau \gamma_{\mu\nu} = \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} - 2K_{\mu\nu}\ . \tag{91}$$

and the evolution Equation (77) for $K_{\mu\nu}$ reduces to

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\frac{1}{2}\sigma\partial_\mu\partial_\nu h_{55} - \sigma\bar{R}_{\mu\nu} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right) \tag{92}$$

where we used

$$D_\mu D_\nu N = \partial_\mu(\partial_\nu N) - \Gamma^\lambda_{\mu\nu}\partial_\lambda N \approx \partial_\mu\partial_\nu N = \frac{1}{2}\sigma\partial_\mu\partial_\nu h_{55}\ . \tag{93}$$

The coupled Equations (91) and (92) provide the initial value problem for the metric and extrinsic curvature on $\mathcal{M}$ in the linearized theory, given initial conditions for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ that satisfy the constraints.

The relationship between $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ can be clarified somewhat in the linearized theory. Using (58), (86) and (87) for $K_{\alpha\beta}$, n_α, and n^α we evaluate

$$K_{\alpha\beta} = -\left(\delta^{\alpha'}_\alpha - \sigma n^{\alpha'} n_\alpha\right)\left(\delta^{\beta'}_\beta - \sigma n^{\beta'} n_\beta\right)\left(\frac{1}{2}\partial_{\beta'} h_{55}\delta^5_{\alpha'} - \sigma\Gamma^5_{\beta'\alpha'}\right) \quad\longrightarrow\quad K_{\mu\nu} = \sigma\Gamma^5_{\mu\nu} \tag{94}$$

showing explicitly that the extrinsic curvature contains 5-components of the 5D Christoffel symbol not present in the 4D intrinsic curvature. Similarly, rewriting (91) as

$$\frac{1}{2}\left(\partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} - \partial_5\gamma_{\mu\nu}\right) = K_{\mu\nu} \tag{95}$$

we again recognize the LHS as $\eta_{55}\Gamma^5_{\mu\nu}$. Thus, $K_{\mu\nu}$ replaces the "velocity" $\partial_5\gamma_{\mu\nu}$ with a "momentum" extracted from $\Gamma^5_{\mu\nu}$ components that do not explicitly appear in the initial value problem, converting the second order wave equation to a pair of first order equations. In [10] we showed that in the 4+1 canonical ADM formalism, the momentum precisely conjugate to $\gamma_{\mu\nu}$ is

$$\pi^{\mu\nu} = -\sigma\sqrt{\gamma}(K^{\mu\nu} - \gamma^{\mu\nu}K)\ . \tag{96}$$

The linearized evolution equation for $K_{\mu\nu}$ and the constraints can be understood by splitting the Einstein equations into spacetime and 5-parts. The Bianchi identity for the symmetry-broken linearized Einstein tensor is

$$\nabla_\alpha G^{\alpha\beta} = \nabla_\alpha\left(R^{\alpha\beta} - \frac{1}{2}\bar{\eta}^{\alpha\beta}R\right) = \partial_\alpha\left(R^{\alpha\beta} - \frac{1}{2}\bar{\eta}^{\alpha\beta}R\right) + o\left(h^2_{\alpha\beta}\right) = 0 \tag{97}$$

which we rewrite as

$$\frac{1}{c_5}\partial_\tau G^{5\beta} = -\partial_\mu G^{\mu\beta} + o\left(h^2_{\alpha\beta}\right)\ . \tag{98}$$

The RHS must contain $g_{\alpha\beta}$, $\partial_\tau g_{\alpha\beta}$, and $\partial^2_\tau g_{\alpha\beta}$, and so the components of $G^{5\beta}$ on the LHS may be at most first order τ-derivatives. Therefore, the five field equations

$$G_{5\beta} = R_{5\beta} - \frac{1}{2}\bar{\eta}_{5\beta}R = R_{5\beta} = \frac{8\pi G}{c^4}T_{5\beta} \tag{99}$$

define constraints among the initial conditions for the second order field equations: The metric, its first-order τ-derivative, and the source current $j_\beta = T_{5\beta}$. The ten field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad \longrightarrow \quad R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}\right) \tag{100}$$

are unconstrained and contain second τ-derivatives of the metric. From the linearized 5D Ricci tensor (24) with no gauge fixing, we split the spacetime part as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \partial_5\frac{1}{2}\sigma\left(\partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - \partial_5 h_{\mu\nu}\right) - \frac{1}{2}\partial_\mu\partial_\nu\sigma h_{55} \tag{101}$$

where

$$\bar{R}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial^\lambda h_{\lambda\nu} + \partial_\nu\partial^\sigma h_{\mu\sigma} - \partial^\lambda\partial_\lambda h_{\mu\nu} - \partial_\mu\partial_\nu\eta^{\lambda\sigma}h_{\lambda\sigma}\right) \tag{102}$$

contains the terms belonging to the 4D Ricci tensor on $\mathcal{M}$, and the last term in (101) is from the 5-piece of $\eta^{\alpha\beta}h_{\alpha\beta}$. Comparing with (95) we recognize the extrinsic curvature $K_{\mu\nu}$ in the second term of (101) and using (100) to replace $R_{\mu\nu}$ we arrive at

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = \frac{1}{2}\partial_\mu\partial_\nu\eta^{55}h_{55} - \sigma\bar{R}_{\mu\nu} + \sigma\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}\right) . \tag{103}$$

which differs from the evolution equation (92) by the sign of the first term on the RHS. Equation (103) may be approached by seeking a gauge condition that resolves this sign reversal, but this similarly shows that consistency of the linearized 4+1 formalism requires the h_{55} term in (103) to vanish.

Applying the 5D Lorenz gauge condition (25), expanded as

$$\partial^\lambda h_{\alpha\lambda} = \frac{1}{2}\partial_\alpha\bar{\eta}^{\lambda\sigma}h_{\lambda\sigma} + \frac{1}{2}\partial_\alpha\eta^{55}h_{55} - \partial^5 h_{\alpha 5} \tag{104}$$

to the 5-component of (27) we obtain

$$R_{5\beta} = \frac{1}{2}\left(-\partial_5\partial^5 h_{\beta 5} - \partial^\lambda\partial_\lambda h_{5\beta}\right) = -\frac{1}{2}\partial^\gamma\partial_\gamma h_{5\beta} . \tag{105}$$

Comparing with (99) for $\beta = \mu, 5$ this provides

$$-\partial^\gamma\partial_\gamma h_{5\mu} = \frac{16\pi G}{c^4}T_{5\mu} = -\frac{16\pi G}{c^4}p_\mu \qquad -\partial^\gamma\partial_\gamma h_{55} = \frac{16\pi G}{c^4}T_{55} = \frac{16\pi G}{c^4}\kappa \tag{106}$$

and as expected, the RHS of these expressions match the RHS of constraints (79) and (81). Again using (95) to evaluate

$$\partial_\nu K = -\frac{1}{2}\partial_\nu\partial^5 h_{55} \qquad \partial_\mu K^\mu_\nu = \frac{1}{2}\left(-\partial_5\partial^\mu h_{\mu\nu} + \partial^\mu\partial_\mu h_{5\nu} + \frac{1}{2}\partial_\nu\partial_5 h - \partial_\nu\partial^5 h_{55}\right) \tag{107}$$

we find

$$\partial_\mu K^\mu_\nu - \partial_\nu K = \frac{1}{2}\left(\partial_5\partial^5 h_{\nu 5} + \partial^\mu\partial_\mu h_{5\nu}\right) = \partial^\gamma\partial_\gamma h_{5\nu} . \tag{108}$$

which combined with the first of (106), provides the momentum constraint (81).

In the linearized theory we discard the terms $K^2 - K^{\mu\nu}K_{\mu\nu}$ in the Hamiltonian constraint (79) and must evaluate $\bar{R}$. Recalling the definition of $\bar{R}^\delta_{\gamma\alpha\beta}$ in the second of (57), we insert the projectors

$$\left[D_\beta, D_\alpha\right]X_\gamma = P^{\beta'}_\beta P^{\alpha'}_\alpha P^{\gamma'}_\gamma\left[\nabla_{\beta'}, \nabla_{\alpha'}\right]X_{\gamma'} \qquad \longrightarrow \qquad \bar{R}^\delta_{\gamma\alpha\beta} = P^{\beta'}_\beta P^{\alpha'}_\alpha P^{\gamma'}_\gamma R^\delta_{\gamma'\alpha'\beta'} \tag{109}$$

and using the idempotent $P_{\alpha\alpha'}$ as the metric on the projected hypersurface find

$$\bar{R} = P^{\gamma\beta} P^{\alpha}_{\delta} R^{\delta}_{\gamma\alpha\beta} \approx P^{\gamma\beta} P^{\alpha}_{\delta} \left(\partial_\alpha \Gamma^{\delta}_{\gamma\beta} - \partial_\beta \Gamma^{\delta}_{\gamma\alpha} \right) \tag{110}$$

for the projected Ricci scalar, neglecting terms of $o\left(h^2_{\alpha\beta}\right)$. Using the (86) and (87) in the projectors, we are eventually led to

$$\bar{R} = \frac{1}{2}\left(-\partial_\gamma \partial^\gamma h + 2\partial_\gamma \partial^\gamma h^5_5 \right) = -\frac{1}{2} \partial_\gamma \partial^\gamma \left(\bar{h} - h^5_5 \right) . \tag{111}$$

Returning now to the wave equations, in the notation of the 4+1 decompostion, the spacetime field Equation (42) is

$$-\partial^\gamma \partial_\gamma h_{\mu\nu} = \frac{16\pi G}{c^4} \left(S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S \right) \qquad \longrightarrow \qquad -\partial^\gamma \partial_\gamma \bar{h} = -\frac{16\pi G}{c^4} S \tag{112}$$

where $\bar{h} = \eta^{\mu\nu} h_{\mu\nu}$. Combining the trace with the second of (106) we obtain

$$-\frac{1}{2} \partial_\gamma \partial^\gamma \left(\bar{h} - h^5_5 \right) = -\frac{1}{2} \partial_\gamma \partial^\gamma \left(\bar{h} - \sigma h_{55} \right) = -\frac{8\pi G}{c^4} (S + \sigma\kappa) \tag{113}$$

leading us to

$$\bar{R} = -\frac{8\pi G}{c^4} (S + \sigma\kappa) . \tag{114}$$

which modifies the Hamiltonian constraint (79) found for unbroken 5D symmetry.

We have seen that breaking the 5D symmetry of the Einstein equations on $\mathcal{M}_5$ to O(3,1) produces two modifications in the 4+1 formalism. First, in the evolution equation for $K_{\mu\nu}$ we see that $S + \sigma\kappa \longrightarrow S$, stating that the energy density S, but not the mass density κ, acts as a source for evolution of the extrinsic curvature. Second, in the Hamiltonian constraint we see that $\sigma\kappa \longrightarrow S + \sigma\kappa$, so that the energy density and mass density contribute independently to the Ricci scalar.

We recall from (19) and (75) that

$$\kappa = n^\alpha n^\beta T_{\alpha\beta} = T_{55} = c_5^2 \rho(x, \tau) \tag{115}$$

representing the mass density in spacetime [27]. From conservation of the mass–energy–momentum tensor and (75)

$$\nabla_\alpha T^{\alpha\beta} = \frac{1}{c_5} \partial_\tau T^{\alpha 5} + \nabla_\mu T^{\alpha\mu} = 0 \qquad \longrightarrow \qquad \partial_\tau T^{55} = -\nabla_\mu T^{5\mu} = \nabla_\mu p^\mu \tag{116}$$

relating the flow of energy–momentum into spacetime to the time variation of the local mass density $\rho(x, \tau)$. Because of the factor c_5^2 in (115) the second of (106) shows that only a very large scalar mass density will contribute to the perturbed metric. Nevertheless, from the definition (18) of the event current, $\rho(x, \tau)$ and hence κ are non-vanishing.

The significance of κ may also be considered by modifying the second of (100) as

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left[S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} (S + \sigma\kappa) \right] \tag{117}$$

which is the 5D symmetric form presented in [10]. Again using (75) we see that the modification introduces the full 5D trace $\eta^{\alpha\beta} T_{\alpha\beta} = S + \sigma\kappa$. Writing

$$\Lambda(x, \tau) = \sigma \frac{4\pi G}{c^4} \kappa \tag{118}$$

permits us to rearrange (117) as

$$R_{\mu\nu} + \Lambda\eta_{\mu\nu} = \frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right) \tag{119}$$

where the mass density plays the familiar role of a scalar (but local) cosmological term, not derived from the energy–momentum tensor $S_{\mu\nu}$. Once again, the factor of c_5^2 will generally result in a small Λ, that depends on the motions of the events contributing to the mass density.

6. Summary

In the canonical SHP formalism, the block universe $\mathcal{M}(\tau)$ consists of spacetime events $x^\mu(\tau)$ that occur at a universal time τ. The τ-evolution of these events is generated by a scalar Hamiltonian. This structure describes an evolving spacetime formulated as an initial value problem in a natural way. Just as Stueckelberg characterized particle trajectories as defined by τ-evolving events, the 4+1 formalism constructs the structure of spacetime by integrating a coupled pair of first order differential equations in $\gamma_{\mu\nu}$, $K_{\mu\nu}$, and $T_{\alpha\beta}$, specified at some τ, and tested for consistency by constraint equations. As in SHP electrodynamics, one finds field equations by writing a familiar 5D theory whose symmetry is restricted to tensor and scalar representations of O(3,1). One then finds the of 4+1 equations of motion by foliation of 5D pseudo-spacetime, so that geometrical structures and the field equations can be projected onto the resulting 4D hypersurfaces.

This paper clarifies the breaking of the 5D symmetry in the field equations, at the interface between the Ricci tensor (geometry) and the matter/energy source represented by $T_{\alpha\beta}$. We also derive the 4+1 equations of motion, for the linearized field equations, by decomposing the 5D Ricci tensor into spacetime components $R_{\mu\nu}$ providing 10 unconstrained equations for the metric, and $R_{5\mu}$ leading to 5 constraints on the initial conditions. The Bianchi identity establishes the significance of this separation.

This 4+1 formalism has the advantage of employing the external time τ as evolution parameter, manifestly preserving the 4D spacetime symmetries at each step. Spacetime geometries are obtained from specific spacetime event trajectories, with possible coordinate time $x^0(\tau)$ reversal, generally thought of as closed timelike curves. In a forthcoming paper, the 4+1 formalism will be derived using a *quintrad frame* defining a vielbein field that permits a more direct decomposition to an initial value problem for the 4D metric extrinsic curvature. Examples given in [10] include the absence of τ-evolution for a standard Schwarzschild metric, and the perturbation associated with a τ-varying mass. Future directions include solution of the weak field derived from a single arbitrarily moving event with varying mass, and calculation in the full nonlinear framework of more complex scenarios, such as black hole collisions, in which the use of x^0 as both an evolving solution and an evolution parameter may produce considerable difficulties.

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